Asymptotics of class numbers

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Programme

- Quadratic forms, class numbers and Pell's equation.
- Known results for the asymptotic behaviour of class numbers (of quadratic forms).
- An asymptotic result for class numbers corresponding to fundamental discriminants.
- An asymptotic result for class numbers of quadratic fields.

Class numbers

Let

$$q(x,y) = ax^2 + bxy + cy^2$$

be a primitive binary quadratic form with coefficients in $\mathbb{Z}$ and

$$d := b^2 - 4ac$$

its discriminant. We have an equivalence relation given by

$$\gamma^t \begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix} \gamma = \begin{pmatrix} a' & b'/2 \\ b'/2 & c' \end{pmatrix}, \quad \gamma \in \mathrm{SL}_2(\mathbb{Z}),$$

and h_d denotes the class number (in the narrow sense) of primitive quadratic forms of discriminant d.

Some known results

We have to treat the case $d > 0$ and $d < 0$ separately.

Gauß

The mean value of h_d behaves as $2\pi\sqrt{|d|}/(7\zeta(3))$ as $|d| \to \infty$, $d \equiv 0 \mod 4$.

Let $\epsilon_d^+ = \frac{1}{2}(t + u\sqrt{d})$ be the fundamental solution of Pell's equation $t^2 - du^2 = 4$.

Gauß

The mean value of $h_d \log \epsilon_d^+$ behaves as $2\pi^2\sqrt{d}/(7\zeta(3))$ as $d \to \infty$, $d \equiv 0 \mod 4$.

Some known results

Lipschitz (1865), Mertens (1874), Vinogradov (1917–1963)

$$\sum_{0\leq -d<N} h_d = \frac{4\pi}{21\zeta(3)}N^{3/2} - \frac{2}{\pi^2}N + O\left(N^{2/3+\epsilon}\right).$$

Siegel (1944)

$$\sum_{d\leq N} h_d \log \epsilon_d^+ = \frac{\pi^2}{18\zeta(3)}N^{3/2} + O(N\log N).$$

Shintani (1975) improved the error term using the theory of prehomogeneous vector spaces.

Some known results

If we restrict to fundamental discrimininants we have

Datskowsky (1993), Taniguchi (2007)

$$\sum_{\substack{[K:\mathbb{Q}]=2,\\ d(K)\leq N}} h(K)\mathrm{Reg}(K) \sim \frac{\zeta(2)}{3} \prod_{p>2} \left(1 - p^{-2} - p^{-3} + p^{-4}\right) N^{3/2}.$$

Method: Prehomogeneous vector spaces.

Goldfeld/Hoffstein (1985) obtained a similar result using Eisenstein series of half-integral weight.

Changing the order of summation

Goal: determine the asymptotic behaviour of h_d as $d \to \infty$.

Problem: When does the negative Pell equation $t^2 - du^2 = -4$ have a solution?

Obvious: $p \equiv 3 \bmod 4$ divides $d \Longrightarrow$ no solutions.

Further results by Fouvry, Klüners.

We can change the order of summation.

Class numbers ordered by the size of the regulator

Sarnak (1982)

Norms of conjugacy classes of primitive hyperbolic transformations of $\mathrm{SL}_2(\mathbb{Z}) = {\epsilon_d^+}^2$, $d \in \mathcal{D}$, multiplicity $= h_d$. Furthermore, the lengths of closed geodesics on $\mathrm{SL}_2(\mathbb{Z}) \setminus \mathbb{H}^2 = 2 \log \epsilon_d^+$, multiplicity $= h_d$.

Using this identification and the Selberg trace formula we get

Sarnak (1982)

Let $\mathrm{Li}(N) := \int_2^N \frac{dt}{\log t}$. Then

$$\sum_{\epsilon_d^+ \leq N} h_d = \mathrm{Li}(N^2) + O\left(N^{3/2} \log^2 N\right).$$

For fundamental discriminants

Let $\mathcal{D} := \{d \in \mathbb{N} : \ d \equiv 0, 1 \bmod 4, \ d \neq \square\}$ and $\mathcal{D}_F$ be the set of all fundamental discriminants.

Raulf (2009)

Let $\mathcal{D}_F(N) := \{d \in \mathcal{D}_F : \ \epsilon_d^+ \leq N\}$. Then there exist a constant c_F such that

$$\sum_{d \in \mathcal{D}_F(N)} h_d \sim c_F \, \mathrm{Li}(N^2), \quad N \to \infty.$$

The contant c_F can be explicitly determined.

