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ERDŐS – KO – RADO TYPE THEOREMS WITH UPPER BOUNDS ON THE MAXIMUM DEGREE

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ABSTRACT

Let X be a finite set of cardinality n and let $\mathcal{F}$ be a family of r-subsets of X. Suppose that any two members of $\mathcal{F}$ intersect and for some given positive constant c, every element of X is contained in less than $c|\mathcal{F}|$ members of $\mathcal{F}$. Our main results are

(Theorem 2.) *If there exists a* k*-uniform projective plane and*

$$\frac{k}{k^2-k+1} < c < \frac{1}{k-1},$$

then

$$\max|\mathcal{F}| = (k^2-k+1)\binom{n}{r-k} + O\left(\binom{n}{r-k-1}\right).$$

(Theorem 3.) *There exist a function* $C(c) > 0$ *and an integer* $k(c)$ *such that for every* c $(0 < c < 1)$

$$\max|\mathcal{F}| = C(c)\binom{n}{r-k} + O\left(\binom{n}{r-k-1}\right).$$

(Proposition 8.) *The function* $C(c)$: $(0, 1) \to \mathbf{R}$ *is piecewise constant or a rectangular hyperbola arc.*

1. INTRODUCTION

1.1. Let X be a finite set of n elements. A family $\mathcal{F}$ of the subsets of X is intersecting if any two members of $\mathcal{F}$ intersect.

Erdős, Ko and Rado [3] proved, that if $\mathcal{F}$ is an intersecting set-system of the r-tuples of X and $n \geqslant 2r$ then $|\mathcal{F}| \leqslant \binom{n-1}{r-1}$. Equality holds in the case $n > 2r$ if the members of $\mathcal{F}$ have a common element.

Let c be a real number, $0 < c \leqslant 1$. The degree of the point x in the set-system $\mathcal{F}$ is denoted by $d_{\mathcal{F}}(x)$ or simply $d(x) =: |\{F: x \in F \in \mathcal{F}\}|$. Erdős, Rothschild and Szemerédi (see [2]) raised the following question: How large can the intersecting set-system $\mathcal{F}$ of r-tuples of X be if each point has degree at most $c|\mathcal{F}|$. The class of such systems we denote by $\mathcal{F}(n, r, c)$, $f(n, r, c)$ denotes the maximum size of such an $\mathcal{F}$. The Erdős – Ko – Rado's theorem mentioned above can be formulated as follows:

Theorem A [3]. $f(n, r, 1) = \binom{n-1}{r-1}$ *if* $n > n_0(r)$. $|\mathcal{F}| = f(n, r, 1)$ *iff for some* $x \in X$ $\mathcal{F} = \mathcal{F}[x] =: \{F: F \subset X, |F| = r, x \in F\}$.

1.2. Let $D \subset X$, $|D| = k^2 - k + 1$. If a finite geometry of order $k - 1$ exists on this $k^2 - k + 1$ elements then let $\mathcal{P}_k$ denote the set-system consisting of its lines. $\mathcal{P}_2$ denote the set-system consisting of the three edges of a triangle. We put

$$\mathcal{F}[\mathcal{P}_k] =: \{F \subset X: |F| = r, F \cap D \in \mathcal{P}_k\},$$

$$\mathcal{F}[\overline{\mathcal{P}_k}] =: \{F \subset X: |F| = r, F \cap D \text{ contains some line of } \mathcal{P}_k\}.$$

(In what follows we use the notations $\mathcal{F}[\mathcal{B}]$ and $\mathcal{F}[\overline{\mathcal{B}}]$ in the same sense.) Then:

$$|\mathcal{F}[\mathcal{P}_k]| = (k^2 - k + 1)\binom{n - (k^2 - k + 1)}{r - k}$$

and

$$\frac{\max d(x)}{|\mathcal{F}[\mathcal{P}_k]|} = \frac{k}{k^2 - k + 1} \qquad (n > (k^2 - k + 1)r).$$

Relying on this Erdős, Rothschild and Szemerédi conjectured, that

$$f\left(n, r, \frac{k}{k^2 - k + 1}\right) = O\left(\frac{1}{n^k} \binom{n}{r}\right),$$

but if $c < \dfrac{k}{k^2 - k + 1}$, then

$$f(n, r, c) = O\left(\frac{1}{n^{k+1}} \binom{n}{r}\right),$$

furthermore if no k-uniform projective plane exists then

$$f\left(n, r, \frac{k}{k^2 - k + 1}\right) = O\left(\frac{1}{n^{k+1}} \binom{n}{r}\right).$$

They proved this conjecture for $k = 1, 2$ (unpublished). P. Frankl more generally solved the cases $\frac{3}{7} < c < \frac{3}{5}$ and $\frac{2}{3} < c < 1$, thus verifying the conjecture for $k = 3$.

Theorem B [5].

(i) *If* $\frac{2}{3} < c < 1$ *and* $n > n_0(r, c)$ *then*

$$f(n, r, c) = 3\binom{n-3}{r-2} + \binom{n-3}{r-3}.$$

If $\mathcal{F} \in \mathcal{F}(n, r, c)$ *then* $|\mathcal{F}| = f(n, r, c)$ *holds iff* $\mathcal{F} = \mathcal{F}[\overline{\mathcal{P}_2}]$.

(ii) *If* $\frac{3}{7} < c < \frac{1}{2}$ *and* $n > n_0(r, c)$ *then*

$$f(n, r, c) = |\mathcal{F}[\overline{\mathcal{P}_3}]| = 7\binom{n-7}{r-3} + O\left(\binom{n}{r-4}\right).$$

Equality holds iff $\mathcal{F} = \mathcal{F}[\overline{\mathcal{P}_3}]$.

1.3. The problem in 1.1 can also be considered in the following way. Knowing the cardinality of $\mathcal{F}$ what can we say on its maximal degree.

So Theorem C (i) and (iii) are mere consequences of Theorem B.

Let $c(n, r, N)$ denote the greatest real number satisyfing the following statement. If $\mathcal{F}$ is an intersecting set-system of r-tuples of an n-element set and $|\mathcal{F}| \geq N$, then $\max d_{\mathcal{F}}(x) > c|\mathcal{F}|$.

Theorem C [5]. *If* $\epsilon > 0$ *is fixed and* $n \to \infty$, *then*

(i) $c\left(n, r, (3+\epsilon)\binom{n}{r-2}\right) = 1 - o(1)$,

(ii) $c\left(n, r, \binom{n}{r-2}\epsilon\right) = \frac{2}{3} - o(1)$,

(iii) $c\left(n, r, (7+\epsilon)\binom{n}{r-3}\right) = \frac{1}{2} - o(1)$,

(iv) $c\left(n, r, \binom{n}{r-3}\epsilon\right) = \frac{3}{7} - o(1)$.

The intention of this paper is to generalize Theorems B and C, to answer the problems raised above by working out a generalization of known methods. The paper also determines the extremal set-systems for infinitely many further values of c.

2. RESULTS

2.1. We are going to prove the following theorem conjectured by P. Frankl relying on C (i) and C (iii).

Theorem 1. *If* $\epsilon > 0$ *is fixed and* $n \to \infty$, *then*

$$c\left(n, r, \epsilon\binom{n}{r-k}\right) \geq \frac{k}{k^2 - k + 1} - o(1).$$

The following is a generalization of Theorem B and the conjecture 1.2.

Theorem 2. *If there exists* k*-uniform projective plane and*

$$\frac{k}{k^2 - k + 1} < c < \frac{1}{k-1},$$

then

$$f(n, r, c) = (k^2 - k + 1)\binom{n}{r-k} + O\left(\binom{n}{r-k-1}\right).$$

For $n > n_0(r, c)$ $|\mathcal{F}| = f(n, r, c)$ can occur iff there exists an $\mathcal{P}_k$ for which

$$\mathcal{F} \supset \mathcal{F}[\overline{\mathcal{P}_k}].$$

If $k = 2, 3$ then here equality can be written, but for $k \geq 4$ $\mathcal{F} - \mathcal{F}[\overline{\mathcal{P}_k}]$ consists of sets intersecting all lines of $\mathcal{P}_k$ and containing none of them. Thus

$$|\mathcal{F} - \mathcal{F}[\overline{\mathcal{P}_k}]| < O\left(\binom{n}{r - k - \alpha\sqrt{k}}\right) \qquad (\alpha > 0.5).$$

Finally if $n > n_0(r, c)$ then

$$f\left(n, r, \frac{k}{k^2 - k + 1}\right) = |\mathcal{F}[\mathcal{P}_k]|.$$

2.2. The results of Theorems 1 and 2 will be better understood in view of a more general result which is our proper aim. To state this we need some (well-known) definitions.

Let $\mathcal{H}$ denote a finite set-system on a ground-set X. $\nu(\mathcal{H})$ or simply ν denotes the maximum number of disjoint edges in $\mathcal{H}$ i.e. $\nu(\mathcal{H}) = \max\{w\colon \exists E_1, \ldots, E_w \in \mathcal{H},\ E_i \cap E_j = \phi\}$. The *rank* of $\mathcal{H}$ denoted by R is the greatest cardinality of edges, i.e. $R(\mathcal{H}) = \max_{E \in \mathcal{H}} |E|$.

