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SET-SYSTEMS WITH PRESCRIBED CARDINALITIES FOR PAIRWISE INTERSECTIONS

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Suppose that $\mathscr{A}$ is a finite set-system on N points, and for every two different $A, A' \in \mathscr{A}$ we have $|A \cap A'| = 0$ or r. Then we prove that

$$|\mathscr{A}| \leq \binom{\lfloor N/r \rfloor}{2} + \lfloor N/r \rfloor + (N - r\lfloor N/r \rfloor)$$

whenever $N > N_0(r)$. The extremal family is unique and consists of $2r$, r and 1-elements sets only. The assumption $N > N_0(r)$ can not be omitted. We state some further results and problems.

1. Introduction

1.1. The following well-known theorem was proved by H.J. Ryser in 1968 (see [13]):

Let X be a finite set and λ a positive integer. Let $\mathscr{A}$ be a family of subsets of X such that $A, B \in \mathscr{A}$ $(A \neq B)$ implies $|A \cap B| = \lambda$. Then

$$|\mathscr{A}| \leq |X|. \tag{1}$$

One can weaken the restriction imposed on the cardinality on the intersections as follows. Given a set $\Lambda = \{\lambda_1, \lambda_2, \dots, \lambda_s\}$ of integers, we replace the condition $|A \cap B| = \lambda$ by $|A \cap B| \in \Lambda$ for all $A, B \in \mathscr{A}$. What can be said about $|\mathscr{A}|$? A set-system of this type is called *Λ-system*. Let $f(N, \Lambda)$ denote the maximal cardinality of a Λ-system, where N stands for $|X|$. This problem was posed in [15]. The purpose of this paper is to determine the order of magnitude of $f(N, \Lambda)$ for some particular sets Λ.

1.2. The most investigated case is that of uniform set-systems $\mathscr{A}$. Let

$$f(N, k, \Lambda) = \max\{|\mathscr{A}| : |A \cap B| \in \Lambda \text{ for all } A, B \in \mathscr{A}, \text{ and } |A| = k \text{ for all } A \in \mathscr{A}\}$$

As a first result, we mention the following theorem of D.K. Ray-Chaudhuri and

R.M. Wilson [14]:

$$f(N, k, \Lambda) \leq \binom{N}{|\Lambda|}.$$

But most results deal with the situation when N is very large compared to k.

The well-known Erdös–Ko–Rado theorem [7] can also be formulated in the above terms:

$$f(N, k, \{i, i+1, \ldots, k-1\}) = \binom{N-i}{k-i} \quad \text{if } N > N_0(k).$$

P. Frankl has a general method for obtaining sharp upper bounds on $f(N, k, \Lambda)$ for a very broad class of Λ. (See [5, 9, 10, 11]). For further results see the theorems of M. Deza, P. Erdös, P. Frankl, G. Katona and N.M. Singhi [4, 6, 8, 12].

1.3. Here we are going to need only the following theorem of L Babai and P. Frankl [1], which is a stronger version of an earlier theorem of M. Deza, P. Erdös and N.M. Singhi [6]:

Suppose that the greatest common divisor of the members of Λ does not divide k. Then

$$f(N, k, \Lambda) \leq N. \tag{2}$$

2. Results

2.1. *The maximal* $\{0, r\}$*-system*

In this paper we investigate the (non-uniform) cases $\Lambda = \{0, r\}$ (cf. [6]) and $\{0, 1, 3\}$ or $\{0, 2, 3\}$.

Example 1. If N is a multiple of r then let $X = S_1 \cup S_2 \cup \cdots \cup S_{N/r}$, where $|S_1| = |S_2| = \cdots = |S_{N/r}| = r$. Let $\mathcal{A}$ consist of the $2r$-sets $S_i \cup S_j$, and r-sets S_i. If r does not divide N, then $X = S_1 \cup S_2 \cup \cdots \cup S_{\lfloor N/r \rfloor} \cup S_0$, and we can join to $\mathcal{A}$ the $N - r\lfloor N/r \rfloor$ one point sets in S_0. This shows

$$f(N, \{0, r\}) \geq \binom{\lfloor N/r \rfloor}{2} + \lfloor N/r \rfloor + (N - r\lfloor N/r \rfloor). \tag{3}$$

We show that this evident lower bound is best possible if N is large enough.

Theorem 1. *If $N > N_0(r)$ ($N > 1000r^5$), then the family $\mathcal{A}$ in Example 1 is maximal, i.e. we have equality in (3). The extremal family is unique.*

If $N < 2r^2 - r$, then Example 1 is not maximal, since in this case its cardinality is less than N. However trivially $f(N, \{0, r\}) \geq N$ for every N. The following construction disproves the earlier conjecture of the author $f(N, \{0, r\}) = \max\{N;$ the cardinality of Example 1$\}$.

Example of P. Frankl (unpublished). Suppose that $H = (h_{ij})$ is an Hadamard matrix of rank $4r$, i.e. $h_{ij} = \pm 1$, $(h_i, h_j) = 4r\delta_{ij}$ and $h_{1i} = 1$ $(1 \leq i, j \leq 4r)$. Let $\mathcal{H} = \{B_i^e: 2 \leq i \leq 4r,\ e = +1 \text{ or } -1,\ B_i^e = \{j: h_{ij} = e\}\}$, then $\mathcal{H}$ is a $2r$-uniform $\{0, r\}$-system with $8r-2$ subsets.

