Shifted convolution sums and small-scale mass equidistribution.

Budapest via Zoom (we all feel at home)

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⚠ WARNING ⚠

Very much work in progress!!

Shifted convolution sums

$\lambda(n) = \lambda_\pi(n)$ related to an automorphic representation π.

$$A_\pi(X,h) := \sum_{n \leq X} \lambda_\pi(n)\lambda_\pi(n+h)$$

$$A_\pi^W(X,h) := \sum_{n \in \mathbb{N}} \lambda_\pi(n)\lambda_\pi(n+h)W\left(\frac{n+h/2}{X}\right), \quad W \text{ smooth.}$$

Basic questions: Size? Uniformity? Averages? Asymptotics?

Has been studied by many incl. several conference participants, organisers etc.

'Classical' examples

Example (Additive divisor sums)

$\lambda(n) = d(n) =$ *number of divisors of n.*

- Comes from $\frac{d}{ds}E(z,s)|_{s=1/2}$.
- Used e.g. to study moments of the Riemann zeta function.

Example (Hyperbolic lattice counting)

$$\lambda(n) = r(n) = \#\{n = a^2 + b^2\},\ h = 4$$

- Comes from a theta series. $\theta(z) = \sum_{n\in\mathbb{Z}^2} e(\|n\|^2 z)$
- Shifted convolution sum is essentially the hyperbolic lattice counting problem, i.e. counting translates γi for $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ of hyperbolic distance from i less than X.

More general cases has been used to study subconvexity of $L(s,\pi)$, quantum ergodicity etc.

We consider f a cuspidal Hecke eigenform of weight k for the full modular group $\mathrm{SL}_2(\mathbb{Z})$

$$f(z) = \sum_{n=1}^{\infty} \lambda_f(n) n^{\frac{k-1}{2}} e(nz), \quad \lambda_f(1) = 1.$$

Recall

- $\lambda_f(n)$ real
- $\lambda_f(n)$ multiplicative satisfying

$$\lambda_f(n)\lambda_f(m) = \sum_{d|(m,n)} \lambda_f\left(\frac{mn}{d^2}\right)$$

- $|\lambda_f(p)| \leq 2$

Want to understand

$$A_f(X,h) := \sum_{n\leq X} \lambda_f(n)\lambda_f(n+h)$$

$$A_f^W(X,h) := \sum_{n\in\mathbb{N}} \lambda_f(n)\lambda_\pi(n+h) W(\frac{n+h/2}{X}), \quad W \text{ smooth.}$$

Pointwise bounds

Theorem (A. Good '83 (fixed h), M. Jutila '96)

$$A_f(h,X) = O_f(X^{2/3+\varepsilon}),$$

uniformly for $1 \leq h \leq X^{2/3}$.

This uses Good's bound

$$\sum_{0<t_j\leq T} \left|\left\langle \varphi_j, y^k\left|f(z)\right|^2\right\rangle\right|^2 e^{\pi t_j} + \text{Eisenstein contribution} \ll_f T^{2k}.$$

Conjecture: $A_f(h,X) = O(X^{1/2+\varepsilon}),\ \text{uniformly for } 1 \leq h \leq X^{1/2-\varepsilon}$

Theorem (Folklore?)

Assume the Ramanujan-Petersson conjecture for Maass forms . Then

$$A_f^W(h,X) = O_f(X^{1/2+\varepsilon}),$$

uniformly for $1 \leq h \leq X^{1/2-\varepsilon}$.

Averages over X

Theorem (Nordentoft-Petridis-R., 2020)

$$\left(\frac{1}{X}\int_X^{2X} |A_f(h,x)|^2\, dx\right)^{1/2} = O_f(X^{1/2+\varepsilon}), \qquad h \leq X^{1/2}.$$

Uses also Good's bound and a spectral large sieve inequality due to Jutila.

Similar results for:
$\lambda(n) = d(n)$: Faĭziev (1985), Ivić and Motohashi (1994)
$\lambda(n) = r(n)$, $h = 4$: Chamizo (1996), Cherubini (2018)

Averages over f

From now on only smooth sums.
Let H_k be a Hecke basis for $S_k(1)$ (normalized to have $\|f\| = 1$).

