

## THE GREATEST ANGLE AMONG $n$ POINTS IN THE $d$ -DIMENSIONAL EUCLIDEAN SPACE

P. ERDŐS and Z. FÜREDI

*Mathematical Institute of the Hungarian Acad. Sci., 1053 Budapest, Reáltanoda u. 13-15,  
Hungary*

There exists a pointset  $\mathcal{P}$  of cardinality at least  $1.15^d$  in  $E_d$  such that all angles determined by the triples of  $\mathcal{P}$  are less than  $\pi/2$ . This disproves the old conjecture that  $|\mathcal{P}| \leq 2d - 1$ .

### 1. Introduction

Many decades ago Erdős conjectured that if there are given  $2^d + 1$  points in a  $d$ -dimensional Euclidean space at least one of the angles determined by the points is greater than  $\pi/2$ . A very simple and ingenious proof for this conjecture was given by Danzer and Grünbaum. The following problems remained open: Determine the largest  $\alpha_d$  for which  $2^d + 1$  points in  $E_d$  always determine an angle  $\geq (\pi/2) + \alpha_d$ . We can make no contribution to this problem at present. The second problem states: Denote by  $f(d)$  the largest integer for which there are  $f(d)$  points in  $E_d$  all the angles of which are  $< \pi/2$ .  $f(2) = 3$  is trivial and Croft proved  $f(3) = 5$ . It was often conjectured that  $f(d) < Cd$ . We prove that  $f(d)$  tends to infinity exponentially and at the moment cannot prove that  $f(d) < (2 - \varepsilon)^d$ . As a matter of fact we cannot even prove that  $f(d) < 2^d - 1$  ( $d > 2$ ). We prove that for every  $\varepsilon > 0$  there exists a  $\delta$ , so that one can have  $(1 + \delta)^d$  points in  $E_d$  all the angles of which are  $< (\pi/3) + \varepsilon$ .

Erdős and Szekeres proved that  $2^n$  points in  $E_2$  always determine an angle  $> \pi(1 - 1/n)$  and Szekeres proved that this result is best possible in the following sense: One can give, for every  $\varepsilon > 0$ ,  $2^n$  points in  $E_2$  no angle of which is greater than  $\pi(1 - (1/n)) + \varepsilon$ . It does not seem easy to get a result of the same precision for higher dimension but we will get inequalities giving estimates for the number of points in  $E_d$  which give an angle  $> \pi - \varepsilon$ .

### 2. Strictly antipodal polytopes

Let us denote by  $E_d$  the  $d$ -dimensional Euclidean space. Let  $\mathcal{P} \subset E_d$  be a pointset, and let  $a, b \in \mathcal{P}$ . We shall say that  $a$  and  $b$  are an *antipodal pair* of  $\mathcal{P}$

provided there exists a pair of parallel (distinct) supporting hyperplanes of  $\mathcal{P}$  such that  $a$  belongs to one of them and  $b$  to the other. A pointset  $\mathcal{P}$  is said to be *antipodal* provided each two of its points forms an antipodal pair of  $\mathcal{P}$ . (Clearly, in this case  $\mathcal{P}$  is the vertex set of the polytope  $\text{conv } \mathcal{P}$ , the convex hull of  $\mathcal{P}$ .) Let us denote by  $G(d)$  the maximal number of vertices in an antipodal  $d$ -polytope. Danzer and Grünbaum [4] showed the conjecture of Klee [11] that  $G(d) = 2^d$ .

**Theorem 2.1** [4]. *If the pointset  $\mathcal{P} \subset E_d$  is an antipodal, then  $|\mathcal{P}| \leq 2^d$ . Equality holds if and only if  $\mathcal{P}$  consists of the vertices of a  $d$ -dimensional parallelotope.*

This result implies also that  $F(d) = 2^d$  is the answer to the following problem of Erdős [5, 6]: What is the maximal possible number  $F(d)$  of points in  $E_d$  such that all angles determined by the triples of them are less than or equal to  $90^\circ$ ?

