Boundary Crossings and the Distribution Function of the Maximum of Brownian Sheet

Endre Csáki[1]
Alfréd Rényi Institute of Mathematics
Hungarian Academy of Sciences
P. O. Box 127
H–1364 Budapest
Hungary
csaki@math-inst.hu

Davar Khoshnevisan[2]
University of Utah
Department of Mathematics
155 South 1400 East JWB 233
Salt Lake City, UT 84112
U. S. A.
davar@math.utah.edu

Zhan Shi[3]
Laboratoire de Probabilités
Université Paris VI
4 Place Jussieu
F–75252 Paris Cedex 05
France
zhan@proba.jussieu.fr

July 12, 1999

[1]Supported, in part, by the Hungarian National Foundation for Scientific Research, Grant No. T 019346 and T 029621 and by the joint French-Hungarian Intergovernmental Grant "Balaton" No. F25/97.

[2]Supported, in part, by a grant from NSF and by NATO grant No. CRG 972075.

[3]Supported, in part, by the joint French-Hungarian Intergovernmental Grant "Balaton" No. F25/97 and by NATO grant No. CRG 972075.

Abstract

Our main intention is to describe the behavior of the (cumulative) distribution function of the random variable $M_{0,1} := \sup_{0 \le s,t \le 1} W(s,t)$ near 0, where W denotes one–dimensional, two–parameter Brownian sheet. A remarkable result of Florit and Nualart asserts that $M_{0,1}$ has a smooth density function with respect to Lebesgue's measure; cf. [13]. Our estimates, in turn, seem to imply that the behavior of the density function of $M_{0,1}$ near 0 is quite exotic and, in particular, there is no clear-cut notion of a two–parameter reflection principle.

We also consider the supremum of Brownian sheet over rectangles that are away from the origin. We apply our estimates to get an infinite dimensional analogue of Hirsch's theorem for Brownian motion.

Keywords Tail probability, Quasi-sure analysis, Brownian sheet

1991 AMS Mathematics Subject Classification 60G60; 60G17.

1 Introduction

While it is simple and completely classical, the following boundary crossing problem is still illuminating to this day. Given a standard linear Brownian motion $B = \{B(t);\ t \geq 0\}$, we let $T_a = \inf\{s \geq 0 :\ B(s) \geq a\}$ denote the first passage time to $a \in \mathbb{R}_+$ and recall that as $n \to \infty$,

$$\mathbb{P}\{T_a > n\} \sim \left(\frac{2}{\pi n}\right)^{1/2} a. \tag{1.1}$$

This is a very well understood 'Tauberian' phenomenon and, together with its numerous extensions, can be shown by a variety of techniques. For example, see the treatment of FELLER [11, Ch. III] and [12, Ch. VI, XIV.5]. One way to verify (1.1) is by relating T_a to the supremum of B as follows:

$$\mathbb{P}\{T_a > n\} = \mathbb{P}\{\sup_{0 \leq s \leq n} B(s) \leq a\}.$$

At this point, one can use ANDRÉ's reflection principle, Brownian scaling and L'Hospital's rule to derive Eq. (1.1) readily. It is interesting to point out that modern applications of (1.1) and its refinements still abound in the literature; see [21, 25] for two striking classes of examples.

In the context of a more general random field B, the argument of the previous paragraph relates "boundary crossing problems" to the cumulative distribution function of $\sup_{0 \leq s \leq n} B(s)$ (henceforth, written as the c.d.f. of $\sup_{0 \leq s \leq n} B(s)$). Even when B is a Gaussian random field, outside a handful of examples, neither this c.d.f., nor its behavior near 0, are known; cf. [3] for a list and for detailed references. However, it *is* well known that the tail of the distribution of the maximum of a Gaussian process plays an important rôle in the structure and regularity of its sample paths; cf. [3, 22] for two textbook treatments. Such large deviation estimates are quite well–understood and, in certain cases, can be shown to a surprising degree of accuracy. For this, and for other interesting applications, see [1, 2, 5, 7, 10, 15, 16, 17, 22, 26, 30, 31].

Our main intention for writing this article is to understand boundary crossing problems for a two–parameter Brownian sheet $W = \{W(s,t);\ s,t \geq 0\}$. In light of our argument leading to Eq. (1.1), such boundary crossing issues translate to, and should be interpreted as, the estimation of the c.d.f. of $\sup_{(s,t)\in[0,1]^2} W(s,t)$ near 0. While very good asymptotic results of a large deviations type are found in [24], the analysis of the lower tails of $\sup_{(s,t)\in[0,1]^2} W(s,t)$ requires more subtle methods, as we shall see below.

Other than the results of this paper, we are aware of the following discovery of FLORIT AND NUALART regarding the c.d.f. of the maximum of W: the law of $\sup_{(s,t)\in[0,1]^2} W(s,t)$ is absolutely continuous with respect to Lebesgue's measure on $\mathbb{R}$ and has a C^∞ density. Our Theorem 1.1 below strongly suggests that the behavior of this density function near 0 is very exotic.

Throughout, we let $W := \{W(s,t);\ (s,t) \in \mathbb{R}^2_+\}$ designate a standard Brownian sheet. That is, W is a centered, real–valued Gaussian process with continuous samples and whose covariance is given by

$$\mathbb{E}\{W(s_1,t_1)W(s_2,t_2)\} = \min(s_1,s_2) \times \min(t_1,t_2), \qquad s_1,s_2,t_1,t_2 \geq 0.$$

We are interested in the distribution function of the maximum of W over a compact set (say, a rectangle along coordinates) in $\mathbb{R}^2_+$. To expedite the exposition, for all $0 \leq a < b$, we define

$$M_{a,b} := \sup_{(s,t)\in[a,b]\times[0,1]} W(s,t), \tag{1.2}$$

$$\xi^*_{a,b} := \frac{2\pi}{\pi + 2\arcsin\sqrt{a/b}}. \tag{1.3}$$

We shall soon see that the degree of regularity of the c.d.f. of $M_{a,b}$ depends on whether or not $a > 0$. Equivalently, the behavior of the c.d.f. of $M_{a,b}$ will be shown to depend on whether or not the rectangle $[a,b] \times [0,1]$ contains the origin. First, let us look at this c.d.f. when the rectangle in question is bounded away from the origin.

Theorem 1.1 *For all $b > a > 0$, there exists a finite constant $\xi_{a,b} \geq \xi^*_{a,b}$, such that*

$$\lim_{\varepsilon\to 0} \frac{\log \mathbb{P}(M_{a,b} < \varepsilon)}{\log \varepsilon} = \xi_{a,b} \,. \tag{1.4}$$

Remark 1.2 It is important to note that the constant $\xi_{a,b}$ of Eq. (1.4) is *strictly* greater than 1. This observation will lead us to a new class of exceptional sets for Brownian motion in the sense of WILLIAMS; cf. [29]. In fact, Theorem 1.1 yields a quasi-sure analogue of a theorem of HIRSCH for Brownian motion; see Section 6 for details. □

Remark 1.3 Roughly speaking, Theorem 1.1 states that the decay of the distribution function of the maximum of W over a rectangle that is bounded away from the origin satisfies a power law. □

Remark 1.4 Theorem 1.1 and Theorem 1.5 below are not related to the small ball problem for the Brownian sheet: the lack of absolute values around W in (1.2) is critical (as it is in a 1–parameter setting.) □

Next, we look at the distribution function of the maximum of W over a rectangle that contains the origin. By scaling, we may restrict our attention to the supremum of W over $[0,1]^2$ which, you may recall, we denote by $M_{0,1}$.

