New Directions in the Subconvexity Problem

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Sub-convexity problem

- To non-trivially bound the size of L-functions $L(s,\pi)$ on the critical line $\mathrm{Re}(s) = 1/2$.
- In general, functional equation $\implies$ convexity bound (trivial bd):

$$L(1/2+it,\pi) \ll_\varepsilon C_\pi(t)^{1/4+\varepsilon}.$$

Here $C_\pi(t)$ is the analytic conductor.

- Sub-convexity Problem: Improve the exponent. (Indeed RH $\implies$ Lindelöf: $L(1/2+it,\pi) \ll_\varepsilon C_\pi(t)^\varepsilon$)
- Example: Weyl-Hardy-Littlewood: (say $t > 2$)

$$\zeta(1/2+it) \ll t^{1/4+\varepsilon} \text{ (convexity) } \rightsquigarrow \zeta(1/2+it) \ll t^{1/6+\varepsilon}.$$

- Same bound holds for Dirichlet L-function (χ prim mod q)

$$L(1/2+it,\chi) \ll_\chi t^{1/6+\varepsilon}, \quad \text{Note that: } C_\chi(t) = qt.$$

This is subconvex in the t-aspect.

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Sub-convexity problem: different flavours

- It is more challenging to improve the exponent of q.
- Example: Burgess

$$L(1/2, \chi) \ll_\varepsilon q^{3/16+\varepsilon}.$$

Level aspect subconvexity.

- Later Heath-Brown:

$$L(1/2 + it, \chi) \ll_\varepsilon (qt)^{3/16+\varepsilon}.$$

- Two directions:
  1) Improve the exponent.
  2) Extend to higher degree.

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Higher degree

Deg	L-function	Results
2	$GL(2)$	'completely' G, DFI, MV, et al.
4	$GL(2) \times GL(2)$	KMV, HM, M (level), MV
		S, LLY (weight/spectral)
8	$GL(2) \times GL(2) \times GL(2)$	BR (spectral), V (level)
3	$GL(3)$	M. (t aspect), BB (spectral/for generic)
3	$GL(3) \times GL(1)$	M. (twist aspect)
6	$GL(3) \times GL(2)$	$\star$

$GL(3) \times GL(2)$ Rankin-Selberg L-functions

Let $f \in S_k(N, \psi)$ Hecke modular/Maass form. Let F be a Hecke-Maass form for $SL(3, \mathbb{Z})$ with Langlands parameter α_j.
Rankin-Selberg convolution of f and F is defined by (for $\mathrm{Re}(s) > 1$)

$$L(s, f \otimes F) = \prod_{p \text{ prime}} \prod_{\substack{i=1,2 \\ j=1,2,3}} (1 - \alpha_{f,i}(p)\alpha_{F,j}(p)p^{-s})^{-1}$$

$$= \sum_{m,n=1}^{\infty} \frac{\lambda_F(m,n)\lambda_f(n)\psi(m)}{(m^2 n)^s}.$$

This has an integral representation

$$\int_{\Gamma_0(N)\backslash\mathbb{H}} f(z)\overline{F(\iota(z))}y^{s-1}\mathrm{d}x\mathrm{d}y$$

which gives analytic continuation and functional equation.
A deep result of Kim-Shahidi says that this L-function is automorphic.

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We want to study the size of this L-function on the central line:

$$L(1/2+it, f \otimes F) \ll \begin{cases} \text{in terms of } t & \text{when } f,\ F \text{ are fixed} \\ \text{in terms of } k,\ N & \text{when } t,\ F \text{ are fixed} \\ \text{in terms of } \alpha_j & \text{when } t,\ f \text{ are fixed.} \end{cases}$$

From functional equation and convexity principle we get

$$L(1/2+it, f \otimes F) \ll \begin{cases} t^{3/2+\varepsilon} & \text{when } f,\ F \text{ are fixed} \\ (k^6 N^3)^{1/4+\varepsilon} & \text{when } t,\ F \text{ are fixed} \\ (\prod_j (1+|\alpha_j|)^2)^{1/4+\varepsilon} & \text{when } t,\ f \text{ are fixed.} \end{cases}$$

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Note that

$$L(1/2+it, f \otimes \text{Eis ser.}) \approx L(1/2+it, f)^3$$
$$L(1/2+it, \text{Eis ser.} \otimes F) \approx L(1/2+it, F)^2.$$

Hence from subconvex bounds for $L(s, f \otimes F)$ we can derive subconvex bounds for $L(s, f)$ and $L(s, F)$.