$\tau = \tau(\mathcal{H})$ denotes the minimum number of covering points in $\mathcal{H}$ i.e. $\tau(\mathcal{H}) = \min\{|T|\colon T \subset X,\ E \cap T \neq \phi$ for all $E \in \mathcal{H}\}$. Clearly

(1) $\qquad \nu \leq \tau \leq R\nu.$

A function $w\colon \mathcal{H} \to \mathbf{R}$ is a *fractional matching of* $\mathcal{H}$ if

(2)
$$\begin{cases} w(E) \geq 0 \text{ for all edges } E \in \mathcal{H}, \\ \sum_{E \ni x} w(E) \leq 1 \text{ for each point } x \in X. \\ \text{Introducing the notation } |w| = \sum_{E \in \mathcal{H}} w(E) \text{ let} \\ \nu^*(\mathcal{H}) =: \max\{|w|\colon w \text{ is a fractional matching of } \mathcal{H}\}. \end{cases}$$

For instance if $\mathcal{H}$ is an union of ν disjoint $\mathcal{P}_R$ then

(3) $\nu^*(\mathscr{H}) = \left(R - 1 + \frac{1}{R}\right)\nu$

A function $t\colon X \to \mathbf{R}$ is *a fractional cover of* $\mathscr{H}$ if

(4)
$$\begin{cases} t(x) \geqslant 0 \quad \text{for each point} \quad x \in X, \\ \sum_{x \in E} t(x) \geqslant 1 \quad \text{for each edge} \quad E \in \mathscr{H} \\ \text{Introducing the notation} \quad |t| = \sum_{x \in X} t(x) \quad \text{let} \\ \tau^*(\mathscr{H}) =: \{\min |t|\colon t \text{ is a fractional cover of } \mathscr{H}\}. \end{cases}$$

It is a well-known fact (a consequence of the Duality Theorem in linear programming) that $\nu^*(\mathscr{H})$ and $\tau^*(\mathscr{H})$ exist, are finite and equal to each other:

(5) $\nu \leqslant \nu^* = \tau^* \leqslant \tau.$

From (1) and (5) we have

(6) $\nu^* = \tau^* \leqslant R\nu.$

On the right-hand side of (1) equality can occur (there exists a $\mathscr{H}$ for all R and ν satisfying $\tau = R\nu$), but as L. Lovász showed ([8])

$$\nu^* < R\nu$$

for all hypergraphs. L. Lovász also investigated the expression $\sup \nu^*(\mathscr{H})$ when the values R and ν of the rank and the matching number are given. Let it be denoted by $\nu^*(R, \nu)$ i.e. $\nu^*(R, \nu) = \sup \{\nu^*(\mathscr{H})\colon R(\mathscr{H}) \leqslant R, \nu(\mathscr{H}) \leqslant \nu\}$ and we put $\nu^*(R, 1) = \nu^*(R)$.

The following sharper results will be used:

Theorem D [6].

(i) *If* $R(\mathscr{H}) \leqslant R$, $\nu(\mathscr{H}) \leqslant \nu$ *and* $\mathscr{H}$ *is not the union of* ν *disjoint* $\mathscr{P}_R$ *then*

$$\nu^*(\mathscr{H}) \leqslant \left(R - 1 + \frac{1}{R}\right)\nu - \frac{1}{R}.$$

Thus $\nu^*(R,\nu) = \left(R - 1 + \frac{1}{R}\right)\nu$ *if there exists a* $\mathcal{P}_R$.

(ii) *If no* $\mathcal{P}_R$ *exists then* $\nu^*(R,\nu) \leq (R-1)\nu$.

(iii) $|\{\nu^*(\mathcal{H})\colon R(\mathcal{H}) \leq R,\ \nu(\mathcal{H}) \leq \nu\}| < \infty$,

so if the rank and the matching number is bounded then $\nu^*(\mathcal{H})$ *has only finitely many values.*

Proposition E [6].

(i) $\nu^*(R,\nu) = \max\{\nu^*(\mathcal{H})\colon R(\mathcal{H}) \leq R,\ \nu(\mathcal{H}) \leq \nu\}$, *in other words: there exists an hypergraph with the given bounds on the rank and matching number for which* $\nu^*(\mathcal{H})$ *takes its supremum.*

(By D (i) *and* (3), *the union of disjoint* R*-uniform projective planes are such hypergraphs. If* $\mathcal{P}_R$ *does not exist then the value of* $\nu^*(R,\nu)$ *is unknown.)*

(ii) $\nu^*(R,\nu) < \nu^*(R+1,\nu)$

(iii) $(R - o(R))\nu < \nu^*(R,\nu) \leq \left(R - 1 + \frac{1}{R}\right)\nu$.

*(*E (i) *is a consequence of* D (iii)*).*

2.3. The sequence $\left\{\frac{1}{\nu^*(R)}\right\}_{R=1,2,\ldots} = \left\{1, \frac{2}{3}, \frac{3}{7}, \frac{4}{13}, \ldots\right\}$ is strictly monoton decreasing by E (ii) and E (iii), and it tends to zero. Thus one can find a natural number k $(k \geq 2)$ for any real number $0 < c < 1$ satisfying $\frac{1}{\nu^*(k)} \leq c < \frac{1}{\nu^*(k-1)}$.

Theorem 3 (Main Theorem). *If* $\frac{1}{\nu^*(k)} \leq c < \frac{1}{\nu^*(k-1)}$, *then exists a* $C(c) > 0$ *such that*

$$f(n,r,c) = C(c)\binom{n}{r-k} + O\left(K(r,c)\binom{n}{r-k-1}\right).$$

As in Theorems A, B and 2 here also holds, that if $n > n_0(r,c)$ and $\mathcal{F} \in \mathcal{F}(n,r,c)$ is maximal then there exists an $D \subset X$, the size of which depends only on k (e.g. $|D| < 4^k \cdot k^2$) and an intersecting set-system $\mathcal{B}$ of rank k on D, such that $\mathcal{F} \cap \mathcal{F}[\overline{\mathcal{B}}]$ is the essential part of $\mathcal{F}$, i.e.

$$|\mathscr{F} \cap \mathscr{F}|\overline{\mathscr{B}}|| = C(c)\binom{n}{r-k} + O\left(K(r,c)\binom{n}{r-k-1}\right),$$

$$|\mathscr{F} - \mathscr{F}[\overline{\mathscr{B}}]| < L(r)\binom{n}{r-k-1}.$$

($\mathscr{B}$ will be ν-critical, see chapter 3.)

The determination of $C(c)$ is a finite problem. This means the following: solving a problem of linear programming, to be defined later, for each of fhe finitely many $\mathscr{B}$-s under consideration we can get $C(c)$. We do this in the proof of Theorem 11 in [7].

2.4. Corollaries 4 and 5 are trivial consequences of Theorem 3. They answer the question of 1.3 and are extensions of C (ii), (iv) resp. (i), (iii).

Corollary 4. *If* $\epsilon > 0$ *fixed and* n *tends to infinity then*

$$c\left(n, r, \epsilon\binom{n}{r-k}\right) = \frac{1}{\nu^*(k)} - o(1).$$

Corollary 5. *If* $\epsilon > 0$ *fixed and* n *tends to infinity then*

$$c\left(n, r, (k^2 - k + 1 + \epsilon)\binom{n}{r-k}\right) \geq \frac{1}{k-1} - o(1).$$

2.5. The Propositions 6-10 investigate the function C.

Proposition 6. *The function* C: $(0, 1) \to \boldsymbol{R}$ *is right continuous, and monoton increasing in the interval* $\left[\frac{1}{\nu^*(k)}, \frac{1}{\nu^*(k-1)}\right)$.

Proposition 7. *There exists a function* $L(k)$ *for which*

$$C(c) < \frac{L(k)}{1 - c\nu^*(k-1)}$$

in the interval $\left[\frac{1}{\nu^*(k)}, \frac{1}{\nu^*(k-1)}\right)$.

Thus C can be majorated by a hyperbola in this interval. We do not whether C realy tends to infinity in any of these intervals, however we have the following.

Proposition 8. *The interval* $\left[\frac{1}{\nu^*(k)}, \frac{1}{\nu^*(k-1)}\right)$ *can be split into finitely many intervals such that on any of these intervals* C *is either* *constant or a hyperbola of type* $\frac{1}{u - vc}$ $(u, v > 0)$.

Note that C is piecewise constant in all known cases. (c.f. Theorem B, Theorem 2, see also Theorem F and Theorem 11 below.) It seems very likely that it is not always so. It is very likely that some hyperbolas occur in the interval $\left[\frac{4}{13}; \frac{3}{7}\right)$, although $C(c) = 13$ in the interval $\left[\frac{4}{13}; \frac{1}{3}\right)$, according to Theorem 2.

In what follows we use the following notations. Let $\mathscr{B}$ be a set system, k any natural number then $\mathscr{B}_{<k}, \mathscr{B}_k, \mathscr{B}_{>k} = \{B \in \mathscr{B}: |B| \lesseqgtr k\}$ respectively.

Proposition 9. $\lim C(c) = \infty$ *when* c *tends to* $\frac{1}{\nu^*(k-1)}$ *from the left iff there exists an intersecting set system* $\mathscr{B}$ *of rank* k *such that*

(a) $\nu^*(\mathscr{B}_{\leqslant k-1}) = \nu^*(k-1)$

(b) $\nu^*(\mathscr{B}) > \nu^*(k-1)$.

