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On Automorphisms of Line-graphs

PETER L. ERDÖS AND Z. FÜREDI

Let α be an automorphism of the line-graph of the r-uniform hypergraph $\mathscr{H}$ with n points. If the valencies of $\mathscr{H}$ $v(x_1) \leq v(x_2) \leq \cdots \leq v(x_n)$ and

$$v(x_2) \geq v(n, r) \quad \text{and} \quad v(x_{2r}) > v(n, r) = \binom{n-1}{r-1} - \binom{n-r-1}{r-1} + 1,$$

then for $n > 4r$ α is induced by an automorphism of $\mathscr{H}$ (i.e. a permutation of $V(\mathscr{H})$).

Two examples show that the valency conditions of above theorem cannot be weakened in any point of $\mathscr{H}$.

1. Introduction, Results

Let $X = \{x_1, \ldots, x_n\}$ denote the points of an r-uniform hypergraph $\mathscr{H}$. Let $L(\mathscr{H})$ be the line-graph of $\mathscr{H}$, i.e. the underlying set of $L(\mathscr{H})$ is the edges of $\mathscr{H}$ and the pair (E_1, E_2), $E_1 \neq E_2$, $E_i \in \mathscr{H}$ is an edge of $L(\mathscr{H})$ iff $E_1 \cap E_2 \neq \emptyset$. Let $\mathrm{Aut}(L(\mathscr{H}))$ be its automorphisms. Denote the set of r-tuples of X by K_n^r.

Every permutation $\alpha \in \mathrm{Aut}\, \mathscr{H}$ of X induces an automorphism a_α of $L(\mathscr{H})$ in a natural way, namely $a_\alpha(E) = \{\alpha(x) : x \in E\}$ for every $E \in \mathscr{H}$.

C. Berge and J. C. Fournier proved the following theorem [1, 2].

PROPOSITION 1. *If $a \in \mathrm{Aut}(L(K_n^r))$ and $n > 2r$, then there exists a permutation α on X which induces a (i.e. $a = a_\alpha$).*

The condition $n > 2r$ cannot be omitted.

This question is strongly connected with the problem of reconstructing an r-graph $\mathscr{H}$ from its line-graph $L(\mathscr{H})$. For graphs the following theorem of Whitney [5] is well known.

PROPOSITION 2. *If every vertex of the graph $\mathscr{G}$ has valency >3, then it can be reconstructed from its line-graph.*

Therefore, under the condition of Proposition 2 if $a \in \mathrm{Aut}\, L(\mathscr{G})$, then there exists a permutation α of vertices of $\mathscr{G}$ for which $a = a_\alpha$.

The following theorem is an extension of Proposition 1 and generalized Proposition 2 to r-graphs. Henceforth the *valency* or degree of the point x of the hypergraph $\mathscr{H}$ is, as usual, $v(x) =: |\{E : x \in E \in \mathscr{H}\}|$.

THEOREM 1. *If every vertex of the r-uniform hypergraph $\mathscr{H}$ has valency greater than*

$$v(n, r) = \binom{n-1}{r-1} - \binom{n-r-1}{r-1} + 1$$

and $n > 2r$, then for every $a \in \mathrm{Aut}(L(\mathscr{H}))$ there exists a permutation α on X such that $a = a_\alpha$.

For graphs, Theorem 1 seems slightly weaker than Proposition 2 but, as a matter of fact, they are equivalent. The reason for this is that $v(n, 2) = 3$ is independent of n while $v(n, r)$ tends to infinity for fixed $r \geq 3$.

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The following example shows that the claim of Theorem 1 is sharp in the sense that $v(n, r)$ cannot be replaced by a smaller integer if n is great enough.

EXAMPLE 1. Let $F_1=\{x_1, \dots, x_r\}$, $F_2=\{x_{r+1}, \dots, x_{2r}\}$ and $\mathcal{H}_1=\{F_1, F_2\}\cup\{F\subset X: |F|=r, F\cap F_1=\emptyset \text{ iff } F\cap F_2=\emptyset\}$. Then

$$v(x_i)=v(n, r) \quad \text{if } i\leq 2r$$

and

$$v(x_i)=\binom{n-1}{r-1}-2\binom{n-r-1}{r-1}+2\binom{n-2r-1}{r-1} \quad \text{if } i>2r.$$

Thus, if $n>2r^2$, then $v(x_i)>v(n, r)$ for $i>2r$. Finally the following automorphism a_1 of $L(\mathcal{H}_1)$ cannot be induced by any permutation of X.

