ORTHOGONALITY RELATIONS FOR COEFFICIENTS OF AUTOMORPHIC L-FUNCTIONS

Dorian Goldfeld

Department of Mathematics
Columbia University
New York, NY 10027

June 4, 2020

Historical Motivation

Orthogonality relation for Dirichlet characters

Assume $n, a, q \in \mathbb{Z}$ with n, a coprime to $q \geq 1$. Then

$$\frac{1}{\phi(q)} \sum_{\chi \pmod{q}} \chi(n)\,\overline{\chi(a)} = \begin{cases} 1, & \text{if } n \equiv a \pmod{q}, \\ 0, & \text{otherwise.} \end{cases}$$

Orthogonality relation for Dirichlet characters

Assume $n, a, q \in \mathbb{Z}$ with n, a coprime to $q \geq 1$. Then

$$\frac{1}{\phi(q)} \sum_{\chi \pmod q} \chi(n)\,\overline{\chi(a)} = \begin{cases} 1, & \text{if } n \equiv a \pmod q, \\ 0, & \text{otherwise.} \end{cases}$$

- **1837** *Dirichlet proved infinitely many primes in arithmetic progression using this orthogonality relation.*

Historical Motivation

Orthogonality relation for Dirichlet characters

Assume $n, a, q \in \mathbb{Z}$ with n, a coprime to $q \geq 1$. Then

$$\boxed{\frac{1}{\phi(q)} \sum_{\chi \pmod{q}} \chi(n)\,\overline{\chi(a)} = \begin{cases} 1, & \text{if } n \equiv a \pmod{q}, \\ 0, & \text{otherwise.} \end{cases}}$$

- **1837** *Dirichlet proved infinitely many primes in arithmetic progression using this orthogonality relation.*
- *Countless other applications since then.*

Orthogonality Relation for $GL(1, \mathbb{A}_{\mathbb{Q}})$

Orthogonality relation for Dirichlet characters is orthogonality relation for $GL(1, \mathbb{A}_{\mathbb{Q}})$

• An automorphic representation for $GL(1, \mathbb{A}_{\mathbb{Q}})$ is a one dimensional representation, i.e., a Dirichlet character lifted to the adele ring.

• $\chi(n)$ can be interpreted as the n^{th} coefficient of a $GL(1, \mathbb{A}_{\mathbb{Q}})$ L-function.

Upper half plane $\mathfrak{h}^n$

Upper half plane $\mathfrak{h}^n$

Upper half plane: $\mathfrak{h}^n := GL(n,\mathbb{R})/\big(O(n,\mathbb{R})\cdot\mathbb{R}^\times\big)$

Upper half plane $\mathfrak{h}^n$

Upper half plane: $\mathfrak{h}^n := GL(n,\mathbb{R})/\big(O(n,\mathbb{R})\cdot\mathbb{R}^\times\big)$

$$\boxed{g \in \mathfrak{h}^n \implies g = xy}$$

Upper half plane $\mathfrak{h}^n$

Upper half plane: $\mathfrak{h}^n := GL(n,\mathbb{R})/\big(O(n,\mathbb{R})\cdot\mathbb{R}^\times\big)$

$$\boxed{g \in \mathfrak{h}^n \implies g = xy}$$

with

$$x \in U_n(\mathbb{R}) = \begin{pmatrix} 1 & & & * \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}$$

Upper half plane $\mathfrak{h}^n$

Upper half plane: $\mathfrak{h}^n := GL(n,\mathbb{R})/\big(O(n,\mathbb{R})\cdot\mathbb{R}^\times\big)$

$$\boxed{g \in \mathfrak{h}^n \implies g = xy}$$

with

$$x \in U_n(\mathbb{R}) = \begin{pmatrix} 1 & & & * \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}$$

$$y = \begin{pmatrix} y_1\cdots y_{n-1} & & & \\ & \ddots & & \\ & & y_1 & \\ & & & 1 \end{pmatrix}, \qquad \big(y_i > 0, i = 1\ldots n\big).$$

Upper half plane $\mathfrak{h}^n$

Upper half plane: $\mathfrak{h}^n := GL(n,\mathbb{R})\big/\big(O(n,\mathbb{R})\cdot\mathbb{R}^\times\big)$

$$\boxed{g \in \mathfrak{h}^n \implies g = xy}$$

with

$$x \in U_n(\mathbb{R}) = \begin{pmatrix} 1 & & & * \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}$$

$$y = \begin{pmatrix} y_1\cdots y_{n-1} & & & \\ & \ddots & & \\ & & y_1 & \\ & & & 1 \end{pmatrix}, \qquad \big(y_i > 0, i = 1\ldots n\big).$$

Example 1

The classical upper half plane $\mathfrak{h}^2$ can be realized with matrices:

$$\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix} \mapsto x + iy, \qquad \big(x \in \mathbb{R},\ y > 0\big).$$