The conditions of Proposition 9 are never satisfied if there exists a $\mathscr{P}_{k-1}$.

Proposition 10. *If there exists a* $k-1$ *uniform finite projective plane,* $\mathscr{P}_{k-1}$, *then* $C(c)$ *is bounded on the interval* $\left[\frac{1}{\nu^*(k)}; \frac{1}{\nu^*(k-1)}\right)$, *namely*

$$C(c) < k^{k+2}.$$

(This upper bound may possibly be improved.) It seems to be very likely that conditions of Proposition 9 are never satisfied, more precisely we conjecture that $\lim C(c) < \infty$ if c tends to $\frac{1}{\nu^*(k-1)}$ from the left.

2.6. Case $\frac{1}{2} \leqslant c < \frac{2}{3}$.

Theorem F [4]. *If* $\frac{1}{2} < c < \frac{3}{5}$ *and* $n > n_0(r, c)$ *then*

$$f(n, r, c) = |\mathscr{F}[\overline{\mathscr{H}_1}]| = 10\binom{n-5}{r-3} + 5\binom{n-5}{r-4} + \binom{n-5}{r-5}.$$

If $\mathcal{F} \in \mathcal{F}(n, r, c)$ and the cardinality of $\mathcal{F}$ is maximal, then there exists a 3-uniform, intersecting set system $\mathcal{H}_1$ with 10 members on a 6-element set such that

$$\mathcal{F} = \mathcal{F}[\overline{\mathcal{H}_1}]$$

There exists exactly one such $\mathcal{H}_1$. (see Figure 1)

Theorem 11 [7]. *If* $\frac{3}{5} < c < \frac{2}{3}$ *and* $n > n_0(r, c)$ *then*

$$f(n, r, c) = 10\binom{n-5}{r-3} + 5\binom{n-5}{r-4} + \binom{n-5}{r-5}.$$

If $\mathcal{F} \in \mathcal{F}(n, r, c)$ and cardinality of $\mathcal{F}$ is maximal, then $\mathcal{F} = \mathcal{F}[\overline{\mathcal{H}_i}]$ for some $1 \leqslant i \leqslant 6$. (see Figure 1)

So $f(n, r, c)$ is constant on the whole interval $\left(\frac{1}{2}; \frac{2}{3}\right)$. Theorem 11 differs of Theorem F in that in case $\frac{3}{5} < c < \frac{2}{3}$ five more extremal systems are allowed. E.g.

$$\mathcal{F}[\overline{\mathcal{H}_2}] = \{F \subset X: |F| = r, \ |F \cap D| \geqslant 3\} \ (|D| = 5).$$

So we have completely different extremums. Such occurrences are not rare in combinatorics, even in the Erdős – Ko – Rado type theorems, see e.g. the theorem of Hilton – Milner for $r = 3$.

The following is a consequence of Theorem 11 and a strengthening of Theorem C (ii).

Corollary 12 [7]. *If* $\epsilon > 0$ *is fixed and* n *tends to infinity, then*

$$c\left(n, r, (10 + \epsilon)\binom{n}{r-3}\right) = \frac{2}{3} - o(1).$$

This is also an improvement of a theorem of P. Frankl [5].

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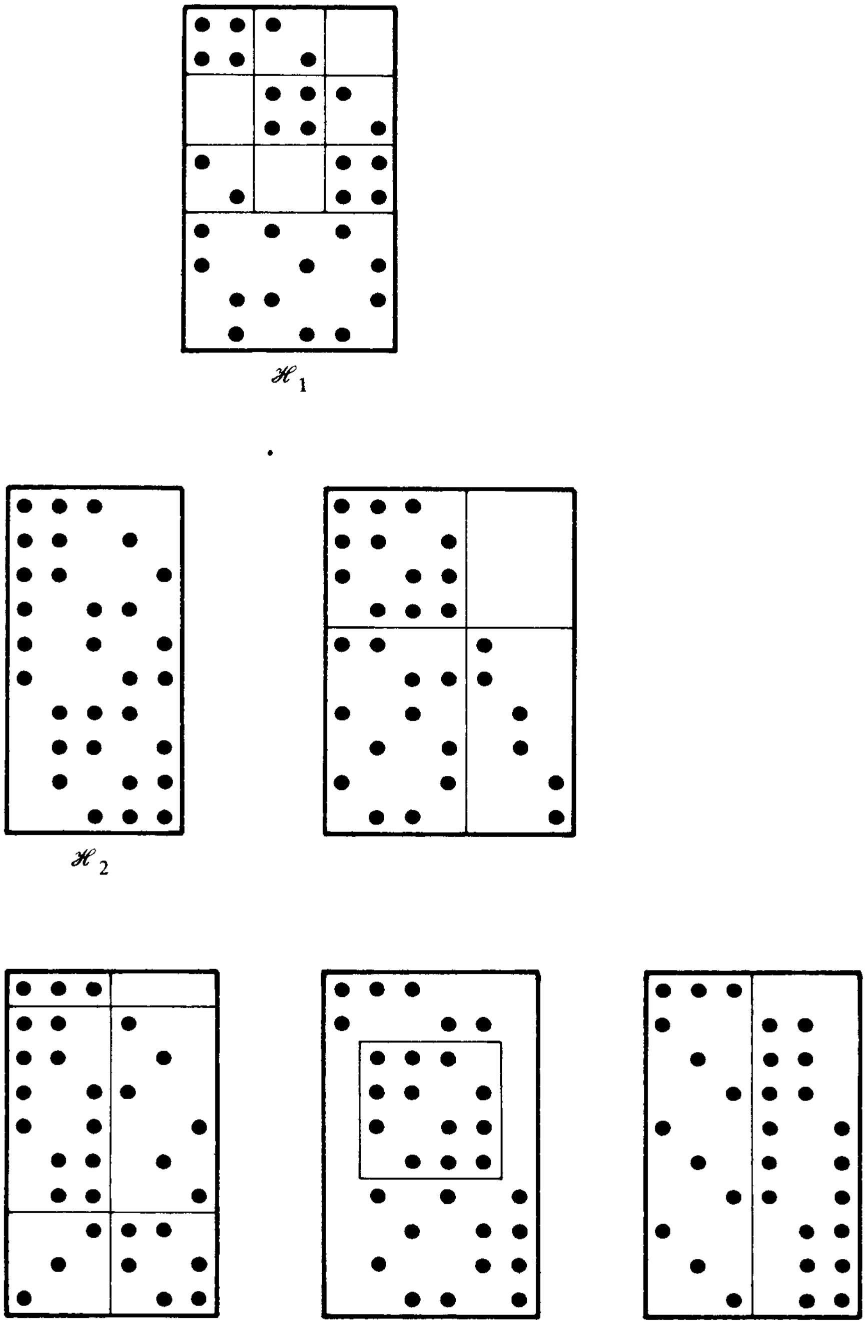


Figure 1

3. LEMMAS AND THE PROOF OF THE MAIN THEOREM

3.1. By contraction of an *edge* E *in a hypergraph* $\mathcal{H}$ we mean the following operation: we substitute an edge E by a smaller non-empty edge $E' \subsetneqq E$, and thus we get the hypergraph $(\mathcal{H} - \{E\}) \cup \{E'\}$.

A hypergraph $\mathcal{H}$ is ν-*critical* if it has no multiple edges and contracting any of its edges increases $\nu(\mathcal{H})$. We can get a ν-critical hypergraph from any hypergraph by contracting its edges as far as possible and deleting all but one copies of the appearing multiple edges.

We are going to use the following theorem:

Theorem G [4], [8]. *There exists a function* $L(R, \nu)$ *such that if the hypergraph* $\mathcal{H}$ *is* ν*-critical and its rank is* R, *then*

$$|E(\mathcal{H})| \leqslant L(R, \nu) < (R\nu)^R$$

and

$$|V(\mathcal{H})| \leqslant \frac{R}{2}\binom{R\nu + R - 1}{R}.$$

3.2. A set system $\mathcal{B}$ is called the *nucleus* of the hypergraph $\mathcal{H}$ if $\phi \notin \mathcal{B}$ and for every $H \in \mathcal{H}$ there exists a $B = B(H) \in \mathcal{B}$ such that B is contained in H. By 3.1, $\mathcal{H}$ has a ν-critical nucleus, and its cardinality is at most $(R\nu)^R$.

Lemma 13. *Suppose that* $\mathcal{H}$ *is a system of* r*-tuples of an* n*-element set,* $\nu(\mathcal{H}) = \nu$ *and* $1 \leqslant k \leqslant r$. *Then there exists a* ν*-critical set system* $\mathcal{B}^*$ *of rank* k *such that*

$$|\mathcal{H} - \mathcal{F}[\overline{\mathcal{B}^*}]| \leqslant L(r, \nu)\binom{n-k-1}{r-k-1}. \tag{7}$$

Proof. Let $\mathcal{B}$ a ν-critical nucleus of $\mathcal{H}$, and $\mathcal{B}^*$ a ν-critical nucleus of $\mathcal{B}_{\leqslant k}$. Then

$$\begin{aligned}|\mathcal{H} - \mathcal{F}[\overline{\mathcal{B}^*}]| &\leqslant |\{H \in \mathcal{H}:\ \exists B \in \mathcal{B}_{>k} \text{ such that } B \subset H\}| \leqslant \\ &\leqslant \sum_{B \in \mathcal{B}_{>k}} \binom{n-|B|}{r-|B|} \leqslant |\mathcal{B}_{>k}|\binom{n-k-1}{r-k-1} \leqslant \\ &\leqslant L(r, \nu)\binom{n-k-1}{r-k-1}.\end{aligned}$$

In what follows we call the $\mathscr{B}^*$ of the Lemma 13 the *ν-critical nucleus of rank* k *of* $\mathscr{H}$. Of course $\mathscr{B}^*$ is not unique. (In what follows we only use (7).)