Let $X = S_1 \cup S_2 \cup \cdots \cup S_{\lfloor N/4r \rfloor} \cup S_0$ where $|S_1| = |S_2| = \cdots = |S_{\lfloor N/4r \rfloor}| = 4r$ and $|S_0| = N - 4r\lfloor N/4r \rfloor$. Put the above set-system $\mathcal{H}$ on every S_i for $i \geq 1$ and consider the set-system consisting of the one-point sets of S_0. These set-systems form a $\{0, r\}$-system $\mathcal{A}$, and

$$|\mathcal{A}| = (8r-2)\lfloor N/4r \rfloor + (N - 4r\lfloor N/4r \rfloor).$$

If $N < 4r^2 - 2r$, then the cardinality of $\mathcal{A}$ is greater than the cardinality of Example 1.

2.2. *The stability of the extremum*

Visibly, the $2r$-sets play the leading part in the maximal $\{0, r\}$-family. This property of the extremum is fairly stable in the following sense. If a $\{0, r\}$-family $\mathcal{A}$ does not contain $2r$-sets, then $|\mathcal{A}| \leq \frac{1}{3}|\text{Example } 1| + O(N)$. More generally:

Theorem 2. *If $\mathcal{A}$ is a $\{0, r\}$-family and for all $A \in \mathcal{A}$ $|A| \neq r, 2r, \ldots, kr$ then*

$$|\mathcal{A}| \leq \frac{N(N-r)}{r^2 k(k+1)} \tag{4}$$

whenever N is large enough ($N > 1000r^5k^7$). Equality holds in (4) *if and only if $\mathcal{A}$ has the structure of Example* 2 *(see below).*

A $(k+1)$-uniform set-system $\mathcal{S}$ over the underlying set Y with v elements is an $S(v, k+1, 2)$ *Steiner-system* if for any 2-tuple $\{y_1, y_2\}$ of elements of Y, there is exactly one member of $\mathcal{S}$ containing $\{y_1, y_2\}$. By a well-known theorem of R.M. Wilson [16] if $(v-1)/k$ and $v(v-1)/k(k+1)$ are integers and $v > v_0(k)$, then there exists an $S(v, k+1, 2)$.

Example 2. Suppose that N/r is an integer and there exists an $S(N/r, k+1, 2)$ Steiner-system $\mathcal{S}$ over the underlying-set $Y = \{y_1, y_2, \ldots, y_{N/r}\}$. Let $X = S_1 \cup S_2 \cup \cdots \cup S_{N/r}$ where $|S_i| = r$, $|X| = N$ and

$$\mathcal{A} = \{S_{i_1} \cup S_{i_2} \cup \cdots \cup S_{i_{k+1}}: \{y_{i_1}, y_{i_2}, \ldots, y_{i_{k+1}}\} \in \mathcal{S}\}.$$

Then $\mathcal{A}$ is a $\{0, r\}$-family consisting of $(k+1)r$-sets only and with cardinality $N(N-r)/r^2k(k+1)$.

(In general, the optimal system can be determined if N/r or $(v-1)/k$ or $v(v-1)/k(k+1)$ are not integers, too.)

2.3. *Some further Λ*

There is only one Λ for which $f(N, \Lambda)$ is exactly known. This is a very simple case:

Proposition 1. $f(N, \{0, 1, \ldots, r\}) = \binom{N}{1} + \binom{N}{2} + \cdots + \binom{N}{r+1}$.

Proof. Let $\mathscr{A}$ be a $\{0, 1, \ldots, r\}$ family over X, and

$$\mathscr{A}' = \{A \in \mathscr{A}: |A| \leq r\}, \qquad \mathscr{A}'' = \{A \in \mathscr{A}: |A| \geq r+1\}.$$

Clearly $|\mathscr{A}'| \leq \binom{N}{1} + \binom{N}{2} + \cdots + \binom{N}{r}$. Each $(r+1)$-subsets of X is contained in at most one member of $\mathscr{A}''$, thus $|\mathscr{A}''| \leq \binom{N}{r+1}$. □

Proposition 2. $N^3/1500 < f(N, \{0, 1, 3\}) < N^3/6$.

Proof. The Example 3 gives the lower bound (see below). For the proof of the upper bound consider an arbitrary $\{0, 1, 3\}$-system $\mathscr{A}$. The number of 1-element subsets in $\mathscr{A}$ is not greater than N. Set $\mathscr{A}[x, y] = \{A \in \mathscr{A}: \{x, y\} \subset A\}$ $(x, y \in X)$. Since $\mathscr{A}[x, y]$ is a $\{3\}$-family, by (1) $|\mathscr{A}[x, y]| \leq (N-2)$. Thus the number of members of $\mathscr{A}$ with more than 2 elements is at most

$$\tfrac{1}{3} \sum |\mathscr{A}[x, y]| \leq \tfrac{1}{3} \binom{N}{2} (N-2). \qquad \square$$

Example 3. Let $N = 2^t - 1$ and let X be the points of the t-dimensional vector space over GF(2) except $\mathbf{0}$. Set

$$\mathscr{A} = \{\{S - \{\mathbf{0}\}\}: S \text{ is a 3-dimensional subspace of } X\}.$$

Then $\mathscr{A}$ is a $\{0, 1, 3\}$-system with cardinality $N(N-1)(N-3)/168$.

It would be suspected $f(N, \Lambda)$ has order of magnitude $N^{|\Lambda|}$ in general. However the following result shows that this is not the case.

Theorem 3. $\frac{1}{2}(N+1)(N-4) \leq f(N, \{0, 2, 3\}) < 50N^2$.

The set-system consisting of all 2, 3 and 4-subsets containing two fixed points yields the lower estimation. The proof of the upper bound to be presented in Chapter 5 can be improved a little bit by more precise computation of details. For instance the coefficient 50 can be replaced by 5.7, but I could not eliminate the gap of order N^2 between the upper and lower bounds.

