Some extensions of Duke's theorems

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based on joint works with

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Duke's Theorems

Duke's Theorems (1988) are three equidistribution theorems for the set of representations of large integers by a fixed ternary (integer valued) quadratic form $q(a, b, c)$.
Given $d \in \mathbb{Z} - \{0\}$, the set of representation of d by q is

$$\mathcal{R}_q(d) := \{(a, b, c) \in \mathbb{Z}^3, q(a, b, c) = d\}.$$

Given $l \in \mathbb{R}$, the l-level set of q is

$$V_{q,l}(\mathbb{R}) := \{(x, y, z) \in \mathbb{R}^3, \ q(x, y, z) = l\}.$$

For $l = \text{sign}(d)$, one has

$$\frac{1}{|d|^{1/2}} \mathcal{R}_q(d) \subset V_{q,l}(\mathbb{R}).$$

Duke's Theorems

Theorem (Duke)

For q either of the quadratic forms

$$E_3(a,b,c) = -(a^2+b^2+c^2),\ \Delta(a,b,c) = b^2 - 4ac$$

As $d \to \infty$ such that

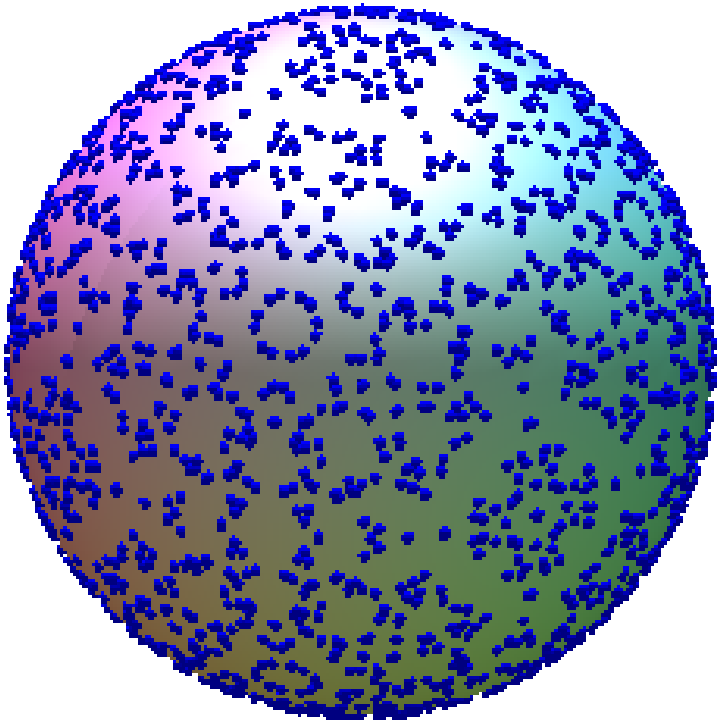
- *if $q = E_3$: $d < 0,\ d \neq 0,1,4 \mod 8$,*
- *if $q = \Delta,\ d \equiv 1 \mod 4$,*

the set $|d|^{-1/2}.\mathcal{R}_q(d)$ become equidistributed on $V_{q,\pm 1}(\mathbb{R})$ ($\pm 1 = \mathrm{sign}(d)$) wrt to the unique (up to scalars) $\mathrm{SO}_q(\mathbb{R})$-invariant measure $\mu_{q,\pm 1}$.

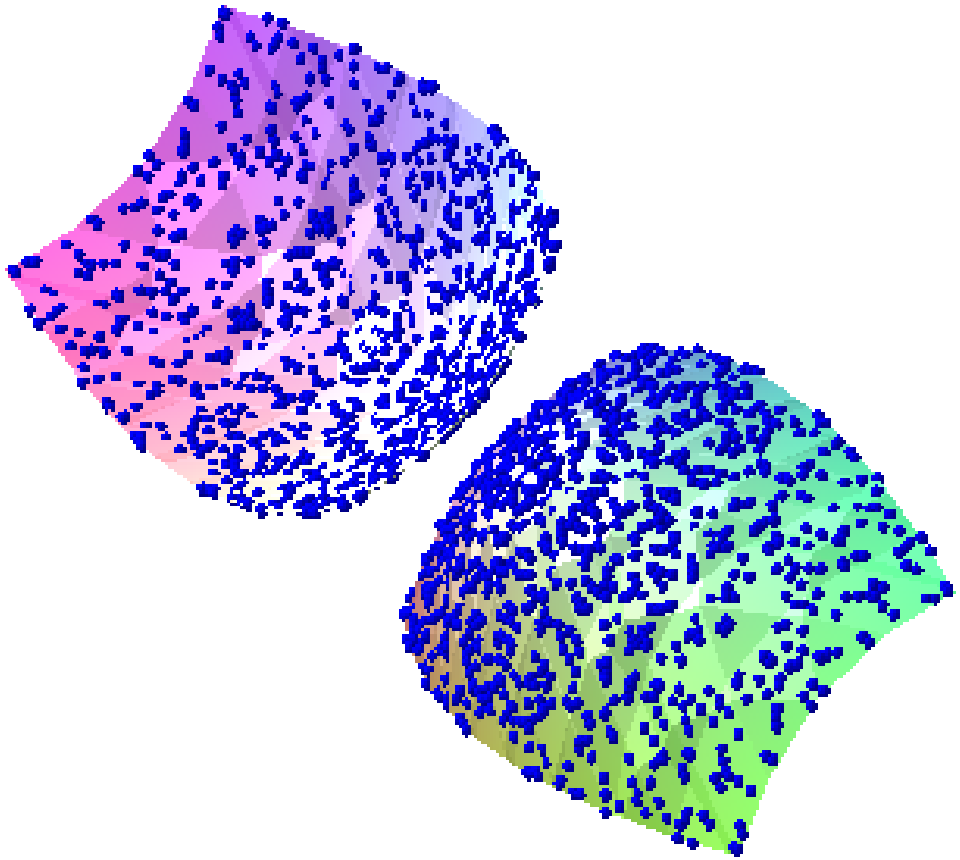
Remark

For E_3 this was also proven by Fomenko-Golubeva at about the same time.