$$a_1(F)=:\begin{cases} F_2 & \text{if } F=F_1 \\ F_1 & \text{if } F=F_2 \\ F & \text{otherwise.} \end{cases}$$

EXAMPLE 2. Let $r\geq 3$ and

$$F_3=\{x_1\}\cup\{x_3, x_4, \dots, x_{r+1}\},$$
$$F_4=\{x_2\}\cup\{x_3, x_4, \dots, x_{r+1}\}$$

and

$$\mathcal{H}_2=\{F_3, F_4\}\cup\{F\subset X: |F|=r, F\cap F_3=\emptyset \quad \text{iff } F\cap F_4=\emptyset\}.$$

Then

$$v(x_1)=v(x_2)=v(n, r)-1,$$

$$v(x_3)=v(x_4)=\cdots=v(x_{r+1})=\binom{n-1}{r-1}>v(n, r),$$

and if $i>r+1$, $n\geq 3r$ then

$$v(x_i)=\binom{n-1}{r-1}-2\binom{n-r-2}{r-2}>v(n, r).$$

Finally the following automorphism a_2 of $L(\mathcal{H}_2)$ cannot be induced by any permutation of X.

$$a_2(F)=\begin{cases} F_4 & \text{if } F=F_3 \\ F_3 & \text{if } F=F_4 \\ F & \text{otherwise.} \end{cases}$$

As we have seen, the valency condition of Theorem 1 cannot be weakened in every point of $\mathcal{H}$. The following theorem shows that it can be done in fewer than $2r$ points.

THEOREM 2. *Let $\mathcal{H}$ be an r-uniform hypergraph on $|X|=n\geq 4r$ points, and $v(x_1)\leq v(x_2)\leq\cdots\leq v(x_n)$. If $v(x_2)\geq v(n, r)$ and $v(x_{2r})>v(n, r)$, then for every $a\in \operatorname{Aut}(L(\mathcal{H}))$ there exists a permutation α on X such that $a=a_\alpha$.*

Examples 1 and 2 show that the conditions of Theorem 2 cannot be weakened in any point of $\mathcal{H}$.

Let a set system be called intersecting if the pairwise intersections are non-empty. The value of $v(n, r)$ in the theorems comes from the following theorem of Hilton and Milner [3].

THEOREM 3. *If $\mathcal{C}$ is an r-uniform intersecting set system on X, and $\bigcap\mathcal{C} = \varnothing$ then*

$$|\mathcal{C}| \leq v = v(n, r) = \binom{n-1}{r-1} - \binom{n-r-1}{r-1} + 1.$$

For $n > 2r$ and $r \neq 3$ the equality holds iff there are a point $x \in X$ and r-tuple $D \subset X (x \notin D)$ for which

$$\mathcal{C} = \mathcal{C}_{x,D} = \{E: x \in E \subset X: |E| = r, E \cap D \neq \varnothing\} \cup \{D\}. \tag{1}$$

If $n > 2r$ and $r = 3$, then there is another extremum

$$\mathcal{C} = \mathcal{C}_D = \{E: E \subset X, |E| = 3, |E \cap D| \geq 2\}. \tag{2}$$

2. PROOF OF THEOREM 1

Let us suppose that $\mathcal{H}$ satisfies the conditions of Theorem 1. Let $a \in \mathrm{Aut}(L(\mathcal{H}))$. The proof is constructive. First we define a permutation α on X, then show that $a = a_\alpha$.

Let $\mathcal{C}_i$ denote the system of edges of $\mathcal{H}$ containing point x_i. Similarly $\mathcal{C}_p$ denotes the system of edges containing p.

LEMMA 1. *If $\mathcal{C}$ is an r-uniform set system on X and $|\mathcal{C}| \geq v - 1$, then $|\bigcap\mathcal{C}| \leq 1$.*

Indeed, if $n > r$, then

$$v - 1 = \binom{n-1}{r-1} - \binom{n-r-1}{r-1} > \binom{n-2}{r-2}.$$

PROOF OF THEOREM 1. Since $a(\mathcal{C}_i) = \{a(E): E \in \mathcal{C}_i\}$ is an intersecting set system and $|a(\mathcal{C}_i)| = |\mathcal{C}_i| > v(n, r)$, thus, according to Theorem 3, there is a common point of sets of $a(\mathcal{C}_i)$. This point is unique by Lemma 1. Let us denote it by $a(x_i)$. Therefore if

$$x_i \in E \quad \text{then } \alpha(x_i) \in a(E) \tag{3}$$

for every $x_i \in X$.