Upon formally taking $a = 0$ and $b = 1$ in Theorem 1.1, one may be tempted to think that for small ε, $\mathbb{P}(M_{0,1} < \varepsilon)$ also behaves like a power of ε; cf. Remark 1.3. However, the covariance structure of W has a "kink" in the origin which forces $M_{0,1}$ to be much larger than $M_{1,2}$, say. A more precise statement follows.

Theorem 1.5 *There exist finite constants $c_1, c_2 > 0$, such that for all sufficiently small $\varepsilon > 0$,*

$$\exp\Big(-c_1 \left\{ \log(1/\varepsilon)\right\}^2\Big) \le \mathbb{P}(M_{0,1} < \varepsilon) \le \exp\left(-\frac{c_2 \left\{ \log(1/\varepsilon)\right\}^2}{\log\log(1/\varepsilon)}\right). \tag{1.5}$$

Remark 1.6 The second inequality in (1.5) shows that the distribution of $M_{0,1}$ decays faster than any power function. It also suggests that the density function of $M_{0,1}$ near 0 has unusual behavior. In fact, if f denotes the density of $M_{0,1}$, one may guess from (1.5) that as $\varepsilon \to 0^+$, $\log f(\varepsilon)$ is of the same rough order as $-\big\{\log(1/\varepsilon)\big\}^2$. Furthermore, Theorem 1.5 implies that the law of the maximum of Brownian sheet is incomparable to that of the absolute value Brownian sheet at any given time point. As such, there can never be a two–parameter reflection principle for Brownian sheet. □

Remark 1.7 An important property of the 2–parameter Brownian sheet is that, locally and away from the axes, it looks like 2–parameter additive Brownian motion; cf. [9, 18, 20]. Recall that the latter is defined as the 2–parameter process $A := \big\{A(s,t);\ s,t \ge 0\big\}$, where $A(s,t) := B_1(s) + B_2(t)$ and where B_1 and B_2 are independent standard Brownian motions. It is not hard to check the following directly: as $\varepsilon \to 0^+$,

$$\mathbb{P}\Big\{ \sup_{(s,t)\in[0,1]^2} A(s,t) < \varepsilon \Big\} \sim \frac{1}{\pi}\varepsilon^2.$$

Comparing this with Theorem 1.5, we see that the nonpower decay law of the latter theorem is indeed caused by a "kink" near the axes. □

As an interesting consequence of Theorem 1.5, we mention the following boundary crossing estimate for the samples of 2–parameter Brownian sheet.

Corollary 1.8 *There exist two finite constants $\gamma_1 > 0$ and $\gamma_2 > 0$, such that with probability one,*

1. *for all $R > 0$ large enough,*

$$\sup_{0\le s,t\le R} W(s,t) \ge R \cdot \exp\Big\{ -\gamma_1 \sqrt{\log\log R \cdot \log\log\log R}\Big\}; \text{ and}$$

2. *there exists a random sequence $R_1, R_2, \ldots$, tending to infinity, such that for all $k \ge 1$,*

$$\sup_{0\le s,t\le R_k} W(s,t) \le R_k \cdot \exp\Big\{ -\gamma_2 \sqrt{\log\log R_k}\Big\}.$$

The proof of Corollary 1.8 can also be modified to imply the following local version. We leave the details to the interested reader.

Corollary 1.9 *There exist two finite constants $\gamma_3 > 0$ and $\gamma_4 > 0$, such that with probability one,*

1. *for all $\varepsilon > 0$ small enough,*
$$\sup_{0 \le s,t \le \varepsilon} W(s,t) \ge \varepsilon \cdot \exp\Big\{ -\gamma_3 \sqrt{\log\log(1/\varepsilon) \cdot \log\log\log(1/\varepsilon)} \Big\}; \text{ and}$$

2. *there exists a random sequence $\varepsilon_1, \varepsilon_2, \ldots$, tending to zero, such that for all $k \ge 1$,*
$$\sup_{0 \le s,t \le \varepsilon_k} W(s,t) \le \varepsilon_k \cdot \exp\Big\{ -\gamma_4 \sqrt{\log\log(1/\varepsilon_k)} \Big\}.$$

Our methods rely on exploiting the relationships between Brownian sheet (viewed as infinite-dimensional Brownian motion) and (Euclidean) Brownian motion. In the proof of Theorem 1.5, the lower bound for $\mathbb{P}(M_{0,1} < \varepsilon)$ is obtained by using the comparison method of SLEPIAN applied to a sequence of nearly independent Brownian motions extracted from W. The proof of the corresponding upper bound is much harder and is at the heart of this article; it is done by first coupling the Brownian sheet to a sequence of independent Brownian motions, and then by using a variation of a theorem of KESTEN on the collision time of several Brownian particles. In the proof of Theorem 1.1, we relate the tail of $M_{a,b}$ to the first exit time of a planar Brownian motion from a cone. This, in turn, allows us to use an estimate of SPITZER [28] on the winding angle of planar Brownian motion.

This paper is organized as follows: Section 2 is devoted to the presentation of the mentioned theorem of KESTEN on several Brownian particles. Theorem 1.1 is proved in Section 4. The proof of Theorem 1.5 is divided in two parts: we prove its upper bound in Section 3 and the lower bound in §5. As an application of our estimates, in §6, we obtain a quasi-sure version of HIRSCH's theorem for Brownian motion and in a final Section 7, we present a proof for Corollary 1.8. While the latter argument is standard in spirit, it needs care in a few spots and we include it at the risk of one or two more (admittedly too terse) paragraphs.

2 A Variation on a Theorem of Kesten

Throughout this section, $\{W_k(t);\ t \ge 0\}$ $(k = 0, 1, 2, \cdots)$ denote independent Brownian motions, all starting from 0. The following was raised by BRAMSON AND GRIFFEATH [6], but was originally formulated for random walks: for any integer $N \ge 1$, let
$$\tau_N = \inf\Big\{ t > 0 : \max_{1 \le k \le N} W_k(t) = W_0(t) + 1 \Big\}.$$
Then, when is $\mathbb{E}(\tau_N)$ finite?

