JOURNAL OF COMBINATORIAL THEORY, Series A 32, 66–72 (1982)

An Intersection Problem Whose Extremum Is the Finite Projective Space

Z. FÜREDI

Mathematical Institute of the Hungarian Academy of Sciences, 1053 Budapest, Reáltanoda 13-15, Hungary

Communicated by the Managing Editors

Received January 15, 1981

Suppose that $\mathscr{A}$ is a finite set-system of N elements with the property $|A \cap A'| = 0, 1$ or k for any two different $A, A' \in \mathscr{A}$. We show that for $N > k^{14}$

$$|\mathscr{A}| \leqslant \frac{N(N-1)(N-k)}{(k^2-k+1)(k^2-k)(k^2-2k+1)} + \frac{N(N-1)}{k(k-1)} + N + 1,$$

where equality holds if and only if $k = q+1$ (q is a prime power) $N = (q^{t+1}-1)/(q-1)$ and $\mathscr{A}$ is the set of subspaces of dimension at most two of the t-dimensional finite projective space of order q.

1. Introduction

Let $0 \leqslant \lambda_1 < \lambda_2 < \cdots < \lambda_s$ be integers, $\Lambda = \{\lambda_1, ..., \lambda_s\}$. A finite set-system $\mathscr{A}$ on the N element set X is called a Λ-system if for any two $A, A' \in \mathscr{A}$, $A \neq A'$ there exists an i such that $|A \cap A'| = \lambda_i$. Concerning Λ-systems we recollect two general theorems

(1) (Frankl and Wilson [4]) *For a Λ-system $\mathscr{A}$*

$$|\mathscr{A}| \leqslant \binom{N}{0} + \binom{N}{1} + \cdots + \binom{N}{s}.$$

(2) (Ray-Chaudhury and Wilson [7]) *For an r-uniform Λ-system* (*i.e.*, *for every $A \in \mathscr{A}$ we have $|A| = r$*) $|\mathscr{A}| \leqslant \binom{N}{s}$.

These theorems are generalizations of a generalization by Ryser of an old theorem of Erdös and de Bruijn.

(3) (Ryser [8]) *If for any two different $A, A' \in \mathscr{A}$ we have $|A \cap A'| = \lambda$ ($\neq 0$) then $|\mathscr{A}| \leqslant N$.*

(4) (Deza [2]) *If for any two different $A, A' \in \mathscr{A}$ we have $|A \cap A'| = \lambda$ and $|\mathscr{A}| > \max_{A \in \mathscr{A}} |A|^2$ then $|\bigcap_{A \in \mathscr{A}} A| = \lambda$.*

0097-3165/82/010066-07$02.00/0

Though estimations (1), (2) and (3) are valid for every N and Λ, for special Λ's they can be improved (see, e.g., [3, 5]). The aim of this paper is to investigate the case $\Lambda = \{0, 1, k\}$.

2. Results

We give two examples of $\{0, 1, k\}$-systems.

Example 1 (The finite projective space). Let q be a prime power and $k = q + 1$, $N = q^t + q^{t-1} + \cdots + q + 1$. We write $\mathscr{P}_k^t$ for the at most two dimensional subspaces of the t-dimensional projective space of order q. Clearly for $A \in \mathscr{P}_k^t$, $|A| = k^2 - k + 1$, k, 1 or 0 and for $A, A' \in \mathscr{P}_k^t$, $A \cap A' \in \mathscr{P}_k^t$, thus $\mathscr{P}_k^t$ is a $\{0, 1, k\}$-system.

Example 2 (The finite affine space). Let k be a prime power and $N = k^t$. We write $\mathscr{F}_k^t$ for the at most two dimensional linear manifolds of the t-dimensional vector space over the finite field of order k. If $A \in \mathscr{F}_k^t$ then $|A| = k^2, k$, 1 or 0.

The case $k = 2$ is easy (see, e.g., [5]).

$$\max \{|\mathscr{A}| : \mathscr{A} \text{ is a } \{0, 1, 2\}\text{-system}\} = \binom{N}{3} + \binom{N}{2} + N + 1,$$

where $\mathscr{A}$ is maximal if and only if $\mathscr{A} = \{A \subset X : |A| \leqslant 3\}$. From now on we suppose that $k > 2$.

Theorem 1. *Suppose that $\mathscr{A}$ is a finite set-system on N elements such that $A, A' \in \mathscr{A}$, $A \neq A'$ implies $|A \cap A'| \in \{0, 1, k\}$. Then for every sufficiently large N ($N > k^{14}$)*

$$|\mathscr{A}| \leqslant \frac{N(N-1)(N-k)}{(k^2-k+1)(k^2-k)(k^2-2k+1)} + \frac{N(N-1)}{k(k-1)} + N + 1. \qquad (5)$$

Equality holds in (5) if and only if $k = q + 1$ for some prime power q, $N = q^t + q^{t-1} + \cdots + q + 1$ and $\mathscr{A} \cong \mathscr{P}_k^t$ (see Example 1).