On the other hand if $i \neq j$, then $\alpha(x_i) \neq \alpha(x_j)$. Suppose the contrary. Then $a(\mathcal{C}_i) \cup a(\mathcal{C}_j)$ is an intersecting set system and hence $\mathcal{C}_i \cup \mathcal{C}_j$ is an intersecting system, too.

By Theorem 3 there is a point p of X such that $p \in \bigcap(\mathcal{C}_i \cup \mathcal{C}_j)$. If, e.g. $p \neq x_i$, then $\{x_i, p\} \subset \bigcap\mathcal{C}_i$, and it contradicts Lemma 1. So we proved that α is a permutation of X, and, by (3), $a = a_\alpha$.

REMARK. As a matter of fact we proved the following assertion. In the class of those r-uniform hypergraphs $(Y, \mathcal{F})$ for which $\min_{y \in Y} v_{\mathcal{F}}(y) > v(|Y|, r)$, the line-graphs of the hypergraphs are isomorphic iff the hypergraphs themselves are isomorphic. Since $v(n, 2) = 3$ for every n, it is exactly Proposition 2.

Let us remark, if r is fixed and n tends to infinity, then almost all hypergraphs are in the above class.

3. Proof of Theorem 2 (Sketch)

The detailed proof would contain some parts easier to prove than to understand their proofs. These proofs are left to the reader.

Lemma 2. *If the conditions of Theorem 2 are satisfied, and* $|\mathscr{C}_x| = v(n, r)$ *then either*

$$|\bigcap a(\mathscr{C}_x)| = 1 \tag{4}$$

or there exist an $F_x \in \mathscr{C}_x$ *and an* $x' \notin a(F_x)$ *so that*

$$a(\mathscr{C}_x) = \mathscr{C}_{x', a(F_x)}. \tag{5}$$

This states that the case (2) in Theorem 3 cannot be realized. The lemma can be proved indirectly.

According to the valency of the points of X let us divide the set X into three disjoint parts; $X_{<v}, X_v, X_{>v} = \{x \in X: v(x) \lesseqqgtr v(n, r)\}$. (Naturally $X_{<v} = \emptyset$ or $\{x_1\}$.) The definition of the map α on the $X \backslash X_{<v}$ is similar to definition of α in the proof of Theorem 1. If $|\mathscr{C}_x| \geq v$ and $\bigcap a(\mathscr{C}_x) \neq \emptyset$ then let $\alpha(x) = \bigcap a(\mathscr{C}_x)$. The points $x \in (X \backslash X_{<v})$ are called regular for which claim (3) is realized. In particular, every point of $X_{>v}$ is regular. For a non-regular point x of X_v let $\alpha(x) = x'$ as was defined in Lemma 2. By Lemma 1, if $x, y \in X \backslash X_{<v}, x \neq y$, then $\alpha(x) \neq \alpha(y)$. Moreover, the restriction of α on $X_{>v}$ is a permutation of $X_{>v}$.

Lemma 3. *If* $x \in X \backslash X_{<v}$ *is a regular point, then*

$$x \in E \quad \text{iff } \alpha(x) \in a(E). \tag{6}$$

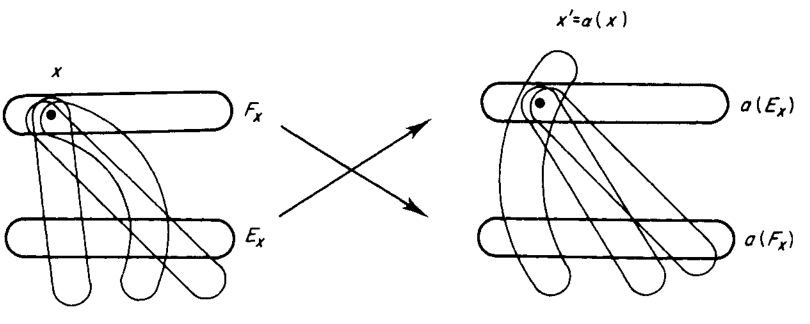
For points of $X_{>v}$, (6) is evidently true. If $x \in X_v$ is a regular point, then $\alpha(x) \in X_v$, so $|a(\mathscr{C}_x)| = |\mathscr{C}_{\alpha(x)}|$. On the other hand, by the regularity, $a(\mathscr{C}_x) \subset \mathscr{C}_{\alpha(x)}$ so they coincide.