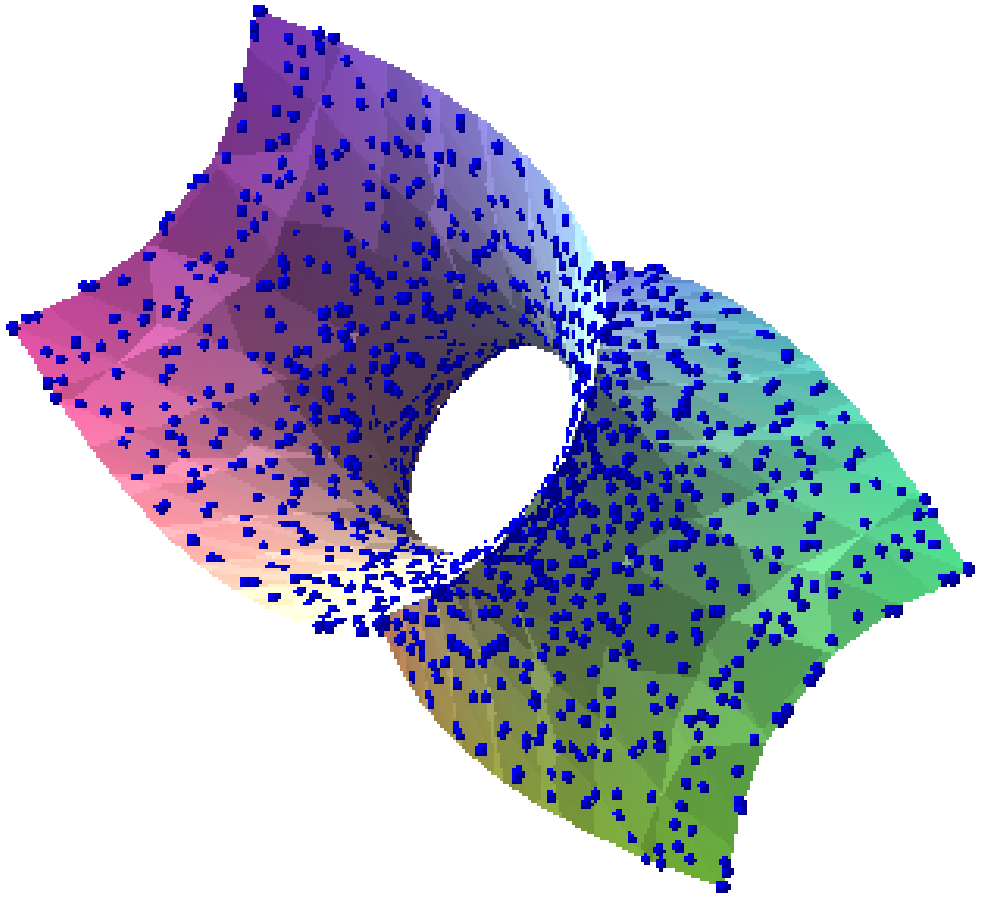
d=-40001



$d = -110003$



$d = 10001$



Duke's Theorems : dual formulation

Set $\mathrm{G} = \mathrm{SO}_q$. Choosing a base point $\mathbf{x} \in V_{q,\pm 1}(\mathbb{R})$ one has

$$V_{q,\pm 1}(\mathbb{R}) \simeq \mathrm{G}(\mathbb{R})/\mathrm{G}_{\mathbf{x}}(\mathbb{R}),\ \mathrm{G}_{\mathbf{x}} = \mathrm{Stab}_{\mathrm{G}}(\mathbf{x})$$