We remark that some condition of the type $N > N_0(k)$ is in fact necessary as the following example shows. If $N < (k-1)^6/2$ then

$$|\{A \subset X : |A| \leqslant 2\}| = \binom{N}{2} + N + 1 > \quad \text{(left hand side of (5))}.$$

THEOREM 2. *Suppose that $\mathscr{A}$ is a $\{0, 1, k\}$-system and $k-1$ is not a prime power. Then for large N $(N > k^{14})$*

$$|\mathscr{A}| \leqslant \frac{N(N-1)(N-k)}{k^2(k^2-1)(k^2-k)} + \frac{N(N-1)}{k(k-1)} + N + 1. \tag{6}$$

Equality holds in (6) *if and only if k is a prime power, $N = k^t$ and $\mathscr{A} \cong \mathscr{P}_k^t$ (see Example 2).*

If neither k nor $k-1$ is a prime power (of course $k \geqslant 15$) then I could not prove whether the function

$$f_{\{0,1,k\}}(N) = \max \left\{ |\mathscr{A}| : \mathscr{A} \text{ is a } \{0, 1, k\}\text{-system, } \left| \bigcup \mathscr{A} \right| \leqslant N \right\}$$

is of order N^3 or not. Upon considering this problem one is led to the question of the existence of some special resolvable block designs. The proof method of Theorems 1 and 2 gives the following theorem.

THEOREM 3. *Suppose that $\mathscr{A}$ is a $\{0, 1, k\}$-system and neither k nor $k-1$ is a prime power then for large N $(N > k^{14})$*

$$|\mathscr{A}| \leqslant \frac{N(N-1)(N-k)}{(k^2+k-1)(k^2+k-2)(k^2-1)} + \frac{N(N-1)}{k(k-1)} + N + 1. \tag{7}$$

Conjecture. If neither k nor $k-1$ is a prime power then

$$f_{\{0,1,k\}}(N) = o(N^3).$$

3. PROOFS

Let $\mathscr{A}$ be a $\{0, 1, k\}$-system, $k \geqslant 3$. We introduce the notation

$$\mathscr{A}_i = \{A \in \mathscr{A} : |A| = i\}, \qquad \mathscr{A}_{>i} = \{A \in \mathscr{A} : |A| > i\} \text{ and so on.}$$

Further for a set $D \subset X$

$$\mathscr{A}[D] = \{A \in \mathscr{A} : D \subset A\}.$$

The theorems will be proved by the method of [3] and [5]. Split $\mathscr{A}$ into three set systems $\mathscr{A} = \mathscr{A}_0 \cup \mathscr{A}_1 \cup \mathscr{A}_{\geqslant 2}$. Evidently

$$|\mathscr{A}_0| \leqslant 1, \qquad |\mathscr{A}_1| \leqslant N. \tag{8}$$

In order to prove (5) we only need consider the case

$$|\mathscr{A}_{>2}| \geqslant \frac{N(N-1)(N-k)}{(k^2-k+1)(k^2-k)(k^2-2k+1)} + \frac{N(N-1)}{k(k-1)}. \tag{9}$$

A pair $\{x, y\}$ of X is called *good* provided

$$|\mathscr{A}_{\leqslant K}[x, y]| > K^2.$$

We shall specify K as $K = \sqrt{N}/k^3$. Any two members of $\mathscr{A}[x, y]$ intersect in exactly k points. Hence Deza's theorem (4) implies that for a good pair $\{x, y\}$ any member of $\mathscr{A}_{\leqslant K}[x, y]$ contains a k element subset M. We call this $M = M(x, y)$ the *nucleus* corresponding to the pair $\{x, y\}$. If $\{u, v\} \subset M$ then the pair $\{u, v\}$ is good, too, and $M(u, v) = M(x, y)$. And what is more

$$\text{If } |M \cap A| > 1 \text{ and } |A| \leqslant K^2 \text{ then } M \subset A. \tag{10}$$

Put $\mathscr{M} = \{M(x, y)\colon \{x, y\} \text{ is good}\}$. Clearly $\mathscr{M}$ is a $\{0, 1\}$-system. Let ℓ denote the number of non-good pairs, then

$$|\mathscr{M}| = \left(\binom{N}{2} - \ell\right)\bigg/\binom{k}{2}. \tag{11}$$

The crucial point of the proof is

LEMMA 1. *If the every pair of the set $A \in \mathscr{A}_{\leqslant K^2}$ is good then either $|A| = k$ and $A \in \mathscr{M}$ or $|A| \geqslant k^2 - k + 1$. In this latter case equality holds if and only if the nuclei contained in A are the lines of a k-uniform finite projective plane.*