3.3.

Lemma 14. *Let* a *be any non-negative real-valued function on the edges of the hypergraph* $(X, \mathscr{B})$. *Then*

$$\max_{x \in X} \Big(\sum_{B \ni x} a(B) \Big) \geqslant \frac{1}{\nu^*(\mathscr{B})} \Big(\sum_{B \in \mathscr{B}} a(B) \Big) = \frac{|a|}{\nu^*(\mathscr{B})}$$

($|a|$ *denotes the sum* $\sum a(B)$*).*

Remark. This lemma is the extension of the well-known inequality

$$\max_{x \in X} d_{\mathscr{B}}(x) \geqslant \frac{|\mathscr{B}|}{\nu^*(\mathscr{B})}.$$

Proof. Let $M =: \max_{x \in X} \sum_{B \ni x} a(B)$. Then the function $\frac{a}{M}$ is a fractional matching of $\mathscr{B}$, thus $\left|\frac{a}{M}\right| = \frac{|a|}{M} \leqslant \nu^*(\mathscr{B})$.

3.4. Let $\mathscr{B}$ be ν-critical nucleus of $\mathscr{H}$ with rank k. We define a function $a(\mathscr{B}, \mathscr{H})$: $\mathscr{B} \to \mathbf{R}$ on the edges of $\mathscr{B}$ as follows. First we choose an edge $B = B(H) \in \mathscr{B}$ to any $H \in \mathscr{H} \cap \mathscr{F}[\overline{\mathscr{B}}]$ such that $B \subset H$. Then let

$$a(B) = \frac{1}{\binom{n-k}{r-k}} \, |\{H \colon H \in \mathscr{H} \cap \mathscr{F}[\overline{\mathscr{B}}], \ B = B(H)\}|.$$

$a(\mathscr{B}, \mathscr{H})$ is not uniquely determined by $\mathscr{B}$ and $\mathscr{H}$ in general but this will cause no confusion.

3.5. Let $0 < c < 1$, and $(X, \mathscr{B})$ a set system with $\max_{B \in \mathscr{B}} |B| = k$.

The optimum value of the linear programming problem (8) is called the *capacity of* $\mathscr{B}$ *belonging to* c.

(8) $$\begin{cases} w\colon \mathscr{B} \to \mathbf{R} \\ w(B) \geq 0 \quad \text{for all} \quad B \in \mathscr{B} \\ w(B) \leq 1 \quad \text{for all} \quad B \in \mathscr{B}_k \\ \sum_{B \ni x} w(B) \leq c(\sum w(B)) = c\,|w|, \quad \text{for all} \quad x \in X. \end{cases}$$

$$\mathrm{Cap}_{\mathscr{B}}(c) =: \max\{|w|\colon w \ \text{satisfies (8)}\}.$$

It may of course occurs $\mathrm{Cap}_{\mathscr{B}}(c) = \infty$ and $\mathrm{Cap}_{\mathscr{B}}(c) = 0$.

The following Main Lemma explains how useful the concept of capacity is and that $\mathrm{Cap}_{\mathscr{B}}(c)$ shows how large a set system $\mathscr{F} \in \mathscr{F}(n, r, c)$ can be if its nucleus of rank k is $\mathscr{B}$.

Lemma 15 (Main Lemma). *If $\mathscr{F} \in \mathscr{F}(n, r, c)$ and $\mathscr{B}$ is its ν-critical nucleus of rank k then*

(9) $$|\mathscr{F}| < \mathrm{Cap}_{\mathscr{B}}(c)\binom{n}{r-k} + K(r, c)\binom{n}{r-k-1}.$$

On the other hand for any intersecting ν-critical set system $\mathscr{B}$ of order k there exists an $\mathscr{F}' \in \mathscr{F}(n, r, c)$ such that

(10) $$|\mathscr{F}'| > \mathrm{Cap}_{\mathscr{B}}(c)\binom{n}{r-k} - K(r, c)\binom{n}{r-k-1}.$$

If $\mathrm{Cap}_{\mathscr{B}}(c) = \infty$ the instead of (10) we have

$$\sup_{n \to \infty} \frac{|\mathscr{F}'|}{\binom{n}{r-k}} = \infty.$$

We are goint to prove this Main Lemma in chapter 5. Before this we discuss the function $\mathrm{Cap}_{\mathscr{B}}(c)$ in chapter 4. Now we only prove some simple properties of this function helping us in obtaining the Main Theorem from Main Lemma.

Lemma 16. $\mathrm{Cap}_{\mathscr{B}}(c) > 0$ *iff* $\dfrac{1}{\nu^*(\mathscr{B})} \leq c$.

This is a consequence of Lemma 14, and the definitions of ν and Cap.

Lemma 17. *If* $\mathscr{B}$ *is an intersecting set system of rank* k *and*

$$\frac{1}{\nu^*(k)} \leqslant \frac{1}{\nu^*(\mathscr{B})} \leqslant c < \frac{1}{\nu^*(k-1)}$$

then

$$\nu^*(\mathscr{B}) \leqslant \mathrm{Cap}_{\mathscr{B}}(c) \leqslant \frac{|\mathscr{B}_k|}{1 - c\nu^*(\mathscr{B}_{<k})}.$$

Proof. The lower bound is obtained from the fact that an optimal fraction matching w satisfies (8) if $c \geqslant \frac{1}{\nu^*(\mathscr{B})}$ and clearly $|w| = \nu^*(\mathscr{B})$.

To prove the upper bound let w be any function satisfying (8). We apply Lemma 14 to the restriction of w on $\mathscr{B}_{<k}$ and then we use that w fulfils (8):

$$\frac{1}{\nu^*(\mathscr{B}_{<k})} (|w| - |\mathscr{B}_k|) \leqslant \frac{1}{\nu^*(\mathscr{B}_{<k})} \Big(\sum_{B \in \mathscr{B}_{<k}} w(B)\Big) \leqslant$$

$$\leqslant \max_{x \in X} \Big(\sum_{\substack{B \ni x \\ B \in \mathscr{B}_{<k}}} w(B)\Big) \leqslant \max_{x \in X} \Big(\sum_{\substack{B \ni x \\ B \in \mathscr{B}}} w(B)\Big) \leqslant c\,|w|.$$

Comparing the first and last part of this inequality we get Lemma 17.

The proof of Theorem 3. Let $\mathscr{F} \in \mathscr{F}(n, r, c)$

$$\frac{1}{\nu^*(k)} \leqslant c < \frac{1}{\nu^*(k-1)}.$$

By Lemma 15 (9)

$$|\mathscr{F}| < \mathrm{Cap}_{\mathscr{B}}(c)\binom{n}{r-k} + K(r, c)\binom{n}{r-k-1},$$

where $\mathscr{B}$ is a ν-critical nucleus of rank k. $\mathscr{B}$ is intersecting. There are only finitely many ν-critical, intersecting set-systems of rank k by Theorem G. Moreover $\mathrm{Cap}_{\mathscr{B}}(c) < \infty$ for all such $\mathscr{B}$-s by Lemma 17. Thus

$$(11) \qquad |\mathscr{F}| < \max \mathrm{Cap}_{\mathscr{B}}(c)\binom{n}{r-k} + K(r, c)\binom{n}{r-k-1},$$

where the maximum extends over all ν-critical, intersecting set-systems of rank k.

Suppose $\mathrm{Cap}_{\mathscr{B}'}(c) = \max \mathrm{Cap}_{\mathscr{B}}(c)$ then Lemma 15 (10) gives an $\mathscr{F}' \in \mathscr{F}(n, r, c)$ such that

(12) $$|\mathscr{F}'| > \mathrm{Cap}_{\mathscr{B}'}(c)\binom{n}{r-k} - K(r, c)\binom{n}{r-k-1}.$$

The inequalities (11) and (12) proves that

$$f(n, r, c) = \max_{\mathscr{F} \in \mathscr{F}(n,r,c)} |\mathscr{F}| = \max \mathrm{Cap}_{\mathscr{B}}(c)\binom{n}{r-k} + $$
$$+ O\left(K(r, c)\binom{n}{r-k-1}\right).$$

This establishes Theorem 3. Moreover we have

Corollary 18. *If* $\frac{1}{\nu^*(k)} \leqslant c < \frac{1}{\nu^*(k-1)}$ *then*

$C(c) = \max \{\mathrm{Cap}_{\mathscr{B}}(c)$: $\mathscr{B}$ *has rank* k, *is* ν*-critical, and intersecting*$\}$.