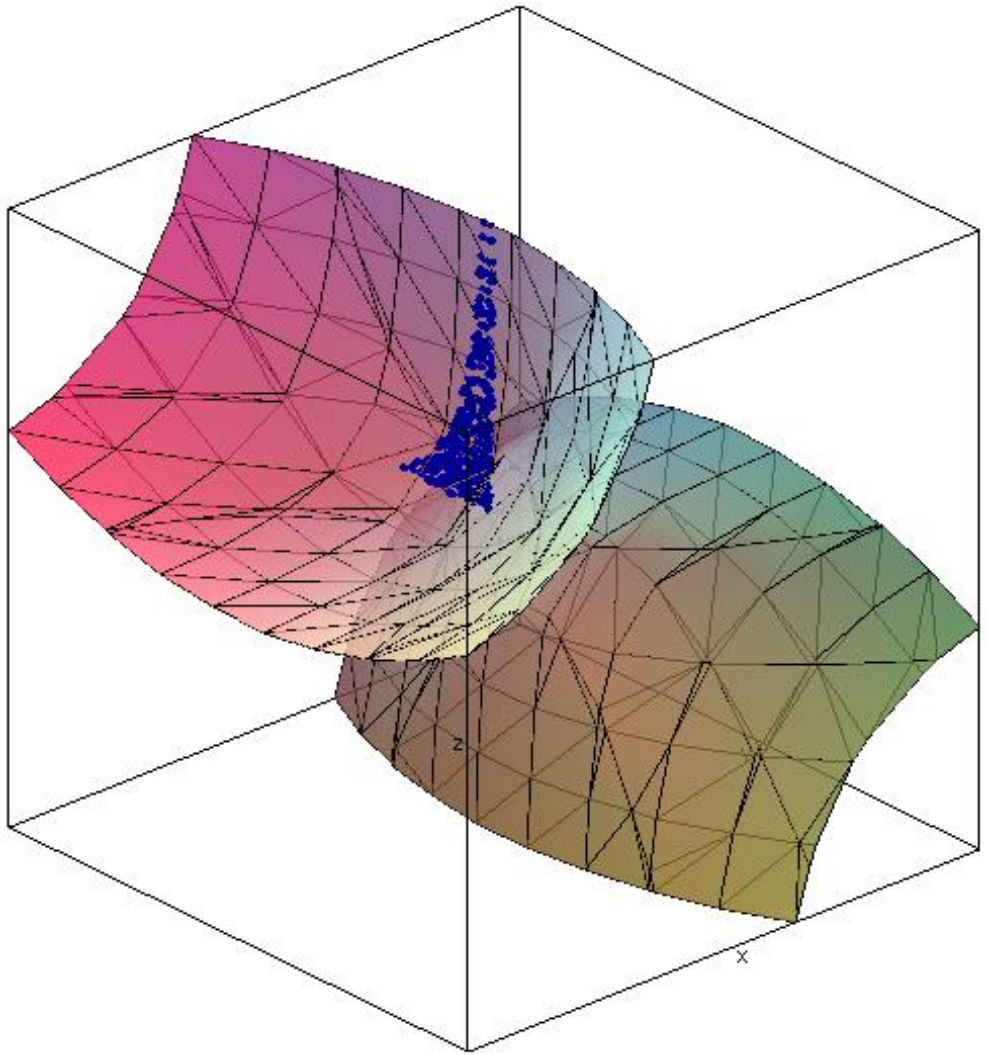
and

$$|d|^{-1/2}(a,b,c) \simeq g_{(a,b,c)}.\mathrm{G}_{\mathbf{x}}(\mathbb{R}).$$

The group $\mathrm{G}(\mathbb{Z})$ acts (on the left) on $\mathcal{R}_q(d)$ and decompose into finitely many orbits (classes)

$$[a,b,c] := \mathrm{G}(\mathbb{Z}).(a,b,c) \simeq \mathrm{G}(\mathbb{Z}).g_{(a,b,c)}.\mathrm{G}_{\mathbf{x}}(\mathbb{R})/\mathrm{G}_{\mathbf{x}}(\mathbb{R})$$

$$d = -194444$$



Duke's Theorems : dual formulation

Theorem (Duality principle)

TFAAE

- *Equidistribution of the finite union of (left)* $\mathrm{G}(\mathbb{Z})$*-orbits*

$$\bigsqcup_{[a,b,c]} \mathrm{G}(\mathbb{Z}).g_{(a,b,c)}\mathrm{G}_{\mathbf{x}}(\mathbb{R})/\mathrm{G}_{\mathbf{x}}(\mathbb{R})$$

  on $\mathrm{G}(\mathbb{R})/\mathrm{G}_{\mathbf{x}}(\mathbb{R}) \simeq V_{q,\pm 1}(\mathbb{R})$.

- *Equidistribution of the finite set of (right)* $\mathrm{G}_{\mathbf{x}}(\mathbb{R})$*-orbits*

$$\{\mathrm{G}(\mathbb{Z})\backslash\mathrm{G}(\mathbb{Z}).g_{(a,b,c)}.\mathrm{G}_{\mathbf{x}}(\mathbb{R}),\ [a,b,c] \in [\mathcal{R}_q(d)]\}$$

  on $\mathrm{G}(\mathbb{Z})\backslash\mathrm{G}(\mathbb{R})$.

Duke's Theorems : dual formulation

- For $q = E_3$ the dual form amounts to equidistribution of collection of circles at the surface of (a quotient of) the 3-sphere (which is a 2-covering of $\mathrm{SO}_q(\mathbb{R})$) and projecting to $\mathbb{R}^3$ we obtain points on the 2-sphere.
- For $q = \Delta$, $\mathrm{SO}_q \simeq \mathrm{PGL}_2$ and $\mathrm{PGL}_2(\mathbb{Z})\backslash \mathrm{PGL}_2(\mathbb{R})$ is identified with the unit tangent bundle of the modular curve

$$X_0(1) = \mathrm{PGL}_2^+(\mathbb{Z})\backslash\mathbb{H} \simeq \mathrm{PGL}_2(\mathbb{Z})\backslash \mathrm{PGL}_2(\mathbb{R})/\mathrm{PSO}_2(\mathbb{R})$$

Duke's Theorems : dual formulation

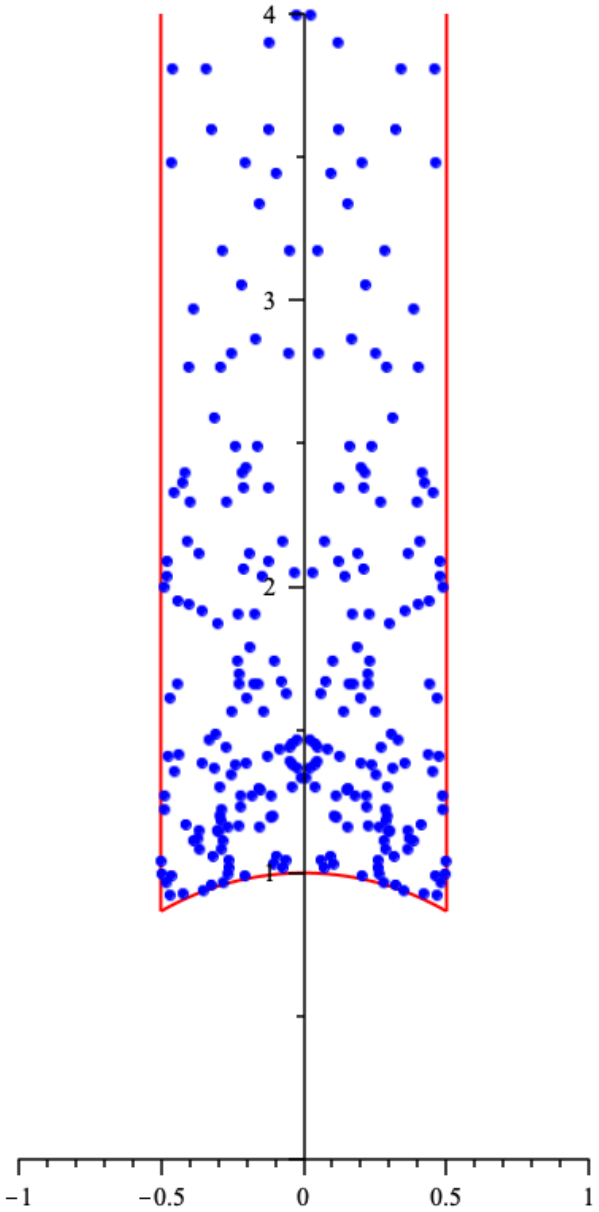
- For $d < 0$, Duke's theorem is equivalent to the equidistribution of Heegner points on the modular surface $X_0(1)$

$$z_{[a,b,c]} = \mathrm{SL}_2(\mathbb{Z}) z_{a,b,c}, \ z = \frac{-b + i|d|^{1/2}}{2a}, \ (a,b,c) \in \mathcal{R}_\Delta(d).$$