Proof of Lemma 1. By (10) the set-system $\mathscr{M}_A = \{M \in \mathscr{M}\colon M \subset A\}$ is a $2-(|A|, k, 1)$ BIBD. (A k-uniform set-system on v points is a $2-(v, k, \lambda)$ BIBD if any pair of its underlying set is contained in λ sets). Then $|\mathscr{M}_A| \binom{k}{2} = \binom{|A|}{2}$. The well-known Fisher inequality (see Hall [6]) states that a non-trivial BIBD has at least as many blocks as the cardinality of the underlying set. Consequently

$$|\mathscr{M}_A| = \binom{|A|}{2}\bigg/\binom{k}{2} \geqslant |A|, \qquad \text{i.e.,} \quad |A| \geqslant k^2 - k + 1.$$

(Or in the trivial case $|A| = k$). Finally a $2-(k^2-k+1, k, 1)$ BIBD is a finite projective plane. Q.E.D.

Going on with the proof of Theorem 1 the Lemma 1 implies that if $A \in \mathscr{A}_{>2}$ then at least one of the following cases holds.

(a) $|A|=k$ and $A\in\mathscr{M}$

(b) A contains at least k^2-k+1 nuclei and then $|A|\geqslant k^2-k+1$

(c) A contains a non-good pair or $|A|>K^2$. (12)

Put

$$\mathscr{A}_{\geqslant 2}=\mathscr{A}_k^{\text{good}}\cup\mathscr{A}_{>k}^{\text{good}}\cup\mathscr{A}_{\leqslant K}^c\cup\mathscr{A}_{K<\ \leqslant K^2}^c\cup\mathscr{A}_{>K^2}^c,$$

where

$$\mathscr{A}_k^{\text{good}}=\{A\in\mathscr{A}: |A|=k \text{ and } A\in\mathscr{M}\},$$

$$\mathscr{A}_{>k}^{\text{good}}=\{A\in\mathscr{A}: A \text{ contains at least } k^2-k+1 \text{ nuclei}\},$$

$$\mathscr{A}^c=\{A\in\mathscr{A}_{\geqslant 2}: \text{neither (12a) nor (12b) holds}\}.$$

Now we give upper bounds for the five parts of $\mathscr{A}_{\geqslant 2}$. From (11)

$$|\mathscr{A}_k^{\text{good}}|\leqslant|\mathscr{M}|=\frac{N(N-1)}{k(k-1)}-\frac{2\ell}{k(k-1)}. \tag{13}$$

Since any nucleus is contained in at most $(N-k)/(k^2-k+1-k)$ sets of cardinality at least k^2-k+1, and every $A\in\mathscr{A}_{>k}^{\text{good}}$ contains at least k^2-k+1 nuclei we have

$$|\mathscr{A}_{>k}^{\text{good}}|\leqslant|\mathscr{M}|\frac{N-k}{k^2-2k+1}\cdot\frac{1}{k^2-k+1}$$
$$=\frac{N(N-1)(N-k)-2\ell(N-k)}{(k^2-k+1)(k^2-k)(k^2-2k+1)}. \tag{14}$$

From the definition of he good pair

$$|\mathscr{A}_{\leqslant K}^c|\leqslant \ell K^2. \tag{15}$$

If $A\in\mathscr{A}_{K<\ \leqslant K^2}^c$ then A contains at least $\binom{K}{2}-(k^2-k)\binom{k}{2}>N/3k^6$ non-good pairs ($K=\sqrt{N}/k^3$, $N>k^{14}$). Ryser's theorem (3) gives:

$$|\mathscr{A}_{K<\ \leqslant K^2}^c|\leqslant\sum_{\{x,y\}\text{non-good}}|\mathscr{A}[x,y]|\frac{3k^6}{N}$$
$$\leqslant \ell N\, 3k^6/N=3k^6\ell. \tag{16}$$

Finally

$$|\mathscr{A}_{>K^2}^c|\leqslant|\mathscr{A}_{>K^2}|<2k^6. \tag{17}$$

(Because more than $2k^6$ set of cardinality more than N/k^6 cannot form a $\{0, 1, 2,..., k\}$-system on N points. See, e.g., [5]). Using estimations (13)–(17) and (9) we have

$$0 \leqslant -2\ell/k(k-1) - 2\ell(N-k)/(k^2-k+1)(k^2-k)(k^2-2k+1) + \ell K^2 + \ell\, 3k^6 + 2k^6.$$

Here the coefficient of ℓ is less than $-2k^6$. So for $\ell > 0$, (5) holds with strict inequality.

