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A NEW GENERALIZATION OF THE ERDŐS—KO—RADO THEOREM

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Dedicated to Paul Erdős on his seventieth birthday

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Let $\mathcal{F}$ be a family of k-subsets of an n-set. Let s be a fixed integer satisfying $k \leqq s \leqq 3k$. Suppose that for $F_1, F_2, F_3 \in \mathcal{F}$ $|F_1 \cup F_2 \cup F_3| \leqq s$ implies $F_1 \cap F_2 \cap F_3 \neq \emptyset$. Katona asked what is the maximum cardinality, $f(n, k, s)$ of such a system. The Erdős—Ko—Rado theorem implies $f(n, k, s) = \binom{n-1}{k-1}$ for $s=3k$ and $n \geqq 2k$. In this paper we show that $f(n, k, s) = \binom{n-1}{k-1}$ holds for $n > n_0(k)$ if and only if $s \geqq 2k$.

Equality holds only if every member of $\mathcal{F}$ contains a fixed element of the underlying set.

Further we solve the problem for $k=3, s=5, n \geqq 3000$. This result sharpens a theorem of Bollobás.

1. Introduction

The simplest version of the Erdős—Ko—Rado theorem is the following

Theorem 1. [4] *Let $\mathcal{F}$ be a collection of k-element subsets of an n-set X. Suppose $F \cap F' \neq \emptyset$ for $F, F' \in \mathcal{F}$. Then for $n > 2k$*

$$|\mathcal{F}| \leqq \binom{n-1}{k-1}, \tag{1}$$

and equality holds iff for some $x \in X$ we have

$$\mathcal{F} = \{F \subset X \,|\, |F| = k, \quad x \in F\}. \tag{2}$$

In Frankl [5] the following is proven

Theorem 2. *Let $\mathcal{F}$ be a collection of k-element subsets of an n-set X, and let $t \geqq 2$. Suppose that, for every $F_1, F_2, ..., F_t \in \mathcal{F}$, $F_1 \cap ... \cap F_t \neq \emptyset$ holds. Then for $n > (t/t-1)k$ (1) holds. Equality is possible only for $\mathcal{F}$ satisfying (2).*

Katona raised the following problem, concerning the case $t=3$ of Theorem 2. What happens if, for some integer s, we require $F_1 \cap F_2 \cap F_3 \neq \emptyset$ only for triples satisfying $|F_1 \cup F_2 \cup F_3| \leqq s$? For which values of s does the condition entail (1)? In this paper we investigate this problem for $n \geqq n_0(k)$, and show that (1) holds whenever $s \geqq 2k$.

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2. Results

Theorem 3. *Let $\mathcal{F}$ be a collection of k-element subsets of the n-set X. Suppose that for any $F_1, F_2, F_3 \in \mathcal{F}$, satisfying $|F_1 \cup F_2 \cup F_3| \leqq 2k$ $F_1 \cap F_2 \cap F_3 \neq \emptyset$ holds. Then there is a number $n_0(k)$ such that, for $n > n_0(k)$*

$$|\mathcal{F}| \leqq \binom{n-1}{k-1},$$

and equality holds only if $\mathcal{F}$ is a family consisting of all the k-subsets containing a fixed element. Moreover $n_0(3)=5$, $n_0(k) \leqq k^2+3k$.

It is somewhat surprising that the extremal family is unchanged in the range $2k \leqq s \leqq 3k$.

However for $s<2k$ the situation is completely different, as it is shown by the following construction.

Let us consider a partition of X into k sets $X_1, \dots, X_k$ with $\left[\frac{n}{k}\right] \leqq |X_i| \leqq \left[\frac{n}{k}\right]+1$. Let us define

$$\mathcal{G} = \{G \subset X \,\big|\, |G \cap X_i| = 1 \quad \text{for} \quad 1 \leqq i \leqq k\}. \tag{3}$$

Suppose now $G_1 \cap G_2 \cap G_3 = \emptyset$ for some $G_1, G_2, G_3 \in \mathcal{G}$. Then obviously for every $1 \leqq i \leqq k$ we have

$$|(G_1 \cup G_2 \cup G_3) \cap X_i| \geqq 2.$$

From this we immediately obtain $|G_1 \cup G_2 \cup G_3| \geqq 2k$, in other words $|G_1 \cup G_2 \cup G_3| \leqq 2k-1$ implies $G_1 \cap G_2 \cap G_3 \neq \emptyset$, i.e. $\mathcal{G}$ satisfies the condition of Katona. $|\mathcal{G}| \geqq \left[\frac{n}{k}\right]^k$ which is of greater order of magnitude than $\binom{n-1}{k-1}$.