Notation:

$$\sum_{f\in H_k}{}^{h} Q(f) := \sum_{f\in H_k} \frac{1}{L(1,\mathrm{sym}^2 f)} Q(f), \qquad \sum_{f\in H_k}{}^{h}{}' Q(f) := \sum_{f\in H_k} L(1,\mathrm{sym}^2 f) Q(f),$$

Theorem (Petridis-Nordentoft-R., 2020)

For $X \ll k^{1/2-\varepsilon}$, $1 \leq h_i \ll X$

$$\frac{2\pi^2}{k-1} \sum_{f\in H_k}{}^{h} A_f^{W_1}(h_1, X)\overline{A_f^{W_2}(h_2, X)} = B_{h_1,h_2}(W_1, W_2)X + O_{W_i,h_i}(1),$$

where

$$B_{h_1,h_2}(W_1, W_2) = \sigma((h_1, h_2)) \int_0^\infty W_1(h_1 y)\overline{W_2(h_2 y)} dy.$$

Averages over f (continued)

Would like a result also for $X \geq k^{1/2}$.

Theorem (Petridis-Nordentoft-R., 2020)

Let $0 < \theta < 1$. *Let* u *be compact support on* $(0, \infty)$, $X = (k-1)^{1-\theta}$, $0 < h_1, h_2 \leq K^{1-\theta-\varepsilon}$. *Then*

$$\sum_{2|k} u\left(\frac{k-1}{K}\right) \frac{2\pi^2}{k-1} \sum_{f \in H_k}{}^{h} A_f^{W_1}(h_1, X(k)) \overline{A_f^{W_2}(h_2, X(k))}$$
$$= B_{h_1,h_2}(W_1, W_2) \frac{K^{2-\theta}}{2} \int_0^\infty u(y) y^{1-\theta} dy + O_{W_i,h_i}(K).$$

Idea of proof for both: Interchange sums, use Hecke relations, use Petersson's trace formula. Diagonal part gives the main term. Off-diagonal can be bounded by properties of J_{k-1}.

APPLICATIONS

Quantum ergodicity

Berry's random wave model

Eigenfunctions of the Laplacian on a classically ergodic system should tend to exhibit Gaussian random behavior as the eigenvalue tends to infinity.

In particular; Consider $X = \Gamma\backslash\mathbb{H}$, $\Delta_M\phi_j = \lambda_j\phi_j$, $\|\phi_j\|_2 = 1$. Then

$$|\phi_j|^2\, d\mu(z) \xrightarrow{weak*} \frac{1}{\mathrm{vol}\,(\Gamma\backslash\mathbb{H})} d\mu(z)$$

for a density one subsequence.
Solved by Schnirelmann, Colin de Verdière, Zelditch, Lindenstrauss, Soundararajan.

Shrinking sets

Basic question: Small scale equidistribution

Consider shrinking sets $B_H \subseteq \Gamma\backslash\mathbb{H}$ and functions ψ_H supported on B_H. Which conditions can we impose on B_H, ψ_H and the relation between λ, H such that we still have

$$\int_{\Gamma\backslash\mathbb{H}} \psi_H(z) \left|\phi_j(z)\right|^2 d\mu(z) = \frac{1}{\mathrm{vol}\,(\Gamma\backslash\mathbb{H})} \int_{\Gamma\backslash\mathbb{H}} \psi_H d\mu(z) + o(\langle\left|\psi_H\right|, 1\rangle), \text{ as } \lambda \to \infty.$$

for a density one subsequence of eigenfunctions?

Han, Hezari and Rivière: OK for $\phi_H = 1_{B_r(z_0)}$ where $r \gg 1/\log(\lambda)^\delta$ for some $\delta > 0$.
Young: On GLH OK for $\phi_H = 1_{B_r(z_0)}$ where $r \gg \lambda^{-\delta}$ for $\delta < 1/6$. (Full sequence)
Humphries: On GLH OK for $\phi_H = 1_{B_r(z_0)}$ for 'almost all' $z_0 \in \Gamma\backslash\mathbb{H}$ down to $r \gg \lambda^{-\delta}$ for $\delta < 1/2$

When the set B_H is of length scale below the *Planck scale* $(1/\sqrt{\lambda})$ we do *not* expect equidistribution.
Humphries: In general we do not have equidistribution if $r \ll \lambda^{-1/2}\log^A(\lambda)$.