This can be viewed as a random pursuit problem. Assume that a Brownian prisoner escapes, running along the path of W_0. In his/her pursuit, there are

N independent Brownian policemen. These policemen run along the paths of $W_1, \cdots, W_N$, respectively. At the outset, the prisoner is ahead of the policemen by some fixed distance (1 unit, in our model). Then, τ_N represents the capture time when the fastest of the policemen catches the prisoner. Thus, the question of Bramson and Griffeath is whether the expected capture time is finite. A more animated interpretation is "*How many Brownian policemen does it take to arrest a Brownian prisoner?*" Based on computer simulations, Bramson and Griffeath conjectured that $\mathbb{E}(\tau_4) < \infty$. By a simple monotonicity argument, if this were true, $\mathbb{E}(\tau_N)$ would be finite for any $N \geq 4$. While this problem still remains open, Kesten found the following partial answer in [19]: there exists $N_0 < \infty$ such that $\mathbb{E}(\tau_N) < \infty$ for all $N \geq N_0$. What Kesten actually demonstrated was an upper bound for the tail of the distribution of τ_N. (For the exact statement of Kesten's theorem, see the comments after Lemma 2.1 below). Of course, estimating the tail of τ_N is the same as estimating the law of $\max_{1\leq k\leq N} \sup_{0\leq t\leq T}(W_k(t) - W_0(t))$. In fact, for any $T > 0$,

$$\mathbb{P}(\tau_N > T) = \mathbb{P}\left(\max_{1\leq k\leq N} \sup_{0\leq t\leq T} (W_k(t) - W_0(t)) < 1\right).$$

It turns out that the boundary crossing problem discussed in Introduction is closely related to (a variation of) the random pursuit problem for Brownian particles. More precisely, we need to estimate the following, for $\delta > 0$:

$$\mathbb{P}\left(\max_{1\leq k\leq N} \sup_{0\leq t\leq T} (W_k(t) - \delta W_0(t)) < 1\right).$$

Let us first introduce some notation. Throughout, $\Phi(\cdot)$ denotes the standard Gaussian distribution function:

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} \exp\left(-\frac{u^2}{2}\right) \mathrm{d}u, \qquad x \in \mathbb{R}.$$

We shall also frequently use the following function:

$$h(d,L) = \Phi\left(-\frac{1+d+de^{-L}}{\sqrt{1-e^{-2L}}}\right), \qquad (d,L) \in (0,\infty)^2. \tag{2.1}$$

We mention that $h(d,L)$ is the same as the "constant" $C_1(d,L)$ in [19, p. 65].

Below is the main estimate of this section. This will be applied in Section 3 to prove the upper bound in Theorem 1.5.

Lemma 2.1 *Let $0 < \beta < 1/2$, $\gamma > 0$, $d > 0$, $L > 0$, $\delta > 0$, $N \geq 1$ and $T > 1$. Assume that*

$$\Phi(-d) < \beta, \qquad \frac{d}{\delta} \geq \frac{e}{\sqrt{2\pi}\,\beta} \vee \sqrt{\frac{8\gamma}{\beta}}. \tag{2.2}$$

Then

$$\mathbb{P}\left(\max_{1\leq k\leq N} \sup_{0\leq t\leq T} (W_k(t) - \delta W_0(t)) < 1\right) \leq T^{-\gamma} + (I_1 + I_2)^N, \tag{2.3}$$

where

$$I_1 \quad := \quad \exp\left\{-\frac{(1-2\beta)h(d,L)\,\log T}{2L} + 3h(d,L)\right\}, \tag{2.4}$$

$$I_2 \quad := \quad \exp\left\{-\frac{\beta\,\log T}{4}\left(\frac{\Phi(-d)}{\beta} - 1 + \log\frac{\beta}{\Phi(-d)}\right)\right\}, \tag{2.5}$$

and $h(d,L)$ *is defined in* (2.1).

When $\delta = 1$, this is implicitly proved by Kesten in [19]. For arbitrary $\delta > 0$, we can use his method with some modifications. First, let

$$U_k(t) = \frac{W_k(e^{2t})}{e^t}, \qquad t \in \mathbb{R}, \tag{2.6}$$

which are the associated Ornstein–Uhlenbeck processes. Let us recall two technical lemmas. The first, estimates the probability that the sojourn time of an Ornstein–Uhlenbeck process is far from being typical.

Lemma 2.2 *Let* $\beta > 0$, $\gamma > 0$ *and* $T > 0$. *Then, for any* $r > 0$ *such that* $\Phi(-r) < \beta$,

$$\begin{aligned} &\mathbb{P}\left(\int_0^T 1_{\{U_0(t) > -r\}}\ \mathrm{d}t \le (1-\beta)T\right) \\ &\qquad \le \exp\left\{-\frac{\beta T}{2}\left(\frac{\Phi(-r)}{\beta} - 1 + \log\frac{\beta}{\Phi(-r)}\right)\right\}. \end{aligned} \tag{2.7}$$

In particular, using the estimate $\Phi(-r) \le e^{-r^2/2}/(\sqrt{2\pi}\,r)$, *we immediately get*

$$\mathbb{P}\left(\int_0^T 1_{\{U_0(t) > -r\}}\ \mathrm{d}t \le (1-\beta)T\right) \le e^{-2\gamma T}, \tag{2.8}$$

provided that

$$r \ge \frac{e}{\sqrt{2\pi}\,\beta} \vee \sqrt{\frac{8\gamma}{\beta}}. \tag{2.9}$$

The second technical lemma that we need is a boundary crossing estimate for the typical values of an Ornstein–Uhlenbeck process.

Lemma 2.3 *Fix* $0 < \beta < 1/2$, $L > 0$ *and* $T > 0$. *Let* S_0 *be a deterministic measurable subset of* $[0,T]$ *such that* $|S_0| \ge (1-\beta)T$, *where* $|S_0|$ *denotes Lebesgue's measure of* S_0. *Then, for each* $d > 0$ *and for all* $1 \le k \le N$,

$$\begin{aligned} &\mathbb{P}\left(U_k(t) > -d - 1 \ \textit{ for all } t \in S_0\right) \\ &\qquad \le \exp\left\{-\frac{(1-2\beta)h(d,L)\,T}{L} + 3h(d,L)\right\} \\ &\qquad\qquad + \mathbb{P}\left(\int_0^T 1_{\{U_0(t) > -d\}}\ \mathrm{d}t \le (1-\beta)T\right), \end{aligned} \tag{2.10}$$

where $h(d,L)$ *is as in* (2.1).

Remark 2.4 Eq. (2.7) is due to KESTEN [19] whose Lemma 1 is stated as Eqs. (2.8)–(2.9), above. Eq. (2.10) is not exactly Lemma 2 of [19]; see the extra condition (2.8) in [19]. This condition was used only at the last displayed formula in [19, p. 64]. □

Proof of Lemma 2.1. We recall U_k from Eq. (2.6), fix $0 < \beta < 1/2$, $L > 0$, $\gamma > 0$ and $0 < \delta < 1$. We also choose $d > 0$ such that

$$\frac{d}{\delta} \geq \frac{e}{\sqrt{2\pi}\,\beta} \vee \sqrt{\frac{8\gamma}{\beta}},$$

(so that Eq. (2.9) is satisfied with (d/δ) in place of r). For $T > 1$, define

$$\begin{aligned} S_0 &= \left\{ t \in [0,T] : \ U_0(t) > -\frac{d}{\delta} \right\}, \\ E &= \{ |S_0| \leq (1-\beta)T \}. \end{aligned}$$

By (2.8), $\mathbb{P}(E^c) \leq e^{-2\gamma T}$. On the other hand, since W_k $(1 \leq k \leq N)$ are independent, we can use (2.10) to see the following upper bound for the conditional probability for $1 \leq k \leq N$:

$$\begin{aligned} &\mathbb{P}\left(U_k(t) > -d - 1 \text{ for all } t \in S_0 \mid E \right) \\ &\qquad \leq \exp\left\{ -\frac{(1-2\beta) h(d,L)\, T}{L} + 3h(d,L) \right\} + \\ &\qquad\quad + \mathbb{P}\left\{ \int_0^T 1_{\{U_0(t) > -d\}} \, \mathrm{d}t \leq (1-\beta) T \right\}. \end{aligned}$$