Discriminants in progression

For $m \in \mathbb{N}$, $a \in \mathbb{N}_0$ let $\mathcal{D}_{m,a} := \{d \in \mathcal{D} : \ d \equiv a \bmod m\}$. We will study the asymptotic behaviour of

$$\sum_{\substack{d \in \mathcal{D}_{m,a}, \\ \epsilon_d^+ \leq N}} h_d \log \epsilon_d^+ . \tag{1}$$

The behaviour of (1) implies the theorem since

$$\begin{aligned} \sum_{\substack{d \in \mathcal{D} \text{ sqf}, \\ \epsilon_d^+ \leq N}} h_d \log \epsilon_d^+ &= \sum_{\epsilon_d^+ \leq N} \sum_{m^2 \mid d} \mu(m)\, h_d \log \epsilon_d^+ \\ &= \sum_{m \geq 1} \mu(m) \sum_{\substack{d \equiv 0 \bmod m^2, \\ \epsilon_d^+ \leq N}} h_d \log \epsilon_d^+ . \end{aligned}$$

Methods for studying class numbers

For determining the asymptotic behaviour of class numbers we have two main tools:

1. Selberg's trace formula, i. e. the theory of automorphic forms, (Sarnak)
2. the class number formula (Barban, Sarnak).

The problem rewritten

We can study

$$\sum_{\substack{(\epsilon_d^+)^n \leq N, \\ d \equiv a \bmod m}} h_d \log \epsilon_d^+$$

or, using partial summation,

$$\sum_{1 \leq u \leq N} \sum_{\substack{2 < t \leq N,\, d(t,u) := \\ (t^2-4)/u^2 \in \mathcal{D}_{m,a}}} \underbrace{\frac{h_{d(t,u)} \log \epsilon_{d(t,u)}^+}{\sqrt{d(t,u)}}}_{= L(1, \chi_{d(t,u)})}.$$

- Fix u and study the behaviour for each u separately.
- Here it suffices to consider small us, the others are absorbed in the error term.

A theorem of Barban and Sarnak

In order to deduce the asymptotic behaviour we adapt

Barban, Sarnak

Let ϕ be a polynomial satisfying certain conditions,

$$c_{8n}(\phi) := \sum_{8n \leq x < 16n} \left(\frac{\phi(x)}{n} \right)$$

and for $\gamma \in (0, 1)$ set

$$I(\phi(l), N) := \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} \Gamma(s-1) L(s, \phi(l)) N^{s-1} ds.$$

Then for $\epsilon > 0$ we have

$$\sum_{0 \leq l \leq N} L(1, \phi(l)) = N \sum_{n=1}^{\infty} \frac{c_{8n}(\phi)}{8n^2} - \sum_{0 \leq l \leq N} I(\phi(l), N) + O_{\phi,\epsilon}\left(N^{1/2+\epsilon}\right).$$

The right polynomials

For $m \in N$ we divide $\mathcal{D}_m$ into three sets taking into account whether $d \equiv 0 \bmod 4$ or $1, 5 \bmod 8$ and we set $v_r(m) := \mathrm{lcm}(r, m)$, $r = 4, 8$.

For $u \in \mathbb{N}$ let

$$F_*(m,u) := \left\{2 < t \leq v_r(m)u^2 + 2 : \exists d \in \mathcal{D}_* \ s.t. \ t^2 - du^2 = 4\right\}.$$

For $t_0 \in F_*(m,u)$ we set

$$\phi_{m,u,t_0}(l) := \frac{(t_0 + v_r(m)u^2 \cdot l)^2 - 4}{u^2}.$$

- $\phi_{m,u,t_0}(l) \in \mathcal{D}_*$.
- For $\phi = \phi_{m,u,t_0}$ the contribution of $O_{\epsilon,\phi}$ and $I(\phi(l), N)$ to the asymptotics is small.

Calculating the leading coefficients

We begin by determining $c_{8n}(\phi)$, $n \in \mathbb{N}$. This is basically the problem of evaluating sums of the form

$$\sum_{x \bmod p} \left(\frac{\phi_{m,u,t_0}(x)}{p} \right)^{e(p)}, \quad p \text{ a prime.}$$

The results depends on whether $e(p) \equiv 0, 1 \bmod 2$ and on the relation between p and $d_0(t_0, u)$. Furthermore,

$$n \mapsto c_{8n}\left(\phi_{m,u,t_0}\right)/8$$

is multiplicative. Euler product $\Longrightarrow$

$$\mathcal{C}\left(2, \phi_{m,u,t_0}\right) := \sum_{n \geq 1} \frac{c_{8n}\left(\phi_{m,u,t_0}\right)}{8n^2}.$$