Now let the notion of an antipodal pair be modified by defining a pair  $a, b \in \mathcal{P}$  as  $k$ -*antipodal* provided there exist parallel (distinct) supporting hyperplanes of  $\mathcal{P}$ , each of which intersects  $\text{conv } \mathcal{P}$  in a set of dimension at most  $k$ , such that  $a$  belongs to one of the hyperplanes, and  $b$  to the other. In analogy to the above, we define  $k$ -antipodal polytopes and the numbers  $G_k(d)$ . Clearly, a  $d$ -polytope is antipodal if and only if it is  $(d-1)$ -antipodal.

A number of interesting problems concern 0-antipodality, which is called *strict antipodality*. While it is easy to show that  $G_0(2) = 3$ , the proof of  $G_0(3) = 5$  is rather involved (Grünbaum [11]). For  $d \geq 4$ , it is known that  $G_0(d) \geq 2d - 1$ , and it has been conjectured that  $G_0(d) = 2d - 1$  (Danzer-Grünbaum [4], Grünbaum [11, 12]). In this section we disprove this conjecture, giving a construction with more than  $1.15^d$  points. (See Theorem 2.2.)

As in the case of antipodal pairs,  $G_0(d)$  may be considered as the affine variant of the following Euclidean problem due to Erdős [6]: Determine the maximal possible number  $f(d)$  of points in  $E_d$  such that all angles determined by triples of them are acute. Examples show that  $f(d) \geq 2d - 1$ , and clearly  $G_0(d) \geq f(d)$ . In contrast to the situation in the case  $G(d)$ , it is not known whether  $G_0(d) = f(d)$  for  $d \geq 4$ . (Direct proofs of  $f(3) = 5$  were given by Croft [3] and Schütte [15].) The following theorem implies that  $f(d) > 1.15^d$ .

**Theorem 2.2.** *There exists a pointset  $\mathcal{P}$  in  $E_d$  of cardinality  $1.15^d$  such that all angles determined by triples of  $\mathcal{P}$  are acute.*

**Proof.** We select the points of  $\mathcal{P}$  from the vertices of the  $d$ -dimensional cube. As usual, the  $d$ -dimensional 0-1 vectors correspond to the subsets of a  $d$ -element set  $X$ . More precisely, if  $a \in \{0, 1\}^d$  then let  $A = A(a) = \{i: a_i = 1\}$ ,  $X = \{1, 2, \dots, d\}$ .

**Lemma 2.3.** *The points  $a, b, c \in \{0; 1\}^d$  determine a right angle at the point  $c$  if and only if*

$$A \cap B \subset C \subset A \cup B, \quad (2.4)$$

where the sets  $A, B, C \subset X$  are associated with the vertices  $a, b, c$ .

(This lemma is a trivial consequence of Pythagoras' Theorem.) As the angles determined by the triples of the cube are less than or equal to  $\pi/2$ , the construction of the desired  $\mathcal{P}$  will be completed if we find a set system  $\mathcal{F}$  over  $X$  no three different members of which satisfy (2.4), and whose cardinality is greater than  $1.15^d$ . Let  $h(d)$  denote the greatest cardinality of such  $\mathcal{F}$ , i.e.,  $h(d) := \max\{|\mathcal{F}| : \mathcal{F} \subset 2^X, \text{ for all } A \neq B \neq C \in \mathcal{F}, A \cap B \not\subset C \text{ or } C \not\subset A \cup B\}$ .