If, in addition, $\Phi(-d) < \beta$, we can apply Lemma 2.2 to $r = d$ for the last probability term, to arrive at:

$$\begin{aligned} &\mathbb{P}\big(U_k(t) > -d - 1, \text{ for all } t \in S_0 \mid E \big) \\ &\qquad \leq \exp\left\{ -\frac{(1-2\beta) h(d,L)\, T}{L} + 3h(d,L) \right\} \\ &\qquad\quad + \exp\left\{ -\frac{\beta T}{2} \left(\frac{\Phi(-d)}{\beta} - 1 + \log \frac{\beta}{\Phi(-d)} \right) \right\} \\ &\qquad = I_1 + I_2, \end{aligned}$$

where I_1 and I_2 are defined in (2.4) and (2.5), respectively. Therefore,

$$\begin{aligned} &\mathbb{P}\left(U_k(t) + 1 > \delta U_0(t), \text{ for all } 1 \leq k \leq N, \ 0 \leq t \leq T \right) \\ &\qquad \leq \mathbb{P}(E^c) + \prod_{k=1}^{N} \mathbb{P}\left(U_k(t) > -d - 1 \text{ for all } t \in S_0 \mid E \right) \\ &\qquad \leq e^{-2\gamma T} + (I_1 + I_2)^N. \end{aligned} \tag{2.11}$$

Observe that for any $a > 1$,

$$\{W_k(s) + 1 > \delta W_0(s), \text{ for all } 1 \le k \le N,\ 0 \le s \le a\}$$
$$\subset \left\{U_k(t) + 1 > \delta U_0(t), \text{ for all } 1 \le k \le N,\ 0 \le t \le \frac{\log a}{2}\right\}.$$

This, in conjunction with (2.11), yields Lemma 2.1 by changing W_k into $-W_k$. □

3 Proof of Theorem 1.5: Upper Bound

For the sake of clarity, we prove the upper and lower bounds in Theorem 1.5 separately. This section is devoted to the proof of the upper bound. The lower bound will be proved in Section 5.

Let $\varepsilon < 1/100$ and $N \ge 1$ (the value of N will be chosen later on). We write

$$\begin{aligned} M &= M(\varepsilon) = \lfloor \log^2(1/\varepsilon) \rfloor, \\ \delta &= \delta(\varepsilon) = \frac{2}{M^{1/4}}, \\ a_k &= a_k(\varepsilon) = M^k, \qquad 1 \le k \le N. \end{aligned}$$

Define

$$W_j(t) = \sqrt{a_N}\left(W(\frac{j}{a_N}, t) - W(\frac{j-1}{a_N}, t)\right), \qquad t \ge 0. \tag{3.1}$$

It is clear that $\{W_j(t);\, t \in \mathbb{R}_+\}$ ($j = 1, 2, \cdots, a_N$) are independent (one-parameter) Brownian motions. By enlarging the underlying probability space if need be, we can add to this list yet another independent Brownian motion and label it W_0. Define, for $1 \le k \le N$,

$$\begin{aligned} X_k(t) &= a_k^{-1/2} \cdot \sum_{j=1}^{a_k} W_j(t), \\ Y_k(t) &= \left\{1 + \delta^2\right\}^{-1/2} \cdot \left\{W_k(t) - \delta W_0(t)\right\}, \qquad t \ge 0. \end{aligned}$$

Observe that among the original $(a_N + 1)$ Brownian motions $W_0, W_1, \cdots, W_{a_N}$, only $(N+1)$ of them have made contribution to $\{Y_k\}_{1 \le k \le N}$. This "selectiveness" allows us to compare the maxima of X_k and Y_k via the following argument. First, it is easily seen that for each k, both $\{X_k(t);\ t \ge 0\}$ and $\{Y_k(t);\ t \ge 0\}$ are Brownian motions. Thus, $\mathbb{E}\{X_k^2(t)\} = \mathbb{E}\{Y_k^2(t)\} = t$. It is also possible to compare the covariances. Indeed, for $1 \le k \ne \ell \le N$ and $(s,t) \in \mathbb{R}_+^2$,

$$\begin{aligned} \mathbb{E}\{X_k(t) X_\ell(s)\} &= \frac{(a_k \wedge a_\ell)(s \wedge t)}{\sqrt{a_k a_\ell}} \\ &= \frac{s \wedge t}{M^{|k-\ell|/2}} \end{aligned}$$

$$\begin{aligned} &\leq \frac{s \wedge t}{M^{1/2}} \\ &\leq \frac{\delta^2 (s \wedge t)}{1+\delta^2} \\ &= \mathbb{E}\{Y_k(t) Y_\ell(s)\}. \end{aligned}$$

So we can apply SLEPIAN's lemma (see [27]), to get the following inequality: for any $T > 0$ and $x > 0$,

$$\mathbb{P}\left(\max_{1\leq k\leq N}\sup_{0\leq t\leq T} X_k(t) < x\right) \leq \mathbb{P}\left(\max_{1\leq k\leq N}\sup_{0\leq t\leq T} Y_k(t) < x\right). \tag{3.2}$$

Now let us return to our study of the Brownian sheet $\{W(s,t);\ (s,t)\in\mathbb{R}_+^2\}$. In view of Eq. (3.1),

$$\begin{aligned} \mathbb{P}(\,M_{0,1} < \varepsilon) &\leq \mathbb{P}\left(\max_{1\leq k\leq N}\sup_{0\leq t\leq 1}\sum_{j=1}^{a_k} W_j(t) < \sqrt{a_N}\,\varepsilon\right) \\ &= \mathbb{P}\left(\max_{1\leq k\leq N}\sup_{0\leq t\leq 1}\sqrt{a_k}\,X_k(t) < \sqrt{a_N}\,\varepsilon\right) \\ &\leq \mathbb{P}\left(\max_{1\leq k\leq N}\sup_{0\leq t\leq 1} X_k(t) < \sqrt{a_N}\,\varepsilon\right). \end{aligned}$$

Applying Eq. (3.2) to $x := \sqrt{a_N}\ \varepsilon$ gives that

$$\mathbb{P}(M_{0,1} < \varepsilon) \leq \mathbb{P}\left(\max_{1\leq k\leq N}\sup_{0\leq t\leq 1} Y_k(t) < \sqrt{a_N}\,\varepsilon\right).$$

We can choose

$$T := \frac{1}{(1+\delta^2) a_N\,\varepsilon^2}.$$

Then, by the definition of Y_k's,

$$\mathbb{P}(M_{0,1} < \varepsilon) \leq \mathbb{P}\left(\max_{1\leq k\leq N}\sup_{0\leq t\leq T}(W_k(t) - \delta W_0(t)) < 1\right). \tag{3.3}$$