Theorem

For $m,\ u \in \mathbb{N}$ and $a = 0$

$$\mathcal{C}\left(2, \phi_{m,u,t_0}\right) = \frac{3}{4}\zeta(2)\mathcal{P}(m,u),\ \ t_0 \in F_4(m,u),$$

$$\mathcal{C}\left(2, \phi_{m,u,t_0}\right) = \frac{3}{2}\zeta(2)\mathcal{P}(m,u),\ \ t_0 \in F_{8,1}(m,u),$$

$$\mathcal{C}\left(2, \phi_{m,u,t_0}\right) = \frac{1}{2}\zeta(2)\mathcal{P}(m,u),\ \ t_0 \in F_{8,5}(m,u).$$

where

$$\mathcal{P}(m,u) :=$$

$$\prod_{p \geq 3}\left(1 - p^{-2} - p^{-3}\right) \prod_{\substack{p \geq 3,\, p \mid u, \\ (p,m)=1}} \frac{1 - p^{-3}}{1 - p^{-2} - p^{-3}} \prod_{\substack{p \geq 3, \\ p \mid m}} \frac{1 - p^{-2}}{1 - p^{-2} - p^{-3}}.$$

Partial summation, summation over t_0 and $u \Longrightarrow$

Theorem

$$\sum_{\substack{d \in \mathcal{D}_{4,m} \\ \epsilon_d^+ \leq N}} h_d \log \epsilon_d^+ \sim \frac{3\zeta(2)}{8 v_4(m)} \sum_{u=1}^{\infty} \frac{|F_4(m,u)| \mathcal{P}(m,u)}{u^3} \times N^2.$$

The size of $|F_4(m,u)|$

For $m \in \mathbb{N}$ let $\tau(m) := \sharp\{p \geq 3 : \ p|m\}$, $d_2(m) := \max\{k \geq 0 : \ 2^k|m\}$ and we write u in the form $u = 2^r u_0$, $2 \nmid u_0$. Consider the congruences

$$\begin{aligned} t^2 &\equiv 4 \bmod p^{e(p)}, \ p|(mu), p \geq 3, \\ t^2 &\equiv 4 \bmod 2^{2d_2(u)+d_2(v_4(m))}. \end{aligned}$$

Lemma

For m, $u \in \mathbb{N}$, $u = 2^r u_0$, $2 \nmid u_0$,

$$|F_4(m,u)| = \begin{cases} 2^{1+\tau(um)}, & r = 0, d_2(v_4(m)) = 2, \\ 2^{1+\tau(um)}, & r = 0, d_2(v_4(m)) = 3, \\ 2^{2+\tau(um)}, & 2r + d_2(v_4(m)) = 4, \\ 2^{3+\tau(um)}, & 2r + d_2(v_4(m)) \geq 5, \end{cases}$$

The final result

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If $m \equiv 0 \bmod 2$, then

$$\sum_{\substack{d\in\mathcal{D}_m,\\ \epsilon_d^+\leq N}} h_d \log \epsilon_d^+ \sim \frac{2^{\tau(m)}}{56m} \prod_{\substack{p\geq 3,\\ p\mid m}} \left(1 - p^{-3}\right)^{-1} C(2,m) N^2.$$

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If $m \equiv 1 \bmod 2$, then

$$\sum_{\substack{d\in\mathcal{D}_m,\\ \epsilon_d^+\leq N}} h_d \log \epsilon_d^+ \sim \frac{2^{\tau(m)-1}}{m} \prod_{\substack{p\geq 3,\\ p\mid m}} \left(1 - p^{-3}\right)^{-1} N^2.$$

A known result and some remarks

Corollary

$$\sum_{\substack{d\in\mathcal{D},\\ \epsilon_d^+\le N}} h_d \log \epsilon_d^+ \sim \frac{N^2}{2}.$$

- By partial summation one can remove the regulators in the formulae.
- Similar formulae can be proved for the progression $d \equiv a \bmod m$, $m \in \mathbb{N}$, $a \in \mathbb{N}_0$.