**Lemma 2.5.**  $h(d) > (2/\sqrt{3})^{d-1}$ .

**Proof.** Let us choose independently the coordinates of the  $d$ -dimensional 0-1 vectors  $a_1, a_2, \dots, a_{2m}$  with probability  $\text{Prob}(a_{ij} = 0) = 1/2$ ,  $\text{Prob}(a_{ij} = 1) = 1/2$ ,  $1 \leq i \leq 2m$ ,  $1 \leq j \leq d$  and  $m = \lfloor (2/\sqrt{3})^{d-1} \rfloor$ . Then,

$$\text{Prob}(a, b, c \text{ hold for (2.4)}) = (3/4)^d. \quad (2.6)$$

Indeed, (2.4) means that for all  $1 \leq i \leq d$  neither  $a_i = b_i = 1, c_i = 0$  or  $a_i = b_i = 0, c_i = 1$  hold. The independency of the coordinates yields (2.6). Hence the expectation number

$$\begin{aligned} E(\text{the number of the triples } (a, b, c) \text{ satisfying (2.4)}) &= \\ &= 2m(2m-1)(2m-2)(3/4)^d < m. \end{aligned}$$

Hence the expected number of vectors to remain after the omission of the points which are vertices of a right angle is greater than  $2m - m = m$ , and for that set of vectors the conditions are already satisfied.  $\square$

Finally, since  $f(d) \geq \max\{2d-1, h(d)\}$ , (2.2) follows.  $\square$

A bit more complicated random process gives

$$h(d) > (\sqrt{2} - o(1))^d \sim 1.189 \dots^d. \quad (2.7)$$

Instead of Lemma 2.5, we can use the following theorem due to Frankl and the authors [9] to prove  $G_0(d) \geq f(d) > (1+c)^d$ .

**Theorem 2.8** [9]. *There exists a set-system  $\mathcal{F}$  over a  $d$ -element underlying set in which no set is covered by the union of two others and  $|\mathcal{F}| > 1.13^d$ .*

We have the following.

**Conjecture 2.9.** *There exists an absolute constant  $c > 0$  (not depending on  $d$ ) such that*

$$f(d) \leq G_0(d) < (2-c)^d. \quad (?)$$

The following fact says that it is not possible to choose more than  $2\sqrt{3}^d$  points from the vertices of a cube such that all angles among them are less than  $\pi/2$ . Even more, denote by  $G_0(\{0; 1\}^d)$  the greatest cardinality of a strictly antipodal  $\mathcal{P} \subset \{0; 1\}^d$ .

**Fact 2.10.**  $h(d) \leq G_0(\{0; 1\}^d) < \sqrt{2}(\sqrt{3})^d$ .

**Proof.** If  $\{a, b\}, \{c, d\} \subset \mathcal{P}$  are distinct pairs, then  $a + b \neq c + d$  (since  $a + b = c + d$  implies that the points  $a, c, b, d$  form a parallelogram which is contrary to the strict antipodality of  $\mathcal{P}$ ). Thus

$$\binom{|\mathcal{P}|+1}{2} = |\{a+b : a, b \in \mathcal{P}\}| \leq |\{0; 1; 2\}^d| = 3^d. \quad \square$$

Finally, we mention some more problems. Slightly generalizing the question about  $G(d)$  one is led to the problem of determining  $e(d, n)$ , the maximal number of antipodal pairs among the vertices of a  $d$ -polytope with  $n$  vertices. It is not hard to show that  $e(2, n) = \lfloor 3n/2 \rfloor$  (Grünbaum [11]), and that

$$e(3, n) \geq \lfloor \frac{1}{2}n \rfloor \lfloor \frac{1}{3}(n+1) \rfloor + \lfloor \frac{1}{3}n \rfloor + \lfloor \frac{1}{3}(3n+1) \rfloor.$$

**Conjecture of Grünbaum [12].** *Does the relation*

$$\lim \frac{e(d, n)}{n^2} = \frac{1}{2} - \frac{1}{2^{d-1}} \quad (?) \quad (2.11)$$

hold for all  $d \geq 2$ .