To complete the proof of the upper bound in Theorem 1.5, let us choose our parameters:

$$\beta := 1/3\ ,\ L := 1\ ,\ \gamma := \log(1/\varepsilon)\ ,\ d := 10.$$

Note that condition (2.2) is satisfied. Moreover, $h(d, L)$ defined in (2.1) is a finite and positive (absolute) constant. Finally, we choose

$$N := \frac{\log(1/\varepsilon)}{2\log\log(1/\varepsilon)},$$

so that $\log T \geq c_1 \log(1/\varepsilon)$ for some universal constant $c_1 > 0$. According to (2.4) and (2.5), $I_1 \leq \exp(-c_2 \log(1/\varepsilon))$ and $I_2 \leq \exp(-c_3 \log(1/\varepsilon))$, where c_2 and c_3 are positive universal constants. Therefore, by (2.3),

$$\mathbb{P}\left(\max_{1\leq k\leq N} \sup_{0\leq t\leq T} (W_k(t) - \delta W_0(t)) < 1 \right) \leq \exp\left(-c_4 \frac{\log^2(1/\varepsilon)}{\log\log(1/\varepsilon)} \right),$$

for some universal constant $c_4 > 0$. In view of (3.3), this yields the upper bound in Theorem 1.5. □

4 Proof of Theorem 1.1

Given $b > a > 0$, we define

$$\alpha_n = \mathbb{P}\left(\sup_{(s,t)\in[a,\,b]\times[0,\,e^n]} W(s,t) < 1 \right).$$

Since W has positive correlations, SLEPIAN's inequality ([27]) shows that

$$\alpha_{n+m} \geq \alpha_n\, \mathbb{P}\left(\sup_{(s,t)\in[a,\,b]\times[e^n,\,e^{n+m}]} W(s,t) < 1 \right).$$

Write $\widetilde{W}(s,t) := W(s,t+e^n) - W(s,e^n)$, so that

$$\alpha_{n+m} \geq \alpha_n\, \mathbb{P}\left(\sup_{(s,t)\in[a,\,b]\times[0,\,e^{n+m}-e^n]} \left(\widetilde{W}(s,t) + W(s,e^n)\right) < 1 \right).$$

Clearly, $\{\widetilde{W}(s,t);\, (s,t) \in \mathbb{R}_+^2\}$ is a Brownian sheet that is independent of $\{W(s,t);\, (s,t) \in \mathbb{R}_+ \times [0,e^n]\}$. Consequently, the probability term on the right hand side is bounded below by

$$\mathbb{P}\left(\sup_{(s,t)\in[a,\,b]\times[0,\,e^{n+m}-e^n]} W(s,t) \leq 1 + e^{n/2} \right) \cdot \mathbb{P}\left(\sup_{s\in[a,\,b]} W(s,e^n) < -e^{n/2} \right).$$

Since $\{e^{-n/2}W(s,e^n);\, s \in \mathbb{R}_+\}$ is a standard (one-parameter) Brownian motion, we have

$$\mathbb{P}\left(\sup_{s\in[a,\,b]} W(s,e^n) < -e^{n/2} \right) = \mathbb{P}\left(\sup_{s\in[a,\,b]} W_0(s) < -1 \right) := c_5,$$

where $\{W_0(t);\, t \in \mathbb{R}_+\}$ is a (one-parameter) Brownian motion, and $c_5 \in (0,1)$ is a positive, finite constant depending only on a and b. Accordingly,

$$\alpha_{n+m} \quad \geq \quad \alpha_n\, \mathbb{P}\left(\sup_{(s,t)\in[a,\,b]\times[0,\,e^{n+m}-e^n]} W(s,t) < 1 + e^{n/2} \right) c_5$$

$$
\begin{aligned}
&= c_5\, \alpha_n \mathbb{P}\left(\sup_{(s,t)\in[a,\,b]\times[0,\,e^m]} W(s,t) < \frac{(1+e^{n/2})e^{m/2}}{(e^{n+m}-e^n)^{1/2}} \right) \\
&\geq c_5\, \alpha_n \mathbb{P}\left(\sup_{(s,t)\in[a,\,b]\times[0,\,e^m]} W(s,t) < 1 \right) \\
&= c_5\, \alpha_n \alpha_m .
\end{aligned}
$$

This shows that $\{-\log(c_5\,\alpha_n)\}_{n\geq 1}$ is sub-additive, so that

$$
\varrho := \lim_{n\to\infty} \frac{-\log(c_5\,\alpha_n)}{n} = \inf_{n\geq 1} \frac{-\log(c_5\,\alpha_n)}{n},
$$

exists, and lies in $[0,\infty)$. Of course, $\varrho = -\lim_{n\to\infty}(\log \alpha_n)/n$. A simple argument using the monotonicity of $T \mapsto \sup_{(s,t)\in[a,\,b]\times[0,\,T]} W(s,t)$ yields that

$$
\varrho = -\lim_{T\to\infty} \frac{1}{\log T} \log \mathbb{P}\left(\sup_{(s,t)\in[a,\,b]\times[0,\,T]} W(s,t) < 1 \right).
$$

This implies the existence of the limit in (1.4) by scaling, with $\xi_{a,b} = 2\varrho$.

It remains to check that $\xi_{a,b} \geq \xi^*_{a,b}$, where $\xi^*_{a,b}$ is the constant in (1.3). We observe that

$$
\mathbb{P}\left(M_{a,b} < \varepsilon\right) \leq \mathbb{P}\left(\sup_{t\in[0,\,1]} W(a,t) < \varepsilon,\ \sup_{t\in[0,\,1]} W(b,t) < \varepsilon \right). \tag{4.1}
$$

Define

$$
\begin{aligned}
B_1(t) &= \frac{W(a,t)}{\sqrt{a}}, \\
B_2(t) &= \frac{W(b,t)-W(a,t)}{\sqrt{b-a}}, \qquad t \in \mathbb{R}_+ .
\end{aligned}
$$

Clearly, $\{B_1(t);\, t \in \mathbb{R}_+\}$ and $\{B_2(t);\, t \in \mathbb{R}_+\}$ are two independent (one-parameter) Brownian motions. The probability expression on the right hand side of (4.1) can be written as

$$
\begin{aligned}
&= \mathbb{P}\left(\sup_{t\in[0,\,1]} \left(\sqrt{a}\, B_1(t)\right) < \varepsilon,\ \sup_{t\in[0,\,1]} \left(\sqrt{a}\, B_1(t) + \sqrt{b-a}\, B_2(t)\right) < \varepsilon \right) \\
&= \mathbb{P}\left(\sup_{t\in[0,\,\varepsilon^{-2}]} \left(\sqrt{a}\, B_1(t)\right) < 1,\ \sup_{t\in[0,\,\varepsilon^{-2}]} \left(\sqrt{a}\, B_1(t) + \sqrt{b-a}\, B_2(t)\right) < 1 \right).
\end{aligned}
$$

The above is precisely the probability that the planar Brownian motion (B_1, B_2) stays in the cone $\{(x,y) \in \mathbb{R}^2 :\ \sqrt{a}\, x < 1,\ \sqrt{a}\, x + \sqrt{b-a}\, y < 1\}$ during the entire time period $[0,\ \varepsilon^{-2}]$.

It is known that if $D \subset \mathbb{R}^2$ is an open cone containing the origin, with angle θ, then for all $T \geq 1$,

$$
\mathbb{P}\Big\{ \big(B_1(t), B_2(t)\big) \in D,\ \text{for all } t \in [0,T] \Big\} \leq c_6 T^{-\pi/(2\theta)}, \tag{4.2}
$$

where c_6 is a positive finite constant. Spitzer [28] stated a slightly weaker version of this, though his argument actually yields (4.2). In this stated form, the above can be found in the work of Bañuelos and Smits regarding exit times from general cones; cf. [4].

