Bessel functions outside $GL(2)$.

Jack Buttcane

University of Maine
Orono, ME

05 June 2020

- What are Bessel functions outside of $GL(2)$?
- What do we know about these Bessel functions?
- Work in progress.
- Applications.

What are Bessel functions outside of $GL(2)$?

Let

- $G = PGL(n, \mathbb{R}) = GL(n, \mathbb{R})/\mathbb{R}^\times$,
- $U(\mathbb{R})$ the upper triangular unipotent matrices,
- Y the diagonal matrices,
- Y^+ the positive diagonal matrices,
- $K = PO(n, \mathbb{R})$.

What are Bessel functions outside of $GL(2)$?

Define characters

- of Y:

$$p_\mu\begin{pmatrix} a_1 & & \\ & \ddots & \\ & & a_n \end{pmatrix} = \prod_{i=1}^{n} |a_i|^{\mu_i}, \qquad \chi_\delta\begin{pmatrix} a_1 & & \\ & \ddots & \\ & & a_n \end{pmatrix} = \prod_{i=1}^{n} \operatorname{sgn}(a_i)^{\delta_i},$$

What are Bessel functions outside of $GL(2)$?

Define characters

- of Y:
$$p_\mu\begin{pmatrix} a_1 & & \\ & \ddots & \\ & & a_n \end{pmatrix} = \prod_{i=1}^n |a_i|^{\mu_i}, \qquad \chi_\delta\begin{pmatrix} a_1 & & \\ & \ddots & \\ & & a_n \end{pmatrix} = \prod_{i=1}^n \operatorname{sgn}(a_i)^{\delta_i},$$
- of $U(\mathbb{R})$: $\psi_y(x) = \psi_I(yxy^{-1}) = e\left(y_1x_1 + \ldots + y_{n-1}x_{n-1}\right)$,
$$y = \begin{pmatrix} y_1\cdots y_{n-1} & & & & \\ & y_1\cdots y_{n-2} & & & \\ & & \ddots & & \\ & & & y_1 & \\ & & & & 1 \end{pmatrix}, \qquad x = \begin{pmatrix} 1 & x_n & & * \\ & \ddots & \ddots & \\ & & 1 & x_1 \\ & & & 1 \end{pmatrix},$$

What are Bessel functions outside of $GL(2)$?

Define characters

- of Y:
$$p_\mu\begin{pmatrix} a_1 & & \\ & \ddots & \\ & & a_n \end{pmatrix} = \prod_{i=1}^n |a_i|^{\mu_i}, \qquad \chi_\delta\begin{pmatrix} a_1 & & \\ & \ddots & \\ & & a_n \end{pmatrix} = \prod_{i=1}^n \operatorname{sgn}(a_i)^{\delta_i},$$
- of $U(\mathbb{R})$: $\psi_y(x) = \psi_I(yxy^{-1}) = e\left(y_1x_1 + \ldots + y_{n-1}x_{n-1}\right)$,
$$y = \begin{pmatrix} y_1\cdots y_{n-1} & & & & \\ & y_1\cdots y_{n-2} & & & \\ & & \ddots & & \\ & & & y_1 & \\ & & & & 1 \end{pmatrix}, \qquad x = \begin{pmatrix} 1 & x_n & & * \\ & \ddots & \ddots & \\ & & 1 & x_1 \\ & & & 1 \end{pmatrix},$$
- normalizations: $\rho = \left(\frac{n-1}{2}, \frac{n-3}{2}, \ldots, -\frac{n-1}{2}\right)$, use $p_{\rho+\mu}$, assume $\displaystyle\sum_{i=1}^n \mu_i = 0$, extend to G by $p_{\rho+\mu}(xyk) = p_{\rho+\mu}(y)$.

What are Bessel functions outside of $GL(2)$?

Spherical Jacquet-Whittaker function:

$$W(g,\mu,\psi) = \int_N p_{\rho+\mu}(w_l x g)\overline{\psi(x)}dx, \qquad w_l = \begin{pmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{pmatrix}.$$

What are Bessel functions outside of $GL(2)$?