Watson formula relates the period integrals appearing in QUE problem to $L(s, f \otimes \text{Sym}^2\phi)$, which is a $GL(2) \times GL(3)$ Rankin-Selberg L-function.

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Third moment: Non-negativity

$\{f_j\}$ an orthogonal Hecke basis of Maass forms for $SL(2,\mathbb{Z})$.

- Conrey-Iwaniec: For χ quadratic mod q

$$\sum_{-R<|t_j|<R} L(1/2, f_j \otimes \chi)^3 + \int_{-R}^{R} |L(1/2+it,\chi)|^6 \ll R^A q.$$

- Li (2011): For F self-dual (i.e. $F = \mathrm{Sym}^2 f$), $T^{3/8} < M < T^{1/2}$ we have

$$\sum_{T-M<|t_j|<T+M} L(1/2, F\otimes f_j) + \int_{T-M}^{T+M} |L(1/2+it,F)|^2 \ll TM.$$

Lapid $\implies$ $L(1/2, F\otimes f_j) \geq 0$. One gets subconvex bounds

$$L(1/2, F\otimes f_j) \ll |t_j|^{11/8}, \quad \text{and} \quad L(1/2+it, F) \ll t^{11/16}.$$

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$\{f_j\}$ an orthogonal Hecke basis of Maass forms for $\Gamma_0(q)$.

- Blomer: For χ quadratic mod q, and F self-dual

$$\frac{1}{q}\sum_{f_j}\frac{L(1/2,F\otimes f_j\otimes\chi)}{(1+|t_j|)^B}+\frac{1}{q}\int\frac{|L(1/2+it,F\otimes\chi)|^2|t|^2}{(1+|t|)^B}\ll q^{1/4}.$$

Lapid $\implies L(1/2,F\otimes f_j\otimes\chi)\geq 0$. One gets subconvex bounds

$$L(1/2,F\otimes f_j\otimes\chi)\ll q^{5/4},\quad \text{and}\quad L(1/2,F\otimes\chi)\ll q^{5/8}.$$

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Results

Theorem 1

Let f be a $SL_2(\mathbb{Z})$ modular/Maass form, and F be a $SL_3(\mathbb{Z})$ Maass form, then $(t > 2)$

$$L(\tfrac{1}{2} + it, f \otimes F) \ll t^{\frac{3}{2} - \frac{1}{42} + \varepsilon}.$$

- The proof is based on separation of oscillation via circle method
- It is not sensitive to f or F being cuspidal or not. Hence as a corollary we obtain:

$$L(\tfrac{1}{2} + it, \chi) \ll_\chi t^{\frac{1}{4} - \frac{1}{252} + \varepsilon}$$

$$L(\tfrac{1}{2} + it, f) \ll_f t^{\frac{1}{2} - \frac{1}{126} + \varepsilon}.$$

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Related results

- Theorem (Sharma arxiv 2019) Let $\chi \bmod p$ be a primitive Dirichlet character, then
$$L(\tfrac{1}{2}, F \otimes f \otimes \chi) \ll p^{\frac{3}{2}-\frac{1}{32}+\varepsilon}.$$

- Theorem (Kumar upcoming) Let f be a $SL_2(\mathbb{Z})$ modular/Maass form with weight/spectral parameter k. Then
$$L(\tfrac{1}{2}, F \otimes f) \ll k^{\frac{3}{2}-\delta},$$
for some $\delta > 0$.