Conjecture. Let $\mathcal{F}$ be a family of k-subsets of $X, |X|=n$. Suppose $F_1, F_2, F_3 \in \mathcal{F}$ $|F_1 \cup F_2 \cup F_3| \leqq 2k-1$ implies $F_1 \cap F_2 \cap F_3 \neq \emptyset$. Then for $n \geqq n_0(k)$ and $\mathcal{G}$ defined above

$$|\mathcal{F}| \leqq |\mathcal{G}|,$$

with equality iff $\mathcal{F} = \mathcal{G}$.

Theorem 4. *If $k=3$ and $n \geqq 3000$ then $f(n, 3, 5) = \left[\frac{n}{3}\right]\left[\frac{n+1}{3}\right]\left[\frac{n+2}{3}\right]$.*

This result is a sharpening of the following theorem.

Theorem 5. (Bollobás [1]) *Let $\mathcal{F}$ be a family of 3-subsets of $X, |X|=n$. Suppose that for $F_1, F_2, F_3 \in \mathcal{F}$ we have $F_1 \triangle F_2 \not\subset F_3$ ($\triangle$ denotes the symmetric difference). Then $|\mathcal{F}| \leqq |\mathcal{G}|$ with equality holding only if $\mathcal{F}$ is isomorphic to $\mathcal{G}$.*

Thus Bollobás excludes the configuration when F_1, F_2, F_3 are three different 3-subset of a 4-set, while Theorem 4 permits it. However Bollobás's result holds for every n while we assume $n \geqq 3000$, and our theorem is definitely not true for $n \leqq 10$.

As for $k=3, s \leqq 4$, trivially $f(n, 3, s) = \binom{n}{3}$ holds, therefore Katona's problem is solved for $k=3$, except when $s=5$, $n<3000$.

3. The proof of Theorem 3

When $k=2$, $\mathcal{F}$ is a simple graph containing no triangles or path of length 3, so it is the union of vertex disjoint stars, thus Theorem 3 is true. From now on assume that $k \geqq 3$.

We proceed in a similar way as in Frankl [6]. First we prove that (1) holds asymptotically. Let $m(n, k, 1)$ denote the maximum number of k-subsets of an n-set, such that no two intersect in a singleton.

Then we have:

Lemma 1. *If $\mathcal{F}$ satisfies the conditions of Theorem 3, then*

$$|\mathcal{F}| \leqq \binom{n}{k-1} + m(n, k, 1). \tag{4}$$

Proof. Let $\mathcal{F}_0$ be the family of those subsets of $\mathcal{F}$ which contain a $(k-1)$-subset not contained in any other member of $\mathcal{F}$, i.e. $\mathcal{F}_0 = \{F \in \mathcal{F} \mid \exists G \subset F,\ |G| = k-1,\ G \subset F' \in \mathcal{F}$ implies $F' = F\}$, and define $\mathcal{F}_1 = \mathcal{F} - \mathcal{F}_0$.

Clearly $|\mathcal{F}_0| \leqq \binom{n}{k-1}$. Hence it suffices to prove $|\mathcal{F}_1| \leqq m(n, k, 1)$. Suppose the contrary, then we can find $F_1, F_2 \in \mathcal{F}_1$ such that $|F_1 \cap F_2| = 1$. Let $F_1 \cap F_2 = \{x\}$. As $F_1 \notin \mathcal{F}_0$ there is an $F_3 \in \mathcal{F}$, $F_1 \neq F_3$ such that $(F_1 - \{x\}) \subset F_3$. But in this case $F_1 \cap F_2 \cap F_3 = \emptyset$ and $|F_1 \cup F_2 \cup F_3| \leqq |F_1 \cup F_2| + 1 = 2k$, a contradiction. ∎