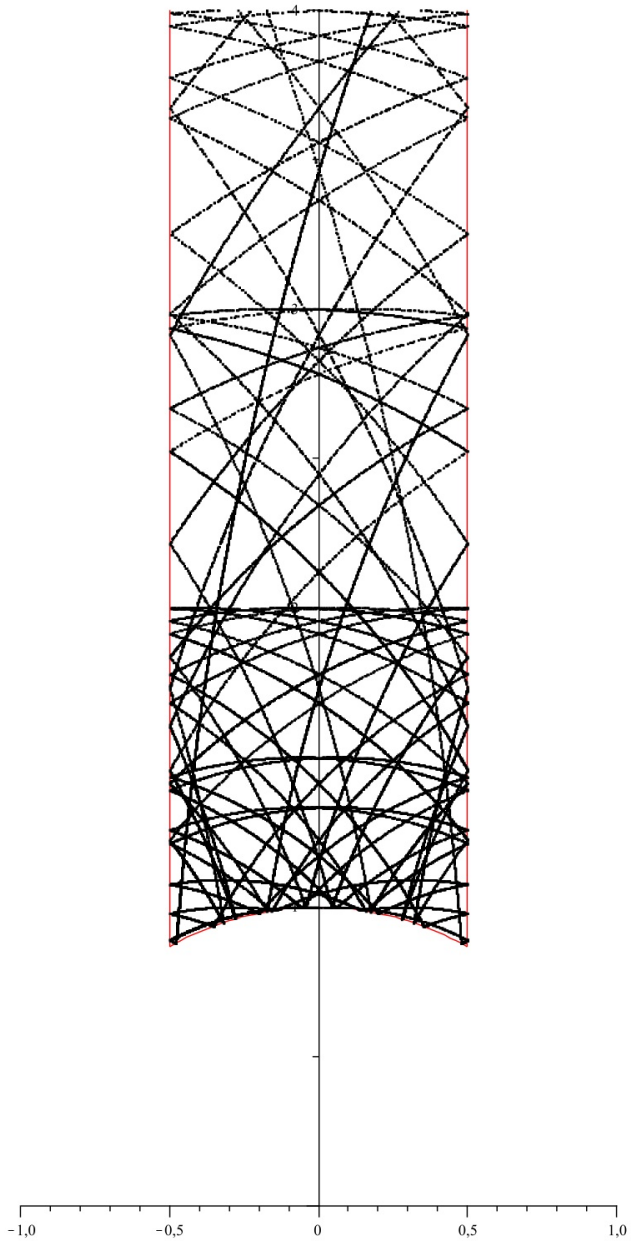
- For $d > 0$, this is equivalent to the equidistribution of closed geodesics $\gamma_{[a,b,c]}$ on the unit tangent bundle of $X_0(1)$ with diametral "end-points"

$$x^{\pm}_{a,b,c} = \frac{-b \pm d^{1/2}}{2a}, \ (a,b,c) \in \mathcal{R}_\Delta(d).$$

$$d = -418916$$



$$d = 2308$$



Duke's Theorems

Analogous equidistribution results hold for *any* integral valued ternary quadratic form q.

Theorem (Duke–Schulze-Pillot)

As $d \to \infty$ such that d is everywhere locally representable by q, $(d, \mathrm{disc}(q)) = 1$, the squarefree part of d avoid a finite set of integers depending on q, the set $|d|^{-1/2}\mathcal{R}_q(d)$ is non-empty and equidistribute on $V_{q,\pm 1}(\mathbb{R})$.

Remark

This solved Hilbert's 11th problem over $\mathbb{Q}$ in the (remaining) case of definite ternary forms : for these the set of genus classes (global isometry classes of forms locally isometric to q) is of size > 1 as soon as $\mathrm{disc}(q) \gg 1$.

Duke's Theorems : proofs

The proofs use the dual formulation. Goal : show that the Weyl sums attached to (non-characters) automorphic forms satisfy :

$$\frac{1}{\mathrm{vol}(d)} \sum_{[a,b,c]} \int_{\mathrm{G}_{\mathbf{x}}(\mathbb{R})} \varphi(g_{(a,b,c)}.t)\, dt \to 0, \;\; d \to \infty,$$

$$\mathrm{vol}(d) = \sum_{[a,b,c]} \int_{\mathrm{G}_{\mathbf{x}}(\mathbb{R})} dt.$$

In Duke's original proof, the Weyl sums are shown to be related to the Fourier coefficients of metaplectic forms (theta series) which have to be bounded non-trivially.
Such bounds were obtained by Iwaniec in the definite case and by Duke in the indefinite case.

Waldspurger's formula & subconvexity

A different and more powerful proof comes from Waldspurger's formula. For this is it useful to first adelize the problem (Clozel-Ullmo) :

- $\exists$ a quat. algebra B, s.t. $(\mathbb{Q}^3, q) \sim (\mathrm{B}^0, \lambda\,\mathrm{Nr}_{\mathrm{B}})$; wlogwma
$$\mathrm{G} = \mathrm{PB} = Z_{\mathrm{B}}^{\times}\backslash \mathrm{B}^{\times}.$$
- Equidistribution takes place on an adelic quotient
$$[\mathrm{G}]/K_f,\ [\mathrm{G}] := \mathrm{G}(\mathbb{Q})\backslash \mathrm{G}(\mathbb{A}),\ K_f \subset \mathrm{G}(\mathbb{A}_f).$$
- The set of orbits$^{(\star)}$ $\bigsqcup_{[a,b,c]} \mathrm{G}(\mathbb{Z})\backslash \mathrm{G}(\mathbb{Z}).g_{(a,b,c)}\mathrm{G}_{\mathbf{x}}(\mathbb{R})$ correspond to the projection to $[\mathrm{G}]/K_f$ of an adelic orbit
$$[\mathrm{T}.g] = \mathrm{T}(\mathrm{PB})\backslash \mathrm{T}(\mathbb{A}).g \subset [\mathrm{G}],\ \mathrm{g} = \mathrm{g}_{\infty}.\mathrm{g}_{\mathrm{f}} \in \mathrm{G}(\mathbb{A}).$$
$$\mathrm{T} = \mathrm{G}_{(a,b,c)} \simeq \mathrm{res}_{K/\mathbb{Q}}\mathbb{G}_m/\mathbb{G}_m,\ K = \mathbb{Q}(\sqrt{-d/\lambda}).$$

Waldspurger's formula & subconvexity

Let B/F, $\mathrm{G} = \mathrm{PB}$, $\mathrm{T} = \mathrm{G}_{\mathbf{x}}$, $\mathbf{x} \in \mathrm{B}^0$, $-\mathrm{Nr}_{\mathrm{B}}(\mathbf{x}) = d = x^2 \notin F^2$, $K = F(x)$.

