Non-vanishing of Dirichlet L-functions

Rizwan Khan

(Joint work with Ngo and Milićević and Ngo)

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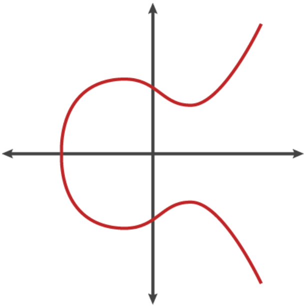
Question: Is $L(1/2) = 0$?

Birch and Swinnerton-Dyer Conjecture



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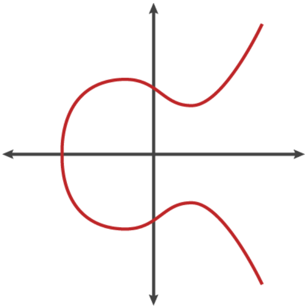

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Part of the Birch and Swinnerton-Dyer Conjecture:

E has infinitely many rational points $\iff L(1/2, E) = 0.$

Landau-Siegel zeros

$H_k(\Gamma) =$ Hecke basis of cusp forms of weight k for $\Gamma = SL_2(\mathbb{Z})$.

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Non-vanishing proportion of $\frac{1}{2} - \epsilon$ for any $\epsilon > 0$.

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- A Dirichlet character of modulus q is homomorphism

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For χ primitive, define $L(s, \chi) = \sum_{n \geq 1} \frac{\chi(n)}{n^s}$

Conjecture: $L(1/2, \chi)$ is never 0.

Non-vanishing problem

$\mathcal{F}_p :=$ set of primitive Dirichlet characters mod p.
Problem: Prove that $L(1/2, \chi) \neq 0$ for a positive proportion of characters χ in $\mathcal{F}_p$, as $p \to \infty$.

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K-Ngo (2016), (K-Milićević-Ngo, 2020): proportion$= \frac{5}{13} = 0.3846$.

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Need to evaluate first and second mollified moments:

$\frac{1}{p}\sum_{\chi \bmod p} L(1/2,\chi)\mathcal{M}(1/2,\chi)$

$\frac{1}{p}\sum_{\chi \bmod p} |L(1/2,\chi)|^2|\mathcal{M}(1/2,\chi)|^2$

Tool: Orthogonality

$$\frac{1}{p}\sum_{\chi \bmod p} \chi(n)\overline{\chi}(m) = \begin{cases} 1 \text{ if } n \equiv m \bmod p \\ 0 \text{ otherwise} \end{cases}$$

Tool: Poisson summation

$$\sum_{\substack{n \in \mathbb{Z} \\ n \equiv a \bmod p}} f(n) = \sum_{n \in \mathbb{Z}} e\left(\frac{an}{p}\right) \hat{f}\left(\frac{n}{p}\right)$$

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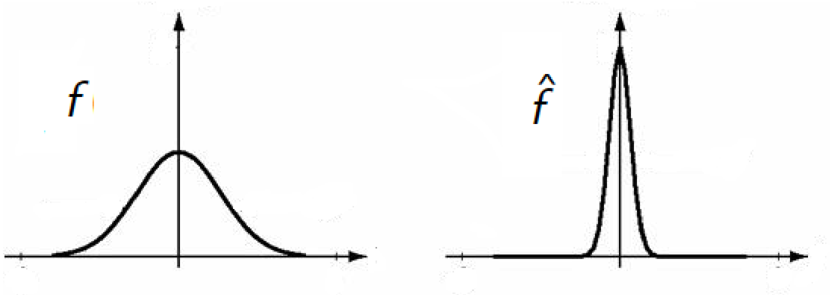

A sum of length N is traded for a sum of length p/N.

Tool: Approximate functional equation

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$$|L(1/2, \chi)|^2 = 2 \sum_{n_1 n_2 \le p} \frac{\chi(n_1)\overline{\chi}(n_2)}{(n_1 n_2)^{1/2}}.$$

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Cross terms without $\tau(\chi)$:

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Contain distinguished terms $n_1 m_1 = n_2 m_2$, giving the main term:

$$\sum_{\substack{n_1 n_2\le p\\ m_1,m_2\le M\\ n_1 m_1 = n_2 m_2}} c_{m_1}c_{m_2}\frac{1}{(n_1 n_2 m_1 m_2)^{1/2}}$$

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The hard part is with

$$\tau(\chi)=\sum_{a \bmod p}\chi(a)e\left(\frac{a}{p}\right) \qquad \text{(where } e(x)=e^{2\pi i x}\text{)}$$

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$\sum_{a \bmod p} \rightsquigarrow S(\overline{m_2}, n_1 n_2 \overline{m_1}, p)$.

Sums of Kloosterman sums

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By now, can use results bilinear forms $\sum_{n,m} \alpha_n \alpha_m S(n, m, p)$

of Kowalski, Michel, Sawin.

Sums of products of Kloosterman sums

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Suppose $(b_1, b_2, \ldots, b_k) \in (\mathbb{F}_p^*)^k$.

If some b_i occurs with odd multiplicity,

$$\sum_{a \bmod p} S(b_1, a, p) S(b_2, a, p) \cdots S(b_k, a, p) \ll p^{(k+1)/2}.$$

Beats Weil by factor $p^{1/2}$.

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For $m_1 = m_2$ (rare), apply Weil. For $m_1 \neq m_2$, apply FGKM.

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Just fails! $\ell \in \mathcal{L} \rightsquigarrow a \bmod p$ wasteful step. ($|\mathcal{L}| \asymp p^{3/4}$)

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&\le p^{9/16} \Big(\sum_{a \bmod p} \Big| \sum_{m\sim p^{1/4}} c_m S(m, a, p) \Big|^4 \Big)^{1/4} \\
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\end{aligned}$$

First idea (K-Ngo): try again

$$S = \sum_{n_1,n_2\sim p^{1/4}} \sum_{m_1,m_2\le p^{1/4}} c_{m_1}c_{m_2}S(\overline{m_2}, n_1n_2\overline{m_1}, p).$$

Write $\ell = n_1n_2\overline{m_1}$, $m = \overline{m_2}$, and apply Hölder.

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If some m_i distinct, apply FGKM. Other cases rare – use Weil.

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Other Hölder exponents give weaker results.

Further ideas (K-Milićević-Ngo)

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$$\mathcal{M}(1/2,\chi) = \frac{3}{5} \sum_{m_1 \le p^{3/8-\epsilon}} \frac{c_{m_1}\chi(m_1)}{{m_1}^{1/2}} + \frac{2}{5} \frac{\tau(\overline{\chi})}{p^{1/2}} \sum_{m_2 \le p^{1/4-\epsilon}} \frac{c_{m_2}\overline{\chi}(m_2)}{{m_2}^{1/2}}.$$

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Final proportion of non-vanishing: $\dfrac{5}{13}$.

End

Thank you