The problem of determining $m(n, k, 1)$ was raised by Erdős and Sós (see [2]), who determined $m(n, 3, 1)$, in particular they proved $m(n, 3, 1) \leqq n$, and conjectured $m(n, k, 1) = \binom{n-2}{k-2}$ for $n \geqq 2k$. This was proved by Frankl [7] for $n > n_0(k)$. Since $\binom{n}{k-1} = \binom{n-1}{k-1} + \binom{n-1}{k-2}$, Lemma 1 yields

Corollary 1. *If $\mathcal{F}$ satisfies the conditions of Theorem 3, then for $n > n_0(k)$*

$$|\mathcal{F}| \leqq \binom{n-1}{k-1} + \binom{n-1}{k-2} + \binom{n-2}{k-2} < (1 + 3k/n)\binom{n-1}{k-1}. \tag{5}$$

In the proof of Lemma 1 we used only:

Proposition 0. *If $F_1, F_2 \in \mathcal{F}$ and $F_1 \cap F_2 = \{x\}$ then there are no sets F_1' or F_2' in $\mathcal{F}$ satisfying $(F_1 - \{x\}) \subset F_1'$ or $(F_2 - \{x\}) \subset F_2'$.* ∎

For $x \in X$ let $\mathcal{D}(x)$ denote the family of sets $F \in \mathcal{F}$ with $x \in F$, and $\mathcal{D}_0(x)$ the family of sets $F \in \mathcal{F}$ with $x \in F$ such that $F - \{x\}$ is not contained in any other $F' \in \mathcal{F}$. Let further $|\mathcal{D}(x)| = d(x)$, $|\mathcal{D}_0(x)| = d_0(x)$. Clearly we have

$$\sum_{x \in X} d(x) = k|\mathcal{F}|, \tag{6}$$

$$\sum_{x \in X} d_0(x) \leqq \binom{n}{k-1}. \tag{7}$$

In view of Proposition 0 we have

Proposition 1. *Suppose* $F_1, F_2 \in \mathcal{F}$ *and* $F_1 \cap F_2 = \{x\}$. *Then* $F_1, F_2 \in \mathcal{D}_0(x)$. ∎

Let us set $\mathcal{A}(x) = \{F - \{x\} \mid F \in (\mathcal{D}(x) - \mathcal{D}_0(x))\} = \{F - \{x\} \mid x \in F \in \mathcal{F}$ and $\exists F' \in \mathcal{F}$ with $(F - \{x\}) \subset F'\}$. Then Proposition 1 yields:

Proposition 2. *For* $A, A' \in \mathcal{A}(x)$ *we have* $A \cap A' \neq \emptyset$. ∎

We will use the following theorem of Hilton and Milner:

Theorem 6. [9] *Let* $\mathcal{A}$ *be a collection of r-element subsets of an n-set,* $n \geqq 2r$. *Suppose that* $A \cap A' \neq \emptyset$ *for* $A, A' \in \mathcal{A}$ *and* $|\mathcal{A}| > \binom{n-1}{r-1} - \binom{n-r-1}{r-1} + 1$. *Then there exists an element y such that* $y \in A$ *for every* $A \in \mathcal{A}$. ∎

Proposition 3. *If* $r \geqq 2$, $n \geqq 2r$, *then*

$$\binom{n-1}{r-1} - \binom{n-r-1}{r-1} + 1 \geqq r\binom{n-2}{r-2} + 1. \tag{8}$$

Proof. It follows from

$$\binom{n-1}{r-1} - \binom{n-r-1}{r-1} = \sum_{i=0}^{r-1} \binom{n-2-i}{r-2}. \quad ∎$$

Proposition 4. *If* $|\mathcal{A}(x)| > k\binom{n-3}{k-3}$ *then there exists* $y \in (X - x)$ *such that* $y \in F$ *for every* $F \in \mathcal{D}(x)$.