4. A SURVAY OF THE FUNCTION Cap (c)

4.1. In this chapter $\mathscr{B}$ is a fixed hypergraph of rank k, and $\mathrm{Cap}_{\mathscr{B}}(c) = \mathrm{Cap}\,(c)$.

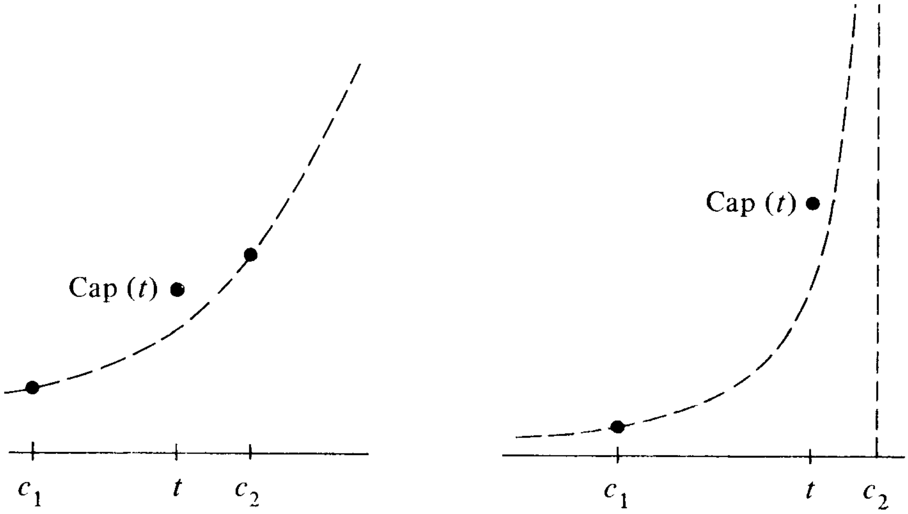
Lemma 19.

(i) *The function* Cap: $(0, 1) \to R \cup \{\infty\}$ *is monoton increasing and is continuous except in the point* $\frac{1}{\nu^*(\mathscr{B})}$.

(ii) *If* $\frac{1}{\nu^*(\mathscr{B})} \leqslant c_1 < t < c_2$ *and* $\mathrm{Cap}_{\mathscr{B}}(c_2) < \infty$ *then*

(13) $$\mathrm{Cap}\,(t) \geqslant \frac{\mathrm{Cap}\,(c_1)\,\mathrm{Cap}\,(c_2)(c_2 - c_1)}{\mathrm{Cap}\,(c_2)(c_2 - t) + \mathrm{Cap}\,(c_1)(t - c_1)}.$$

Lemma 19 means that Cap (t) is not smaller than the value at t of the rectangular hyperbola joining the points $(c_1;\ \mathrm{Cap}\,(c_1))$ and $(c_2;\ \mathrm{Cap}\,(c_2))$. See Figure 2.



$0 < \mathrm{Cap}\,(c_1) < \mathrm{Cap}\,(c_2) < \infty$ $\qquad$ $0 < \mathrm{Cap}\,(c_1),\quad \mathrm{Cap}\,(c_2) = \infty$

Figure 2

If we substitute c_1 by $\frac{1}{\nu^*(\mathscr{B})}$, t by $c-h$, c_2 by c in (13), and use Lemma 17, according to which $\mathrm{Cap}\left(\frac{1}{\nu^*(\mathscr{B})}\right) \geqslant \nu^*(\mathscr{B})$ we get

$$\text{(14)} \qquad \mathrm{Cap}\,(c-h) \geqslant \mathrm{Cap}\,(c) - h\,\frac{\mathrm{Cap}\,(c)^2}{\nu^*(\mathscr{B})\left(c - \frac{1}{\nu^*(\mathscr{B})}\right)}$$

since $\frac{1}{\nu^*(\mathscr{B})} < c - h < c$ and $\mathrm{Cap}\,(c) < \infty$.

We need this Lemma on the interval $\left[\frac{1}{\nu^*(k)}, \frac{1}{\nu^*(k-1)}\right)$. In this interval $\mathrm{Cap}_{\mathscr{B}}(c) < \infty$ by Lemma 17. However the proof will show that the proposition of the lemma can also be formulated for the case when $\mathrm{Cap}\,(c_2) = \infty$.

In this case (13) becomes the simpler formula

$$\text{(15)} \qquad \mathrm{Cap}_{\mathscr{B}}(t) \geqslant \frac{\mathrm{Cap}\,(c_1)(c_2 - c_1)}{c_2 - t}.$$

Also the function $\mathrm{Cap}_{\mathscr{B}}(c)$ is $\left((0,1) \setminus \left\{\frac{1}{\nu^*(\mathscr{B})}\right\}\right) \to \boldsymbol{R} \cup \{\infty\}$ continuous in the sense that $R \cup \{\infty\}$ is the Alexandroff-compactification of the real numbers.

Theorem 20. *The function* $\mathrm{Cap}_{\mathscr{B}}(c)$ *is constant on the interval* $\left(0, \frac{1}{\nu^*(\mathscr{B})}\right)$ *and on an other interval* $[t(\mathscr{B}), 1)$ *(with some* $t(\mathscr{B}) \leqslant 1$*) and on the interval* $\left(\frac{1}{\nu^*(\mathscr{B})}, t(\mathscr{B})\right)$ *it is the union of finitely-many rectangular hyperbola arcs.*

(All these hyperbola arcs are of the type $c \to \frac{1}{u - vc}$, and the number of them is between 0 and $2^{4k|\mathscr{B}|}$. See Figure 3.)

According to Lemma 19 (ii) the hyperbola arcs mentioned in Theorem 20, get less and less steep when c increases. The proofs will be presented in the following order: 19 (ii), 19 (i), 20.

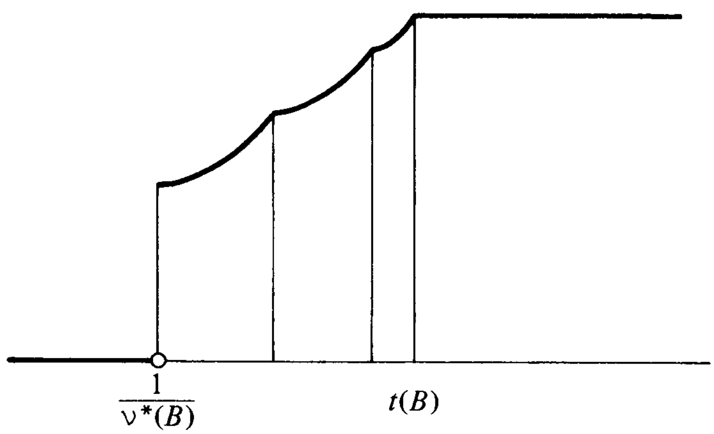


Figure 3

4.2. *The proof of Lemma* 19 (ii).

Suppose that $w_i \colon \mathscr{B} \to R^+$ satisfies (8) substituting c by c_i, and suppose $|w_i| = \mathrm{Cap}\,(c_i)$ $(i = 1, 2)$. Then $\sum_{B \ni x} w_i(B) \leqslant c_i |w_i|$ thus

$$\sum_{B \ni x} \alpha w_1(B) + (1 - \alpha) w_2(B) \leqslant \alpha c_1 |w_1| + (1 - \alpha) c_2 |w_2|,$$

for $0 \leqslant \alpha \leqslant 1$ arbitrary. Hence the function $\alpha w_1 + (1 - \alpha) w_2$ has weight $\alpha |w_1| + (1 - \alpha)|w_2|$, and satisfies (8) substituting c by

$$\frac{\alpha c_1 |w_1| + (1 - \alpha) c_2 |w_2|}{\alpha |w_1| + (1 - \alpha)|w_2|}.$$

From this we get

$$\text{Cap}\left(\frac{\alpha c_1 |w_1| + (1 - \alpha) c_2 |w_2|}{\alpha |w_1| + (1 - \alpha)|w_2|}\right) \geqslant$$
$$\geqslant \alpha \,\text{Cap}\,(c_1) + (1 - \alpha)\,\text{Cap}\,(c_2).$$

Putting

$$\alpha = \frac{(c_2 - t)|w_2|}{|w_2|(c \quad - t) + |w_1|(t - c_1)}$$

we get (13).

Q.E.D.

4.3. *The proof of Lemma* 19 (i)

The function $\text{Cap}_{\mathscr{B}}(c)$ is clearly monoton increasing. This and (14) show that if $c > \frac{1}{\nu^*(\mathscr{B})}$ then $\text{Cap}_{\mathscr{B}}(c)$ is left continuous.

The rightside continuity is proved by a simple argument. In view of the monotonity we only have to prove that if $c_{i+1} < c_i$ and $\lim_{i \to \infty} c_i = c$ then $\text{Cap}\,(c) \geqslant \lim_{i \to \infty} \text{Cap}\,(c_i)$. Let T be a real number such that $T <$ $< \lim \text{Cap}\,(c_i)$ further on let $w_i \colon \mathscr{B} \to \boldsymbol{R}^+$ some function satisfying (8) (with $c = c_i$) and $T \leqslant |w_i| \leqslant 2T$. A function w_i with such an upper bound can be found since when w_i satisfies (8) so does αw_i for all $0 \leqslant \alpha \leqslant 1$.