Analogues for Hecke Eigenforms

Consider instead of ϕ_j the function $y^{k/2}f(z)$ for $f \in H_k$.

$$\Delta_k(y^{k/2}f(z)) = -\frac{k}{2}(1-\frac{k}{2})y^{k/2}f(z),$$

where Δ_k is the weight k Laplacian.
We still expect

$$y^k |f(z)|^2 \, d\mu(z) \xrightarrow{weak*} \frac{1}{\mathrm{vol}\,(\Gamma\backslash\mathbb{H})} d\mu(z)$$

for a density one subsequence.
Solved by Luo-Sarnak, Holowinsky-Soundararajan.

Basic question: Small scale equidistribution

Consider shrinking sets $B_H \subseteq \Gamma\backslash\mathbb{H}$ and functions ψ_H supported on B_H. Which conditions can we impose on B_H, ψ_H and the relation between k, H such that we still have

$$\int_{\Gamma\backslash\mathbb{H}} \psi_H(z) y^k |f(z)|^2 \, d\mu(z) = \frac{1}{\mathrm{vol}\,(\Gamma\backslash\mathbb{H})} \int_{\Gamma\backslash\mathbb{H}} \psi_H d\mu(z) + o(\langle|\psi_H|, 1\rangle), \text{ as } k \to \infty.$$

for a density one subsequence of eigenfunctions?

Can we get information down to the Planck scale i.e. lenght scale $k \asymp \sqrt{-\frac{k}{2}(1-\frac{k}{2})}$.

Shrinking sets/Squeezed sets around ∞

Let

$$B_H(\infty) := \{z \in \Gamma\backslash\mathbb{H} : y > H\}$$

be a horocyclic region and consider, for $0 < \theta < 1$ the operator

$$M_{(k-1)^\theta}\psi(z) = \psi(x + i\frac{y}{(k-1)^\theta})$$

mapping functions supported in $B_1(\infty)$ to functions supported in $B_{(k-1)^\theta}(\infty)$.
We want to consider the small scale equidistribution problem for such test functions.
Note that

$$\left\|M_{(k-1)^\theta}\psi\right\|_2^2 = \frac{1}{(k-1)^\theta}\left\|\psi\right\|_2^2, \quad \left\langle M_{(k-1)^\theta}\psi, 1\right\rangle = \frac{1}{(k-1)^\theta}\left\langle\psi, 1\right\rangle.$$

and that the Planck scale corresponds to θ close to 1. So in order to solve the shrinking set problem we need to show

$$\mu_f(M_{(k-1)^\theta}\psi) = \nu(M_{(k-1)^\theta}\psi) + o_\psi(k^{-\theta}), \text{ as } k \to \infty.$$

where

$$\mu_f(M_{(k-1)^\theta}\psi) = \int_{\Gamma\backslash\mathbb{H}}(M_{(k-1)^\theta}\psi)y^k\left|f(z)\right|^2 d\mu(z), \quad \nu(M_{(k-1)^\theta}\psi) = \frac{1}{\text{vol}\left(\Gamma\backslash\mathbb{H}\right)}\int_{\Gamma\backslash\mathbb{H}} M_{(k-1)^\theta}\psi d\mu(z).$$

Variance - upper bounds

Still $X = \Gamma\backslash\mathbb{H}$, $0 < \theta < 1$
$B \subseteq X$ open
$C_0^\infty(X, B) = \{\text{ smooth compactly supported functions on } X \text{ with support in } B\}$

Theorem (Petridis-Nordentoft-R. 2020)

Fix u compactly supported in $(0, \infty)$. Then for $\psi \in C_0^\infty(X, B_1(\infty))$ we have

$$\sum_{2|k} u\left(\frac{k-1}{K}\right) \sum_{f \in H_k}^{h} \left|\mu_f(M_{(k-1)^\theta}\psi) - \nu(M_{(k-1)^\theta}\psi)\right|^2 \ll K^{2-2\theta-\delta_\theta}$$

for some $\delta_\theta > 0$.