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Related results 2

- Theorem (Kumar-Mallesham-Singh pre-print) Let F be a $SL_3(\mathbb{Z})$ Maass form with Langlands parameters $\alpha_j \sim T$ such that $\alpha_1 - \alpha_2 \sim T^{1-\xi}$ (non-generic)

$$L(\tfrac{1}{2}, F \otimes f) \ll \max\left\{ T^{\frac{3}{2}-\frac{\xi}{4}}, T^{\frac{3}{2}-\frac{1-2\xi}{4}} \right\}.$$

- Blomer-Buttcane (2019) have recently applied $GL(3)$ Kuznetsov formula to evaluate fourth moment and consequently obtained

$$L(\tfrac{1}{2}, F) \ll T^{\frac{3}{4}-\frac{1}{120000}},$$

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Upcoming result

- Sharma pre-print extends work of Blomer-Buttcane and obtains

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Theorem 2, in progress

f and F as above -

$$L(\tfrac{1}{2}, f \otimes F) \ll T^{\frac{3}{2} - \delta}.$$

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Proof of Theorem 1.

- Functional equation $\implies$ Approximate functional equation, which yields

$$L(\tfrac{1}{2}+it, f\otimes F) \ll t^{\varepsilon} \sup_{N\ll t^{3+\varepsilon}} \frac{|S(N)|}{\sqrt{N}} + t^{-2019}$$

with

$$S(N) = \sum_{n,r=1}^{\infty}\sum \lambda_F(n,r)\lambda_f(n)(nr^2)^{it} V\left(\frac{nr^2}{N}\right).$$

- Trivial estimation of $S(N)$ yields convexity. We will establish non-trivial cancellation in

$$S(N) \approx \sum_{n\sim N} \lambda_F(n,1)\lambda_f(n)n^{it}$$

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Sketch of proof

- We write

$$S(N) = \sum_{n,m\sim N}\sum \lambda_F(n,1)\lambda_f(m)m^{it}\delta(n-m)$$
$$= \frac{1}{K}\int W\left(\frac{v}{K}\right)\sum_{n,m\sim N}\sum \lambda_F(n,1)n^{iv}\lambda_f(m)m^{it-iv}\delta(n-m)$$

- δ-method: We use a Fourier expansion of δ

$$\delta(n) = \frac{1}{Q}\sum_{q\ll Q}\frac{1}{q}\sum_{a \bmod q}^{\star} e\left(\frac{an}{q}\right)h\left(\frac{q}{Q},\frac{n}{qQ}\right)$$

where h is a 'reasonably nice function' when Q is chosen wisely.

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- We pick
$$Q = \sqrt{N/K}.$$
- Roughly speaking we have
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Sketch of proof

- Next we apply Voronoi summation formulae.
- $GL(3)$ Voronoi transforms

$$\sum_{n\sim N}\lambda_F(n,1)n^{iv}e\left(\frac{an}{q}\right)\rightsquigarrow\sum_{n\sim K^{3/2}N^{1/2}}\lambda_F(1,n)S(-\bar{a},n;q)\int(\dots),$$

and gives a saving of $N^{1/4}/K^{3/4}$.

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and gives a saving of $(NK)^{1/2}/t$.

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Sketch of proof

- Next we see that the character sum

$$\sum_{a \bmod q}^{\star} S(-\bar{a}, n; q) e\left(\frac{\bar{a}m}{q}\right) \approx qe\left(-\frac{\bar{m}n}{q}\right),$$

  and we save $\sqrt{Q}$.
- We also save $\sqrt{K}$ in the integral over v.
- It remains to save t (plus extra) in the sum

$$\sum_{q \sim Q} \sum_{n \sim K^{3/2} N^{1/2}} \lambda_F(1, n) \sum_{m \sim t^2/K} \lambda_f(m)\, e\left(-\frac{\bar{m}n}{q}\right) \mathcal{I}$$

Sketch of proof

- Next we see that the character sum

$$\sum_{a \bmod q}^{\star} S(-\bar{a}, n; q) e\left(\frac{\bar{a}m}{q}\right) \approx qe\left(-\frac{\bar{m}n}{q}\right),$$

  and we save $\sqrt{Q}$.
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- Now we apply Cauchy, and our job reduces to saving t^2 (plus extra) in

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- Notice the structural advantage. If one had a generic character sum in place of $e(-\bar{m}n/q)$ then one would save $K^{3/2}N^{1/2}/QK^{1/2}$ which would be sufficient if $K > t^{4/3}$!

