DISJOINT r-TUPLES IN AN r-GRAPH WITH GIVEN MAXIMUM DEGREE

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AN *r-uniform hypergraph* (or *r-graph*) $\mathcal{F}$ is a finite set-system of (unordered) r-tuples of a finite ground-set X. We call X the *vertex set* of $\mathcal{F}$, and the r-tuples in $\mathcal{F}$ are called *edges* of $\mathcal{F}$. Thus the 2-graphs are the usual graphs without loops and multiple edges. The *degree* of a vertex in X is the number of edges of $\mathcal{F}$ containing it. As usual, we denote by $D(\mathcal{F})$ the maximum degree of a vertex in $\mathcal{F}$. We denote by $\nu(\mathcal{F})$ the maximum number of pairwise disjoint r-tuples of $\mathcal{F}$.

Given positive integers ν and D denote by $f_r(\nu, D)$ the maximum number of r-tuples contained in an r-graph $\mathcal{F}$ with $\nu(\mathcal{F}) \leq \nu$ and $D(\mathcal{F}) \leq D$. The function $f_2(\nu, D)$ was investigated by Erdös and Rado [**4**], Abbott and Hanson [see **6**] and Sauer [**6**]. The determination of $f_2(\nu, D)$ was completed by Chvàtal and Hanson [**3**]. In particular they proved that if $D > 2\nu$, then $f_2(\nu, D) = \nu D$. Bollobàs conjectured in [**1**] that this last result has the following extension:

Suppose r is such that there exists a finite projective plane of order r-2 (for example $r = P+2$, where P is a prime power) or $r = 2, 3$, and suppose that ν is given. If D is sufficiently large and divisible by $r-1$, then

$$f_r(\nu, D) = \frac{r^2-3r+3}{r-1}\,\nu D.$$

Furthermore, the extremal r-graphs can be obtained as follows. Take ν pairwise-disjoint projective planes, $\mathcal{P}_1, \ldots, \mathcal{P}_\nu$ (or triangles or points if $r = 3$ or 2) each with $(r-2)^2+(r-2)+1 = r^2-3r+3$ points and with $r-1$ points on each line. For each line of each plane $\mathcal{P}_i$ take $D/(r-1)$ r-tuples in such a way that each of these r-tuples intersects $\bigcup_{i=1}^{\nu} \mathcal{P}_i$ exactly in this line. Then, in the obtained r-graph $\mathcal{F}$, every vertex in $\bigcup_{i=1}^{\nu} \mathcal{P}_i$ has degree D and $D(\mathcal{F}) = D$ for $D > \nu r^2$. Clearly $\nu(\mathcal{F}) = \nu$, and $\mathcal{F}$ has $(r^2-3r+3)\nu D/(r-1)$ r-tuples.

In [**2**] Bollobàs proved his conjecture for the case $r = 3$. The aim of this note is to prove this conjecture for all r. The proof is based on his method. Given natural numbers ν and D, let $\mathcal{E}(\nu, D)$ be the set of all r-graphs obtained in above described way.

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THEOREM. *Suppose ν and D are natural numbers, D is divisible by $r-1$, and $D > r^{2r+1}\nu^r$. Let $\mathcal{F}$ be an r-graph such that $D(\mathcal{F}) \leq D$, $\nu(\mathcal{F}) \leq \nu$. Then $|\mathcal{F}| \leq (r^2-3r+3)\nu D/(r-1)$. Equality holds if and only if $\mathcal{F} \in \mathscr{E}(\nu, D)$.*

We need some definitions and lemmas. The set-system $F_1, F_2, \ldots, F_k$ is called a *Δ-system with nucleus N* if, for every $1 \leq i < j \leq k$, we have $F_i \cap F_j = N$. The well-known theorem of Erdös and Rado [**4**] says:

LEMMA 1. *Suppose $R \geq 2$. If the set-system $\mathcal{H}$ is of rank R (i.e., $\max_{H \in \mathcal{H}} |H| = R$) and if $|\mathcal{H}| \geq k^R R!$, then $\mathcal{H}$ contains a Δ-subsystem consisting of k members.*

Besides the Erdös-Rado theorem our proof will be based on the following result of the second author [**5**]. We put this result in the form we need.

LEMMA 2. *Suppose $\mathcal{H}$ is a hypergraph of rank R with multiple edges such that $D(\mathcal{H}) = D$ and $\nu(\mathcal{H}) = \nu$. Then*

$$|\mathcal{H}| \leq (R^2 - R + 1)\nu D/R.$$

Furthermore, either $\mathcal{H}$ consists of isolated vertices and ν disjoint R-uniform finite projective planes (with multiple edges), or $\mathcal{H}$ has at most $((R^2-R+1)\nu-1)D/R$ edges.

(Furthermore, if such a plane does not exist, then $|\mathcal{H}| \leq (R-1)\nu D$.)