Asymptotics for fundamental discriminants

The formulae for class numbers in progressions and the identity

$$\sum_{\substack{d \in \mathcal{D}\,\mathrm{sqf}, \\ \epsilon_d^+ \leq N}} h_d \log \epsilon_d^+ = \sum_{m \geq 1} \mu(m) \sum_{\substack{d \equiv 0 \bmod m^2, \\ \epsilon_d^+ \leq N}} h_d \log \epsilon_d^+$$

yield

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1. $$\sum_{\substack{d \in \mathcal{D}_F, \\ \epsilon_d^+ \leq N}} h_d \log \epsilon_d^+ \sim \frac{25\zeta(3)}{32} \prod_{p \geq 2} \left(1 - 2p^{-2} - p^{-3}\right) N^2,$$

2. $$\sum_{\substack{d \in \mathcal{D}_F, \\ \epsilon_d^+ \leq N}} h_d \sim \frac{25\zeta(3)}{16} \prod_{p \geq 2} \left(1 - 2p^{-2} - p^{-3}\right) \mathrm{Li}(N^2).$$

Quadratic fields and class numbers

Let d be a fundamental discriminant and $K = \mathbb{Q}(\sqrt{d})$. There is a close relationship between h_d, $\log \epsilon_d^+$ and the class number $H(K)$ and the regulator $\mathrm{Reg}(K)$ of K.

If we consider the size of the regulator instead of the size of the discriminant, then we can count the solutions of $t^2 - du^2 = -4$ and we get:

Raulf (2020)

Let $\mathcal{P}_1 := \{p : \ p \text{ prime}, p \equiv 1 \bmod 4\}$. There exists an $\epsilon > 0$ s.t., as $N \to \infty$,

$$\sum_{\mathrm{Reg}(K) \le \log N} H(K) = C \,\mathrm{Li}\left(N^2\right) + O\left(\frac{N^{2-\epsilon}}{\log N}\right),$$

$$C = \frac{25\zeta(3)}{32} \prod_{p \ge 2} \left(1 - 2p^{-2} - p^{-3}\right) + \frac{3}{8} \prod_{p \in \mathcal{P}_1} \frac{1 - 2p^{-2} - p^{-3}}{1 - p^{-3}}.$$

Modifications of the previous proof

We need to look at the negative Pell equation $t^2 - du^2 = -4$. We write

$$\mathcal{D} = \mathcal{D}^+ \cup \mathcal{D}^-,$$

- $\mathcal{D}^+$ contains all those discriminants for which the negative Pell equation does not have a solution,
- $\mathcal{D}^-$ those for which the negative Pell equation does have a solution.
- $d \in \mathcal{D}^- \implies h_d = H(K)$ and $\log \epsilon_d^+ = 2\text{Reg}(K)$,
- $d \in \mathcal{D}^+ \implies h_d = 2H(K)$ and $\log \epsilon_d^+ = \text{Reg}(K)$.

Thus

$$2 \sum_{\text{Reg}(K) \leq \log N} H(K)\text{Reg}(K) = \sum_{\substack{d \in \mathcal{D}_F^+, \\ \epsilon_d^+ \leq N}} h_d \log \epsilon_d^+ + \sum_{\substack{d \in \mathcal{D}_F^-, \\ \epsilon_d \leq N}} h_d \log \epsilon_d^+$$

Modifications of the previous proof

If ϵ_d is the fundamental solution of $t^2 - du^2 = -4$, then all its solutions are given by ϵ_d^{2k+1}, $k \in \mathbb{Z}$. We are interested in the behaviour of

$$\sum_{\substack{(d,k),\\ d\in\mathcal{D}_{m,a},\, k\geq 1,\\ \epsilon_d^k\leq N}} \frac{h_d \log \epsilon_d^+}{\sqrt{d}} = \sum_{1\leq u\leq N} \sum_{\substack{2<t\leq N,\, d(t,u):=\\ (t^2+4)/u^2\in\mathcal{D}_{m,a}}} \frac{h_{d(t,u)} \log \epsilon_{d(t,u)}^+}{\sqrt{d(t,u)}}.$$

Modifications of the previous proof

For $m, u \in \mathbb{N}$ and $a \in \mathbb{N}_0$ we set

$$F_*^-(m,a,u) := \left\{2 < t \leq v_*(m)u^2 + 2 : \exists\, d \in \mathcal{D}_* \text{ s.t. } t^2 - du^2 = -4\right\}$$

and for $t_0 \in F_*^-(m,a,u)$ we define

$$\phi_*^-(l) := \frac{\left(t_0 + v_*(m)u^2 \cdot l\right)^2 + 4}{u^2}.$$

This influences the calculation of the leading term in the following way:

$$\sum_{x \bmod p} \left(\frac{\phi^-_{m,u,t_0}(x)}{p}\right)^e$$

depends, as before, on the parity of e, on the relation between p and $d_0(t_0, u)$ and additionally on whether $p \equiv 1$ or $3 \bmod 4$.