Proof. In view of Propositions 2 and 3 we can find $y \in (X - x)$ such that $y \in A$ for every $A \in \mathcal{A}(x)$. Suppose that for some $F \in \mathcal{D}_0(x)$ we have $y \notin F$. In view of Proposition 1 for every $A \in \mathcal{A}(x)$ we have $A \cap F \neq \emptyset$, yielding $|\mathcal{A}(x)| \leqq k\binom{n-3}{k-3}$, a contradiction. ∎

Call the point $x \in X$ *good* if there exists a $y \neq x$ such that $x \in F \in \mathcal{F}$ entails $y \in F$. If x is good then fix one such y and denote it by $f(x)$.

Corollary 2. *If* $|\mathcal{A}(x)| > k\binom{n-3}{k-3}$ *then x is good.* ∎

We assume from now on that $|\mathcal{F}| \geqq \binom{n-1}{k-1}$ and that there is no vertex $z \in X$ which is contained in every member of $\mathcal{F}$.

Lemma 2. *If x is good then* $d(x) \leqq \binom{n-2}{k-2} - (k-1)\binom{n-k-2}{k-3}$.

Proof. By the indirect assumption there exists $F_0 \in \mathcal{F}$ with $f(x) \notin F_0$. As x is good and $f(x) \notin F_0$ we have $x \notin F_0$. Let us consider k-subsets of the following form: $F = G \cup \{y\} \cup \{x, f(x)\}$, where $y \in F_0$, $G \subset X - (F_0 \cup \{x, f(x)\})$, $|G| = k - 3$. The total number of such k-sets is $k\binom{n-k-2}{k-3}$. As for given G_0 and $y_1, y_2 \in F_0$ the intersection of the three sets F_0, $G_0\cup\{y_1, x, f(x)\}$, $G_0\cup\{y_2, x, f(x)\}$ is empty and their union of cardinality less than $2k$, so at most one set of the form $G_0\cup\{y, x, f(x)\}$ $(y\in F_0)$ belongs to $\mathcal{F}$. Consequently, at most $\binom{n-k-2}{k-3}$ sets of the form $\{G\cup\{y, x, f(x)\} \mid y\in F_0,\ G\subset (X-F_0\cup\{x, f(x)\}),\ |G|=k-3\}$ belong to $\mathcal{F}$. This means that at least $(k-1)\binom{n-k-2}{k-3}$ sets are missing from the $\binom{n-2}{k-2}$ possible k-sets containing $\{x, f(x)\}$. ▮

Lemma 3. *If $n>k^2-k$ then there exists at least one good vertex x.*

Proof. Suppose the contrary then using (6), (7) and Corollary 2 we deduce

$$k\binom{n-1}{k-1} \leqq k|\mathcal{F}| = \sum_{x\in X} d(x) = \sum_{x\in X} d_0(x) + \sum_{x\in X} |\mathcal{A}(x)| \leqq \binom{n}{k-1} + nk\binom{n-3}{k-3}.$$

It is easy to see that the right hand side is less than $k\binom{n-1}{k-1}$ for $n>k^2-k$, a contradiction. ▮

We prove Theorem 3 for $k=3$, $n\geqq 5=n_0(3)$. For $n=5, 6$ it follows from Theorem 2. We apply induction on n.

Let $\mathcal{F}$ be a family satisfying the assumptions but not the statement. As $n\geqq 7$, by Lemmas 2 and 3, wa can find $x\in X$ with $d(x)<(n-2)$. Then $\mathcal{F}-\mathcal{D}(x)$ is a family satisfying the assumptions on $X-\{x\}$. We may use the induction hypothesis $|\mathcal{F}-\mathcal{D}(x)|\leqq\binom{n-2}{2}$, yielding $|\mathcal{F}|<\binom{n-2}{2}+(n-2)=\binom{n-1}{2}$ which concludes the proof.

