Zeros of Rankin–Selberg L-functions and effective forms of multiplicity one for GL_n

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Preliminaries

- $\mathfrak{F}_m$ = cuspidal automorphic repesentations of $\mathrm{GL}_m(\mathbb{A}_\mathbb{Q})$[1] with (normalized) unitary central character
- q_π = conductor of $\pi \in \mathfrak{F}_m$
- $\pi = (\otimes_p \pi_p) \otimes \pi_\infty$
- Satake parameters: $\pi_p \leftrightarrow \mathrm{Diag}(\alpha_{1,\pi}(p), \ldots, \alpha_{m,\pi}(p))$
- Langlands parameters: $\pi_\infty \leftrightarrow \mu_{1,\pi}(\infty), \ldots, \mu_{m,\pi}(\infty)$

[1] We restrict to $\mathbb{Q}$ for notational simplicity only

L-functions

- $L(s,\pi)=\prod_p L(s,\pi_p)=\prod_p\prod_{j=1}^m(1-\alpha_{j,\pi}(p)p^{-s})^{-1},\quad \mathrm{Re}(s)>1$
- $L(s,\pi_\infty)=\prod_{j=1}^m\Gamma_{\mathbb{R}}\Big(\frac{s+\mu_{j,\pi}(\infty)}{2}\Big),\quad \Gamma_{\mathbb{R}}(s):=\pi^{-s/2}\Gamma(s/2)$

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- Analytic continuation for $L(s,\pi)L(s,\pi_\infty)$
- Analytic conductor $C(\pi):=q_\pi\prod_{j=1}^m(1+|\mu_{j,\pi}(\infty)|)$

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Theorem (Jacquet–Langlands, Piatetski-Shapiro, Shalika)

Let $\pi, \pi' \in \mathfrak{F}_m$. If $\pi_p \simeq \pi'_p$ for all but finitely many p, then $\pi = \pi'$.

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Conjecture (Ramakrishnan)

It suffices that $\pi_p \simeq \pi'_p$ for a density $1 - \frac{1}{2m^2}$ subset of the primes.

Effective multiplicity one

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Yes, if $Y \geq Q^{O_m(\log Q)}$.

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Theorem (Liu–Wang, '09)

Yes, if $Y \gg_{m,\varepsilon} Q^{2m+\varepsilon}$

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Theorem (GRH + GRC for ramified primes)

Yes, if $Y \gg_m (\log Q)^2$.

- GRH: $L(s, \pi' \times \widetilde{\pi}')$ and $L(s, \pi \times \widetilde{\pi}')$
- GRC = generalized Ramanujan conjecture ($|\alpha_{j,\pi}(p)| \le 1$)

Rankin–Selberg L-functions

- $\pi \in \mathfrak{F}_m$, $\pi' \in \mathfrak{F}_{m'}$

$$L(s, \pi \times \pi') \doteq \prod_{p \nmid q_\pi q_{\pi'}} \prod_{j=1}^{m} \prod_{j'=1}^{m'} (1 - \alpha_{j,\pi}(p) \alpha_{j',\pi'}(p) p^{-s})^{-1}, \ \mathrm{Re}(s) > 1$$

- Analytic continuation and functional equation
- Pole at $s = 1$ if and only if $\pi' = \widetilde{\pi}$

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- Equality of smoothed prime number theorems:

$$\sum_{p^\ell} \Lambda_{\pi\times\widetilde{\pi}'}(p^\ell)\phi\Big(\frac{p^\ell}{Y}\Big) \doteq \sum_{p^\ell} \Lambda_{\widetilde{\pi}\times\widetilde{\pi}'}(p^\ell)\phi\Big(\frac{p^\ell}{Y}\Big)$$

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$$\int_{2-i\infty}^{2+i\infty} \frac{L'}{L}(s, \pi \times \widetilde{\pi}')\widehat{\phi}(s) Y^s ds \doteq \int_{2-i\infty}^{2+i\infty} \frac{L'}{L}(s, \pi' \times \widetilde{\pi}')\widehat{\phi}(s) Y^s ds$$

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- If $Y = (\log Q)^A$ and $[1-\delta, 1] \times [-\log Q, \log Q]$ is zero-free,

$$\sum_{p^\ell} \Lambda_{\pi \times \widetilde{\pi}'}(p^\ell) \phi\Big(\frac{p^\ell}{Y}\Big) \ll_{m,\phi} Y^{1-\delta} \log Q,$$

$$\sum_{p^\ell} \Lambda_{\pi' \times \widetilde{\pi}'}(p^\ell) \phi\Big(\frac{p^\ell}{Y}\Big) \gg_{m,\phi} Y.$$

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Can we do better "on average"?