3. The proof of Theorem 1

3.1. *Lemmas*

Lemma 1. *If $\mathcal{A}$ is an $\{r\}$-family over X (i.e. $|A'\cap A''|=r$ for all $A'\neq A''\in\mathcal{A}$), $|X|=N$ and $\min\{|A|: A\in\mathcal{A}\}>r$, then*

$$|\mathcal{A}|\leq\max\left\{\frac{N}{\min|A|-r};\ \max|A|^2\right\}.$$

This lemma is an improvement of the theorem of Ryser mentioned in (1) and really is a reformulation of the following theorem due to M. Deza [3]:

> If $\mathcal{A}$ is a finite set-system and for any different members A_1, A_2 of $\mathcal{A}$ $|A_1\cap A_2|=r$ holds and $|\mathcal{A}|\geq\max|A|^2-\max|A|+2$ then $|\bigcap_{A\in\mathcal{A}} A|=r$. (5)

(I.e. $\mathcal{A}$ is a Δ-system. The set-system $\mathcal{A}$ is a *Δ-system* if the parts of the sets outside $\bigcap\mathcal{A}$ are disjoint. The $\bigcap\mathcal{A}$ is called the *nucleus* of the Δ-system.)

Lemma 2. *If $\mathcal{B}$ is a set-system over X and for every $B'\neq B''\in\mathcal{B}$, $|B'\cap B''|\leq r$ and $|\mathcal{B}|>(\min|B|)/r$, then*

$$|X|>\frac{1}{2r}\min|B|^2.$$

This implies that the number of sets having more than $\sqrt{2rN}$ elements in a $\{0, r\}$-system is at most $\sqrt{2N/r}$.

Proof. Let $\mathcal{B}=\{B_1, B_2,\ldots, B_i,\ldots\}$ and $\min|B|=K$. The inequality $|B_i\cap B_j|\leq r$ implies $|B_i-(\bigcup_{j<i}B_j)|\geq|B_i|-(i-1)r\geq K-(i-1)r$. Using $|\mathcal{B}|>\min|B|/r$, it follows that

$$|X|\geq\left|\bigcup_{i=1}^{\lfloor K/r\rfloor+1}B_i\right|\geq\left(\left[\frac{K}{r}\right]+1\right)K-\left(1+2+\cdots+\left[\frac{K}{r}\right]\right)r>K^2/2r.\qquad\square$$

3.2. *A simple upper bound for $f(N,\{0, r\})$*

Split $\mathcal{A}$ into four parts as follows

$$\mathcal{A}_1=\{A\in\mathcal{A}: |A|<2r\},$$
$$\mathcal{A}_2=\{A\in\mathcal{A}: 2r\leq|A|<\sqrt{N/r}\},$$
$$\mathcal{A}_3=\{A\in\mathcal{A}: \sqrt{N/r}\leq|A|\leq\sqrt{2Nr}\},$$
$$\mathcal{A}_4=\{A\in\mathcal{A}: \sqrt{2Nr}<|A|\}.$$

First, we shall estimate their cardinalities separately. (For a set-system $\mathcal{B}$ denote $\{B\in\mathcal{B}: x\in B\}$ by $\mathcal{B}[x]$.) We shall apply the Ryser theorem mentioned in (1) several times. E.g. $\mathscr{A}[x]$ satisfies the assumptions of (1) thus

$$|\mathscr{A}[x]| \leq N. \tag{6}$$

Define the relation $\sim$ on the members of $\mathscr{A}_1$ as follows. For $A, A' \in \mathscr{A}$ let $A \sim A'$ iff $A \cap A' \neq \emptyset$. Obviously, $\sim$ is an equivalence relation. Denote the equivalence classes by $\mathscr{A}_1^{(1)}, \mathscr{A}_1^{(2)}, \ldots, \mathscr{A}_1^{(k)}$. By definition, $(\bigcup \mathscr{A}_1^{(i)}) \cap (\bigcup \mathscr{A}_1^{(j)}) = \emptyset$ whenever $i \neq j$. Furthermore $\mathscr{A}_1^{(i)}$ satisfies the assumptions of (1) thus $|\mathscr{A}_1^{(i)}| \leq |\bigcup \mathscr{A}_1^{(i)}|$, hence

$$|\mathscr{A}_1| = \sum |\mathscr{A}_1^{(i)}| \leq \sum \left|\bigcup \mathscr{A}_1^{(i)}\right| \leq |X| = N.$$

Applying Lemma 1 for $\mathscr{A}_2[x]$

$$|\mathscr{A}_2[x]| \leq \max\left\{\frac{N}{r} : \max_{A \in \mathscr{A}_2} |A|^2\right\} = \frac{N}{r}. \tag{7}$$

Thus

$$|\mathscr{A}_2| \leq \frac{1}{\min\limits_{A \in \mathscr{A}_2} |A|} \sum_{x \in X} |\mathscr{A}_2[x]| \leq \frac{1}{2r} \frac{N}{r} N = \frac{N^2}{2r^2}. \tag{8}$$

Finally, applying (6) for $\mathscr{A}_3 \cup \mathscr{A}_4$ we get

$$|\mathscr{A}_3 \cup \mathscr{A}_4| \leq \frac{1}{\min |A|} \sum_{x \in X} |(\mathscr{A}_3 \cup \mathscr{A}_4)[x]|$$
$$\leq \frac{1}{\sqrt{N/r}} N N = \sqrt{r}\sqrt{N} N.$$

Summing up these inequalities

$$|\mathscr{A}| = |\mathscr{A}_1| + |\mathscr{A}_2| + |\mathscr{A}_3 \cup \mathscr{A}_4| \leq N + \frac{N^2}{2r^2} + \sqrt{r}\sqrt{N} N. \tag{9}$$

Hence we already proved Theorem 1 as far as the order of magnitude is concerned.