From now on we assume that $k\geqq 4$. Let us suppose $n>k^2-k$. Suppose the statement of the theorem is false for $\mathcal{F}$. Then by $n>k^2-k$ there exists $x\in X$ with $d(x)\leqq\binom{n-2}{k-2}-(k-1)\binom{n-k-2}{k-3}$. Let $X_1=X-\{x\}$, and $\mathcal{F}_1=\mathcal{F}-\mathcal{D}(x)$. If $X_i, \mathcal{F}_i$ are defined with $|\mathcal{F}_i|>\binom{|X_i|-1}{k-1}$ then let $x\in X_i$ with $d(x)\leqq\binom{|X_i|-2}{k-2}-(k-1)\cdot \cdot\binom{|X_i|-k-2}{k-3}$ (such a vertex exists certainly for $|X_i|>k^2-k$.)

Let $X_{i+1}=X_i-\{x\}$, $\mathcal{F}_{i+1}=\mathcal{F}_i-\mathcal{D}(x)$. Let j be the index for which $|X_j| =k^2-k$, i.e., $j=n-k^2+k$. Then we have

$$(9)\qquad |\mathcal{F}_j| \geqq |\mathcal{F}|-\sum_{i=0}^{j-1}\left(\binom{n-2-i}{k-2}-(k-1)\binom{n-k-2-i}{k-3}\right) \geqq \binom{n-1}{k-1}-\sum_{i=0}^{j-1}\binom{n-2-i}{k-2}$$
$$+(k-1)\sum_{i=0}^{j-1}\binom{n-k-2-i}{k-3} = \binom{k^2-k-1}{k-1}+(k-1)\left(\binom{n-k-1}{k-2}-\binom{k^2-2k-1}{k-2}\right).$$

On the other hand $\mathcal{F}_j$ is a family of k-subsets of the (k^2-k)-element set X_j, thus by Lemma 1

$$(10)\qquad |\mathcal{F}_j| \leqq \binom{k^2-k}{k-1}+m(k^2-k, k, 1).$$

Proposition 5. *For* $n>2k$ *we have*

$$m(n, k, 1) \leqq \frac{n}{k}\binom{n-2}{k-2}.$$

Proof. Let $\mathcal{G}$ be a family of k-subsets of an n-set which does not contain two members intersecting in a singleton. Then for every vertex x, $\mathcal{G}_x=\{G-\{x\}: x\in G\in\mathcal{G}\}$ is an intersecting family of $(k-1)$-subsets of an $(n-1)$-set. Thus by the Erdős—Ko—Rado theorem (Theorem 1) we have $|\mathcal{G}_x|\leqq\binom{n-2}{k-2}$. Therefore

$$|\mathcal{G}| = \frac{1}{k}\sum_x |\mathcal{G}_x| \leqq \frac{n}{k}\binom{n-2}{k-2}. \quad \blacksquare$$

Combining (10) with Proposition 5, we obtain

$$|\mathcal{F}_j| \leqq \binom{k^2-k}{k-1}+(k-1)\binom{k^2-k-2}{k-2}. \tag{11}$$

However, for $n\geqq k^2+3k$, (11) contradicts (9), which concludes the proof of Theorem 3.

Remark 1. Proposition 4 remains true for $|\mathcal{A}(x)|>\binom{n-2}{k-2}-\binom{n-k-1}{k-2}+1$. Using this one can prove Lemma 3 for $n>k^2/(1.5\log k)$ and in this way the upper bound $n_0(k)\leqq k^2+3k$ can be improved to $n_0(k)<k^2/\log k$. But it is still far from the real value of $n_0(k)$ which we conjecture to be $\lceil 3k/2\rceil$. We can prove this for $k=4, 5$.

4. The proof of Theorem 4.

With the family $\mathcal{F}$ let us associate the graph $\mathcal{A}$ whose vertex set is X and whose edges are all the 2-sets which are contained in some $F\in\mathcal{F}$.

Let us recall now a result of Erdős [3]. For simplicity we state it only for a special case.

Theorem 7. [3] *Let $\mathcal{F}$ be a family of 3-subsets of $X, |X|=n$. Suppose that $\mathcal{A}$ contains no complete subgraph on 4 vertices. Then for $\mathcal{F}$ the assertion of Theorem 4 holds.*

Let s be the greatest number for which $\mathcal{A}$ contains a complete subgraph on s vertices.