Spherical Jacquet-Whittaker function:

$$W(g,\mu,\psi) = \int_N p_{\rho+\mu}(w_l x g)\overline{\psi(x)}dx, \qquad w_l = \begin{pmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{pmatrix}.$$

Since $p_{\rho+\mu}$ is an eigenfunction of the Casimir operators, so is the Whittaker function, with the same eigenvalues.

Call μ the (Harish-Chandra/Langlands) spectral parameters.

What are Bessel functions outside of $GL(2)$?

Conjecture (Interchange of Integrals)

If $f(\mu)$ is homorphic with rapid decay on an open tube domain containing $\mathrm{Re}(\mu)=0$, and $y, t \in G$, then

$$\int_{U(\mathbb{R})} \int_{\mathrm{Re}(\mu)=0} f(\mu) W(yw_l xt, \mu, \psi_I) d\mu \, \overline{\psi_I(x)} dx$$
$$= \int_{\mathrm{Re}(\mu)=0} f(\mu) \widetilde{K}_{w_l}(y, t, \mu) d\mu,$$

where

$$\widetilde{K}_{w_l}(y,t,\mu) := \lim_{R\to\infty} \int_{U(\mathbb{R})} h\left(\frac{\|x\|}{R}\right) W(yw_l xt, \mu, \psi_I) \overline{\psi_I(x)} dx$$

is smooth in t and y, entire in μ and polynomially bounded in the coordinates of $y, t, y^{-1}, t^{-1}, \mu$ for $\mathrm{Re}(\mu)$ in some fixed, compact set and h smooth and compactly supported with $h(0)=1$.

Convergence!

What are Bessel functions outside of $GL(2)$?

It follows from work of Shalika that

$$\widetilde{K}_{w_l}(y, t, \mu) = K_{w_l}(y, \mu) W(t, \mu, \psi_I)$$

for some function $K_{w_l}(y, \mu)$, and this is called the long-element, spherical Bessel function for $GL(n)$.

What are Bessel functions outside of $GL(2)$?

It follows from work of Shalika that

$$\widetilde{K}_{w_l}(y, t, \mu) = K_{w_l}(y, \mu) W(t, \mu, \psi_I)$$

for some function $K_{w_l}(y, \mu)$, and this is called the long-element, spherical Bessel function for $GL(n)$.

$K_{w_l}(y, \mu)$ has the bi-$U(\mathbb{R})$-invariance property

$$K_{w_l}(xy, \mu) = K_{w_l}(y(w_l x w_l), \mu) = \psi_I(x) K_{w_l}(y, \mu),$$

and is an eigenfunction of the Casimir operators with eigenvalues matching $p_{\rho+\mu}$.

What are Bessel functions outside of $GL(2)$?

It follows from work of Shalika that

$$\widetilde{K}_{w_l}(y,t,\mu) = K_{w_l}(y,\mu) W(t,\mu,\psi_I)$$

for some function $K_{w_l}(y,\mu)$, and this is called the long-element, spherical Bessel function for $GL(n)$.

$K_{w_l}(y,\mu)$ has the bi-$U(\mathbb{R})$-invariance property

$$K_{w_l}(xy,\mu) = K_{w_l}(y(w_l x w_l),\mu) = \psi_I(x) K_{w_l}(y,\mu),$$

and is an eigenfunction of the Casimir operators with eigenvalues matching $p_{\rho+\mu}$.

There are Bessel functions $K_w(y,\mu,\delta)$ attached to the other elements of the Weyl group and to non-spherical $\chi_\delta \neq 1$, as well.

What do we know about these Bessel functions?

Conjecture (Asymptotics)

Dropping the μ integral and ignoring issues of convergence, replacing the Whittaker function in the definition of $K_w(y, \mu, \delta)$ with its first-term asymptotics as $Y \ni y \to 0$ yields the first-term asymptotics of $K_w(y, \mu, \delta)$.

What do we know about these Bessel functions?

