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ON FINITE SET-SYSTEMS WHOSE EVERY INTERSECTION IS A KERNEL OF A STAR

Z. FÜREDI

Math. Inst. Hungarian Acad. Sci., Budapest 1395, Pf. 428, Hungary

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Let k, t be positive integers and let $\mathcal{F}$ be a set-system which consists of k-element sets. In this paper it is proved that one can choose a subsystem $\mathcal{F}^* \subset \mathcal{F}$ containing a positive proportion of the members of $\mathcal{F}$, (i.e. $|\mathcal{F}^*| > c(k, t)\,|\mathcal{F}|$) and having the property that every pairwise intersection is a kernel of a t-star in $\mathcal{F}^*$ (i.e. $\forall F, F' \in \mathcal{F}^*$, $F \cap F' = A$, $\exists F_1, \ldots, F_t \in \mathcal{F}^*$ such that $F_i \cap F_j = A$ for $1 \le i < j \le t$).

This result is used to obtain some new bounds for the maximum cardinality of a k-graph with prescribed cardinalities for pairwise intersections.

1. Intersections and stars

The sets $F_1, F_2, \ldots, F_t$ are said to form a *t-star* (or *Δ-system*) with *kernel* A if $F_i \cap F_j = A$ for all $1 \le i < j \le t$. Erdös and Rado [6] proved that if the set-system $\mathcal{F}$ has *rank* k (i.e. $|F| \le k$ for all $F \in \mathcal{F}$) and $|\mathcal{F}| > \Phi(k, t)$, then $\mathcal{F}$ contains a subsystem $\mathcal{F}' \subset \mathcal{F}$ which is a t-star. (Here $\Phi(k, t)$ is a constant depending only on k and t, $(t-1)^k \le \Phi(k, t) \le k!\,(t-1)^k$. The determination of the asymptotic value of $\Phi(k, t)$ is a favourite open problem of Erdös [4].)

We say that *every intersection is a kernel of a t-star* in the set-system $\mathcal{F}$ if for every $F, F' \in \mathcal{F}$ there exist $F_1, \ldots, F_t \in \mathcal{F}$ such that $F_i \cap F_j = F \cap F'$ for all $1 \le i < j \le t$. The following theorem conjectured by P. Frankl is a generalization of the Erdös–Rado's result.

Theorem 1. *For every pair of positive integers k and t, there exists a positive real number $c = c(k, t)$ with the following property: If $\mathcal{F}$ is a set-system of rank k, then we can choose a subsystem $\mathcal{F}^* \subset \mathcal{F}$, $|\mathcal{F}^*| > c\,|\mathcal{F}|$ so that every intersection is a kernel of a t-star in $\mathcal{F}^*$.*

We shall obtain extremely small values for c ($c(k, t) > (tk2^k)^{-2^k}$). Obviously we have $c(k, t) \le (1/\Phi(k, t)) < (t-1)^{-k}$.

A set-system of rank k is called a *k-graph* if each of its members has exactly k elements. A k-graph $\mathcal{F}$ is *k-partite* if there exist sets $X_1, \ldots, X_k$ satisfying

$X_i \cap X_j = \emptyset$ and $|X_i \cap F| = 1$ for all $F \in \mathcal{F}$, $1 \leq i \leq k$. Given a k-partite k-graph with parts $X_1, \ldots, X_k$, define the *natural homomorphisms* or *projection* $\pi : (\bigcup X_i) \to \{1, 2, \ldots, k\}$ by $\pi(x) = i$ for all $x \in X_i$. We use the notation $\pi(A) = \{\pi(a) : a \in A\}$ for the set A and $\pi(\mathcal{A}) = \{\pi(A) : A \in \mathcal{A}\}$ for the set-system $\mathcal{A}$ on $(\bigcup X_i)$. The following statement was proved by Erdös [3] for $k = 2$ and by Erdös and Kleitman [5] for all k. Every k-graph $\mathcal{F}$ contains a k-partite subgraph $\mathcal{F}'$ such that $|\mathcal{F}'| \geq (k!/k^k)\,|\mathcal{F}|$. We are going to connect this result with Theorem 1. Denote by $\mathcal{M}(F, \mathcal{F})$ the *system of pairwise intersections* in F, where $\mathcal{F}$ is a k-graph and $F \in \mathcal{F}$. In other words $\mathcal{M}(F, \mathcal{F}) = \{F \cap F' : F' \in \mathcal{F}\}$.