If $s=3$ then Theorem 7 yields the statement of our theorem. For $s\geqq 4$ we will proceed in a similar way as with the proof of Theorem 3. Let $t=\min(s, 5)$. Let $x_1, \dots, x_t$ be the vertices of a complete subgraph of $\mathcal{A}$. By Turán's theorem [9] we have for $t=s=4$

$$|\mathcal{A}| \leqq \frac{3}{8}n^2, \tag{12}$$

Let $\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3$ be the collection of members B of $\mathcal{F}$ for which $|B\cap\{x_1, \dots, x_t\}|$

$=1, 2, 3$, respectively. Obviously, we have

(13) $$|\mathcal{B}_3| \leqq \binom{t}{3},$$

and

(14) $$|\mathcal{B}_2| \leqq \binom{t}{2}(n-t).$$

For $1 \leqq i < j \leqq t$, let us choose $z_{i,j} \in X$ such that $\{x_i, x_j, z_{i,j}\} \in F$. This is possible since $\{x_i, x_j\} \in \mathcal{A}$. Let Z be the set of different $z_{i,j}$'s. Of course $|Z| \leqq \binom{t}{2}$.

Proposition 6. *If for* $1 \leqq i < j \leqq t$, and for $y_1, y_2 \in X$ both $\{x_i, y_1, y_2\}$ and $\{x_j, y_1, y_2\}$ *belong to* $\mathcal{F}$, *then either* y_1 *or* y_2 *belongs to* Z.

Proof. Let us write $z = z_{i,j}$. By definition $\{x_i, x_j, z\} \in \mathcal{F}$. Then $|\{x_i, x_j, z\} \cup \{x_i, y_1, y_2\} \cup \{x_j, y_1, y_2\}| \leqq 5$, consequently the intersection of the 3 sets is non-empty, i.e., $z = y_1$ or $z = y_2$, as desired. ∎

Proposition 7.

(15) $$|\mathcal{B}_1| < 2\binom{t}{2}(n-t) + \frac{t-1}{8} n^2.$$

Proof. Our first claim is that the first term is an upper bound for the number of $F \in \mathcal{F}$ with $|F \cap \{x_1, \dots, x_t\}| = 1$, $F \cap Z \neq 0$. For $z \in Z$ let $m(z)$ denote the multiplicity of Z, i.e. the number of pairs (i, j), $1 \leqq i < j \leqq t$ with $z = z_{i,j}$. For $z \in Z$ and $y \in X - \{x_1, \dots, x_t\}$ let $D(z, y)$ denote the set of x_i, $1 \leqq i \leqq t$ such that $\{z, y, x_i\} \in \mathcal{F}$. If $y \notin Z$ then by Proposition 6 for $x_i, x_j \in D(z, y)$ we have $z = z_{i,j}$. If $y \in Z$ then the only other possibility is $y = z_{i,j}$. Thus for $y \notin Z$ we have $m(z) \geqq \binom{|D(z, y)|}{2}$, in particular $2m(z) \geqq |D(z, y)|$ holds. Similarly, if $y \in Z$ then $2m(z) + 2m(y) \geqq |D(z, y)|$. Summing up these inequalities for all pairs $z \in Z$, $y \in X - \{x_1, \dots, x_t\}$, considering the pairs with $y \in Z$ only once and taking into consideration $\sum_{z \in Z} m(z) = \binom{t}{2}$ we obtain our first claim. In view of Proposition 6 and (12) the second term is an upper bound for $|\mathcal{A}|$, which is at least the number of $F \in \mathcal{F}$ with $|F \cap \{x_1, \dots, x_t\}| = 1$, $F \cap Z \neq 0$. ∎

Now summing (13), (14) and (15) we obtain

(16) $$\min_{1 \leqq i \leqq t} d(x_i) \leqq \frac{1}{t} \sum_{i=1}^{t} d(x_i) = \frac{1}{t}(3|\mathcal{B}_3| + 2|\mathcal{B}_2| + |\mathcal{B}_1|)$$

$$\leqq \begin{cases} \frac{3}{32} n^2 + 6n - 21 & \text{for} \quad t = 4 \\ \frac{1}{10} n^2 + 8n - 37 & \text{for} \quad t = 5 \end{cases} \leqq \frac{1}{10} n^2 + 8n - 37 \quad (\text{if } n \geqq 8).$$