3.3. *The 2r-sets play the main role in the extremal family*

Suppose that N is large enough ($N > 1000r^5$) and that the set-system $\mathscr{A}$ is maximal, i.e. $|\mathscr{A}| = f(N, \{0, r\})$. Then by Example 1

$$|\mathscr{A}| \geq \frac{N^2}{2r^2} - \frac{N}{2r^2}(r-2) > \frac{N^2}{2r^2} - \frac{N}{2r}. \tag{10}$$

Split $\mathscr{A}_2$ into three parts

$$\mathscr{A}_2' = \{A \in \mathscr{A}_2 : |A| = 2r\},$$
$$\mathscr{A}_2'' = \{A \in \mathscr{A}_2 : 2r < |A| < 3r\},$$
$$\mathscr{A}_2''' = \{A \in \mathscr{A}_2 : 3r \leq |A|\}.$$

By the theorem recalled in (2):

$$|\mathscr{A}_2''| \leq (r-1)N. \tag{11}$$

By (7) we get

$$N\frac{N}{r} \geq \sum_{x\in X} |\mathscr{A}_2[x]| = \sum_{A\in\mathscr{A}_2} |A| \geq |A_2' \cup \mathscr{A}_2''|\, 2r + |\mathscr{A}_2'''|\, 3r.$$

Consequently

$$N^2/2r^2 \geq |\mathscr{A}_2| + \tfrac{1}{2}\, |\mathscr{A}_2'''|.$$

By (10)

$$|\mathscr{A}_1| + |\mathscr{A}_2| + |\mathscr{A}_3 \cup \mathscr{A}_4| + N/2r > N^2/2r^2 > |\mathscr{A}_2| + \tfrac{1}{2}\, |\mathscr{A}_2'''|.$$

Thus

$$|\mathscr{A}_2'''| < 2\left(|\mathscr{A}_1| + |\mathscr{A}_3 \cup \mathscr{A}_4| + \frac{N}{2r}\right) \leq 2\left(N + \frac{N}{2r} + \sqrt{r}\,\sqrt{N}\,N\right). \tag{12}$$

Thus from (10)–(12):

$$\begin{aligned} |\mathscr{A}_2'| &\geq \max |\mathscr{A}| - |\mathscr{A}_1| - |\mathscr{A}_2''| - |\mathscr{A}_2'''| - |\mathscr{A}_3 \cup \mathscr{A}_4| \\ &> \left(\frac{N^2}{2r^2} - \frac{N}{2r}\right) - N - (r-1)N - N\left(2 + \frac{1}{r}\right) - 2\sqrt{r}\,\sqrt{N}\,N - \sqrt{r}\,\sqrt{N}\,N \\ &> \frac{N^2}{2r^2} - 4\sqrt{r}\,\sqrt{N}\,N. \end{aligned} \tag{13}$$

This means that the greatest part of a maximal $\mathscr{A}$ consists of $2r$-sets.

3.4. *The structure of a maximal* $\{0, r\}$*-family is similar to Example* 1

Let us denote by S the set of those points of X which have small degree in $\mathscr{A}_2'$ i.e.

$$S =: \{x \in X : |\mathscr{A}_2'[x]| < \sqrt{2rN}\},$$

and s denotes the cardinality of S. Applying (13) we get

$$\begin{aligned} |\mathscr{A}_2'| &= \frac{1}{2r} \sum_{x\in X} |\mathscr{A}_2'[x]| \\ &= \frac{1}{2r} \sum_{x\in X-S} |\mathscr{A}_2'[x]| + \frac{1}{2r} \sum_{x\in S} |\mathscr{A}_2'[x]| \\ &< \frac{1}{2r}\left((N-s)\frac{N}{r} + s\sqrt{2rN}\right\} \\ &= \frac{N^2}{2r^2} - \frac{s}{2r}\left(\frac{N}{r} - \sqrt{2rN}\right). \end{aligned}$$

Since if $N > 1000r^5$, then $\sqrt{2Nr} < N/5r$, comparing this with (13) we get

$$s < 10r^2\sqrt{r}\sqrt{N}. \tag{14}$$

Remark that $s<\frac{1}{2}N$ because N is large enough. Let us define the following equivalence relation over the points of $X-S$.

$$x \sim y \iff |\mathscr{A}_2'[x] \cap \mathscr{A}_2'[y]| \geq 2,$$

i.e. $x \sim y$ iff there exist two different members A, A' of $\mathscr{A}_2'$ such that $\{x, y\} \subset A \cap A'$. Since $\sqrt{2rN} > 4r^2$, by the Deza theorem (see (5)) $\mathscr{A}_2'[x]$ and $\mathscr{A}_2'[y]$ is a Δ-system, i.e.

$$A \cap A' = \bigcap_{A_i \in \mathscr{A}_2'[x]} A_i = \bigcap_{A_j \in \mathscr{A}_2'[y]} A_j.$$

Consequently $\mathscr{A}_2'[x] = \mathscr{A}_2'[y]$.