Theorem (Waldspurger)

Let $\varphi : [\mathrm{G}] \to \mathbb{C}$ is an (factorable) automorphic form which is not a character, $\pi = \langle \mathrm{G}(\mathbb{A}).\varphi \rangle \in \mathrm{Aut}(\mathrm{G})$ the autorep it generates, $\pi^{JL} \in \mathrm{Aut}(\mathrm{PGL}_2)$ the Jacquet-Langlands correspondant and π_K^{JL} its base change to $\mathrm{PGL}_{2,K}$, one has

$$\frac{|\int_{[\mathrm{T}]} \varphi(tg)\,dt|^2}{\langle \varphi, \varphi \rangle} = c.\frac{L(\pi_K^{JL}, 1/2)}{\mathrm{Nr}_{F/\mathbb{Q}}(d_K)^{1/2}} \prod_v \int_{\mathrm{T}(\mathbb{Q}_v)} \frac{\langle g_v.\varphi_v, t_v.g_v.\varphi_v \rangle}{\langle \varphi_v, \varphi_v \rangle}\,dt_v.$$

Here $c = c_{K,\varphi} > 0$ satisfies $c = |d_K|^{o(1)}$.

Waldspurger's formula & subconvexity

To the adelic torus orbit $[\mathrm{T}.g] = \mathrm{T}(F)\backslash\mathrm{T}(\mathbb{A}).g$ is associated a *discriminant* (related to the volume of the intersection of $\mathrm{T}(\mathbb{A}).g$ with a fixed compact of $\mathrm{G}(\mathbb{A})$).

$$D_{[\mathrm{T}.g]} = \prod_v D_{\mathrm{T}(F_v).g_v} > 0.$$

Theorem (Einsiedler-Lindenstrauss-M-Venkatesh)

$\exists \eta > 0$ *absolute s.t.*

$$\frac{\int_{[\mathrm{T}]} \varphi(tg)\,dt}{\langle \varphi, \varphi \rangle^{1/2}} \ll_{F,\varphi} D_{[\mathrm{T}.g]}^{-\eta}$$

Waldspurger's formula & subconvexity

The proof consists in bounding the righthand side in Waldspurger's formula

1. The local integrals $\int_{\mathrm{T}(\mathbb{Q}_v)} \langle g_v.\varphi_v, t_v.g_v.\varphi_v \rangle dt_v$ are bounded using the *decay of matrix coefficients* (as t_v varies).
2. The global L-function is bounded using *subconvex bounds* (due to DFI for $\mathbb{Q}$ and Venkatesh for F)

$$L(\pi_K^{JL}, 1/2) = L(\pi^{JL}, 1/2)L(\pi^{JL}.\chi_K, 1/2) \ll_\pi \mathrm{Nr}_{F/\mathbb{Q}}(d_K)^{1/2-\eta}.$$

This bound essentially resolves Duke's theorem in general.

Waldspurger's formula & subconvexity

To conclude it remains to deal with the case of characters :

$$\varphi = \chi_{\mathrm{B}} : g \in \mathrm{B}^{\times}(\mathbb{A}) \mapsto \chi(\mathrm{Nr}_{\mathrm{B}}(g)),\ \chi^2 = 1,\ \chi \neq 1.$$

In that case, the period is easy to compute : let K_χ be the quadratic field whose χ is the "Legendre symbol"

$$\int_{[\mathrm{T}]} \chi(\mathrm{Nr}(tg))dt = \chi(\mathrm{Nr}(g)) \int_{[\mathrm{T}]} \chi(\mathrm{Nr}(t))dt = \begin{cases} \chi(\mathrm{Nr}(g)) & \text{if } K = K_\chi \\ 0 & \text{if } K \neq K_\chi \end{cases}.$$

Therefore Duke's theorem hold for the quotient

$$\mathrm{G}(F)\backslash \mathrm{G}(\mathbb{A})/K_f$$

($K_f \subset \mathrm{G}(\mathbb{A}_f)$ open compact) as long as the quadratic fields K involved are not amongst the finite set of quadratic fields K_χ such that χ_{B} is K_f-invariant.

Waldspurger's formula & subconvexity

The two approaches toward Duke's theorem (via Fourier coefficients of theta series or via L-functions) are connected via another formula of Waldspurger.
However the L-function approach has advantages : the realization of the set of (classes of) representations as a torus orbit $[\mathrm{T}.g]$ provides it with a group-like (torsor) structure : that was already identified by Gauss for the discriminant and sums of three squares in the *Disquisitiones*).

- This allows to refine Duke's theorem by restricting the average along $[\mathrm{T}]$ via harmonic analysis : for any Hecke character $\chi_{\mathrm{T}} : [\mathrm{T}] \to \mathbb{C}^{\times}$, Waldspurger established the more general formula relating the twisted period $|\int_{[\mathrm{T}]} \varphi(tg)\chi_{\mathrm{T}}(t)\,dt|^2$ to $L(\pi_K^{JL}.\chi_{\mathrm{T}}, 1/2)$ and a product of twisted local integrals and non-trivial bounds for these are known (M, M-V).