Conjecture (Asymptotics)

Dropping the μ integral and ignoring issues of convergence, replacing the Whittaker function in the definition of $K_w(y, \mu, \delta)$ with its first-term asymptotics as $Y \ni y \to 0$ yields the first-term asymptotics of $K_w(y, \mu, \delta)$.

Conjecture (Uniqueness)

The function $K_w(y, \mu, \delta)$ is uniquely determined by its first-term asymptotics, bi-$U(\mathbb{R})$-invariance and eigenvalues under the Casimir operators.

What do we know about these Bessel functions?

The uniqueness conjecture is true in the long element case:

- $p_{-\rho}(g) K_{w_l}(gg^T, \mu, \delta)$ is the spherical Whittaker function $W(g, 2\mu, \psi_I^2)$, up to a function $C(\mu, \delta)$

What do we know about these Bessel functions?

The uniqueness conjecture is true in the long element case:

- $p_{-\rho}(g)K_{w_l}(gg^T, \mu, \delta)$ is the spherical Whittaker function $W(g, 2\mu, \psi_I^2)$, up to a function $C(\mu, \delta)$
- so for $y \in Y^+$, $K_{w_l}(y, \mu, \delta) = C(\mu, \delta)p_{\rho/2}(y)W(\sqrt{y}, 2\mu, \psi_I^2)$

What do we know about these Bessel functions?

The uniqueness conjecture is true in the long element case:

- $p_{-\rho}(g) K_{w_l}(gg^T, \mu, \delta)$ is the spherical Whittaker function $W(g, 2\mu, \psi_I^2)$, up to a function $C(\mu, \delta)$
- so for $y \in Y^+$, $K_{w_l}(y, \mu, \delta) = C(\mu, \delta) p_{\rho/2}(y) W(\sqrt{y}, 2\mu, \psi_I^2)$
- from Hashizume's solution of the Whittaker differential equations:

$$K_{w_l}(y, \mu, \delta) = \sum_{w \in W} C_w(\mu, \delta, \operatorname{sgn}(y)) J_{w_l}(y, \mu^w),$$

$$J_{w_l}(y, \mu) = p_{\rho+\mu}(y) \sum_{m \in \mathbb{N}_0^{n-1}} a_m(\mu)(4\pi^2 y)^m$$

$$\left(\sum_{j=0}^{n-1} (m_j - m_{j+1})^2 - \sum_{j=1}^{n-1} m_{n-j}(\mu_{j+1} - \mu_j) \right) a_m(\mu) = \sum_{j=1}^{n-1} a_{m-e_j}(\mu)$$

$$a_0(\mu) = 1, \qquad m_0 = m_n = 0.$$

What do we know about these Bessel functions?

The uniqueness conjecture is true in the long element case:

- $p_{-\rho}(g) K_{w_l}(gg^T, \mu, \delta)$ is the spherical Whittaker function $W(g, 2\mu, \psi_I^2)$, up to a function $C(\mu, \delta)$
- so for $y \in Y^+$, $K_{w_l}(y, \mu, \delta) = C(\mu, \delta) p_{\rho/2}(y) W(\sqrt{y}, 2\mu, \psi_I^2)$
- from Hashizume's solution of the Whittaker differential equations:

$$K_{w_l}(y, \mu, \delta) = \sum_{w \in W} C_w(\mu, \delta, \operatorname{sgn}(y)) J_{w_l}(y, \mu^w),$$

$$J_{w_l}(y, \mu) = p_{\rho+\mu}(y) \sum_{m \in \mathbb{N}_0^{n-1}} a_m(\mu)(4\pi^2 y)^m$$

$$\left(\sum_{j=0}^{n-1}(m_j - m_{j+1})^2 - \sum_{j=1}^{n-1} m_{n-j}(\mu_{j+1} - \mu_j)\right) a_m(\mu) = \sum_{j=1}^{n-1} a_{m-e_j}(\mu)$$

$$a_0(\mu) = 1, \qquad m_0 = m_n = 0.$$

- The asymptotics determine $C_w(\mu, \delta, \operatorname{sgn}(y))$.