Thus the set $X-S$ can be partitioned into r-elements equivalence classes $S_1, S_2, \ldots, S_t$ where S_i's are the nuclei.

$$\text{If} \quad x \in S_i \quad \text{and} \quad x \in A \in \mathscr{A}_2', \quad \text{then} \quad S_i \subset A. \tag{15}$$

But (15) holds for all $A \in (\mathscr{A}_1 \cup \mathscr{A}_2 \cup \mathscr{A}_3)$. Indeed, if $A \in (\mathscr{A}_1 \cup \mathscr{A}_2 \cup \mathscr{A}_3)$, i.e. $|A| \leq \sqrt{2rN}$ and $x \in A \cap S_i$ but $S_i \not\subset A$, then A intersects all members of $\mathscr{A}_2'[x]$ outside S_i. However the parts of the members of $\mathscr{A}_2'[x]$ lying outside S_i are pairwise disjoint, thus this leads to the contradiction $|A| > |\mathscr{A}_2'[x]| \geq \sqrt{2rN}$. Thus

$$\text{If } x \in S_i \text{ and } x \in A \in (\mathscr{A}_1 \cup \mathscr{A}_2 \cup \mathscr{A}_3), \text{ then } S_i \subset A. \tag{16}$$

3.5. *$s \leq r-1$, i.e. the sets S_i fill X as far as possible*

Now split $\mathscr{A}$ into four classes according to the number of elements of $A \cap S$ and $A \cap (X-S)$.

$$\mathscr{A}^{(1)} = \{A \in \mathscr{A} - \mathscr{A}_4 : |A \cap (X-S)| > r\} \cup \{A \in \mathscr{A} - \mathscr{A}_4 : A \subset (X-S)\},$$

$$\mathscr{A}^{(2)} = \{A \in \mathscr{A} - \mathscr{A}_4 : |A \cap (X-S)| = r \text{ and} |A \cap S| = r\},$$

$$\mathscr{A}^{(3)} = \{A \in \mathscr{A} - \mathscr{A}_4 : |A \cap S| > 0 \text{ but } |A \cap S| \neq r\},$$

$$\mathscr{A}^{(4)} = \mathscr{A}_4$$

It is easy to check that every $A \in \mathscr{A}$ belongs to one or more of the classes $\mathscr{A}^{(i)}$.

Since there are no two different sets A, B in $\mathscr{A}^{(1)}$ such that $A \cap (X-S) = B \cap (X-S)$, we have $|\mathscr{A}^{(1)}| = |\{A \cap (X-S) : A \in \mathscr{A}^{(1)}\}|$. Moreover $X-S$ is disjoint union of the sets $S_1, S_2, \ldots, S_t$, we can apply Proposition 1:

$$|\mathscr{A}^{(1)}| \leq \binom{(N-s)/r}{2} + \frac{N-s}{r} \leq \frac{N^2}{2r^2} - s\frac{N}{r^2} + \frac{s^2}{2r^2} + \frac{N}{2r}. \tag{17.1}$$

Consider the traces of $\mathscr{A}^{(3)}$ on S. These are all different and form a $\{0, r\}$-system on S, thus by (9)

$$|\mathscr{A}^{(3)}| \leq f(s, \{0, r\}) < \frac{s^2}{2r^2} + \sqrt{r}\sqrt{s}\,s + s. \tag{17.2}$$

Finally (as $\mathscr{A}^{(2)} \subset \mathscr{A}_2'$)

$$|\mathscr{A}^{(2)}| \leq \frac{1}{r} \sum_{x \in S} |\mathscr{A}^{(2)}[x]| \leq \frac{1}{r} \sum_{x \in S} |\mathscr{A}_2'[x]| \leq \frac{1}{r} s\sqrt{2rN}, \tag{17.3}$$

and by Lemma 2

$$|\mathscr{A}^{(4)}| = |\mathscr{A}_4| < \sqrt{2N/r}. \tag{17.4}$$

Summing (17.1)–(17.4) and comparing the result with (10) we get

$$\frac{N^2}{2r^2} - \frac{N}{2r} + \frac{N}{r^2} \leq |\mathscr{A}| < \frac{N^2}{2r^2} - s\frac{N}{r^2} + \frac{s^2}{2r^2} + \frac{N}{2r}$$

$$+ \frac{s^2}{2r^2} + \sqrt{r}\sqrt{s}\, s + s + s\sqrt{\frac{2N}{r}} + \sqrt{\frac{2N}{r}}$$

Hence multiplying by r^2 we get

$$s(N - s - r^2\sqrt{r}\, s - r^2 - r\sqrt{2rN}) < (r-1)N + r\sqrt{2rN}.$$

By (14) we can see that the coefficient of s on the left hand side is $N - o(N)$ thus we get $s < r$ provided $N > 400r^5$ i.e.

$$s \leq r - 1. \tag{18}$$

3.6. *The extremum has no large sets*

The proof of theorem will be complete showing $\mathscr{A}_4 = \emptyset$. Suppose, indirectly that $B \in \mathscr{A}_4$, $|B| > \sqrt{2rN}$. Let

$$\mathscr{S}_1 = \{S_i : 0 < |S_i \cap B| < r\}, \qquad \mathscr{S}_2 = \{S_j : |S_j \cap B| = r\}.$$

(16) and (18) imply that

$$|\mathscr{A}_1| \leq \lfloor N/r \rfloor - |\mathscr{S}_1| + s = \lfloor N/r \rfloor + (N - r\lfloor N/r \rfloor) - |\mathscr{S}_1|,$$

$$|\mathscr{A}_2 \cup \mathscr{A}_3| \leq \binom{\lfloor N/r \rfloor}{2} - \binom{|\mathscr{S}_2|}{2},$$

$$|\mathscr{A}_4| \leq \sqrt{2N/r},$$

that is

$$|\text{Ex. } 1| \leq |\mathscr{A}| \leq |\text{Ex. } 1| + \sqrt{2N/r} - |\mathscr{S}_1| - \binom{|\mathscr{S}_1|}{2}. \tag{19}$$