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Proof of Theorem 2.

- F an $SL(3,\mathbb{Z})$ form with Langlands parameter $\alpha_j \sim T$ (generic).
- Approximate functional equation reduces the problem to showing cancellation in
$$S(N) \approx \sum_{n \sim N} \lambda_F(n,1)\lambda_f(n)$$
with $N \sim T^3$.
- We write
$$S(N) = \sum\sum_{n,m \sim N} \lambda_F(n,1)\lambda_f(m)\delta(n-m)$$
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- We use modular forms $S_k(q,\psi)$ with $k \sim K$ and $q \sim Q$. Petersson formula gives an expansion for $\delta(n-m)$

$$\frac{1}{Q^2KV}\int_{v\sim V}\left(\frac{n}{m}\right)^{iv}\sum_{\substack{k\sim K\\ k \text{ odd}}}\sum_{q\sim Q}\sum_{\psi \bmod q}\sum_{g\in H_k(q,\psi)} w_g^{-1}\lambda_g(n)\overline{\lambda_g(m)}$$

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$$\sum_{n\sim T^3}\lambda_F(n,1)\lambda_g(n)n^{iv}\rightsquigarrow i^{-3k}g_\psi^3 q^{3iv}\,\gamma_{3\times 2}\sum_{n\sim (TQ)^3}\lambda_F(n,1)\overline{\lambda_g(nq^3)}n^{-iv}$$

We take $V\ll K\ll T$.

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$$\gamma=(e/2)^{2iv}|\alpha_1\alpha_2\alpha_3|^{2iv}k^{-4iv}e^{\frac{ik^2}{4}\sum_{j=1}^{3}\frac{1}{\alpha_j}}$$

- The k sum and v integral:

$$\int_{v\sim V}\sum_{k\sim K}\rightsquigarrow e\left(\frac{2\sqrt{mn}}{c\sqrt{q}}+\frac{|\alpha_1\alpha_2\alpha_3|\sqrt{qm}}{\sqrt{n}}\Big(\sum_{j=1}^{3}\frac{1}{\alpha_j}+\frac{c\sqrt{q}}{2\pi\sqrt{mn}}\Big)\right).$$

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- Next we save Q in the ψ sum

$$\sum_{\psi} S_{\psi}(nq, m; cq) g_{\psi} \approx S(n, m\bar{q}; c)\, q^2 \mathbf{1}_{c \equiv m \bmod q}$$

- So far we have saved

$$\frac{1}{Q^{3/2}} \frac{T^3}{QK^2} \frac{\sqrt{QK}}{\sqrt{Q^2V}} \sqrt{K}\sqrt{V} Q = \frac{T^3}{Q^2K}$$

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where $c = m + rq$.

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- Next we apply Voronoi summation. To match the phase of

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with the size of the spectral parameter, we take $V=K^2/T$. Now Voronoi gives

$$\sum_{n\sim(TQ)^3}\lambda_F(n,1)S(n,m\bar{q};c)e\Big(\frac{2\sqrt{mn}}{c\sqrt{q}}+\dots\Big)$$

$$\rightsquigarrow\sum_{n\sim(QV)^3}\lambda_F(n,1)e\Big(\frac{\bar{m}qn}{c}\Big)\int(\dots)$$

which saves $(T/V)^{3/2}=(T/K)^3$. It remains to save $Q^2K^4/T^3\approx K^2$.

Sketch of proof

- Now we apply Cauchy inequality, and our job reduces to saving Q^4K^8/T^6 (plus extra) in

$$\sum_{n\sim(QV)^3}\left|\sum_{q\sim Q}\sum_{m\sim Q^2K^4/T^3}\sum_{r\sim QV}\lambda_f(m)\,e\left(\frac{\bar{m}n}{r}\right)\,\mathcal{I}\right|^2$$

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Thank You!