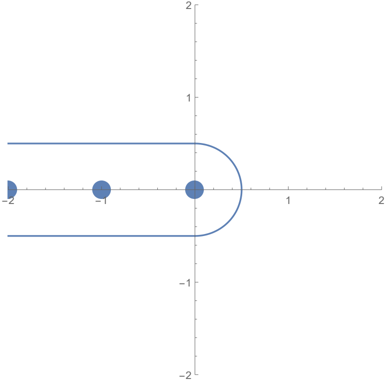
What do we know about these Bessel functions?

Stade has given (multidimensional) Mellin-Barnes integrals for the spherical Whittaker functions, so we know the $a_m(\mu)$ are multidimensional hypergeometric functions at 1.

What do we know about these Bessel functions?

Stade has given (multidimensional) Mellin-Barnes integrals for the spherical Whittaker functions, so we know the $a_m(\mu)$ are multidimensional hypergeometric functions at 1.

Then $J_{w_l}(y, \mu)$ has a "bad" (multidimensional) Mellin-Barnes integral over a contour(s)



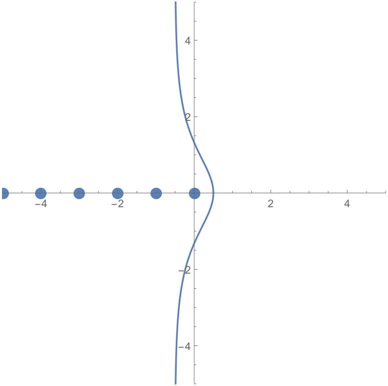
What do we know about these Bessel functions?

The $J_{w_l}(y,\mu)$ function has exponential growth at $y_i = \infty$, so the (multidimensional) contour(s) cannot be unbent.

What do we know about these Bessel functions?

The $J_{w_l}(y,\mu)$ function has exponential growth at $y_i = \infty$, so the (multidimensional) contour(s) cannot be unbent.

In the sum over the Weyl group, there must be cancellation in the integrand at $\pm i\infty$, so we can unbend the contour(s), but can we find that cancellation algebraically?



What do we know about these Bessel functions?

Conjecture

For $y \in Y$, let $v = \operatorname{sgn}(y)$ and define

$$W_v = \{w \in W \,|\, wv = vw\},$$

then

$$\sum_{w \in W_v} C_w(\mu, \delta, v) J_{w_l}(y, \mu^w)$$

has a Mellin-Barnes integral (using vertical contours).

What do we know about these Bessel functions?

What about the other Weyl elements?

What do we know about these Bessel functions?

What about the other Weyl elements?

- The long element case of the interchange of integrals is the most difficult (highest dimension), the others are only difficult because there are many things to keep track of.

What do we know about these Bessel functions?

What about the other Weyl elements?

- The long element case of the interchange of integrals is the most difficult (highest dimension), the others are only difficult because there are many things to keep track of.
- The positive sign case is no longer the spherical Whittaker function, so we lose Hashizume's (and Harish-Chandra's) and Stade's work, so we have to work out the differential equations and power series solutions ourselves.

What do we know about these Bessel functions?

What about the other Weyl elements?

- The long element case of the interchange of integrals is the most difficult (highest dimension), the others are only difficult because there are many things to keep track of.
- The positive sign case is no longer the spherical Whittaker function, so we lose Hashizume's (and Harish-Chandra's) and Stade's work, so we have to work out the differential equations and power series solutions ourselves.
- The trivial Bessel function is 1; the Voronoi Bessel function (probably) also shows up.

Work in progress.

Theorem (B, in progress)

The interchange of integrals conjecture is true in the spherical and non-spherical cases for all Weyl elements on $GL(2)$, $GL(3)$, $GL(4)$ and $Sp(4)$.

Theorem (B, in progress)

The interchange of integrals conjecture is true in the spherical and non-spherical cases for all Weyl elements on $GL(2)$, $GL(3)$, $GL(4)$ and $Sp(4)$.

Stationary phase trick:

Suppose $\phi(x) = \sum_i \phi_i(x)$ where $C\left|\phi_i'(x)\right| \asymp A_i$ and $C^j\left|\phi_i^{(j)}(x)\right| \ll A_i$.