Applying (4.2) to $\theta = \pi/2 + \arcsin\sqrt{a/b} := \theta_{a,b}$ gives

$$\mathbb{P}\left(M_{a,b} < \varepsilon\right) \le c_6\, \varepsilon^{\pi/\theta_{a,b}}.$$

As a consequence, $\xi \ge \pi/\theta_{a,b} = \xi^*_{a,b}$. Theorem 1.1 is now proved. □

5 Proof of Theorem 1.5: Lower Bound

Fix any constant $\xi > \xi_{1,2}$, where $\xi_{1,2}$ is the finite constant defined in (1.4). According to Theorem 1.1, for all $\varepsilon \in (0,1)$,

$$\mathbb{P}\left(\sup_{(s,t)\in[1,\,2]\times[0,\,1]} W(s,t) < \varepsilon \right) \ge c_7\, \varepsilon^{\xi},$$

where $c_7 > 0$ is a finite constant depending only on ξ. By scaling, for all integer $j \ge 0$ such that $2^{j/2}\varepsilon < 1$,

$$\mathbb{P}\left(\sup_{(s,t)\in[2^{-j},\,2^{-j+1}]\times[0,\,1]} W(s,t) < \varepsilon \right) \ge c_7\, (2^{j/2}\varepsilon)^{\xi}.$$

Let $j_0 = j_0(\varepsilon) = \max\{j \ge 0 : 2^{j/2}\varepsilon < 1\}$. Since the Brownian sheet W has positive covariances, we can apply Slepian's lemma ([27]) to arrive at the following:

$$\begin{aligned}\mathbb{P}(M_{0,1} < \varepsilon) \;&\ge\; \mathbb{P}\left\{ \sup_{(s,t)\in[0,\,2^{-j_0}]\times[0,\,1]} W(s,t) < \varepsilon \right\} \\ &\qquad \times \prod_{j=1}^{j_0} \mathbb{P}\left\{ \sup_{(s,t)\in[2^{-j},\,2^{-j+1}]\times[0,\,1]} W(s,t) < \varepsilon \right\} \\ &\ge\; \mathbb{P}\left\{ M_{0,1} < 2^{j_0/2}\varepsilon \right\} \prod_{j=1}^{j_0} \left\{ c_7\, (2^{j/2}\varepsilon)^{\xi} \right\}. \end{aligned} \tag{5.1}$$

By definition, $2^{j_0/2}\varepsilon \ge 2^{-1/2}$, so that

$$\mathbb{P}\left\{ M_{0,1} < 2^{j_0/2}\varepsilon \right\} \ge \mathbb{P}\left\{ M_{0,1} < 2^{-1/2} \right\} := c_8.$$

Therefore, the expression on the right hand side of (5.1) is

$$\ge\; c_8\, \exp\left\{ \sum_{j=1}^{j_0} \left(\log c_7 - \xi \log(1/\varepsilon) + \frac{j\xi \log 2}{2} \right) \right\}$$

$$= \quad c_8 \exp\left(j_0 \log c_7 - \xi j_0 \log(1/\varepsilon) + \frac{j_0(j_0+1)\xi \log 2}{4}\right).$$

As $\varepsilon \to 0^+$, $j_0 \sim 2(\log(1/\varepsilon))/\log 2$. Thus,

$$\liminf_{\varepsilon \to 0} \frac{\log \mathbb{P}(M_{0,1} < \varepsilon)}{\{\log(1/\varepsilon)\}^2} \geq -\frac{\xi}{\log 2}, \tag{5.2}$$

which completes the proof of the lower bound in Theorem 1.5. □

Remark 5.1 The estimate in (5.2) says that

$$\liminf_{\varepsilon \to 0} \frac{\log \mathbb{P}(M_{0,1} < \varepsilon)}{\{\log(1/\varepsilon)\}^2} \geq -\frac{\xi_{1,2}}{\log 2}.$$

Furthermore, since $\xi^*_{1,2} = 4/3$, we can deduce that $\xi_{1,2}$ is a finite constant that is greater than (or equal to) $4/3$. □

6 Quasi-sure Version of Hirsch's Theorem

Using the Brownian sheet $\{W(s,t);\ (s,t) \in \mathbb{R}^2_+\}$, we can define the Ornstein–Uhlenbeck process $\{O_s;\ s \in \mathbb{R}_+\}$ via

$$O_s(t) = \frac{W(e^s, t)}{e^{s/2}}, \qquad t \in \mathbb{R}_+. \tag{6.1}$$

The process $\{O_s;\ s \in \mathbb{R}_+\}$ takes its values in the space of continuous functions $\Omega = \mathcal{C}(\mathbb{R}_+, \mathbb{R})$ and is, in fact, a stationary ergodic diffusion whose stationary measure is WIENER's measure $\mathbb{W}$; see MALLIAVIN [23].

For any Borel set $A \subset \Omega = \mathcal{C}(\mathbb{R}_+, \mathbb{R})$, define

$$\mathsf{Cap}(A) = \int_0^\infty \mathbb{P}^{\mathbb{W}}\{O_s \in A \text{ for some } s \in [0,t]\} e^{-t}\, \mathrm{d}t, \tag{6.2}$$

where

$$\mathbb{P}^{\mathbb{W}}\{A\} = \int_\Omega \mathbb{P}\{A \,|\, O_0 = x\} \mathbb{W}(\,\mathrm{d}x).$$

It is known that Eq. (6.2) defines a natural capacity on the Wiener space (see [14, 23]) in the sense of CHOQUET and is the 1–*capacity* of the Ornstein–Uhlenbeck process on Wiener space (or the FUKUSHIMA–MALLIAVIN capacity on Wiener space).

When $\mathsf{Cap}(A) > 0$, we say that A happens quasi-surely. A Borel set $A \subset \Omega$ is called *exceptional*, if $\mathsf{Cap}(A) > 0$ whereas $\mathbb{W}(A) = 0$. It is an interesting problem, going back to WILLIAMS, to find exceptional sets; cf. [29]. Various classes of such exceptional sets have been found in the literature. See for example, [14] and the references of [20].