Since $(r-1)\,|\mathscr{S}_1| \leq (r-1)\sqrt{2N/r} \leq (r-1)\,|B|/r$ we get $|\mathscr{S}_2| > |B|/r^2$ thus from (19)

$$\sqrt{2N/r} \geq \binom{|\mathscr{S}_2|}{2} > \frac{|\mathscr{S}_2|^2}{4} \geq \frac{1}{4r^4}\, 2rN$$

which is a contradiction if N is large enough. □

4. The proof of Theorem 2

4.1. This part coincides with 3.1. (We will use the same lemmas as in the proof of Theorem 1.)

4.2. Split the set-system $\mathcal{A}$ according to the cardinalities of its members:

$$\mathcal{A}_1=\{A\in\mathcal{A}: |A|<(k+1)r\},$$
$$\mathcal{A}_2=\{A\in\mathcal{A}: (k+1)r\leq|A|\leq\sqrt{N/kr}\},$$
$$\mathcal{A}_3=\{A\in\mathcal{A}: \sqrt{N/kr}<|A|\leq\sqrt{2Nr}\},$$
$$\mathcal{A}_4=\{A\in\mathcal{A}: \sqrt{2Nr}<|A|\}.$$

Estimate their cardinalities analogously to 3.2.

$$|\mathcal{A}_1|<(k+1)rN,$$
$$|\mathcal{A}_2|\leq\frac{1}{\min_{A\in\mathcal{A}_2}|A|}\sum_{x\in X}|\mathcal{A}_2[x]|\leq\frac{1}{(k+1)r}N\frac{N}{kr},$$
$$|\mathcal{A}_3\cup\mathcal{A}_4|\leq\sqrt{kr}\sqrt{N}N.$$

4.3. By similar arguments as 3.3., suppose that $|\mathcal{A}|$ is maximal, and split $\mathcal{A}_2$ into three parts.

$$|\mathcal{A}|>N^2/r^2k(k+1)-(2/r)N; \tag{20}$$
$$\mathcal{A}_2'=\{A\in\mathcal{A}_2: |A|=(k+1)r\},$$
$$\mathcal{A}_2''=\{A\in\mathcal{A}_2: (k+1)r<|A|<(k+2)r\},$$
$$\mathcal{A}_2'''=\{A\in\mathcal{A}_2: |A|\geq(k+2)r\}.$$

Then $|\mathcal{A}_2''|\leq(r-1)N$. We get

$$\frac{|\mathcal{A}_2'''|}{k+1}+|\mathcal{A}_2|\leq\frac{N^2}{r^2k(k+1)}\leq|\mathcal{A}_1|+|\mathcal{A}_2|+|\mathcal{A}_3\cup\mathcal{A}_4|+\frac{2}{r}N$$

Thus $|\mathcal{A}_2'''|<(k+1)(rN+\sqrt{kr}\sqrt{N}N)$. Hence

$$|\mathcal{A}_2'|>N^2/r^2k(k+1)-(k+2)\sqrt{kr}\sqrt{N}N.$$

4.4. In a similar way we can define the set S as in 3.4

$$S:=\{x\in X: |\mathcal{A}_2'[x]|<\sqrt{2rN}\},\qquad s:=|S|.$$

Carrying out similar calculation we get

$$s<2k(k+1)(k+2)\sqrt{k}\sqrt{r}r^2\sqrt{N}. \tag{21}$$

Split $X-S$ the equivalence classes $S_1, S_2, \dots, S_t$.

$$x\sim y \quad\Leftrightarrow\quad |\mathcal{A}_2'[x]\cap\mathcal{A}_2'[y]|\geq 2$$

It holds true that $|S_i|=r$ and if $x\in S_i$, $x\in A\in(\mathcal{A}_1\cup\mathcal{A}_2\cup\mathcal{A}_3)$, then $S_i\subset A$.

4.5. Similarly by 3.5 let

$$\mathcal{A}^{(1)}=\{A\in(\mathcal{A}-\mathcal{A}_4)\colon |A\cap(X-S)|\geqslant(k+1)r\},$$
$$\mathcal{A}^{(2)}=\{A\in(\mathcal{A}-\mathcal{A}_4)\colon |A\cap(X-S)|=kr \text{ and } |A\cap S|=r\},$$
$$\{A\in(\mathcal{A}-\mathcal{A}_4)\colon |A\cap(X-S)|\leqslant(k-1)r \text{ and} |A\cap S|=r\}=\emptyset,$$
$$\mathcal{A}^{(3)}=\{A\in(\mathcal{A}-\mathcal{A}_4)\colon A\cap S\neq\emptyset \text{ and } |A\cap S|\neq r\}\supset\{A\in(\mathcal{A}-\mathcal{A}_4)\colon A\subset S\},$$
$$\mathcal{A}^{(4)}=\mathcal{A}_4.$$

From (20) and from estimates analogous to (17.1)–(17.4) we have

$$\frac{N^2}{r^2k(k+1)}-\frac{2}{r}N<|\mathcal{A}|<\binom{(N-s)/r}{2}\binom{k+1}{2}^{-1}+\frac{1}{r}\sqrt{2Nr}\,s \tag{22}$$
$$+s^2/2r^2+s\sqrt{s}\sqrt{r}+s+\sqrt{2N/r}.$$

From this using (21) we get

$$s<rk(k+1). \tag{23}$$

4.6. Finally using (23), the right hand side of (22) can be estimated by $N(N-r)/r^2k(k+1)$. □

5. The proof of Theorem 3

5.1. $\{0,2,3\}$-*systems with almost equal sets*

Let $\mathcal{A}$ be a $\{0,2,3\}$-system over the underlying set X, $|X|=N$. Suppose that for every $A\in\mathcal{A}$ we have $K\leqslant|A|\leqslant 2K$ for some real K. A point $x\in X$ is called *good* if there exists a point $y\in X$ $(y\neq x)$ with $\mathcal{A}[x]\subset\mathcal{A}[y]$. The aim of this section is to prove the following lemma.