Blomer-Khan-Young: $\int_{\mathbb{R}} f(x/C)e\left(\phi(x)\right)dx$ is negligible unless

$$C\left|\phi'(x)\right| \leqslant C^\epsilon\sqrt{1+\sum_i A_i}.$$

Work in progress.

Theorem (B, in progress)

The interchange of integrals conjecture is true in the spherical and non-spherical cases for all Weyl elements on $GL(2)$, $GL(3)$, $GL(4)$ and $Sp(4)$.

Stationary phase trick:

Suppose $\phi(x) = \sum_i \phi_i(x)$ where $C\left|\phi_i'(x)\right| \asymp A_i$ and $C^j\left|\phi_i^{(j)}(x)\right| \ll A_i$.

Blomer-Khan-Young: $\displaystyle\int_{\mathbb{R}} f(x/C) e\left(\phi(x)\right) dx$ is negligible unless

$$C\left|\phi'(x)\right| \leqslant C^\epsilon \sqrt{1+\sum_i A_i}.$$

For us, this implies the two largest terms A_i are proportional.

Work in progress.

Theorem (B, in progress)

The interchange of integrals conjecture is true in the spherical and non-spherical cases for all Weyl elements on $GL(2)$, $GL(3)$, $GL(4)$ and $Sp(4)$.

Stationary phase trick:

Suppose $\phi(x) = \sum_i \phi_i(x)$ where $C\left|\phi_i'(x)\right| \asymp A_i$ and $C^j\left|\phi_i^{(j)}(x)\right| \ll A_i$.

Blomer-Khan-Young: $\int_{\mathbb{R}} f(x/C) e\left(\phi(x)\right) dx$ is negligible unless

$$C\left|\phi'(x)\right| \leqslant C^\epsilon \sqrt{1 + \sum_i A_i}.$$

For us, this implies the two largest terms A_i are proportional.

On GL(4), there are 6 x's and 6 C's, so knowing that two terms are proportional reduces the number of independent C's.

Work in progress.

Theorem (B, in progress)

The interchange of integrals conjecture is true in the spherical and non-spherical cases for all Weyl elements on $GL(2)$, $GL(3)$, $GL(4)$ and $Sp(4)$.

Stationary phase trick:

Suppose $\phi(x) = \sum_i \phi_i(x)$ where $C\left|\phi_i'(x)\right| \asymp A_i$ and $C^j\left|\phi_i^{(j)}(x)\right| \ll A_i$.

Blomer-Khan-Young: $\displaystyle\int_{\mathbb{R}} f(x/C)e\left(\phi(x)\right)dx$ is negligible unless

$$C\left|\phi'(x)\right| \leqslant C^\epsilon\sqrt{1+\textstyle\sum_i A_i}.$$

For us, this implies the two largest terms A_i are proportional.

On GL(4), there are 6 x's and 6 C's, so knowing that two terms are proportional reduces the number of independent C's.

Can reduce the 6 parameter problem to two separate one parameter problems for the low, low price of computing the null spaces of 10.5 million matrices, 64 times over. Mathematica!

Work in progress.

Theorem (B, in progress)

Assuming the interchange of integrals, and under some mild assumptions on the meromorphic continuation of the Mellin transform of the Whittaker function, the asymptotics conjecture is true.

Work in progress.

Theorem (B, in progress)

Assuming the interchange of integrals, and under some mild assumptions on the meromorphic continuation of the Mellin transform of the Whittaker function, the asymptotics conjecture is true.

The assumptions are that the Mellin expansion of the Whittaker function, shifted slightly to the left of the zero line, converges absolutely and the integrand (i.e. the Mellin transform) is differentiable there. The difference equations of Dietmar suffice to prove this.

Work in progress.

Theorem (B, in progress)

Assuming the interchange of integrals, and under some mild assumptions on the meromorphic continuation of the Mellin transform of the Whittaker function, the asymptotics conjecture is true.