Let $B_1 \in \mathscr{B}$. The sequence $\{w_i(B_1)\}_{i=1,2,\dots}$ is bounded $(\subset [0, 2T])$ thus there exists an infinite subsequence $\boldsymbol{I}_1 \subseteq N$ such that $\lim \{w_i(B_1)\colon i \in \boldsymbol{I}_1\}$ exists. Let B_2 be an edge of $\mathscr{B} - \{B_1\}$ and $\boldsymbol{I}_2$ an infinite subsequence of $\boldsymbol{I}_1$ such that $\lim \{w_i(B_2)\colon i \in \boldsymbol{I}_2\}$ exists. E.t.c. for all m edges of $\mathscr{B}(|\mathscr{B}| = m)$. Let $w_0(B_j) = \lim \{w_i(B_j)\colon i \in \boldsymbol{I}_m\}$. This function satisfies (8). Thus $\text{Cap}_{\mathscr{B}}(c) \geqslant |w_0| \geqslant T$.

Q.E.D.

4.4. *Proof of Theorem* 20

We use the simple fact that the optimum of the linear programming problem (8) can be obtained on a vertex of the $|X|$-dimensional polytope defined by (8). So there exists a w such that $\mathrm{Cap}_{\mathscr{B}}(c) = |w|$ and w is the unique solution of the system of equations:

$$(16)\qquad \begin{cases} w(B) = 0 & \text{for some } B \in \mathscr{B} \\ w(B) = 1 & \text{for some } B \in \mathscr{B}_k \\ \sum_{B \ni x} w(B) = c\,|w| & \text{for some } x \in X, \text{ and} \\ \sum w(B) = \frac{1}{c}\, c\,|w|. \end{cases}$$

We consider $c|w|$ as a new variable (y). Solving this system of equations by Cramer's Rule, the value of $y = c|w|$ is defined as the ratio of two determinations. The determinant in the numerator contains only $0, 1, -1$, the one in the denominator has also $0, 1, -1$ only expect in the last row, where one single $-\frac{1}{c}$ occurs. Thus

$$c\,|w| = \frac{\det[0, 1, -1]}{\det\left[0, 1, -1, -\frac{1}{c}\right]} = \frac{A}{B + D\frac{1}{c}}$$

$$|w| = \frac{A}{D + Bc}$$

and A, B, D are integers and depend on the choose of (16) only.

(8) contains at most $2|\mathscr{B}| + |X|$ inequalities, (16) can be chosen on at most $2^{2k|\mathscr{B}|}$ ways. So far we have that the value of $\mathrm{Cap}_{\mathscr{B}}(c)$ is obtained by one of the finitely many hyperbolas $\frac{1}{u - vc}$. $\mathrm{Cap}_{\mathscr{B}}$ is continuous by Lemma 19 (i), thus it cannot change arbitrarily from a hyperbola to another, this change must allways take place on a common point of some hyperbola arcs. There are at most $2^{4k|\mathscr{B}|}$ common points.

Finally Lemma 19 (ii) shows that once the graph of the function $\mathrm{Cap}_{\mathscr{B}}$ contains an interval on which it is constant, it can not contain any more hyperbola arcs (for larger values of c).

5. PROOF OF LEMMA 15

5.1. *Proof of* (9)

First we apply Lemma 13.

$$|\mathscr{F} - \mathscr{F}[\overline{\mathscr{B}}]| \leqslant L(r)\binom{n-k-1}{r-k-1}.$$

where we use the notation $L(r) = L(r, 1)$.

If $|\mathscr{F} \cap \mathscr{F}[\overline{\mathscr{B}}]| \leqslant \mathrm{Cap}_{\mathscr{B}}(c)\binom{n-k}{r-k}$, then we are done. Suppose on the contrary. In this case $\mathrm{Cap}_{\mathscr{B}}(c) < \infty$ and $|a(\mathscr{B}, \mathscr{F})|\binom{n-k}{r-k} = |\mathscr{F} \cap \mathscr{F}[\overline{\mathscr{B}}]| > \mathrm{Cap}_{\mathscr{B}}(c)\binom{n-k}{r-k}$. (The definition of $a(\mathscr{B}, \mathscr{F})$ is in 3. 4.) From here we get

(17) $\qquad |a| > \mathrm{Cap}_{\mathscr{B}}(c) \geqslant 0.$

For an arbitrary point $x \in X$ we have

$$\binom{n-k}{r-k} \sum a(B) \leqslant |\{F\colon\ x \in F \in \mathscr{F},\ \text{there exists a}\ B \subset F\}| \leqslant$$
$$\leqslant |\{F\colon\ x \in F \in \mathscr{F}\}| \leqslant c|\mathscr{F}| \leqslant$$
$$\leqslant c|\mathscr{F} \cap \mathscr{F}[\overline{\mathscr{B}}]| + c|\mathscr{F} - \mathscr{F}[\overline{\mathscr{B}}]| \leqslant$$
$$\leqslant c\binom{n-k}{r-k}|a| + cL(r)\binom{n-k-1}{r-k-1} =$$
$$= \binom{n-k}{r-k}|a|\left(c + c\,\frac{r-k}{n-k}\,L(r)\,\frac{1}{|a|}\right).$$

This implies

$$\max_{x \in X} \sum_{B \ni x} a(B) \leqslant |a|\left(c + c\,\frac{r-k}{n-k}\,L(r)\,\frac{1}{|a|}\right).$$

This shows that the function a satisfies (8) with the value $c + \dfrac{c(r-k)L(r)}{|a|(n-k)}$ instead of c. This implies

(18) $\qquad 0 < |a| \leqslant \mathrm{Cap}_{\mathscr{B}}\left(c + \dfrac{(r-k)L(r)}{(n-k)|a|}\right).$

First assume $\mathrm{Cap}_{\mathscr{B}}(c) = 0$.

Then Lemma 16 and (18) show

$$c < \frac{1}{\nu^*(\mathscr{B})} \leqslant c + c\,\frac{(r-k)L(r)}{(n-k)|a|}.$$

So we have

$$|a| \leqslant \frac{r-k}{n-k}\,L(r)\,\frac{c}{\dfrac{1}{\nu^*(\mathscr{B})} - c}$$

and

$$(19)\qquad \begin{aligned} |\mathscr{F}| &\leqslant L(r)\binom{n-k-1}{r-k-1} + |a|\binom{n-k}{r-k} < \\ &< L(r)\,\frac{1}{1 - c\nu^*(\mathscr{B})}\binom{n-k-1}{r-k-1}. \end{aligned}$$

So next assume $\mathrm{Cap}_{\mathscr{B}}(c) > 0$. According to (17) Lemma 17 and Theorem E (iii) we get

$$|a| > \mathrm{Cap}_{\mathscr{B}}|c| \geqslant \nu^*(k) > \frac{k}{2}.$$

As a consequence of Theorem 20 $\mathrm{Cap}_{\mathscr{B}}(c)$ has Lipschitz property (form the right, too), for every c, whenever $\mathrm{Cap}_{\mathscr{B}}(c) < \infty$. So there are constants $L = L_{\mathscr{B}}(c)$, $\epsilon = \epsilon_{\mathscr{B}}(c) > 0$ such that

$$(20)\qquad \mathrm{Cap}_{\mathscr{B}}(c+h) \leqslant \mathrm{Cap}_{\mathscr{B}}(c) + Lh$$

for all $0 \leqslant h < \epsilon$.

(Let $T(B) = \sup\{c\colon c \leqslant 1$ and $\mathrm{Cap}_{\mathscr{B}}(c) < \infty\}$, with other words $T(B) = t(B)$ if $\mathrm{Cap}_{\mathscr{B}}$ is unbounded, and $T(B) = 1$ otherwise. Now if $\epsilon = \frac{1}{2}(T - c)$ then $L < 2\,\frac{\mathrm{Cap}\,(c)}{T - c}$.)

So suppose

$$c\,\frac{(r-k)L(r)}{|a|(n-k)} < c\,\frac{rL(r)}{n}\,\frac{2}{k} < \epsilon$$

that is

(21) $$\frac{2rL(r)c}{k\epsilon} < n,$$

then we can apply in (18) the Lipschitz property formulated by (20):

$$|a| \leqslant \mathrm{Cap}_{\mathscr{B}}(c) + Lc\,\frac{(r-k)L(r)}{(n-k)|a|} <$$
$$< \mathrm{Cap}_{\mathscr{B}}(c) + \frac{r-k}{n-k}\,\frac{2cL(r)}{k}\,L.$$

So we can write

(22) $$|\mathscr{F}| \leqslant |a|\binom{n-k}{r-k} + L(r)\binom{n-k-1}{r-k-1} <$$
$$< \mathrm{Cap}_{\mathscr{B}}(c)\binom{n-k}{r-k} + L(r)\binom{n-k-1}{r-k-1}\left(1+\frac{2cL}{k}\right).$$

Finally Theorem G says that there are finitely many ν-critical, intersecting set systems for any given c and so any given k. According to this there are only finitely many constants occuring in (21) and (22).

So there is a $K(r,c)$ such that arbitrarily $\mathscr{F} \in \mathscr{F}(n,r,c)$ we have

$$|\mathscr{F}| < \mathrm{Cap}_{\mathscr{B}}(c)\binom{n-k}{r-k} + K(r,c)\binom{n-k-1}{r-k-1}.$$

Q.E.D.