Rigidity in families

- Equivalence relation: $\pi \sim_B \pi'$ if $\pi_p \simeq \pi'_p$ for $p \leq (\log Q)^B$

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The implied constant is independent of π'.

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- Weak form of a zero density estimate.

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- Suitable progress toward GRC

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Theorem (Humphries–T)

There exists $\alpha \in (0,1)$ *such that if* $(\log Q)^B = Y \geq C(\pi')^{O(m^4)}$,

$$\sum_{p^\ell \leq Y} \Lambda_{\pi' \times \widetilde{\pi}'}(p^\ell) \gg \begin{cases} Y^{1-\frac{\alpha}{m^3}} & \text{if Siegel zero exists,} \\ Y & \text{otherwise.} \end{cases}$$

Main results: Zero density estimate

Definition

For $\pi \in \mathfrak{F}_m$ and $\pi' \in \mathfrak{F}_{m'}$, we set

$$N_{\pi\times\pi'}(\sigma, T) = \#\{\beta + i\gamma\colon L(\beta + i\gamma, \pi \times \pi') = 0,\ \beta \geq \sigma,\ |\gamma| \leq T\}.$$

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Theorem (T)

Let $\varepsilon > 0$. If $\pi' \in \mathfrak{F}_{m'}$, then

$$\sum_{\pi\in\mathfrak{F}_m(Q)} N_{\pi\times\pi'}(\sigma, T) \ll_{m,m',\varepsilon} T(C(\pi')QT)^{28(m'm)^2(1-\sigma)+\varepsilon}.$$

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All except $O_{m,m',\varepsilon}((C(\pi')Q)^{\varepsilon})$ of the $L(s, \pi \times \pi')$ have no zero in the box

$$\left[1 - \frac{\varepsilon}{224(m'm)^2}\right] \times [-Q^{\varepsilon/2}, Q^{\varepsilon/2}].$$

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Main results: Rigidity in $\mathfrak{F}_m$

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The implied constant is independent of π'.

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Theorem (Soundararajan–T, '19)

If $0 \le \delta < \frac{1}{2}$, *then*

$$\log|L(\tfrac{1}{2},\pi\times\pi')| \le \Big(\frac{1}{4}-\frac{\delta}{10^9}\Big)\log C(\pi\times\pi') + \frac{\delta}{10^7}N_{\pi\times\pi'}(1-\delta,6) + 2\log|L(\tfrac{3}{2},\pi\times\pi')| + O((m'm)^2)$$

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- Apply zero density estimate with $\delta = \frac{\varepsilon}{224(m'm)^2}$ and $T = 6 \ldots$

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- Has the potential to be useful for equidistribution problems

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π' trivial $\implies$ joint work of mine with Asif Zaman

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Define

$$A_\pi(p) = \{\alpha_{1,\pi}(p), \ldots, \alpha_{m,\pi}(p)\}, \quad A_{\pi'}(p) = \{\alpha_{1,\pi'}(p), \ldots, \alpha_{m',\pi'}(p)\}.$$

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- Cauchy's identity: If $\gcd(n, q_\pi q_{\pi'}) = 1$, then

$$\lambda_{\pi\times\pi'}(n) = \sum_{\substack{(\lambda_p)_p \\ |\lambda_p| = \mathrm{ord}_p(n)}} \Big(\prod_p s_{\lambda_p}(A_\pi(p))\Big)\Big(\prod_p s_{\lambda_p}(A_{\pi'}(p))\Big)$$

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&\leq \sum_{\substack{(\lambda_p)_p \\ |\lambda_p| = \mathrm{ord}_p(n)}} \Big|x\prod_p s_{\lambda_p}(A_\pi(p)) + y\overline{\prod_p s_{\lambda_p}(A_{\pi'}(p))}\Big|^2
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Thus

$$\begin{aligned}
(\lambda_{\pi\times\widetilde{\pi}}(n) - |\lambda_\pi(n)|^2)x^2 + 2\mathrm{Re}(\lambda_{\pi\times\pi'}(n) - \lambda_\pi(n)\lambda_{\pi'}(n))xy \\
+ (\lambda_{\pi'\times\widetilde{\pi}'}(n) - |\lambda_{\pi'}(n)|^2)y^2
\end{aligned}$$

is a positive-definite QF with nonpositive discriminant. □