In analogy to the above, we define  $e_k(d, n)$ , the maximal number of  $k$ -antipodal pairs. Regarding  $e_0(d, n)$ , it is easy to prove that  $e_0(2, n) = n$  (Grünbaum [11]), but even the 3-dimensional case seems to be very complicated. Clearly,  $e_0(d, n) \geq d(d, n)$ , the maximal number of diameters of an  $n$ -element set in  $E_d$ . It is known that  $d(2, n) = n$  (Erdős [7]),  $d(3, n) = 2n - 2$  (Grünbaum [10], Heppes [13] and Straszewicz [16]), but for  $d \geq 4$

$$\lim d(d, n)/n^2 = \frac{1}{2} - 1/(2 \lfloor d/2 \rfloor)$$

(Erdős [7]).

Theorem 2.2 yields a new lower bound of the following problem: What is the order of the number  $c_d$  defined as follows: For a  $d$ -dimensional convex body  $K$ , denote by  $c(K)$  the minimal number of translates of  $K$  the union of which covers  $K$ . Then  $c_d$  is defined as the maximum of  $c(K)$  for all  $d$ -dimensional convex bodies  $K$ . It is easily seen that  $c_d \geq G_0(d)$ , but it is not known whether  $c_d = G_0(d)$  for  $d \geq 3$ .

**Corollary 2.12.**  $c_d \geq 1.15^d$ .

The following conjecture would extend Theorem 2.1.

**Conjecture 2.13.** *There exists a positive constant  $\varepsilon$  ( $\varepsilon$  independent of  $d$ ) for which the following is true: If the pointset  $\mathcal{P} \subset E_d$  and  $|\mathcal{P}| \geq 2^d + 1$ , then  $\mathcal{P}$  contains an angle greater than  $\pi/2 + \varepsilon$ .*

### 3. Pointsets with all angles small

Denote by  $\alpha(\mathcal{P})$  the greatest angle determined by the triples of the pointset  $\mathcal{P}$ , and  $\alpha_d(n) = \inf\{\alpha(\mathcal{P}) : |\mathcal{P}| = n, \mathcal{P} \subset E_d\}$ . We can write Theorems 2.1 and 2.2 and Conjecture 2.9 as follows:

$$\alpha_d(2^d + 1) > \pi/2, \quad \alpha_d(2^d) = \pi/2 \quad (\text{cf. 2.1}), \quad (3.1)$$

$$\alpha_d(1.15^d) < \pi/2 \quad (\text{cf. 2.2}), \quad (3.2)$$

$$\alpha_3(5) < \pi/2, \quad \alpha_3(6) = \pi/2 \quad (\text{Croft [3] and Schütte [15]}), \quad (3.3)$$

$$\exists c > 0: \alpha_d((2-c)^d) = \pi/2 \quad (?) \quad (\text{cf. 2.9}). \quad (3.4)$$

We were not able to establish an even weaker version of (3.4).

**Conjecture 3.5.** *For each  $\varepsilon > 0$  there exists a  $c(\varepsilon) > 0$  with*

$$\alpha_d((2-c)^d) > \pi/2 - \varepsilon.$$

But for  $\varepsilon$  large enough we can prove the following.

**Theorem 3.6.** *If  $0 < c < 1$  then*

$$\alpha_d((1+c)^d) > \pi/3 + c/4 - o(1). \quad (3.6a)$$

*Further, there exists a construction showing that*

$$\alpha_d((1+c)^d) < \pi/3 + \sqrt{c}. \quad (3.6b)$$

Clearly  $|\mathcal{P}| \geq 3$  implies  $\alpha(\mathcal{P}) \geq \pi/3$ . What Theorem 3.6 says is that there are exponentially many  $(1+c)^d$  points in  $E_d$  with all angles less than  $61^\circ$ , but that  $1.4^d$  points always determine an angle larger than  $72^\circ$ .