Lemma 3. *If $\mathcal{A}$ is a $\{0,2,3\}$-system and for every $A\in\mathcal{A}$ we have $K\leqslant|A|\leqslant 2K$ and for some $x\in X$ we have $|\mathcal{A}[x]|>\max(8N,8K^3)$, then the point x is good $(K\geqslant 4)$.*

Proof. Suppose indirectly that $\mathcal{A}[x]$ is not good. Define the set $\mathcal{D}$ as the nuclei of Δ-systems with $2K+1$ members of $\mathcal{A}[x]$. (The definition of Δ-system can be found in 3.1 after (5).) Since $\mathcal{A}$ is a $\{0,2,3\}$-system $\mathcal{A}[x]$ is a $\{2,3\}$-system. Hence the members of $\mathcal{D}$ are 2 or 3-elements sets (containing x). We will use only the following property of nuclei

$$\text{If}\quad D\in\mathcal{D},\quad A\in\mathcal{A}[x],\quad \text{then}\quad |D\cap A|\geqslant 2. \tag{24}$$

Thus if the $\mathcal{D}$ has a 2-elements nucleus then x is good. Suppose that $\mathcal{D}$ is 3-uniform, $\mathcal{D}=\{D_1,D_2,\dots,D_t\}$. Clearly, $|D_i\cap D_j|=2$ and $x\in\bigcap D_i$. According to the cardinality of $\mathcal{D}$ we have to investigate the following cases: $t=0$, $t=1$, $t\geqslant 2$

and $|\bigcap D_i| = 2$ (i.e. $\mathscr{D}$ is a Δ-system itself) and finally $t = 3$ and $\bigcap D_i = \{x\}$ ($\mathscr{D}$ is a triangle).

We will use several times the following two facts ((25) and (26)) in order to get an upper bound for $|\mathscr{A}[x]|$ in these four cases.

If x is not good, and $\mathscr{A}[x, y]$ does not contain any Δ-system with $2K+1$ members, then $|\mathscr{A}[x, y]| \leq 4K^2$. (25)

Indeed, let A_0 be a set belonging to $\mathscr{A}[x]$ and not containing y. Then every member of $\mathscr{A}[x, y]$ intersects A_0 in a point different from x. Thus

$$|\mathscr{A}[x, y]| \leq \sum_{\substack{z \in A_0 \\ z \neq x, y}} |\mathscr{A}[x, y, z]| \leq (|A_0| - 1)2K < 4K^2.$$

As the set-system $\mathscr{A}[x, y, z]$ is a Δ-system we get

$$|\mathscr{A}[x, y, z]| \leq \frac{N}{K-3}. \tag{26}$$

In the case $t = 0$ let A_1 be fixed so that $A_1 \in \mathscr{A}[x]$. By (25) we have

$$|\mathscr{A}[x]| \leq \sum_{\substack{y \in A_1 \\ y \neq x}} |\mathscr{A}[x, y]| \leq (|A_1| - 1)4K^2 < 8K^3 \tag{27.1}$$

In the case $t = 1$ write $D_1 = \{x, y, z\}$. By (24) every $A \in \mathscr{A}[x]$ contains either y or z. Let us define $\mathscr{A}[x, y, \neg z]$ as follows $\{A \in \mathscr{A}[x, y]: z \notin A\}$. We can apply (25) for $\mathscr{A}[x, y, \neg z]$ thus we get

$$|\mathscr{A}[x]| = |\mathscr{A}[x, y, z]| + |\mathscr{A}[x, y, \neg z]| + |\mathscr{A}[x, \neg y, z]|$$

$$< \frac{N}{K-3} + 2 \cdot 4K^2 \tag{27.2}$$

Case $t \geq 2$, $\bigcap D_i = \{x, y\}$, $D_i = \{x, y, z_i\}$. There exists an $A_1 \in \mathscr{A}[x]$, $y \notin A_1$ because x is not good. By (24) for all $1 \leq i \leq t$ we have $z_i \in A_1$ hence $t \leq |A_1| - 1 \leq 2K - 1$. Furthermore if $A \in \mathscr{A}[x]$ does not contain y, then A contains all $z_i - s$, too. Thus

$$|\mathscr{A}[x]| \leq \sum_{z \in A_1} |\mathscr{A}[x, y, z]| + |\mathscr{A}[x, z_1, z_2, \ldots, z_t]|$$

$$< (2K - 1)\frac{N}{K-3} + \frac{N}{K-3} \leq 8N. \tag{27.3}$$

Case $t = 3$, $D_1 = \{z, y_2, y_3\}$, $D_2 = \{x, y_1, y_3\}$, $D_3 = \{x, y_1, y_2\}$. Then by (24) all $A \in \mathscr{A}[x]$ contains at least two points from $\{y_1, y_2, y_3\}$. Thus applying (26)

$$|\mathscr{A}[x]| = |\mathscr{A}[D_1]| + |\mathscr{A}[D_2]| + |\mathscr{A}[D_3]| < 3\frac{N}{K-3}. \tag{27.4}$$

Thus (27.1)–(27.4) shows that if the point x is not good, then $|\mathscr{A}[x]| \leq \max(8N, 8K^3)$.