Lemma 4. *If the point* $x \in X_v$ *is non-regular, and* $\alpha(x) \neq x_1$, *then there exists an edge* $E_x \notin \mathscr{C}_x$, *which intersects all edges in* $\mathscr{C}_x$ *except for* F_x *defined in Lemma 2. Moreover*

(i) $\mathscr{C}_x = \{F_x\} \cup \{E \subseteq X: |E| = r, x \in E, E \cap E_x \neq \emptyset\}$,

(ii) *no point in* E_x *is regular (if* $n \geq 4r - 2$*),*

(iii) *the point* $\alpha(x)$ *is regular.*



Proof. (i) is trivial and then $a(\mathscr{C}_x \cup \{E_x\}) = \mathscr{C}_{x', a(F_x)} \cup \{a(E_x)\}$ for an $x' \in a(E_x)$. (ii) is proved indirectly. If y is regular point in E_x, let $H \in \mathscr{C}_x$ be an edge, which contains points x, y and further $(r-2)$ regular points, the α-image of which is not in $a(F_x)$. Such an edge exists, because $n \geq 4r - 2$ and there are at least $2r - 1$ regular points of X, and there are at most r of them the α-image of which is in $a(F_x)$. By (i), $H \in C_x$. But $H \neq F_x$, therefore $a(H) = \{\alpha(z): z \in H\}$, $a(H)$ does not intersect $a(F_x)$. This contradicts to $x \in H \cap F \neq \emptyset$.

(iii) According to $|F_x \cup E_x| = 2r$ and $|X_{<v} \cup X_v| < 2r$, there exists a point $z \in (F_x \cup E_x) \cap X_{>v}$, and, by (ii), $z \in F_x$. So $\alpha(z) \in a(F_x) \cap X_{>v}$, because z is regular. On the other hand we show, if x' is not regular then (ii) is satisfied by x' in place of x which contradicts to $\alpha(z) \in a(F_x)$. Indeed, Lemma 2 states that at least v among the images of the elements of the system $\mathscr{C}_{x'} \cup \{a(F_x)\}$ contain $\alpha(x')$, so $\alpha(x') \notin X_{<v}$.

LEMMA 5. *Suppose $v(x_1) < v$ and there exists a point u for which $\alpha(u) = x_1$. Let $\alpha(x_1) = X \backslash \{\alpha(z): z \neq x_1\}$. Then the cycle of the permutation α containing x_1, i.e. $(x_1, \alpha(x_1), \ldots, \alpha^K(x_1) = u)$, has regular points only except possibility for x_1 and u.*

The proof of this lemma is similar to the proof of the Lemma 4(iii). It is not given here.

PROOF OF THEOREM 2

1. First we show that there are at most r non-regular points. Indeed, if there is no $u \in X_v$ for which $\alpha(u) = x_1$, then α is a permutation of $X - X_{<v}$. Let us look at the cycles of α in X_v. Lemma 4(iii) claims that at least the half of the points are regular in every cycle. Therefore the number of non-regular points is $\leq |X_{<v}| + \frac{1}{2}|X_v| \leq r$.

If there exists $u \in X_v$ for which $\alpha(u) = x_1$ then, by (iii) of Lemma 4 and by Lemma 5, the number of non-regular points is at most

$$2 + \left[\frac{|X_{<v} \cup X_v| - 2}{2}\right] \leq 2 + \left[\frac{2r-3}{2}\right] = r.$$

2. After these the application of Lemma 4(ii) gives that every $x \in X_v$, $\alpha(x) \neq x_1$ is a regular point.

3. Now it can be shown that there is no $u \in X_v$ for which $\alpha(u) = x_1$.

4. It was proven that every point of $X - X_{<v}$ is regular and α is a permutation of $X \backslash X_{<v}$. If $X_{<v} = \emptyset$ then $a = a_\alpha$, by Lemma 3. If $v(x_1) < v$, then let $\alpha(x_1) = x_1$. In this case, if $x_1 \notin E$ then every point of E is regular, so $a(E) = \{\alpha(x): x \in E\}$. Finally, if $x_1 \in E$ then, according to Lemma 3, $a(E)$ contains at most $r - 1$ regular points, so $x_1 \in a(E)$ and $a = a_\alpha$.

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P. L. ERDÖS
Institute of Mathematics and Computer Science, K. Marx University of Economics, 1828 Budapest, Dimitrov tér 8, P.O.B. 489, Hungary

Z. FÜREDI
Mathematical Institute of the Hungarian Academy of Sciences, 1053 Budapest, Realtanoda 13–15, Hungary