**Proof.** We start with the construction.

**Lemma 3.7.** *There exists a  $k$ -uniform set system  $\mathcal{F}$  over  $d$  elements such that  $|F_1 \cap F_2| < \epsilon k$  for each distinct  $F_1, F_2 \in \mathcal{F}$  and  $|\mathcal{F}| > (1 + 0.4\epsilon^2)^d$ .*

**Proof.** We choose  $F_1, F_2, \dots, F_i, \dots$  recursively. Suppose  $F_1, \dots, F_i$  are already chosen. Let  $\mathcal{F}_i = \{F \subset X : |F| = k, |F \cap F_j| \geq x\}$  for all  $1 \leq j \leq i$ . We can select an  $F_{i+1} \notin \mathcal{F}_i$  if  $|\mathcal{F}_i| < \binom{d}{k}$ . Thus this process can be continued for at least as many as

$$\binom{d}{k} / \binom{k}{x} \binom{d-x}{k-x}$$

steps. Putting  $k = d\epsilon/4$  and  $x = k\epsilon$  we prove (3.7) using Stirling's formula.  $\square$

Returning to the proof of (3.6b), put  $\epsilon = \sqrt{2.5c}$ , and let  $\mathcal{F}$  be the set system defined in (3.7). For the distances of the two vertices  $f_1, f_2$  corresponding to  $F_1, F_2 \in \mathcal{F}$  we have

$$\sqrt{2k(1-\epsilon)} < |f_1 - f_2| \leq \sqrt{2k},$$

i.e., the distances defined by  $\mathcal{F}$  are almost equal. Now a simple calculation gives (3.6b).

To prove (3.6a) let  $\mathcal{P} = \{P_1, P_2, \dots, P_m\} \subset E_d$  be a pointset with every angle less than  $\pi/3 + x$ . Hence the ratio of the smallest and largest sides of a triangle  $P_i P_j P_k$  is greater than  $\sin(\pi/3 - 2x)/\sin(\pi/3 + x) \geq 1 - 2x$ . So if the largest distance in  $\mathcal{P}$  is 1, then the smallest is at least  $(1 - 2x)^2 \geq 1 - 4x$  ( $x \leq 1/4$ ).

Let  $S$  be the smallest ball containing  $\mathcal{P}$  with center 0. By the Yung theorem [12], the radius of  $S$  is less than  $1/\sqrt{2} + (2/d) < 1/\sqrt{2}$ . Project the points of  $\mathcal{P}$  from 0 to the surface of  $S$ . The image of  $P_i$  is  $Q_i$ . It is easily seen that  $|P_i P_j| < 1/\sqrt{2}$  implies  $|Q_i Q_j| \geq |P_i P_j|$ . So if  $x$  is small enough ( $x < 0.07$ ) then any two point  $Q_i, Q_j$  have a distance at least  $1 - 4x$ . So we can apply Böröczki's quite sharp estimation [1] about the density of a packing of the  $d$ -dimensional sphere by congruent balls, to get  $|\mathcal{P}| = |\mathcal{Q}| < (1 - 4x)^{-d} 2d$ . However we can give a simple straightforward proof of this last step.

**Lemma 3.8.** *If  $\mathcal{Q}$  is a pointset on the surface of the sphere  $S$  of radius  $1/\sqrt{2}$ , and for all  $Q_1, Q_2 \in \mathcal{Q}$  we have  $|Q_1 Q_2| > 1 - y$  then  $|\mathcal{Q}| < d \cdot 2d(1 - y)^{-d}$ .*

**Proof.** Any ball with radius  $(1-y)/\sqrt{2}$  contains at most  $d+1$  points of  $\mathcal{Q}$ . Taking averages,

$$|\mathcal{Q}| \leq (d+1) \frac{\text{surface area of } S}{\text{maximal surface area of the intersection of } S \text{ with a sphere of radius } (1-y)/\sqrt{2}} \\ < (d+1)(1-y)^{-d}. \quad \square$$