5.2. *The cardinality of a* $\{0,2,3\}$*-system with almost equal sets*

Lemma 4. *If* $\mathscr{A}$ *is a* $\{0,2,3\}$*-system and for all* $A \in \mathscr{A}$ *we have* $K \leqslant |A| \leqslant 2K$, *then* $(K \geqslant 4)$

$$|\mathscr{A}| < 8\frac{N^2}{K^2} + \max\left(\frac{16N^2}{K}\,;\,16NK^2\right). \tag{28}$$

Proof. Let S be the set of points whose degree is greater than $\max(8N, 8K^3)$, i.e. $S = \{x \in X: |\mathscr{A}[x]| > \max(8N, 8K^3)\}$. By Lemma 3 all the points of S are good. Denote by x' the point corresponding to the point $x \in S$. By definition $\mathscr{A}[x] \subset \mathscr{A}[x']$. The point x' is good, too, so there exists a point $(x')' = x''$ for which $\mathscr{A}[x] \subset \mathscr{A}[x'] \subset \mathscr{A}[x'']$. Since $|\mathscr{A}[x]| > N$ we get $x'' = x$. Hence the S splits into pairwise disjoint two-elements sets $S_1, S_2, \ldots, S_p$. For S_i "If $S_i \cap A \neq \emptyset$, then $S_i \subset A$" holds. Split $\mathscr{A}$ into two parts

$$\mathscr{A}_S = \{A \in \mathscr{A}: |A \cap S| > \tfrac{1}{2}K\},$$
$$\mathscr{A}_{\neg S} = \{A \in \mathscr{A}: |A \cap (X - S)| \geqslant \tfrac{1}{2}K\}.$$

If $A \in \mathscr{A}_S$, then A contains at least $\frac{1}{4}K$ sets S_i. Thus

$$\binom{\lceil K/4\rceil}{2}|\mathscr{A}_S| \leqslant \binom{|S|/2}{2} < \frac{N^2}{8}.$$

Moreover

$$\tfrac{1}{2}K\,|\mathscr{A}_{\neg S}| \leqslant |X - S| \max(8N, 8K^3) \leqslant \max(8N^2, 8NK^2).$$

We get (28) summing up the last two inequalities. □

5.3. *The proof of Theorem* 3

Now let $\mathscr{A}$ be an arbitrary $\{0,2,3\}$-system. Split $\mathscr{A}$ according to the cardinality of its members

$$A_0 = \{A \in \mathscr{A}: |A| \leqslant 3\},$$
$$\mathscr{A}_i = \{A \in \mathscr{A}: |A| \leqslant {}^3\sqrt{N}, 2 \cdot 2^i \leqslant |A| \leqslant 2 \cdot 2^{i+1}\},$$
$$\mathscr{A}'_i = \{A \in \mathscr{A}: \sqrt{6N}/2^i \leqslant |A| \leqslant 2\sqrt{6N}/2^i, |A| > {}^3\sqrt{N}\},$$
$$\mathscr{A}'' = \{A \in \mathscr{A}: |A| > \sqrt{6N}\}.$$

We can apply (28) to estimate the cardinalities of $\bigcup \mathscr{A}_i$ and $\bigcup \mathscr{A}'_i$.

$$\sum |\mathscr{A}_i| \leqslant 8N^2 \sum_{i=1}^{\infty} \frac{1}{4 \cdot 2^{2i}} + 16N^2 \sum_{i=1}^{\infty} \frac{1}{2 \cdot 2^i} = (\tfrac{2}{3} + 8)N^2 \tag{29.1}$$

$$\sum |\mathscr{A}'_i| < 8N^2 \sum_{2^i < N^{1/6}} \frac{1}{6N} \cdot 2^{2i} + 16N \sum_{j=1}^{\infty} 6N \frac{1}{2^{2i}}$$
$$= N^2/\sqrt{6}\,\sqrt[3]{N} + 32N^2. \tag{29.2}$$

Finally Lemma 2 can be applied, thus

$$|\mathscr{A}''| < \sqrt{2N/3}, \tag{29.3}$$

Furthermore

$$|\mathscr{A}_0| \leq \sum_{x \in X} |\mathscr{A}[x]| \leq N(N-1) \tag{29.4}$$

Summing up (29.1)–(29.4) we get $|\mathscr{A}| < 42N^2$. □

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Note added in proof

Recently, a rapid development is taking place in the topic of this paper. Some of the newest results:

Theorem of Frankl and Rosenberg [19]. *Let* $0 \leq q < r$, $\Lambda = \{q, q+r, q+2r, \ldots\}$, $k \not\equiv q \pmod r$, *then* $f(N, k, \Lambda) \leq N$.

This theorem generalizes the results mentioned in equation (2) (where $q = 0$) and in Deza and Rosenberg [17] (r is a prime).

Theorem of Frankl and Wilson [18]. *If* $\Lambda = \{\lambda_1, \ldots, \lambda_s\}$, *then*

$$f(N, \Lambda) \leq \binom{N}{s} + \binom{N}{s-1} + \cdots + \binom{N}{0}.$$

Proposition 2 was improved in [20], proving

$$f(N, \{0, 1, 3\}) \leq N(N-1)(N-3)/168 \quad \text{for} \quad N > N_0,$$

where equality holds iff $\mathscr{A}$ is isomorphic to Example 3.

Finally, we have to mention that Proposition 1 appears in a paper due to H.-D.O.F. Gronau [21], as well.

References added in proof

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