Theorem 1′. *For any positive integers k and t, there exists a positive real number $c = c(k, t)$ with the following property: If $\mathcal{F}$ is a k-graph, then we can choose a subsystem $\mathcal{F}^* \subset \mathcal{F}$ such that:*

(i) $|\mathcal{F}^*| > c\,|\mathcal{F}|$.

(ii) *Every pairwise intersection in $\mathcal{F}^*$ is a kernel of a t-star of $\mathcal{F}^*$.*

(iii) *$\mathcal{F}^*$ is k-partite with parts $X_1, \ldots, X_k$,*

(iv) *There exists a set-system $\mathcal{M}$ on the elements $\{1, 2, \ldots, k\}$ such that $\mathcal{M} = \pi\mathcal{M}(F, \mathcal{F}^*)$ for all $F \in \mathcal{F}^*$.*

Remark 1. Note that the set-system $\mathcal{M}$ mentioned above is *closed for intersection* if $t \geq k+1$. This means that for $M', M'' \in \mathcal{M}$ we have $M' \cap M'' \in \mathcal{M}$.

2. Applications. Set-systems with prescribed cardinalities for pairwise intersections

2.1. *Necessary and sufficient condition for $m(n, k, L) = O(n)$*

Let $0 \leq l_1 < l_2 < \cdots < l_s < k < n$ be integers, and X a finite set of cardinality n. We say that the family $\mathcal{F}$ of k-subsets of X is an $(n, k, \{l_1, \ldots, l_s\})$-system if $|F_1 \cap F_2| \in \{l_1, \ldots, l_s\}$ holds for every $F_1 \neq F_2$, $F_1, F_2 \in \mathcal{F}$. Denote $\{l_1, \ldots, l_s\}$ by L and let us denote by $m(n, k, L)$ the maximum cardinality of an (n, k, L)-system.

We say that the $(*)$ condition holds for the numbers $l_1, l_2, \ldots, l_s$ and k if

$(*)$ *There exists a set-system $\mathcal{M}$ on the elements $\{1, 2, \ldots, k\}$ which is closed for intersection, $|\bigcup \mathcal{M}| = k$ and $|M| \in \{l_1, \ldots, l_s\}$ for all $M \in \mathcal{M}$.*

Theorem 2. (a) *If for $\{l_1, \ldots, l_s\}$ and k the $(*)$ condition is satisfied, then $m(n, k, L) > c(k) \cdot n^{k/(k-1)}$.*

(b) *If $(*)$ does not hold, then $m(n, k, L) < c_k \cdot n$.*

($c_k \leq 1/c(k, k+1)$, where $c(k, t)$ is defined in Theorem 1′.)

This theorem implies a result of Deza, Erdös and Singhi [2] ($|L| \leq 2$), Babai and Frankl [1] (if the $\gcd(l_1, \ldots, l_s)$ does not divide k, then $m(n, k, L) \leq n$), Frankl and Rosenberg [9] (if each $l_i \equiv r \pmod m$ but $k \not\equiv r \pmod m$, then $m(n, k, L) \leqslant n$) and Frankl [8] ($|L| \leqslant 3$) in a slightly weaker form.

2.2. *A reduction theorem*

Let $a(n)$, $b(n)$ be two positive real functions over positive integers. If there are positive reals c and c' such that $ca(n) \geqslant b(n) \geqslant c'a(n)$, then we shall write $a(n) \approx b(n)$. It is easy to see (cf. Frankl [7]) that

$$m(n, k, \{l_1, \ldots, l_s\}) \approx m(n, k - l_1, \{0, l_2 - l_1, \ldots, l_s - l_1\}).$$

Hence, if we are interested only in the order of magnitude of $m(n, k, L)$, then we can always assume $l_1 = 0$. The following result is another reduction theorem.

Theorem 3. *If the greatest common divisor d of $l_1, \ldots, l_s$ divides k, then*

$$m(n, k, \{l_1, \ldots, l_s\}) \approx m\left(\frac{n}{d}, \frac{k}{d}, \left\{\frac{l_1}{d}, \ldots, \frac{l_s}{d}\right\}\right).$$

If $\gcd(l_1, \ldots, l_s)$ does not divide k, then, by [1], $m(n, k, L) \leqslant n$.

2.3. *Remark about t-times intersections*

Let us denote by $m_t(n, k, L)$ the maximum cardinality of $\mathscr{F}$ where $\mathscr{F} \subset \binom{X}{k}$, $|X| = n$ and $|F_1 \cap F_2 \cap \cdots \cap F_t| \in L$ holds for every distinct $F_1, \ldots, F_t \in \mathscr{F}$. This question was posed by V.T. Sós [10] in more general form. Theorem 1 implies that $m_t(n, k, L) \approx m(n, k, L)$ holds.