#### 4. The greatest angle among $n$ points

Erdős and Szekeres proved [8] that  $2^n$  points in the plane always determine an angle  $> \pi(1 - (1/n))$  and Szekeres [17] proved that this result is best possible in the following sense: One can give, for every  $\varepsilon > 0$ ,  $2^n$  points in  $E_2$  no angle of which is greater than  $\pi(1 - (1/n)) + \varepsilon$ . I.e.,

$$\alpha_2(2^n) = \pi \left(1 - \frac{1}{n}\right) \quad (4.1)$$

([8] and [17]). So in general,

$$\alpha_2(n) = \pi \left(1 - \frac{1}{\log_2 n}\right) + O\left(\frac{1}{(\log_2 n)^2}\right). \quad (4.2)$$

In this section we are concerned with the  $d$ -dimensional version of this fact.

**Theorem 4.3.** *We have*

$$\pi \left(1 - \frac{4}{\sqrt[d-1]{\log_2 n}}\right) < \alpha_d(n) < \pi \left(1 - \frac{1}{\sqrt[d-1]{\log_2 n}}\right).$$

The proof shows that one cannot hope to get an essentially better result without a new estimation for the sphere-packing problem in  $d$ -dimension, which is far better than the existing ones. The proof used adapted the proofs in [8] and [17]. We need two facts.

(4.4). In the  $d$ -dimensional space there exist more than  $(1/\rho)^{d-1}$  lines going through the point 0, such that any two of them determine an angle greater than  $\rho$ .

(4.5). In the  $d$ -dimensional space there exist fewer than  $(4/\rho)^{d-1}$  lines going through 0, such that any other line going through 0 determines an angle less than  $\rho/2$  with some of them.

The facts (4.4) and (4.5) are in fact equivalent to the well-known sphere-packing and sphere-covering problems. There are much better estimations, but (4.4) and (4.5) are very easy to check, and the better known estimations would not eliminate the difference between the constants in (4.3).

**The proof of the upper bound.** Let  $e_1, e_2, \dots, e_m$ , with  $m > (1/\rho)^{d-1}$ , be a system of lines determined by (4.4). We are going to construct  $2^m$  points recursively such that every angle in the triples is less than  $\pi - \rho$ .

Let  $\mathcal{P}_1 = \{A, B\}$  where  $A, B \in e_1$ . Translating  $\mathcal{P}_1$  in the direction  $e_2$  we get  $A', B'$  and if  $A'$  and  $B'$  are far enough then  $AB'$  and  $A'B$  are almost parallel to  $e_2$ . Then translate the parallelogram  $AA'BB'$  in the direction  $e_3$  far enough ... and so on. After  $m - 1$  translations we get a construction showing  $\alpha_d(2^m) < \pi - \rho$ .

**The proof of the lower bound.** Let  $f_1, \dots, f_m$ , with  $m < (4/\rho)^{d-1}$ , be a set system of lines determined by (4.5), and let  $\mathcal{P} \subset E_d$  be a pointset with more than  $2^m$  points. Consider the complete graph with vertex set  $\mathcal{P}$  and colour its edges in the following way. For  $U, V \in \mathcal{P}$  the edge  $UV$  gets colour  $i$ ,  $1 \leq i \leq m$ , provided the angle between  $UV$  and  $f_i$  is less than  $\rho/2$ .

**Lemma 4.6** (see [8]). *If the edges of the complete graph with more than  $2^m$  vertices are coloured with  $m$  colours then there exists an odd circuit whose edges are of the same colour.*

Lemma 4.6 implies that there are  $U, V, W \in \mathcal{P}$  and an  $f_i$  such that the angles  $(UV, f)$ ,  $(VW, f)$  and  $(WU, f)$  are less than  $\rho/2$ . But then the greater angle in the triangle  $UVW$  is at least  $\pi - \rho$ .  $\square$

One more open problem:

Let  $\mathcal{P}$  be a pointset in  $E_d$  and  $0 < \alpha < \pi$ . Define  $f(\mathcal{P}, < \alpha)$  (resp.  $f(\mathcal{P}, > \alpha)$ ) as the number of angles in  $\mathcal{P}$  smaller (resp. greater) than  $\alpha$ . Put

$$f_d(n, < \alpha) := \min\{f(\mathcal{P}, < \alpha) : |\mathcal{P}| = n, \mathcal{P} \subset E_d\},$$

and

$$f_d(< \alpha) := \lim_{n \rightarrow \infty} f_d(n, < \alpha) / \frac{1}{2}n(n-1)(n-2).$$

$f_d(< \alpha)$  and  $f_d(> \alpha)$  show at least what percent of the angles are smaller (or greater) than  $\alpha$  for a pointset  $\mathcal{P}$  in  $E_d$ . Conway, Croft, Erdős and Guy in [2] investigate  $f_2$  and  $f_3$ . They show, for instance, that

$$\frac{1}{6} \leq f_2(> \frac{1}{2}\pi) \leq \frac{4}{25} \tag{4.7}$$

([2]). In higher dimensions no estimation is known for  $f_d$ . Even for  $d \leq 3$ ,  $f_d(< \alpha)$  is known for only a finite number of  $\alpha$ .

## References

- [1] K. Böröczki, Packing of spheres in spaces of constant curvature, *Acta Math. Acad. Sci. Hungar.* 32 (1978) 243–261.
- [2] J.H. Conway, H.T. Croft, P. Erdős and M.J.T. Guy, On the distribution of values of angles determined by coplanar points, *J. London Math. Soc.* (2) 19 (1979) 137–143.
- [3] H.T. Croft, On 6-point configurations in 3-space, *J. London Math. Soc.* 36 (1961) 289–306.
- [4] L. Danzer and B. Grünbaum, Über zwei Probleme bezüglich konvexer Körper von P. Erdős und von V.L. Klee, *Math. Z.* 79 (1962) 95–99.
- [5] P. Erdős, Problem 4306, *Amer. Math. Monthly* 55 (1948) 431.
- [6] P. Erdős, Some unsolved problems, *Michigan Math. J.* 4 (1957) 291–300.
- [7] P. Erdős, On sets of distances of  $n$  points in Euclidean space, *Magyar Tud. Akad. Mat. Kutató Int. Közl.* 5 (1960) 165–169.
- [8] P. Erdős and G. Szekeres, On some extremum problems in elementary geometry, *Ann. Univ. Sci. Budapest, Eötvös Sect. Math.* III/IV (1960–61) 53–62.
- [9] P. Erdős, P. Frankl and Z. Füredi, Families of finite sets in which no set is covered by the union of two others, *J. Comb. Theory, Ser. A* 33 (1982) to appear.
- [10] B. Grünbaum, A proof of Vázsonyi's conjecture, *Bull. Res. Council Israel* 6A (1956) 77–78.
- [11] B. Grünbaum, Strictly antipodal sets, *Israel J. Math.* 1 (1963) 5–10.
- [12] B. Grünbaum, *Convex Polytopes* (Interscience, London, 1967); *Pure and Applied Mathematics* 16.
- [13] A. Heppes, Beweis einer Vermutung von Vázsonyi, *Acta Math. Acad. Sci. Hungar.* 7 (1957) 463–466.
- [14] V.L. Klee, Unsolved problems in intuitive geometry, Mimeographed notes, Seattle, 1960.
- [15] K. Schütte, Minimale Durchmesser endlicher Punktmengen mit vorgeschriebenem Mindestabstand, *Math. Ann.* 150 (1963) 91–98.
- [16] S. Straszewicz, Sur un problème géométrique de P. Erdős, *Bull. Acad. Polon. Sci. Cl. (III)* 5 (1957) 39–40.
- [17] G. Szekeres, On an extremum problem in the plane, *Amer. J. Math.* 63 (1941) 208–210.