So we have equidistribution all the way down to the Planck scale for this particular case.

Variance - Asymptotics

Let $C_{00}^\infty(X,B) = \{\psi \in C_{00}^\infty(X,B) | \langle \psi, 1\rangle = 0\}$

Theorem (Petridis-Nordentoft-R. 2020)

Fix u compactly supported in $(0,\infty)$. *Then for* $\psi \in C_{00}^\infty(X, B_1(\infty))$ *we have*

$$\sum_{2|k} u\left(\frac{k-1}{K}\right) \sum_{f\in H_k}^{h} \left|\mu_f(M_{(k-1)^\theta}\psi) - \nu(M_{(k-1)^\theta}\psi)\right|^2 = B_\theta(\psi,\psi)\int_0^\infty u(y)y^{-\theta}dy\,\frac{K^{1-\theta}}{2} + O(K^{1-\theta-\delta_\theta})$$

for some $\delta_\theta > 0$. *Here* $B_\theta(\psi_1,\psi_2)$ *is an explicit bilinear form.*

The theorem also holds for $\theta = 0$ (Luo-Sarnak).

Variance - Asymptotics continued

The bilinear form satisfies

- $B_\theta(\Delta\psi_1, \psi_2) = B_\theta(\psi_1, \Delta\psi_2)$ for $\psi_1, \psi_2 \in C_{00}^\infty(X, B_1(\infty)) \cap \text{ cuspidal}$
- There are 4 regimes:
  1. $\theta = 0$ (Luo-Sarnak)
  2. $0 < \theta < 1/2$
  3. $\theta = 1/2$
  4. $1/2 < \theta < 1$

  In each regime B_θ is constant in θ. So there is a *phase transition* at $\theta = 1/2$
- As a special case: For $P_{V_i,h_i}(z) = \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} V(y(\gamma z))e(h_i x(\gamma z))$ with $\operatorname{supp}(V_i) \subseteq (1,\infty)$ and $h_1 h_2 \neq 0$ we have

$$B_{1/4}(P_{V_1,h_1}(z), P_{V_2,h_2}(z)) = \frac{\pi}{4}\sigma((|h_1|, |h_2|)) \int_0^\infty V_1(\frac{y}{|h_1|})\overline{V_2(\frac{y}{|h_2|})}\frac{dy}{y^2}$$

$$B_{1/2}(P_{V_1,h_1}(z), P_{V_2,h_2}(z)) = \frac{\pi}{4}\sigma((|h_1|, |h_2|)) \int_0^\infty V_1(\frac{y}{|h_1|})\overline{V_2(\frac{y}{|h_2|})}e^{-2\pi^2y^2(h_1^2+h_2^2)}\frac{dy}{y^2}$$

$$B_{3/4}(P_{V_1,h_1}(z), P_{V_2,h_2}(z)) = 0$$

Below the Planck scale

Proposition

Let $\theta \geq 1$. *There exist* $\psi \in C_0^\infty(X, B_1(\infty))$ *such that*

$$\mu_k(M_{(k-1)^\theta}\psi) = o(\nu(M_{(k-1)^\theta}\psi)) \quad as\ k \to \infty.$$

In particular we do not have equidistribution of mass in this limit.

Comments

- Extending B_θ to a larger space?
- How to prove it?

$$\mu_f(M_{(k-1)^\theta} P_{V,h}) = \frac{2\pi^2}{(k-1)L(1,\mathrm{sym}^2 f)} \sum_n \lambda_f(n)\lambda_f(n+h) W\left(\frac{n+h/2}{(k-1)^{1-\theta}}\right)\left(\frac{\sqrt{n(n+h)}}{n+h/2}\right)^{k-1}$$

$$+O_m(k^{-1-\theta+\varepsilon})$$

where $W(y) = V((4\pi y)^{-1})$

- Equidistribution of zeroes?
  Relation to Lester-Matomäki-Radziwiłł's work on Sarnak-Ghosh conjecture?