Suppose now that $n>3000$, $|\mathcal{F}|\geqq\left\lfloor\frac{n}{3}\right\rfloor\left\lfloor\frac{n+1}{3}\right\rfloor\left\lfloor\frac{n+2}{3}\right\rfloor$ and the theorem is false. By (16) we can take an $x_1\in X$, $X_1=X-\{x_1\}$ and $\mathcal{F}_1=\{F\in\mathcal{F}|x_1\notin F\}$ such that $d(x_1)\leqq\frac{1}{10}n^2+8n-37$. Then, in view of (16), $|\mathcal{F}_1|>\left[\frac{n-1}{3}\right]\left[\frac{n}{3}\right]\left[\frac{n+1}{3}\right]=|\mathcal{G}_{n-1}|$ and we can argue in the same way for $\mathcal{F}_1$ as we did for $\mathcal{F}$. Let q be the first integer with $|X_q|\leqq 750$. Then $q=n-750$. For the cardinality of $\mathcal{F}_q$ we deduce

$$|\mathcal{F}_q|>|\mathcal{F}|-\sum_{i=0}^{q-1}\left(\frac{1}{10}(n-i)^2+8(n-i)-37\right) \tag{17}$$

$$=|\mathcal{F}|-\frac{1}{10}\frac{n(n+1)(2n+1)}{6}-4n(n+1)+37n+\frac{1}{10}\frac{750\cdot 751\cdot 1501}{6}$$

$$+4\cdot 750\cdot 751-37\cdot 750.$$

Now using the assumption $|\mathcal{F}|\geqq\frac{1}{27}(n^3-3n-2)$ we obtain from (17), for $n>3000$, $|\mathcal{F}_q|>\frac{1}{270}n^3-4.1n^2+16\,000\,000>\binom{750}{3}$, a contradiction, proving the theorem.

5. Concluding remarks

Remark 2. Theorem 4 is not only a sharpening of Theorem 5, but the proof is entirely new.

Remark 3. The problems considered in this paper belong to the so-called Turán-type problems, i.e. what is the maximum number of k-subsets of an n-set if it contains no sub-system isomorphic to one member of a set of k-graphs $\{\mathcal{H}_1,\mathcal{H}_2,\dots,\mathcal{H}_q\}$. This maximum is usually denoted by $\operatorname{ext}(n,\{\mathcal{H}_1,\mathcal{H}_2,\dots,\mathcal{H}_q\})$.

Let us define $\mathcal{H}_1=\{\{x_1,x_2,\dots,x_k\},\{x_1,x_2,\dots,x_{k-1},x_{k+1}\},\{x_{k+1},x_{k+2},\dots,x_{2k}\}\}$, $\mathcal{H}_2=\{\{x_1,x_2,\dots,x_k\},\{x_1,x_2,\dots,x_{k-1},x_{k+1}\},\{x_k,x_{k+1},\dots,x_{2k-1}\}\}$. In this terminology we proved (Theorem 4) for $k=3$ $\operatorname{ext}(n,\{\mathcal{H}_2\})=\left[\frac{n}{3}\right]\left[\frac{n+1}{3}\right]\left[\frac{n+2}{3}\right]$. Moreover, the proof of Theorem 3 yields for $n>n_0(k)$ the stronger result $\operatorname{ext}(n,\{\mathcal{H}_1,\mathcal{H}_2\})=\binom{n-1}{k-1}$.

Refining the argument we could even obtain

Theorem 8. *For* $n>n_1(k)$ *we have* $\operatorname{ext}(n,\{\mathcal{H}_1\})=\binom{n-1}{k-1}$.

Finally a special case of a result of the first author gives

Theorem 9. [8] *Let* $\mathcal{H}=\{H_1,H_2,H_3\}$ *be an arbitrary k-graph satisfying* $|H_1\cup H_2\cup H_3|\geqq 2k$, $H_1\cap H_2\cap H_3\neq\emptyset$. *Then for every* n, $\operatorname{ext}(n,\{\mathcal{H}\})<3en^{k-1}$.

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