Adapting the previous proof

Thus

$$\mathcal{C}(2,\phi^-_{4,m,u,t_0}) = \sum_{n=1}^{\infty} \frac{c_{8n}(\phi^-_{m,u,t_0})}{8n^2}$$
$$= \frac{\zeta(2)\,\mathcal{P}(m,u)}{2} \begin{cases} \frac{3}{2} & \text{if } d_0(t_0,u) \equiv 0 \bmod 4, \\ 3 & \text{if } d_0(t_0,u) \equiv 1 \bmod 8, \\ 1 & \text{if } d_0(t_0,u) \equiv 5 \bmod 8. \end{cases}$$

Here

$$\mathcal{P}(m,u) := \prod_{p\in\mathcal{P}_1} (1-p^{-2}-2p^{-3}) \prod_{p\in\mathcal{P}_3} \left(1-p^{-2}\right) \prod_{\substack{p\in\mathcal{P}_1,\\ p\mid m}} \frac{1-p^{-2}}{1-p^{-2}-2p^{-3}}$$
$$\times \prod_{\substack{p\in\mathcal{P}_1,\, p\mid u,\\ (p,m)=1}} \frac{1-p^{-3}}{1-p^{-2}-2p^{-3}}.$$

It remains to evaluate sums of the form

$$\sum_{u=1}^{\infty} \frac{|F_*^-(m,u)|\mathcal{P}(m,u)}{u^3},$$

i.e. to count the solutions of $t^2 - du^2 = -4$ for fixed u and $d \in \mathcal{D}_*$. Note that if u or m contain a prime $p \equiv 3 \bmod 4$ there are no solutions.

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If $m \in \mathbb{N}$ is not divisible by a prime $p \equiv 3 \bmod 4$, we obtain:

1. If $m \equiv 1 \bmod 2$ we have

$$\sum_{\substack{d \in \mathcal{D}_m, \\ \epsilon_d \leq N}} h_d \log \epsilon_d^+ \sim \left(\frac{2^{\tau(m)-1}}{m} \prod_{\substack{p \in \mathcal{P}_1, \\ p \mid m}} \frac{1}{1 - p^{-3}} \right) N^2.$$

2. For $m \equiv 0 \mod 2$ we have

$$\sum_{\substack{d \in \mathcal{D}_m, \\ \epsilon_d \leq N}} h_d \log \epsilon_d^+ \sim \frac{2^{\tau(m)-1}}{m} \prod_{\substack{p \in \mathcal{P}_1, \\ p \mid m}} \frac{1}{1 - p^{-3}} \times \begin{Bmatrix} 1, & d_2(m) = 1, \\ 2, & d_2(m) = 2, 3 \end{Bmatrix} N^2.$$

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There exists an $\epsilon > 0$ such that, as $N \to \infty$,

$$\sum_{\substack{d \in \mathcal{D}_F, \\ \epsilon_d \leq N}} h_d \log \epsilon_d^+ = \frac{3}{8} \prod_{p \in \mathcal{P}_1} \frac{1 - 2p^{-2} - p^{-3}}{1 - p^{-3}} N^2 + O\left(N^{2-\epsilon}\right).$$

What about powers?

For $m \in \mathbb{N}$, $a \in \mathbb{N}_0$ let $\mathcal{D}_{m,a} := \{d \in \mathcal{D} : \; d \equiv a \bmod m\}$ and

$$\mathcal{D}_{m,a}(N) := \{d \in \mathcal{D}_{m,a} : \; \epsilon_d^+ \leq N\}.$$

Raulf (2009), Raulf (2016)

For $k, m \in \mathbb{N}^*$ and $a \in \mathbb{N}_0$ there exist constants $C(k, m, a)$ such that

$$\sum_{d \in \mathcal{D}_{m,a}(N)} \left(h_d \log \epsilon_d^+\right)^k \sim \frac{C(k, m, a)}{k+1} N^{k+1}, \quad N \to \infty.$$

These contants can be explicitly determined. Sieving process allows to restrict to fundamental discriminants. $\log \epsilon_d^+$ can be removed by partial summation.

Distribution functions

The same method allows us to count the discriminants contributing to the sum. $\Longrightarrow$

Raulf (2016)

As $N \to \infty$ the sequence

$$F_n^*(x) := \frac{1}{|\mathcal{D}_*(N)|} \left| \left\{ d \in \mathcal{D}_*(N) : \frac{h_d \log \epsilon_d^+}{\sqrt{d}} \leq x \right\} \right|$$

converges to a distribution function F^* for each x at which F^* is continuous.