Our Theorem 1.1 allows to give a new class of exceptional sets related to HIRSCH's theorem for Brownian motion. For $f \in \Omega = \mathcal{C}(\mathbb{R}_+, \mathbb{R})$, define

$$f^\star(t) = \sup_{s\in[0,t]} f(s), \qquad t \in \mathbb{R}_+.$$

HIRSCH's theorem states that if $g : \mathbb{R}_+ \mapsto \mathbb{R}_+$ is nonincreasing and if $B := \{B(t);\ t \geq 0\}$ denotes standard Brownian motion, then

$$\liminf_{t\to\infty} \frac{B^\star(t)}{t^{1/2}\ g(t)} = \begin{cases} +\infty, & \text{if } \int_1^\infty t^{-1} g(t)\ dt < \infty \\ 0, & \text{otherwise} \end{cases};$$

see CSÁKI [8]. We say that a function $g : \mathbb{R}_+ \mapsto \mathbb{R}_+$ is an *escape envelope* for $f \in \Omega$, if for all $M > 0$ and for all but finitely many integers $k \geq 1$, $\mathcal{E}^g_{k,M}(f) \neq \varnothing$, where

$$\mathcal{E}^g_{k,M}(f) := \Big\{2^k \leq s < 2^{k+1} :\ f^\star(s) \geq M\ 2^{k/2} g(2^k)\Big\}.$$

HIRSCH's theorem is, in fact, the following:

Theorem 6.1 (Hirsch) *Suppose $g : \mathbb{R}_+ \mapsto \mathbb{R}_+$ is nonincreasing and measurable. Then, g is an escape envelope for $\mathbb{W}$–almost all $f \in \Omega$, if $\int_1^\infty t^{-1} g(t)\ dt < \infty$. Conversely, if $\int_1^\infty t^{-1} g(t)\ dt = +\infty$, then for $\mathbb{W}$–almost all $f \in \Omega$, g is not an escape envelope for f.*

In particular, if $\nu \in (0,1)$, $g(t) := \{\log_+ t\}^{-\nu}$ is *not* an escape envelope for f, for $\mathbb{W}$–almost all $f \in \Omega$. We now present the following quasi-sure version of HIRSCH's theorem that states that this fails to hold quasi–surely. In particular, the following readily provides us with a new class of nontrivial exceptional sets in Ω.

Theorem 6.2 *Given $\nu \in (\frac{1}{2}, 1)$, $g(t) := \{\log_+ t\}^{-\nu}$ is an escape envelope for quasi-every $f \in \Omega$.*

Proof. We are to prove the following:

$$\mathsf{Cap}\Big\{f \in \Omega :\ \mathcal{E}^g_{k,M}(f) = \varnothing \text{ for infinitely many } k\Big\} = 0. \tag{6.3}$$

We now define the *incomplete r–capacity* $\mathsf{C}_r(A)$ of a Borel set $A \subset \Omega$ as

$$\mathsf{C}_r(A) := \mathbb{P}^{\mathbb{W}}\Big\{O_s \in A \text{ for some } s \in [0,r]\Big\}.$$

Since $\mathsf{Cap}(A) = \int_0^\infty e^{-r} \mathsf{C}_r(A)\ dr$, for any $r > 0$,

$$\mathsf{Cap}(A) \leq \mathsf{C}_r(A) + e^{-r}.$$

Subsequently, a Borel–Cantelli argument reveals that Eq. (6.3) is implied by the following: for all $r > 0$ large enough,

$$\sum_{k=1}^\infty \mathsf{C}_r\Big\{f \in \Omega :\ \mathcal{E}^g_{k,M}(f) = \varnothing\Big\} < \infty. \tag{6.4}$$

To this end, let us pick $r > 0$ large enough that

$$\frac{2\pi\nu}{\pi + \arcsin(e^{-r/2})} > 1.$$

If ξ^*_{1,e^r} stands for the constant defined in Eq. (1.3), the above simply means $\nu\xi^*_{1,e^r} > 1$. Since $\xi_{1,e^r} \geq \xi^*_{1,e^r}$, we have *a fortiori*, $\nu\xi_{1,e^r} > 1$. Therefore, we can choose $\xi \in (0, \xi_{1,e^r})$ and $\mu \in (0, \nu)$ such that $\mu\xi > 1$.

Since $\xi < \xi_{1,e^r}$, we can apply Theorem 1.1 to see that for all $M, r > 0$, there exists $t_0 > 0$, such that for all $T > t_0$,

$$\begin{aligned}
\mathbb{P}&\left(\sup_{(s,t)\in[1,e^r]\times[0,T]} W(s,t) < Me^{r/2}T^{1/2}(\log T)^{-\mu} \right) \\
&= \mathbb{P}\left(\sup_{(s,t)\in[1,e^r]\times[0,1]} W(s,t) < Me^{r/2}(\log T)^{-\mu} \right) \\
&\leq (\log T)^{-\mu\xi}.
\end{aligned}$$

By Eq. (6.1), for all $T > t_0$,

$$\begin{aligned}
\mathbb{P}&\Big\{ \sup_{0\leq v\leq T} O_u(v) < MT^{1/2}\big(\log T\big)^{-\mu} \text{ for all } u \in [0,r] \Big\} \\
&\leq \mathbb{P}\Big\{ \sup_{0\leq t\leq T} W(s,t) < Me^{r/2}T^{1/2}\big(\log T\big)^{-\mu} \text{ for all } s \in [1,e^r] \Big\} \\
&\leq \big(\log T\big)^{-\mu\xi}.
\end{aligned}$$

Applying the above with $T := T_k = 2^k$, we see that for all $k > \log t_0$,

$$\mathbb{P}\Big\{ \sup_{0\leq v\leq T_k} O_u(v) < MT_k^{1/2}g(T_k) \text{ for all } u \in [0,r] \Big\} \leq (k\log 2)^{-\mu\xi},$$

which sums, since $\mu\xi > 1$. As the above probability equals the incomplete r–capacity of the collection of $f \in \Omega$, such that $\mathcal{E}^g_{k,M}(f) = \varnothing$, this yields Eq. (6.4) and concludes our proof. □

7 Proof of Corollary 1.8

For all integers $k \geq 1$ and all $\gamma > 0$, define $T_k := e^k$ and

$$\Psi_k(\gamma) := \exp\Big\{ -\gamma\sqrt{\log\log T_k \cdot \log\log\log T_k} \Big\}.$$

By Theorem 1.5, for all $\gamma > 0$ large enough,

$$\sum_k \mathbb{P}\Big\{ \sup_{0\leq s,t\leq T_k} W(s,t) < T_k\Psi_k(\gamma) \Big\} < \infty.$$

By the Borel–Cantelli lemma, for any $\gamma > 0$ large enough, the following holds with probability one: for all k large enough,

$$\sup_{0\le s,t\le T_k} W(s,t) \ge T_k \exp\Big\{ -\gamma\sqrt{\log\log T_k \cdot \log\log\log T_k}\Big\}.$$

Thus, outside the above (implicitly stated) null set, if $R \in [T_k, T_{k+1}]$ is large enough,

$$\sup_{0\le s,t\le R} W(s,t) \ge \frac{R}{e} \exp\Big\{ -\gamma\sqrt{\log\log R \cdot \log\log\log R}\Big\}.$$

Since γ is large but otherwise arbitrary, we obtain half of the corollary. To demonstrate the other (usually harder) half, let us define $S_k := k^k$, $k \ge 1$. For any sequence $\{\lambda_k;\ k \ge 0\}$, consider the (measurable) events:

$$\begin{aligned}\Upsilon_k(\lambda) \quad := \quad \Big\{\omega\in\Omega : \sup_{S_{k-1}\le s\le S_k}\ \sup_{0\le t\le S_k} \big[W(s,t) - W(S_{k-1},t)\big] \\ \le \sqrt{S_k(S_k - S_{k-1})}\,\lambda_k\Big\}.\end{aligned}$$

The elementary properties of Brownian sheet guarantee us that $\Upsilon_1(\lambda), \Upsilon_2(\lambda), \ldots$ are independent events. Moreover,

$$\mathbb{P}\big\{\Upsilon_k(\lambda)\big\} = \mathbb{P}\big\{M_{0,1} < \lambda_k\big\}.$$