Fast backwards : Linnik's ergodic method

This torsor structure was exploited by Linnik and later Skubenko in the 50/60's to prove Duke's theorems under an additional congruence on d

Theorem (Linnik, Skubenko)

For $q = E_3$ or Δ, the set $|d|^{-1/2}\mathcal{R}_q(d)$ equidsitributes on $V_{q,\pm 1}(\mathbb{R})$ for $d \to \infty$ satisfying the previous conditions as in addition

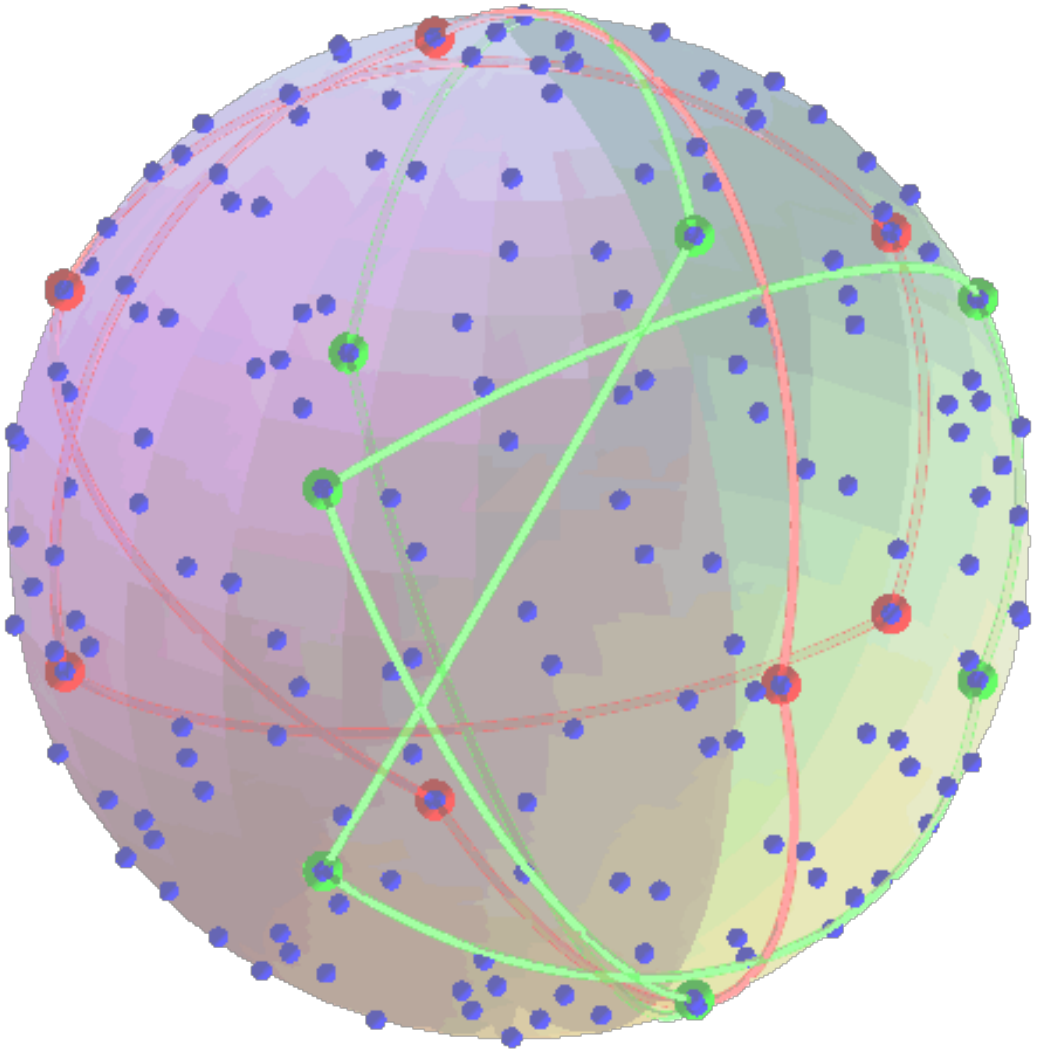
$$5 \text{ split in } \mathbb{Q}(\sqrt{d})$$

Fast backwards : Linnik's ergodic method

The proof of this theorem is via what Linnik's *ergodic method*. Here is a modern reformulation (ELMV)

1. The splitting condition at 5 implies that for any d, $\mathrm{T}_d(\mathbb{Q}_5) \subset \mathrm{G}(\mathbb{Q}_5)$ contains an element conjugate to the matrix (necessarily $\mathrm{B}(\mathbb{Q}_5) \simeq M_2(\mathbb{Q}_5)$) $t_5 := \begin{pmatrix} 5 & 0 \\ 0 & 1 \end{pmatrix}$
2. The sequence of torus orbits $[\mathrm{T}_d.g_d]$ furnishes a sequence of probability measures on $[\mathrm{G}]$ invariant under the action of the group $t_5^{\mathbb{Z}}$. In particular any weak-$\star$ limit of such sequence has this invariance.
3. The *basic lemma* (of Linnik) (a soft form of a case of the Siegel mass formula) implies that the entropy of the t_5-action for any weak limits is maximal.
4. Maximality of entropy $\Longrightarrow$ any weak-$\star$ limit is the Haar measure.

Orbit for the 5-flow



Fast forward : Duke's theorem for products

The group structure allow for further refinements of Duke's theorem.

- We consider a uple of (distinct) quaternion algebras B_i, $i = 1, \cdots, s$ with associated projective groups G_i ; a uple of open-cpt subgroups $K_{f,i} \subset \mathrm{G}_i(\mathbb{A}_f)$. Set

$$\mathbf{G} = \prod_{i=1}^{s} \mathrm{G}_i, \ \mathbf{K}_f = \prod_{i=1}^{s} K_{f,i} \subset \mathbf{G}(\mathbb{A}_f)$$

Fast forward : Duke's theorem for products

- Given a torus $\mathrm{T} \simeq \mathrm{res}_{K/\mathbb{Q}}\mathbb{G}_m/\mathbb{G}_m$, $K = \mathbb{Q}(\sqrt{d})$ with a uple of embeddings

$$\iota_i : \mathrm{T} \hookrightarrow \mathrm{G}_i$$

(take $z_i \in \mathrm{B}_i^0$ s.t. $z_i^2 = d$), which yields a diagonal embedding

$$\iota : t \in \mathrm{T} \hookrightarrow \iota(t) = (\iota_i(t))_{i \leq s} \in \mathbf{T} \subset \mathbf{G}.$$

- The data of a uple $\mathbf{g} = (g_i)_{i \leq s} \in \mathbf{G}(\mathbb{A})$ define an adelic torus orbit $[\mathbf{T}.\mathbf{g}] \subset [\mathbf{G}]$ whose discriminant is defined as $D_{\mathbf{T}.\mathbf{g}} := \min_{i \leq s} D_{\mathrm{T}_i.g_i}$.