3. Proofs

3.1. *Two lemmas*

By virtue of the above-cited theorem of Erdös and Kleitman, there exists a k-partite $\mathscr{F}' \subset \mathscr{F}$, $|\mathscr{F}'| \geqslant (k!/k^k)\,|\mathscr{F}|$. Denote its parts by $X_1, \ldots, X_k$. From now on we will consider only k-partite k-graphs $\mathscr{G}$ with parts $X_1, \ldots, X_k$. Let $\mathscr{M}(\mathscr{G}) = \bigcup \pi\mathscr{M}(F, \mathscr{G})$, where the union is taken over all $F \in \mathscr{G}$. Let us denote by $\mathscr{B}(F, \mathscr{G})$ the set of kernels of t-stars in $F \in \mathscr{G}$, i.e.

$$\mathscr{B}(F, \mathscr{G}) = \{A \subset F : \exists F_1, \ldots, F_t \in \mathscr{G} \text{ such that } F_i \cap F_j = A \text{ for } 1 \leqslant i < j \leqslant t\}.$$

Finally, let $\mathscr{B}(\mathscr{G}) = \bigcup \{\pi\mathscr{B}(F, \mathscr{G}) : F \in \mathscr{G}\}$.

Lemma 1. *Let $\mathscr{G}$ be a k-partite graph with parts $X_1, \ldots, X_k$. Then either*

(α) *there exists a $\mathscr{G}^* \subset \mathscr{G}$, $|\mathscr{G}^*| \geqslant |\mathscr{G}|/(1 + |\mathscr{M}(\mathscr{G})|)$, such that $\mathscr{G}^*$ meets the conditions* (ii) *and* (iv) *in Theorem* 1'; *or*

(β) *there exist a $\mathscr{G}' \subset \mathscr{G}$, $|\mathscr{G}'| \geqslant |\mathscr{G}|/(1 + |\mathscr{M}(\mathscr{G})|)$ and a set $A \in \mathscr{M}(\mathscr{G})$ such that $A \notin \mathscr{B}(\mathscr{G}')$.*

Lemma 2. *Let $\mathscr{G}$ be a k-partite k-graph with parts $X_1, \ldots, X_k$. Suppose that $A \notin \mathscr{B}(\mathscr{G})$. Then there exists a $\mathscr{G}' \subset \mathscr{G}$ such that $|\mathscr{G}'| \geq |\mathscr{G}|/k(t-1)$ and $A \notin \mathscr{M}(\mathscr{G}')$.*

3.2. *Proof of Theorem* 1′

Let $\mathscr{F}_1 = \mathscr{F}'$ and $\mathscr{M}_1 = \mathscr{M}(\mathscr{F}_1)$. Apply Lemma 1. In the case (α) we are ready. In the case (β) we obtain an $\mathscr{F}_1' \subset \mathscr{F}_1$ such that $|\mathscr{F}_1'| \geq |\mathscr{F}_1|/(1+|\mathscr{M}_1|)$, $\mathscr{B}(\mathscr{F}_1') \subsetneq \mathscr{M}(\mathscr{F}_1)$. Now, applying Lemma 2 for $\mathscr{F}_1'$, we obtain a subsystem $\mathscr{F}_2 \subset \mathscr{F}_1'$ such that $|\mathscr{F}_2| \geq |\mathscr{F}_1'|/k(t-1)$ and $\mathscr{M}(\mathscr{F}_2) \subset \mathscr{B}(\mathscr{F}_1') \subsetneq \mathscr{M}(\mathscr{F}_1)$. Proceeding in the same way, we get $\mathscr{F}_2 \supset \mathscr{F}_2' \supset \mathscr{F}_3 \supset \mathscr{F}_3' \supset \cdots$. It ends up in at most $|\mathscr{M}| \leq 2^k$ steps. We may suppose that at the end of the procedure we have a subsystem $\mathscr{F}_s' \subset \mathscr{F}'$ satisfying $|\mathscr{F}_s'| \geq |\mathscr{F}|(k!/k^k)\ (1/(2^k+1)!)\ (1/k(t-1))^{2^k}$. Put $\mathscr{F}^* = \mathscr{F}_s'$. □

3.3. *Proof of Theorems* 2 *and* 3

Theorem 2(b) and Theorem 3 are easy consequences of Theorem 1′. To prove Theorem 2(a) one can give a construction, which is a slightly modified version of a construction of Frankl [8]. The author shall return to these problems in a later paper.

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