5.2. *The proof of* (10)

We have to show a good construction. We only deal with the case then $0 < \mathrm{Cap}_{\mathscr{B}}(c) < \infty$, that is $\frac{1}{\nu^*(k)} \leqslant c < T(\mathscr{B})$.

We distinguish two cases.

Case 1. $\frac{1}{\nu^*(k)} < c < T(\mathscr{B})$. We use the Lipschitz property of $\mathrm{Cap}_{\mathscr{B}}(c)$; Let $\frac{1}{\nu^*(k)} < c-h < c$ (h will be chosen later on). Let $w\colon \mathscr{B} \to \mathbf{R}$ be an optimal solution of (8) when we write $c-h$ instead of c in (8). $|w| = \mathrm{Cap}_{\mathscr{B}}(c-h)$. We construct a set system $\mathscr{F}' \in \mathscr{F}(n,r,c)$ on the n-element set X using this w. We can assume that $\bigcup \mathscr{B} = D \subset X$. By Theorem G $|D| \leqslant 4^k k$ $(< n)$. For each $B \in \mathscr{B}$ we choose a set-system $\mathscr{F}_B$ having $\left[w(B)\binom{n-|D|}{r-k}\right]$ sets. Each member of $\mathscr{F}_B$ has

r elements, contains B and intersects $X-D$ in $r-|B|$ elements. If

$$w(B)\binom{n-|D|}{r-k} \leqslant \binom{n-|D|}{r-|B|} \tag{23}$$

then we can achieve that the members of $\mathscr{F}_B$ are pairwise different and $\mathscr{F}_B$ is almost regular in $X-D$.

(A set system $\mathscr{H}$ is almost regular on a set Y if $|d_{\mathscr{H}}(x) - d_{\mathscr{H}}(y)| \leqslant 1$ for all $x, y \in Y$).

The almost regularity of $\mathscr{F}_B$ is guaranteed by the following theorem due to Zs. Baranyai.

Theorem H [1]. *Suppcse* $0 < m < \binom{n}{p}$, *then there exists an almost regular* p *uniform set system on* n *elements with* m *members.*

Note, that if $n > \mathrm{Cap}_{\mathscr{B}}(c)r + |D|$ then (23) holds. Put $\mathscr{F}' = \bigcup \mathscr{F}_B$.

Now we are going to choose h. We show, that we can do this so that $\mathscr{F}'$ will be in $\mathscr{F}(n, r, c)$. Clearly

$$|\mathscr{F}'| = \sum |\mathscr{F}_B| = \sum \left[w(B)\binom{n-|D|}{r-h}\right] > |w|\binom{n-|D|}{r-h} - |\mathscr{B}|. \tag{24}$$

If $x \in X - D$ then

$$d_{\mathscr{F}'}(x) = \sum_{B \ni x} d_{\mathscr{F}_B}(x) \leqslant \sum_{B \ni x} \left(\left[w(B)\binom{n-|D|}{r-k}\right] \frac{r-k}{n-|D|} + 1\right) \leqslant$$

$$\leqslant |w|\binom{n-|D|-1}{r-k-1} + |\mathscr{B}|.$$

If $x \in D$

$$d_{\mathscr{F}'}(x) = \sum_{B \ni x} d_{\mathscr{F}_B}(x) = \sum_{B \ni x} \left[w(B)\binom{n-|D|}{r-k}\right] <$$

$$< \left(\sum_{B \ni x} w(B)\right)\binom{n-|D|}{r-k} \leqslant (c-h)|w|\binom{n-|D|}{r-k}.$$

Thus we have $\mathscr{F}' \in \mathscr{F}(n, r, c)$ if $d_{\mathscr{F}'}(x) \leqslant c|\mathscr{F}'|$, that is

$$(c-h)|w|\binom{n-|D|}{r-k} \leqslant c|w|\binom{n-|D|}{r-k} - c|\mathscr{B}|,$$

that is

$$h \geqslant \frac{c|\mathscr{B}|}{|w|\binom{n-|D|}{r-k}}.$$

So let

$$h = \frac{c|\mathscr{B}|}{\binom{n-|D|}{r-k}}.$$

If n is large enough $(n > n_0(c, r))$, then this is smaller than $c - \frac{1}{\nu^*(\mathscr{B})}$.

Applying (14) we get from (24):

$$|\mathscr{F}'| > \mathrm{Cap}_{\mathscr{B}}(c-h)\binom{n-|D|}{r-k} - |\mathscr{B}| >$$

$$> \mathrm{Cap}_{\mathscr{B}}(c)\binom{n-|D|}{r-k} -$$

$$- h\frac{\mathrm{Cap}_{\mathscr{B}}(c)^2}{\nu^*(\mathscr{B})\left(c - \frac{1}{\nu^*(\mathscr{B})}\right)}\binom{n-|D|}{r-k} - |\mathscr{B}| =$$

$$= \mathrm{Cap}_{\mathscr{B}}(c)\binom{n-|D|}{r-k} - \left(\frac{c|\mathscr{B}|\mathrm{Cap}_{\mathscr{B}}(c)^2}{\nu^*(\mathscr{B})\left(c - \frac{1}{\nu^*(\mathscr{B})}\right)} + |\mathscr{B}|\right).$$

Using again Theorem G as we have done it several times we get (10).

Case 2. $\frac{1}{\nu^*(\mathscr{B})} = c.$

$\nu^*(\mathscr{B})$ is rational so the optimum of (8) can be obtained by a rational w. Thus there is a natrual number K such that $w(B)K$ is integer for all $B \in \mathscr{B}$. Now let $n = Kr!n_1 + |D| + q$, where $0 \leqslant q < Kr!$

We define a set system $\mathscr{F}'$ on a subset X_1 of X which has $n - q$ elements. We define $\mathscr{F}'$ on a similar way as we did it in the previous case.

So $\mathscr{F}_B$ is an r-uniform set-system on X_1. Each of its members contains B and meets $X_1 - |D|$ in $r - |B|$ elements, and the members of $\mathscr{F}_B$ are pairwise different.

Furtheron $\mathscr{F}_B$ is almost regular on $X_1 - D$, and

$$|\mathscr{F}_B| = w(B)\binom{n - |D| - q}{r - k}.$$

$n - |D| - q$ is divisible by $Kr!$ so this number is an integer. So we get by easy counting $\mathscr{F}' \in \mathscr{F}(n, r, c)$ and that (10) holds for $\mathscr{F}'$.

6. PROOFS

Proposition 15 and Corollary 18 give us a relatively simple and efficient tool to determine $f(n, r, c)$ and $c(n, r, N)$. Their usefulness can be seen for example in the proof of Theorem 1 and 2.

6.1. *Proof of Theorem* 2

Let $\mathscr{F} \in \mathscr{F}(n, r, c)$ and $|\mathscr{F}| = f(n, r, c)$ and $n > n_0(r, c)$. We know by 1.2 that

$$f(n, r, c) \geqslant f\left(n, r \frac{k}{k^2 - k + 1}\right) \geqslant$$

$$\geqslant (k^2 - k + 1)\binom{n}{r - k} + o\left(\binom{n}{r - k - 1}\right).$$

Let $\mathscr{B}$ be a ν-critical nucleus of rank k of $\mathscr{F}$. If $\mathscr{B} \neq \mathscr{P}_k$ then $\mathrm{Cap}_{\mathscr{B}}(c) = 0$ according to Theorem D (i) and Lemma 16, thus in this case $|\mathscr{F}| < K(r, c)\binom{n}{r - k - 1}$, a contradiction.

So $\mathscr{B} = \mathscr{P}_k$ if $|\mathscr{F}| = f(n, r, c)$.

If we use Lemma 13 we get

$$(k^2 - k + 1)\binom{n}{r - k} - K(r, c)\binom{n}{r - k - 1} < f(n, r, c) = |\mathscr{F}| \leqslant$$

$$\leqslant |\mathscr{F} - \mathscr{F}[\overline{\mathscr{P}_k}]| + \sum_{B \in \mathscr{P}_k} |\{F\colon B \subset F \in \mathscr{F}\}| \leqslant$$

$$\leqslant L(r)\binom{n}{r - k - 1} + (k^2 - k + 1)\binom{n - k}{r - k}.$$

(Now $C(c) = k^2 - k + 1$ is immediate).

From this inequalities we see that

$$|\{F\colon B\subset F\in\mathscr{F}\}|>\binom{n}{r-k}-(K(r,c)+L(r))\binom{n}{r-k-1},$$

for all $B\in\mathscr{P}_k$. Moreover for $n>n_0(r,c)$ this implies

(25) $$|\{F\colon B\subset F\in\mathscr{F}\}|>r\binom{n}{r-k-1}$$

for all $B\in\mathscr{P}_k$.

Using (25) we prove that all edges of $\mathscr{F}$ intersect all edges of $\mathscr{B}$. To do this we need the following lemma.

Lemma 21. *If* $\mathscr{H}$ *is a p-uniform set system on* Y, *and* $|\mathscr{H}|>q\binom{|Y|}{p-1}$ *then* $\tau(\mathscr{H})>q$.