Conjecture

As $D_{\mathbf{T}.\mathbf{g}} \to \infty$ the torus orbit $[\mathbf{T}.\mathbf{g}]$ becomes e.d. on $[\mathbf{G}]/\mathbf{K}_f$ for the subspace of functions $\perp$ to the $\mathbf{K}_f$-invariant characters.

Joinings

Theorem (Einsiedler-Lindenstrauss)

Let $v_1 \neq v_2$ be fixed places of $\mathbb{Q}$. The conjecture is true for tori T whose associated quadratic field K splits at v_1, v_2.

- The condition that K splits at v_1, v_2 is similar to Linnik condition in the ergodic method but one needs two places here.
- Duke's theorem for one factor is used as a fuel for the general theorem (in fact as one has already two split places, one could use Linnik's theorem instead).

Joinings

Theorem (Einsiedler-Lindenstrauss)

Let $v_1 \neq v_2$ be fixed places of $\mathbb{Q}$. The conjecture is true for tori T whose associated quadratic field K splits at v_1, v_2.

- Blomer and Brumley have recently obtained a proof for two factors ($s = 2$) without any splitting condition but under several GRHs.
- This is very special case of a much more general theorem classifying *joinings* on products of locally homogeneous spaces invariant under *two* commuting actions (these are *algebraic* measures). In particular the general theorem provides important (but not sufficient) information if some B_i are the same (cf. the work of I. Khayutin).

Characters

To deal with the case of characters, one introduce the following groups

$$\Xi_i(K_{f,i}) = \{\chi_i,\ \chi_i^2 = 1, \chi_i \circ \mathrm{Nr}_{i|K_{f,i}} \equiv 1\}$$

$$\Xi(\mathbf{K}_f) = \prod_i \Xi_i(K_{f,i}).$$

Let Π denote the product map

$$\Pi : (\chi_i)_{i \leq s} \in \Xi(\mathbf{K}_f) \mapsto \prod_i \chi_i$$

Proposition

Under the previous assumptions, if $\ker(\Pi)$ *is trivial, equidistribution holds for tori* T *whose quadratic field* K *is not associated to any quadratic character in the image* $\Pi(\Xi(\mathbf{K}_f))$.

Duke's theorem for products : applications

The first application of the EL Theorem was given by Aka-Einsiedler-Shapira. Here is a simplified version :
Let $(a,b,c) \in \mathcal{R}_{E_3}(d)$ be a primitive representation ; the intersection

$$(a,b,c)^{\perp} \cap \mathbb{Z}^3$$

is a lattice in the $\mathbb{R}$-space $(a,b,c)^{\perp}$ of covolume $|d|^{1/2}$. To this lattice, one can associate a (well defined) $\mathbb{C}^{\times}$-homothety class of lattices $[\Lambda(a,b,c)] = [\mathbb{Z} + \mathbb{Z}.z(a,b,c)]$ in $\mathbb{C}$ and therefore a class on the modular surface

$$[z(a,b,c)] \in \mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H} = \mathrm{PGL}_2(\mathbb{Z})\backslash \mathrm{PGL}_2(\mathbb{R})/\mathrm{PSO}_2(\mathbb{R}),$$

$$z(a,b,c) = z_{(a',b',c')},\ (a',b',c') \in \mathcal{R}_{\Delta}(d) \text{ or } \mathcal{R}_{\Delta}(4d).$$

Duke's theorem for products : applications

Theorem (AES)

Let q_1, q_2 be two odd primes. For $d \to \infty$, satisfying the assumption of the first of Duke's Theorems (for S^2) and such that q_1, q_2 split in $\mathbb{Q}(\sqrt{d})$ the set of pairs

$$\{(|d|^{-1/2}(a,b,c), [z(a,b,c)]),\ (a,b,c) \in \mathcal{R}_{E_3}(d)\}$$

is equidistributed on $S^2 \times X_0(1)$.

The proof is an application of the EL theorem with

$$\mathrm{B}_1 = \mathrm{B}_{2,\infty},\ \mathrm{B}_2 = M_2,\ K_{f,1} = \mathrm{P}\widehat{\mathcal{O}}_{2,\infty}^{\times},\ K_{f,2} = \mathrm{PGL}_2(\widehat{\mathbb{Z}}).$$

The only character is the trivial one.

Reduction of CM elliptic curves

An elliptic curve $E/\mathbb{C}$ has CM if $\mathrm{End}(\Lambda) = \mathcal{O} \subset K \subset \mathbb{C}$. If $E(\mathbb{C}) \simeq \mathbb{C}/\Lambda$ this is equivalent to

$$\mathrm{End}(\Lambda) = \{z \in \mathbb{C},\ z.\Lambda \subset \Lambda\} = \mathcal{O}.$$

$$\mathcal{E}\ell\ell_{\mathcal{O}} = \{\text{Elliptic curves with CM by } \mathcal{O}\}/\sim .$$

- $\mathcal{E}\ell\ell_{\mathcal{O}}$ is a torsor under the action of $\mathrm{Pic}(\mathcal{O})$:

$$\text{for } E \simeq \mathbb{C}/\Lambda,\ \mathfrak{a} \star E \simeq \mathbb{C}/\mathfrak{a}^{-1}\Lambda.$$

- Set $d = \mathrm{disc}(\mathcal{O})$, the map $z \in \mathbb{H} \mapsto \Lambda_z = \mathbb{Z} + \mathbb{Z}.z \mapsto \mathbb{C}/\Lambda_z$ induce

$$\mathcal{E}\ell\ell_{\mathcal{O}} \simeq \{z_{[a,b,c]},\ (a,b,c) \in \mathcal{R}^{\star}_{\Delta}(d)\} \subset X_0(1).$$

So by Duke's second theorem, elliptic curves with CM by $\mathcal{O}$ become ed. on the moduli space of elliptic curves/$\mathbb{C}$ as $\mathrm{disc}(\mathcal{O}) \to \infty$.