Now if $\ell = 0$ then from (15) and (16), $\mathscr{A}^c_{\leqslant K^2} = \varnothing$. We show that $\mathscr{A}^c_{>K^2} = \varnothing$ is true. $\mathscr{M}$ is a $2-(N, k, 1)$ BIBD. If there exists as $A_0 \in \mathscr{A}^c_{>K^2}$ then it contains at most $k^2 - k$ nuclei so A_0 intersects but does not contain at least $((\binom{|A_0|}{2})/\binom{k}{2}) - (k^2 - k)) > 2k^6$ nuclei from $\mathscr{M}$. ($|A_0| > (N/k^6) > k^5$.) These nuclei from $\mathscr{M}$ do not belong to $\mathscr{A}^{\text{good}}_k$ hence instead of (13) we have

$$|\mathscr{A}^{\text{good}}_k| < \frac{N(N-1)}{k(k-1)} - 2k^6. \tag{18}$$

Summing (14), (17) and (18) we get a contradiction to (9).

We have shown that (5) is true for every $N > k^{14}$. Moreover equality can hold in (5) only if $\mathscr{A}^c = \varnothing$ and in (13), (14) the equality holds, too. But if (14) equality holds then every member of $\mathscr{A}^{\text{good}}_{>k}$ is cardinality $k^2 - k + 1$, and each nucleus from $\mathscr{M}$ is contained in exactly $(N-k)/(k^2 - 2k + 1)$ members of $\mathscr{A}^{\text{good}}_{>k}$. So the conditions of the next lemma are satisfied ($q = k - 1$, $\mathscr{L} = \mathscr{M}$, $\mathscr{P} = \mathscr{A}^{\text{good}}_{>k}$).

LEMMA 2. *Suppose that the set-systems $\mathscr{L}$ and $\mathscr{P}$ defined on N elements set X have the following properties*:

(i) *$\mathscr{L}$ is $q + 1$ uniform and for each $x, y \in X$ there exists a uniquely determined $L = L(x, y) \in \mathscr{L}$ with $\{x, y\} \subset L$.*

(ii) *If $x, y \in P \in \mathscr{P}$ then $L(x, y) \subset P$.*

(iii) *$\mathscr{P}$ is $q^2 + q + 1$ uniform and for each $L \in \mathscr{L}$, $x \in X - L$ then there exists a uniquely determined $P \in \mathscr{P}$ with $x \in P$, $L \subset P$.*

Then $N = q^t + \cdots + q + 1$ and $\mathscr{L}$ and $\mathscr{P}$ are the one and two dimensional subspaces of the t-dimensional projective space of order q, resp.

Now q is a prime power, because any at least 3-dimensional finite projective space has to be of order p^α (when p is prime). Finally, Lemma 2 implies that in (5) equality holds only if $\mathscr{A} \cong \mathscr{P}^t_k$. The proof of Theorem 1 is complete.

The proof of Lemma 2. This lemma and its consequence $q = p^\alpha$ was proved by D. Hilbert in 1899. For a proof see Dembowski [1]. Q.E.D.

The proof of Theorem 2 follows the same lines, only instead of Lemma 1 one needs the following well-known result

LEMMA 1′. *If* $\mathcal{M}_A$ *is a* $2-(|A|, k, 1)$ BIBD *and* $|A| \neq k$, $|A| \neq k^2-k+1$, *then* $|A| \geqslant k^2$, *where equality hold if and only if* $\mathcal{M}_A$ *is an affine plane of order* k.

Of course this yields a better estimation in (14), further one has to use an affine analogue of Lemma 2. Since if there exists a $2-(|A|, k, 1)$ BIBD then $|A| \equiv 1 \pmod{k-1}$ thus if $|A| \neq k, k^2-k+1, k^2$ then $|A| \geqslant k^2+k-1$. This can be used in the proof of Theorem 3. We omit the details.

REFERENCES

1. P. DEMBOWSKI, "Finite Geometries," Springer-Verlag, Berlin/New York, 1968.
2. M. DEZA, Solution d'un problème de Erdös-Lovász, *J. Combin. Theory Ser. B* **16** (1974), 166-167.
3. P. FRANKL, Families of finite sets with prescribed cardinalities for pairwise intersections, *Acta Math. Acad. Sci. Hungar.* **35** (1980), 351-360.
4. P. FRANKL AND R. M. WILSON, Intersection theorems with geometric consequences, *Combinatorica*, in press.
5. Z. FÜREDI, Set-systems with prescribed cardinalities for pairwise intersection, *Discrete Math.*, in press.
6. M. HALL, "Combinatorial Theory," Ginn (Blaisdell), Boston, 1967.
7. D. K. RAY-CHAUDHURY AND R. M. WILSON, On *t*-designs, *Osaka J. Math.* **12** (1975), 737-744.
8. H. J. RYSER, An extension of a theorem of the Bruijn and Erdös on combinatorial designs, *J. Algebra* **10** (1968), 246-261.