In particular, if $\lambda_k \downarrow 0$, by Theorem 1.5 there exists a finite $c > 0$ such that,

$$\mathbb{P}\big\{\Upsilon_k(\lambda)\big\} \ge \exp\Big(-c\big\{\log(1/\lambda_k)\big\}^2\Big).$$

Choose $\lambda_k := \exp\big\{-\gamma\sqrt{\log\log S_k}\big\}$ for $\gamma > 0$ to see that $\sum_k \mathbb{P}\big\{\Upsilon_k(\lambda)\big\} = \infty$, for γ small enough. By the Borel–Cantelli lemma for independent events, a.s. infinitely many of $\Upsilon_k(\lambda)$'s must occur. That is, if $\gamma > 0$ is small enough, then almost surely,

$$\begin{aligned}&\sup_{S_{k-1}\le s\le S_k}\ \sup_{0\le t\le S_k} W(s,t) \\ &\quad\le \sup_{0\le t\le S_k} W(S_{k-1},t) + \qquad\qquad (7.1)\\ &\qquad + \sqrt{S_k(S_k - S_{k-1})}\,\exp\Big\{-\gamma\sqrt{\log\log S_k}\Big\}, \quad \text{infinitely often.}\end{aligned}$$

On the other hand, by the law of the iterated logarithm (cf. [24]), there exists a finite random variable Γ such that with probability one, for all $k \ge 1$,

$$\sup_{0\le s\le S_{k-1}}\ \sup_{0\le t\le S_k} W(s,t) \le \Gamma\sqrt{S_k S_{k-1}\log\log S_k}. \qquad (7.2)$$

Since as $k \to \infty$,

$$\sqrt{S_k S_{k-1}\log\log S_k} = o\Big(S_k \exp\Big\{-\gamma\sqrt{\log\log S_k}\Big\}\Big),$$

and since $\gamma > 0$ is small but arbitrary, two applications of (7.2), together with (7.1) complete the proof. □

References

[1] Adler, R.J.: *The Geometry of Random Fields.* Wiley, Chichester, 1981.

[2] Adler, R.J.: The supremum of a particular Gaussian field. *Ann. Probab.* **12** (1984), 436–444.

[3] Adler, R.J.: *Continuity, Extrema and Related Topics for General Gaussian Processes.* IMS Lecture Notes–Monograph Series, Volume 12, Institute of Mathematical Statistics, Hayward, CA., 1990.

[4] Bañuelos, R. and Smits, R.G.: Brownian motion in cones. *Probab. Theory Related Fields* **108**, (1997) 299–319.

[5] Berman, S.M.: *Sojourns and Extremes of Stochastic Processes.* Wadsworth and Brooks/Cole, 1991.

[6] Bramson, M. and Griffeath, D.: Capture problems for coupled random walks. In: *Random Walks, Brownian Motion and Interacting Particle Systems* (R. Durrett and H. Kesten, eds.) pp. 153–188. Birkhäuser, Boston, 1991.

[7] Cabaña, E.M. and Wschebor, M.: The two parameter Brownian bridge–Kolmogorov inequalities and upper and lower bounds for the distribution of the maximum. *Ann. Probab.* **10**, (1982) 289–302.

[8] Csáki, E.: On the lower limits of maxima and minima of Wiener process and partial sums. *Z. Wahrsch. Verw. Gebiete* **43**, (1978) 205–221.

[9] Dalang, R. and Walsh, J. B.: Geography of the level sets of the Brownian sheet. *Probab. Theory Related Fields* **96**, (1993) 153–176.

[10] Dembo, A. and Zeitouni, O.: *Large Deviations: Techniques and Applications.* Second Edition, Springer, New York, 1998.

[11] Feller, W.: *An Introduction to Probability Theory and its Applications.* Volume I, Second Edition, Wiley, New York, 1957.

[12] Feller, W.: *An Introduction to Probability Theory and its Applications.* Volume II, Second Edition, Wiley, New York, 1971.

[13] Florit, C. and Nualart, D.: A local criterion for smoothness of densities and application to the supremum of the Brownian sheet. *Statist. Probab. Lett.* **22**, (1995) 25–31.

[14] Fukushima, M.: Basic properties of Brownian motion and a capacity on the Wiener space. *J. Math. Soc. Japan* **36**, (1984) 161–176.

[15] Goodman, V.: Distribution estimates for functionals of two–parameter Wiener process. *Ann. Probab.* **4** (1975), 977–982.

[16] HOGAN, M.L. AND SIEGMUND, D.: Large deviations for the maxima of some random fields. *Adv. Appl. Math.* **7** (1986), 2–22.

[17] KAHANE, J.-P.: *Some Random Series of Functions.* Second Edition, Cambridge University Press, Cambridge, 1985.

[18] KENDALL, W.S.: Contours of Brownian processes with several–dimensional time. *Z. Wahrsch. Verw. Gebiete* **52**, (1980) 267–276.

[19] KESTEN, H.: An absorption problem for several Brownian motions. *Sem. Stoch. Proc. 91*, pp. 59–72. Birkhäuser, Boston, 1992.

[20] KHOSHNEVISAN, D. AND SHI, Z.: Brownian sheet and capacity. *Ann. Probab.* (to appear)

[21] LAWLER, G.F.: *Intersections of Random Walks.* Birkhäuser, Boston, 1991.

[22] LEDOUX, M. AND TALAGRAND, M.: *Probability in Banach Spaces.* Springer, Berlin, 1991.

[23] MALLIAVIN, P.: Stochastic calculus of variation and hypoelliptic operators. In: *Proc. Intern. Symp. Stoch. Diff. Eq.* (Kyoto 1976), pp. 195–263. Wiley, New York, 1978.

[24] OREY, S. AND PRUITT, W.E.: Sample functions of the N–parameter Wiener process. *Ann. Probab.* **1**, (1973), 138–163.

[25] PEMANTLE, R. AND PERES, Y.: Critical random walk in random environment on trees. *Ann. Probab.* **23**, (1995), 105–140.

[26] SIEGMUND, D.: Approximate tail probabilities for the maxima of some random fields. *Ann. Probab.* **16**, (1988), 487–501.

[27] SLEPIAN, D.: The one-sided barrier problem for Gaussian noise. *Bell System Tech. J.* **41**, (1962) 463–501.

[28] SPITZER, F.: Some theorems concerning 2-dimensional Brownian motion. *Trans. Amer. Math. Soc.* **87**, (1958) 187–197.

[29] WILLIAMS, D.: Appendix to P.-A. Meyer: Note sur les processus d'Ornstein–Uhlenbeck. *Sém. de Probab. XVI.* Lecture Notes in Mathematics, **920**, p. 133. Springer, Berlin, 1982.

[30] WORSLEY, K.J.: Estimating the number of peaks in a random field using the Hadwiger characteristic of excursion sets, with applications to medical images. *Ann. Statist.* **23**, (1995) 640–669.

[31] WORSLEY, K.J.: Boundary corrections for the expected Euler characteristic of excursion sets of random fields, with an application to astrophysics. *Adv. Appl. Probab.* **27**, (1995) 943–959.