Reduction of CM elliptic curves

- A CM elliptic curve is defined of a number field $H_{\mathcal{O}} = K(j(E))$.
- A CM elliptic curve has (potential) good reduction at every prime.
- If p is inert in K and $\mathfrak{p}$ is a place in $\overline{\mathbb{Q}}$ above p, $E \pmod{\mathfrak{p}}$ is a *supersingular* elliptic curve :

$$\mathrm{End}(E \pmod{\mathfrak{p}}) \simeq \mathcal{O}_{p,\infty} \subset \mathrm{B}_{p,\infty}.$$

$$\mathcal{E}\ell\ell_p^{ss} = \{\text{Supersingular elliptic curves}/\overline{\mathbb{F}_p}\}/\sim .$$

We have a reduction modulo p map

$$\mathrm{red}_p : E \in \mathcal{E}\ell\ell_{\mathcal{O}} \mapsto E \pmod{\mathfrak{p}} \in \mathcal{E}\ell\ell_p^{ss}$$

inducing an embedding

$$\iota_p : \mathrm{End}(E) \simeq \mathcal{O} \hookrightarrow \mathrm{End}(E \pmod{\mathfrak{p}}) \simeq \mathcal{O}_{p,\infty}.$$

Reduction of CM elliptic curves

The finite space $\mathcal{E}\ell\ell_p^{ss}$ is identified with an adelic quotient

$$\mathcal{E}\ell\ell_p^{ss} \simeq \mathrm{G}(\mathbb{Q})\backslash\mathrm{G}(\mathbb{A})/\mathrm{G}(\mathbb{R})\mathrm{P}\widehat{\mathcal{O}}_{p,\infty}^{\times},\ \mathrm{G} = \mathrm{PB}_{p,\infty}$$

and the embedding ι_p induces and embedding $\iota_p : \mathrm{T} \hookrightarrow \mathrm{G}$. Under the above identification the image $\mathrm{red}_p(\mathcal{E}\ell\ell_{\mathcal{O}})$ identifies (as a multiset) with the projection of the orbit $[\iota_p(\mathrm{T})]$. Duke's theorem implies

Theorem (M)

As $d \to \infty$ (such that $(d,p) = 1$ and p is inert in K) the map red_p is surjective and $\mathrm{red}_p(\mathcal{E}\ell\ell_{\mathcal{O}})$ (as a multiset) becomes ed. wrt a probability measure μ_p such that $\mu_p(\overline{E}) \approx 1/|\mathrm{End}(\overline{E})^{\times}|$.

Reduction of CM elliptic curves

We can now fix a tuple of primes $p_1, \cdots, p_s$ and for $\mathcal{O}$ such that all the p_i remain inert in K we have a multireduction map

$$\mathrm{red} : E \in \mathcal{E}\ell\ell_{\mathcal{O}} \mapsto (z_E, \mathrm{red}_{p_1}(E), \cdots, \mathrm{red}_{p_s}(E)) \in X_0(1) \times \prod_i \mathcal{E}\ell\ell_{p_i}^{ss}$$

Theorem (ALMW)

Let q_1, q_2 be two primes $\neq p_i$. As $d \to \infty$ – such that $(d, p_i) = 1$, p_i is inert in K for every i, and q_1, q_2 split in K – the set $\mathrm{red}(\mathcal{E}\ell\ell_{\mathcal{O}})$ becomes ed. wrt probability measure $\mu_\infty \otimes \bigotimes_i \mu_{p_i}$.

Reduction of CM elliptic curves

- The proof is an application of the EL theorem for

$$\mathbf{G} = \mathrm{PGL}_2 \times \prod_i \mathrm{G}_{p_i}, \ \mathbf{K}_f = \mathrm{PGL}_2(\widehat{\mathbb{Z}}) \times \prod_i \mathrm{P}\widehat{\mathcal{O}}^{\times}_{p_i,\infty}$$

  with the diagonal embedding for T

$$\iota = (\mathrm{Id}, \iota_{p_1}, \cdots, \iota_{p_s}).$$

- The open compact $\mathbf{K}_f$ is big : this implies that the only character is the trivial one.
- A non obvious point is to verify is that the image $\mathrm{red}(\mathcal{E}\ell\ell_{\mathcal{O}})$ is represented by a diagonal torus orbit $[\iota(\mathrm{T})\mathbf{g}]$. This is a consequence of Serre's "$\mathfrak{a}$-transform", an algebraic version of the $\mathrm{Pic}(\mathcal{O})$-action on complex CM elliptic curves

$$\mathfrak{a} \star \mathbb{C}/\Lambda = \mathbb{C}/\mathfrak{a}^{-1}\Lambda.$$

  This implies that for $\mathfrak{a} \subset \mathcal{O}$

$$(\mathfrak{a} \star E) \,(\mathrm{mod}\, \mathfrak{p}) = \iota_p(\mathfrak{a}) \star (E \,(\mathrm{mod}\, \mathfrak{p})).$$

Reduction of CM elliptic curves : future directions

- Recently, Herrero, Menares and Rivera-Letellier have obtained much more precise version for the equidistribution of the reductions modulo p.
  The set $\mathcal{E}\ell\ell_p^{ss}$ is the indexing set of a disjoint union of p-adic disks of radius 1 ; each disk parametrize the deformations of the (canonical lift of the) formal group of each supersingular curve $\overline{E}$.
  Each CM curve E provide such deformation for $E \pmod{\mathfrak{p}}$ and therefore a point on the corresponding disk. Using the theta series approach, HMR-L have established the equidistribution of the CM points on these disks wrt to a certain probability measure.
  We expect to joint forces to extend this result to several primes simultaneously.

Reduction of CM elliptic curves : future directions

- For his work on Mazur's conjectures, Cornut has proven the equidistribution theorem for simultaneous reductions for elliptic curves with CM by orders

$$\mathcal{O}_{q^k} = \mathbb{Z} + q^k \mathcal{O}_K$$

for K a *fixed* quadratic field, q a *fixed* prime and $k \to \infty$. The proof use Ratner's theory of joinings for unipotent actions. The EL theorem allows to obtain results analogous to those of Vatsal, Cornut and Cornut-Vatsal for orders $\mathcal{O}_{q^k} = \mathbb{Z} + q^k \mathcal{O}_K$ when K is *fixed*, k is *fixed* and $q \to \infty$ and to derivate arithmetic consequences along the lines of C-V (in progress with D. Ramakrishnan).

THANK YOU!