The assumptions are that the Mellin expansion of the Whittaker function, shifted slightly to the left of the zero line, converges absolutely and the integrand (i.e. the Mellin transform) is differentiable there. The difference equations of Dietmar suffice to prove this.

Corollary

The asymptotics conjecture is true for all Weyl elements in the spherical and non-spherical cases:

- on $GL(2)$, $GL(3)$, $GL(4)$ and $Sp(4)$ unconditionally,
- on $GL(n)$, assuming the interchange of integrals conjecture.

Applications - Spectral Kuznetsov trace formula.

Take the Fourier coefficient of a Poincaré series using Wallach's Whittaker inversion formula and Langland's spectral expansion:

$$\int_{\mathcal{B}(\sigma)} \rho_\xi(n)\overline{\rho_\xi(m)} f(\mu_\xi) W(t,\mu_\xi,\chi,\sigma,\psi_I) d\xi$$
$$= \sum_{w \in W} \sum_{c \in A(\mathbb{Z})} p_\rho(c) S_w(\psi_m,\psi_n,c) H_w(f; mcwn^{-1}w^{-1}),$$

Applications - Spectral Kuznetsov trace formula.

Take the Fourier coefficient of a Poincaré series using Wallach's Whittaker inversion formula and Langland's spectral expansion:

$$\int_{\mathcal{B}(\sigma)} \rho_\xi(n)\overline{\rho_\xi(m)} f(\mu_\xi) W(t,\mu_\xi,\chi,\sigma,\psi_I)\, d\xi$$
$$= \sum_{w\in W} \sum_{c\in A(\mathbb{Z})} p_\rho(c) S_w(\psi_m,\psi_n,c) H_w(f; mcwn^{-1}w^{-1}),$$

$$H_w(f;y) = p_{-\rho}(y) \int_{\mathfrak{a}(\sigma)} K_w(y,\mu,\chi) f(\mu) W(t,\mu,\chi,\sigma,\psi_I)\, d^*\mu$$

Applications - Spectral Kuznetsov trace formula.

Take the Fourier coefficient of a Poincaré series using Wallach's Whittaker inversion formula and Langland's spectral expansion:

$$\int_{\mathcal{B}(\sigma)} \rho_\xi(n)\overline{\rho_\xi(m)} f(\mu_\xi) W(t, \mu_\xi, \chi, \sigma, \psi_I) d\xi$$
$$= \sum_{w \in W} \sum_{c \in A(\mathbb{Z})} p_\rho(c) S_w(\psi_m, \psi_n, c) H_w(f; mcwn^{-1}w^{-1}),$$

$$H_w(f; y) = p_{-\rho}(y) \int_{\mathfrak{a}(\sigma)} K_w(y, \mu, \chi) f(\mu) W(t, \mu, \chi, \sigma, \psi_I) d^*\mu$$

General Stone-Weierstrass-type argument to get rid of the spare Whittaker function?
Still needs Stade's formula to convert to Hecke eigenvalues.

More interchanges of integrals (T_T)

Prove the Fourier transform of the zonal spherical function:

$$\int_{U(\mathbb{R})} h_{\mu,\chi,\sigma}(x)\overline{\psi_t(x)}dx = \kappa \operatorname{Tr}\left(\overline{W(I,-\overline{\mu},\chi,\sigma,\psi_{t^\iota})}^T W(I,\mu,\chi,\sigma,\psi_{t^\iota})\right)$$

More interchanges of integrals (T_T)

Prove the Fourier transform of the zonal spherical function:

$$\int_{U(\mathbb{R})} h_{\mu,\chi,\sigma}(x)\overline{\psi_t(x)}dx = \kappa \operatorname{Tr}\left(\overline{W(I,-\overline{\mu},\chi,\sigma,\psi_{t^\iota})}^T W(I,\mu,\chi,\sigma,\psi_{t^\iota})\right)$$

Get Bessel expansion:

$$f(y) = \sum_{\chi,\sigma} \int_{\mathfrak{a}(\sigma)} K_{w_l}(y,\mu,\chi) \int_A f(y')\overline{K_{w_l}(y',\mu)}dy'\, d^\dagger\mu.$$