Proof. If namely there were a Q in Y, with q elements, Q intersecting all edges of $\mathscr{H}$, then

$$|\mathscr{H}|\leqslant\sum_{x\in Q}|\{H\in\mathscr{H}\colon x\in H\}|\leqslant|Q|\binom{|Y|-1}{p-1}<q\binom{|Y|}{p-1}$$

a contradiction.

Returning to the proof of Theorem 2 we apply Lemma 21 to $Y=X-B$, $q=r$, $p=r-k-1$ and to the set system $\mathscr{H}_B=\{F-B\colon B\subset F\in\mathscr{F}\}$. So a set F_0 with r elements may intersect all edges of $\mathscr{H}_B$ only if $B\cap F_0\neq\phi$.

But $\mathscr{F}$ is intersecting and B was arbitrarily, so we get.

(26) $B\cap F_0\neq\phi$ for all $B\in\mathscr{P}_k$ and $F_0\in\mathscr{F}$.

It is a well-known fact that if a set S meets all edges of $\mathscr{P}_k$ *then* $|S|\geqslant k$. This is improved by the following theorem due to J. Pelikán.

Theorem I [9]. *Suppose* $k\geqslant 4$ *and* S *meets all edges of* $\mathscr{P}_k$, *but contains none of them. Then* $|S|>k+\sqrt{\frac{k}{2}}$.

It easy to see there is no such S *in the cases* $k=1,2,3$.

Using Theorem I we obtain from (26) that

$$|\mathscr{F} - \mathscr{F}[\overline{\mathscr{P}_k}]| < 2^{k^2-k+1}\binom{n}{r-k-\sqrt{\frac{k}{2}}} \qquad \text{if } k \geqslant 4 \text{ and}$$

$$|\mathscr{F} - \mathscr{F}[\overline{\mathscr{P}_k}]| = 0 \qquad \text{if } k \leqslant 3.$$

Clearly if $c > \frac{k}{k^2 - k + 1}$, then $\mathscr{F}[\overline{\mathscr{P}_k}] \subset \mathscr{F}$ since otherwise the system $\mathscr{F} \cup \mathscr{F}[\overline{\mathscr{P}_k}]$ would also be in $\mathscr{F}(n, r, c)$. This would contradict to the maximality of $\mathscr{F}$.

On the other hand if $c = \frac{k}{k^2 - k + 1}$ then using the notation $\bigcup_{B \in \mathscr{P}_k} \mathscr{B} = D \subset X$, we have

$$c = \frac{k}{k^2 - k + 1} \geqslant \frac{\max_{x \in D} d(x)}{|\mathscr{F}|} \geqslant \frac{\sum_{x \in D} d(x)}{|D||\mathscr{F}|} =$$

$$= \frac{1}{|\mathscr{F}|} \sum_{F \in \mathscr{F}} \frac{|F \cap D|}{|D|} \geqslant \frac{1}{|\mathscr{F}|} \sum_{F \in \mathscr{F}} \frac{k}{k^2 - k + 1} = \frac{k}{k^2 - k + 1}.$$

So equality stands throughout the formula. Thus $|F \cap D| = k$ for arbitrary $F \in \mathscr{F}$. According to Theorem I $\mathscr{F} \subset \mathscr{F}[\mathscr{P}_k]$ hence $\mathscr{F} = \mathscr{F}[\mathscr{P}_k]$.

Q.E.D.

6.2. *The proof of Theorem* 1 *and Corollary* 4 *and* 5

Theorem 1, follows simple from Corollary 4 and Theorem D (i), namely

$$c\left(n, r, \epsilon\binom{n}{r-k}\right) \geqslant \frac{1}{\nu^*(k)} - o(1) \geqslant \frac{k}{k^2 - k + 1} - o(1).$$

The proof of Corollary 4, runs as follows. Let $\mathscr{F} \in \mathscr{F}(n, r, c_0)$ where $c_0 < \frac{1}{\nu^*(k)}$ arbitrary. Let $\mathscr{B}$ the ν-critical nucleus of rank k of $\mathscr{F}$. Applying (9) we get.

$$|\mathscr{F}| \leqslant \mathrm{Cap}_{\mathscr{B}}(c_0)\binom{n}{r-k} + K(r, c_0)\binom{n}{r-k-1}.$$

$\mathrm{Cap}_{\mathscr{B}}(c_0) = 0$ according to Lemma 16 thus

$$|\mathscr{F}| \leqslant K(r, c_0)\binom{n}{r-k-1} < \epsilon\binom{n}{r-k}$$

if n is large enough. Consequently $c\left(n, r, \epsilon\binom{n}{r-k}\right) \geqslant c_0$ if $n > n(r, c_0)$. c_0 was arbitrary so

$$c\left(n, r, \epsilon\binom{n}{r-k}\right) \geqslant \frac{1}{\nu^*(k)} - o(1).$$

Q.E.D.

Corollary 5, is a simple consequence of Theorem 1 and Theorem D (i) which can be proved on the same way.

6.3. *The proofs of Propositions* 6, 7 *and* 8

These statements are trivial consequences of the analogous statements for the function Cap, using that $C(c) = \max\{\mathrm{Cap}_{\mathscr{B}}(c) \colon \mathscr{B}$ is intersecting, ν-critical and of order $k\}$ if $\frac{1}{\nu^*(k)} \leqslant c < \frac{1}{\nu^*(k-1)}$, by Corollary 18. (We also use the fact that Cap (c), and $C(c)$ too, is finite by Lemma 17). According to all this Proposition 6 follows by the fact $\mathrm{Cap}_{\mathscr{B}}(c)$ is continuous from the right (see Lemma 19 (i)). Proposition 7 follows from the upper bound given in Lemma 17 and finally Proposition 8 follows from the structure of $\mathrm{Cap}_{\mathscr{B}}(c)$ described Lemma 19 (ii).

6.4. *Proofs of Propositions* 9 *and* 10

Proof of Proposition 9. Let $\frac{1}{\nu^*(k)} < c < \frac{1}{\nu^*(k-1)}$. Let $\mathscr{B}$ be an intersecting, ν-critical set system of rank k, such that $\nu^*(\mathscr{B}) > \nu^*(k-1)$. (Such a $\mathscr{B}$ exists, see Proposition E (i).)

$$\mathrm{Cap}_{\mathscr{B}}(c) \leqslant \frac{|\mathscr{B}_k|}{1 - c\nu^*(\mathscr{B}_{<k})} \leqslant \frac{L(k)}{1 - c\nu^*(\mathscr{B}_{<k})},$$

according to Lemma 17 and Theorem G.

$$C(c) = \max \mathrm{Cap}_{\mathscr{B}}(c)$$

by Corollary 18, where the maximum is taken over the $\mathscr{B}$'s mentioned above.

Thus

$$(27)\qquad \lim_{c\to\frac{1}{\nu^*(k-1)}-0} C(c) \leqslant L(k) \max_{\mathscr{B}} \Big(\lim_{c} \frac{1}{1 - c\nu^*(\mathscr{B}_{<k})}\Big).$$

So if $\lim C(c) = \infty$ then there exists a $\mathscr{B}$ satisfying (a) and (b) in Proposition 9.

On the other hand if there exists a $\mathscr{B}_0$ satisfying (a) and (b) in Proposition 9 then $\mathscr{B}_0$ can be reduced into a ν-critical $\mathscr{B}_1$ by the contraction of some edges. We can use the following trivial lemma.

Lemma 21. *If we get $\mathscr{B}_1$ from $\mathscr{B}_0$ by contracting some of its edges then we $\nu^*(\mathscr{B}_1) \geqslant \nu^*(\mathscr{B}_0)$ and $\mathrm{Cap}_{\mathscr{B}_1}(c) \geqslant \mathrm{Cap}_{\mathscr{B}_0}(c)$.*

So $\mathscr{B}_1$ also satisfies (a) and (b) in Proposition 9. But clearly $\mathrm{Cap}_{\mathscr{B}_1}\Big(\frac{1}{\nu^*(k-1)}\Big) = \infty$. $\mathrm{Cap}_{\mathscr{B}_1}$ is $\Big[(0, 1) - \Big\{\frac{1}{\nu^*(\mathscr{B}_1)}\Big\}\Big] \to \boldsymbol{R} \cup \{\infty\}$-continuous (see Remark after Lemma 19.) This proves Proposition 9.

Proof of Proposition 10. Theorem I implies that there is no set with k elements intersecting all edges of $\mathscr{B}_{k-1}$ but containing none of them. So there is no intersecting set system $\mathscr{B}$ of rank exactly k such $\mathscr{B}_{k-1} = \mathscr{P}_{k-1}$.

According to the inequality of Theorem D (i) we have

$$\nu^*(\mathscr{B}_{k-1}) \leqslant k - 2 = \nu^*(k-1) - \frac{1}{k-1} \quad \text{if} \quad \mathscr{B}_{k-1} \neq \mathscr{P}_{k-1}.$$

So (27) implies

$$\lim_{c\to\frac{1}{\nu^*(k-1)}-0} C(c) \leqslant L(k) \max\Big(\lim \frac{1}{1 - c\Big[\nu^*(k-1) - \frac{1}{k-1}\Big]}\Big) \leqslant$$

$$\leqslant L(k)\nu^*(k-1)(k-1).$$

Q.E.D.

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