ON A PROBLEM OF DEZA AND FRANKL

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M. Deza and P. Frankl in [1] posed the following question:

Let $A_1, A_2, \ldots, A_m$ be a family of different sets such that

(i) $$|A_t| = k \quad \text{for every} \quad 1 \le t \le m,$$

(ii) $$|A_{i_1} \cap A_{i_2}| = \lambda_1 \quad \text{for every} \quad 1 \le i_1 < i_2 \le m,$$

(iii) $$|A_{j_1} \cap A_{j_2} \cap A_{j_3}| = \lambda_2 \quad \text{for every} \quad 1 \le j_1 < j_2 < j_3 \le m,$$

and $\lambda_1 > \lambda_2 \ge 0$. A theorem of Deza [2] implies $m < k^2$. Is it true that $m = o(k^2)$?

We prove that the answer is positive, moreover we show that the condition in (iii) is so strong, that conditions (i) and (ii) are essentially superfluous. We prove

THEOREM. *Let r be any integer, $r \ge 3$, and let $A_1 \ldots A_m$ distinct sets, such that*

(iv) $$|A_{i_1} \cap \ldots \cap A_{i_r}| = \lambda$$

for every $1 \le i_1 < \ldots < i_r \le m$. Then either exists a set S with λ elements such that

$$S \subset A_i$$

for every $1 \le i \le m$, or

(v) $$m \le \nu + r - 2$$

where $\nu = \min\{|A_{t_1} \cap \ldots \cap A_{t_{r-2}}| : 1 \le t_1 < \ldots < t_{r-2} \le m\}$.

REMARKS. 1. In our original question the first case cannot occur so in this case $m \le k+1$.

2. The bound (v) of the theorem cannot be improved in general as show by $(\lambda + r - 1)$-subsets of a set on $(\lambda + r)$ elements.

3. The problem of investigating families satisfying

ARS COMBINATORIA, VOL. 13 (1982), pp. 221-222

$|A_{i_1} \cap A_{i_2} \cap \ldots \cap A_{i_r}| = \lambda$ was already by V.T. Sos [4]. That is the problem which is interesting.

Proof. If $|A_{j_1} \cap \ldots \cap A_{j_{r-1}}| = \lambda$ for some $j_1, \ldots, j_{r-1}$, then $S = A_{j_1} \cap \ldots \cap A_{j_{r-1}}$ must be a subset of any other A_i, so the first case holds. In this case there exists no bound on m, as $\{A_1, \ldots, A_m\}$ can be arbitrarily enlarged without violating (iv).

So we may suppose that $\min\{|A_{j_1} \cap \ldots \cap A_{j_{r-1}}| : 1 \le j_1 < \ldots < j_{r-1} \le m\} = \mu > \lambda$. Let $A_1 \cap \ldots \cap A_{r-2} = X$. We may suppose $|X| = \nu$. $|A_i \cap X| \ge \mu$ for any $i > r-2$. Thus $A_i \cap X \ne A_j \cap X$ if $r-2 < i < j \le m$, because $\mu > \lambda$ and (iv) holds. Let $\mathcal{B} = \{X \cap A_i : r-2 < i \le m\}$. The intersection of any two sets in $\mathcal{B}$ has exactly λ elements. A theorem of Ryser [3] states that in this case $|\mathcal{B}|$ is at most the cardinality of the ground set, i.e. $|\mathcal{B}| = m - (r-2) \le |X| = \nu$. Q.E.D.

References

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