Proof of the theorem. Since the edge-set of a hypergraph $\mathcal{H} \in \mathscr{E}(\nu, D)$ is maximal provided that $D(\mathcal{H}) \leq D$ and $\nu(\mathcal{H}) \leq \nu$, it suffices to show that if $|\mathcal{F}| = (r^2-3r+3)\nu D/(r-1)$ then $\mathcal{F} \in \mathscr{E}(\nu, D)$. Thus we suppose that $\mathcal{F}$ has $(r^2-3r+3)\nu D/(r-1)$ r-tuples.

Denote by X the vertex-set of $\mathcal{F}$. We define three hypergraphs $\mathcal{N}$, $\mathcal{F}_{\mathcal{N}}$ and $\mathcal{F}_0$ with vertex-set X as follows.

Let $\mathcal{N}$ be a system of nuclei of those Δ-subsystems of $\mathcal{F}$ which contain at least $r\nu + 1$ different edges of $\mathcal{F}$. Clearly $\varnothing \notin \mathcal{N}$.

Let $\mathcal{F}_0$ be the r-graph obtained from $\mathcal{F}$ by omitting those r-tuples that contain an edge of $\mathcal{N}$. Since $\mathcal{F}_0$ contains no Δ-system with $r\nu+1$ members, we get by Lemma 1 that

$$|\mathcal{F}_0| < (r\nu + 1)^r r!$$

Let us associate with each edge $F \in \mathcal{F} - \mathcal{F}_0$ a nucleus $N \in \mathcal{N}$ such $N \subset F$. Denote by $\mathcal{F}_{\mathcal{N}}$ the hypergraph of the "nuclei with multiplicities"—that is, the hypergraph containing each member $N \in \mathcal{N}$ as many times as it has been associated. Note that N is an edge in $\mathcal{N}$ and W is a set of at most $r\nu$ vertices disjoint from N, then there is an r-tuple in $\mathcal{F}$ that contains N and is also a disjoint from W. Consequently if $N_1, \ldots, N_k$ are disjoint edges of $\mathcal{N}$ (or $\mathcal{F}_{\mathcal{N}}$) then $k \leq \nu$. Now $\mathcal{F}_{\mathcal{N}}$ is a hypergraph of rank $r-1$ and with

$\nu(\mathcal{F}_{\mathcal{N}}) \leq \nu$, $D(\mathcal{F}_{\mathcal{N}}) \leq D$. Thus, by Lemma 2, one of the following two cases holds:

1. *$\mathcal{F}_{\mathcal{N}}$ has at most $((r^2-3r+3)\nu-1)D/(r-1)$ edges.* In this case

$$|\mathcal{F}| = |\mathcal{F}_{\mathcal{N}}| + |\mathcal{F}_0| \leq \frac{r^2-3r+3}{r-1}\nu D - \frac{D}{r-1} + (r\nu+1)^r r! < \frac{r^2-3r+3}{r-1}\nu D,$$

because $D > r^{2r+1}\nu^r$.

2. *$\mathcal{F}_{\mathcal{N}}$ consists of ν disjoint finite projective planes each of order $r-2$*, say $\mathcal{P}_1, \dots, \mathcal{P}_\nu$. Then every r-tuple $F \in \mathcal{F}$ contains a line of one of these planes, since otherwise we can find ν disjoint edges of $\mathcal{F}$ which are disjoint from F as well. (That is, $\mathcal{F}_0 = \emptyset$.)

So the sum of the degrees of $\mathcal{F}$ in the points of $\bigcup_{i=1}^{\nu} \mathcal{P}_i$ is at least $|\mathcal{F}|(r-1) = (r^2-3r+3)\nu D$, and consequently this sum equals $(r^2-3r+3)\nu D$. This implies that any edge $F \in \mathcal{F}$ contains exactly $r-1$ points of $\bigcup_{i=1}^{\nu} \mathcal{P}_i$, and any point of $\bigcup_{i=1}^{\nu} \mathcal{P}_i$ is of degree D.

Now it is easy to see that any line of the projective planes is contained in $D/(r-1)$ edges of $\mathcal{F}$—that is, $\mathcal{F} \in \mathcal{E}(\nu, D)$. Indeed, let P be a line of $\mathcal{P}_1$ (say) and suppose that P is contained in t edges of $\mathcal{F}$. The sum of the degrees of the points of $\mathcal{P}_1$ equals $(r^2-3r+3)D$, so altogether $(r^2-3r+3)D/(r-1)$ edges of $\mathcal{F}$ contain some line of $\mathcal{P}_1$. Further, the sum of the degrees of points of P is $(r-1)D = (r^2-3r+3)D/(r-1) + t(r-2)$. So $t = D/(r-1)$.

Remarks. It can be seen that, in the case when D is not multiple of $r-1$, the extremal r-graph has a structure similar to $\mathcal{E}(\nu, D)$. clearly

$$(r^2-3r+3)\nu\lfloor D/(r-1)\rfloor \leq f_r(\nu, D) \leq (r^2-3r+3)\nu D/(r-1),$$

but we have not been able to find the exact value of $f_r(\nu, D)$.

From the remark at the end of Lemma 2 one concludes that, if there is no projective plane of order $r-2$, then the theorem can be sharpened:

$$f_r(\nu, D) \leq (r-2)\nu D + r^{2r}\nu^r.$$

We think that the bound for D in the theorem can be improved considerably.

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