More interchanges of integrals (T_T)

Prove the Fourier transform of the zonal spherical function:

$$\int_{U(\mathbb{R})} h_{\mu,\chi,\sigma}(x)\overline{\psi_t(x)}dx = \kappa \operatorname{Tr}\left(\overline{W(I,-\overline{\mu},\chi,\sigma,\psi_{t^\iota})}^T W(I,\mu,\chi,\sigma,\psi_{t^\iota})\right)$$

Get Bessel expansion:

$$f(y) = \sum_{\chi,\sigma} \int_{\mathfrak{a}(\sigma)} K_{w_l}(y,\mu,\chi) \int_A f(y')\overline{K_{w_l}(y',\mu)}dy'\, d^\dagger\mu.$$

Get arithmetic Kuznetsov formula:

$$\sum_{c\in A(\mathbb{Z})} p_\rho(c) S_{w_l}(\psi_m,\psi_n,c) F(mcw_l n^{-1} w_l^{-1})$$

$$= \sum_{\chi,\sigma} \int_{\mathcal{B}(\sigma)} \widehat{F}(\mu_\xi,\chi) \frac{\lambda_\xi(n)\overline{\lambda_\xi(m)}}{L(1,\operatorname{Ad}^2\xi)} d\xi,$$

$$\widehat{F}(\mu,\chi) = \frac{1}{|Y/Y^+|} \int_Y p_\rho(y) F(y) \overline{K_{w_l}(y,\mu,\chi)} dy.$$

Applications - Arithmetically-weighted Weyl laws.

From the Spectral Kuznetsov formula, the Mellin-Barnes integrals of the Bessel functions and good bounds for the Kloosterman sums, we get

$$\int_{\substack{\mathcal{B}(\sigma) \\ \mu_\xi \in \Omega}} \frac{1}{L(1, \mathrm{Ad}^2\, \xi)} d\xi = \int_\Omega d_{\mathrm{spec}}\mu + O\left(\int_{\partial\Omega + B(0,1)} d_{\mathrm{spec}}\mu \right)$$

Theorem (Blomer/B)

Arithmetically-weighted Weyl laws with error term for $SL(3, \mathbb{Z})$.

On a set $T\Omega$, we can probably improve the radius in the error term from 1 to $(\log T)^{-\delta}$ for some $\delta \in (0, \frac{1}{2})$.

Applications - Subconvexity.

With much work, we get

Theorem (Blomer,B)

For ϕ a cusp form for $SL(3,\mathbb{Z})$ such that μ_ϕ is in "generic position",

1. If ϕ is spherical, then $L(\frac{1}{2},\phi) \ll \|\mu_\phi\|^{\frac{3}{4}-\frac{1}{120000}}$.
2. If ϕ has weight $d \geqslant 3$, then $L(\frac{1}{2},\phi) \ll \|\mu_\phi\|^{\frac{3}{4}-\frac{1}{140000}}$.

Generic position: $\mu_i, \mu_i - \mu_j \asymp \|\mu\|$.

Applications - Smooth sums of Kloosterman sums.

Immediately from the Arithmetic Kuznetsov formula, we get

Theorem (B)

For $F : \mathbb{R}^2 \to \mathbb{C}$ smooth and compactly supported away from $y_i = 0$ and $X_1, X_2 \geqslant 1$,

$$\sum_{\varepsilon \in \{\pm 1\}^2} \sum_{c \in \mathbb{N}^2} \frac{S_{w_l}(\psi_m, \psi_{\varepsilon n}, c)}{c_1 c_2} F\left(\frac{\varepsilon_1 c_1}{X_1}, \frac{\varepsilon_2 c_2}{X_2}\right)$$
$$\ll_{m,n,F,\epsilon} (X_1 X_2)^{\theta+\epsilon} + X_1^{1+\epsilon}/X_2^2 + X_2^{1+\epsilon}/X_1^2.$$

$\theta = \frac{5}{14}$ is the Ramanujan-Selberg parameter due to Kim-Sarnak.

Thank you!