Invariant Differential Operators

Let $n \geq 2$ and $g = xy \in \mathfrak{h}^n$ with coordinates

$$x = \begin{pmatrix} 1 & x_{1,2} & x_{1,3} & \cdots & x_{1,n} \\ & 1 & x_{2,3} & \cdots & x_{2,n} \\ & & \ddots & & \vdots \\ & & & 1 & x_{n-1,n} \\ & & & & 1 \end{pmatrix}, \qquad y = \begin{pmatrix} y_1 y_2 \cdots y_{n-1} & & & & \\ & y_1 y_2 \cdots y_{n-2} & & & \\ & & \ddots & & \\ & & & y_1 & \\ & & & & 1 \end{pmatrix},$$

$x_{i,j} \in \mathbb{R}$ $(1 \leq i < j \leq n)$, and $y_i > 0$ $(1 \leq i \leq n-1)$.

Invariant Differential Operators

Let $n \geq 2$ and $g = xy \in \mathfrak{h}^n$ with coordinates

$$x = \begin{pmatrix} 1 & x_{1,2} & x_{1,3} & \cdots & x_{1,n} \\ & 1 & x_{2,3} & \cdots & x_{2,n} \\ & & \ddots & & \vdots \\ & & & 1 & x_{n-1,n} \\ & & & & 1 \end{pmatrix}, \qquad y = \begin{pmatrix} y_1 y_2 \cdots y_{n-1} & & & & \\ & y_1 y_2 \cdots y_{n-2} & & & \\ & & \ddots & & \\ & & & y_1 & \\ & & & & 1 \end{pmatrix},$$

$x_{i,j} \in \mathbb{R}$ $(1 \leq i < j \leq n)$, and $y_i > 0$ $(1 \leq i \leq n-1)$.

Ring of Invariant Differential Operators $\mathcal{D}^n$

The ring $\mathcal{D}^n$ consists of all polynomials (with complex coefficients) in the differential operators $\left\{ \frac{\partial}{\partial x_{i,j}}, \frac{\partial}{\partial y_k} \right\}$ which are invariant under $GL(n, \mathbb{R})$ transformations.

Let $n \geq 2$ and $g = xy \in \mathfrak{h}^n$ with coordinates

$$x = \begin{pmatrix} 1 & x_{1,2} & x_{1,3} & \cdots & x_{1,n} \\ & 1 & x_{2,3} & \cdots & x_{2,n} \\ & & \ddots & & \vdots \\ & & & 1 & x_{n-1,n} \\ & & & & 1 \end{pmatrix}, \qquad y = \begin{pmatrix} y_1 y_2 \cdots y_{n-1} & & & & \\ & y_1 y_2 \cdots y_{n-2} & & & \\ & & \ddots & & \\ & & & y_1 & \\ & & & & 1 \end{pmatrix},$$

$x_{i,j} \in \mathbb{R}$ $(1 \leq i < j \leq n)$, and $y_i > 0$ $(1 \leq i \leq n-1)$.

Ring of Invariant Differential Operators $\mathcal{D}^n$

The ring $\mathcal{D}^n$ consists of all polynomials (with complex coefficients) in the differential operators $\left\{ \frac{\partial}{\partial x_{i,j}}, \frac{\partial}{\partial y_k} \right\}$ which are invariant under $GL(n, \mathbb{R})$ transformations.

Example 2

For $n = 2$ the Laplacian $\Delta = -y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right)$ is an invariant differential operator. Here $\left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right) \left(\begin{smallmatrix} y & 0 \\ 0 & 1 \end{smallmatrix}\right) \in \mathfrak{h}^2$.

Power Function

Langlands Parameters

Let $n \geq 2$. The Langlands parameters are complex numbers $\boxed{\alpha = \{\alpha_1, \ldots, \alpha_n\}}$ which satisfy $\alpha_1 + \cdots + \alpha_n = 0$.

• *These parameters classify automorphic representations at the archimedean place.*

Power Function

Langlands Parameters

Let $n \geq 2$. The Langlands parameters are complex numbers $\boxed{\alpha = \{\alpha_1, \ldots, \alpha_n\}}$ which satisfy $\alpha_1 + \cdots + \alpha_n = 0$.

• *These parameters classify automorphic representations at the archimedean place.*

Power Function

Let $n \geq 2$. Let $\alpha = (\alpha_1, \ldots, \alpha_n)$ denote Langlands parameters and let $\rho = (\rho_1, \ldots, \rho_n)$, where $\rho_i = \frac{n+1}{2} - i$. We define a power function $\mathrm{I}(*, \alpha) : \mathfrak{h}^n \to \mathbb{C}$ by

Power Function

Langlands Parameters

Let $n \geq 2$. The Langlands parameters are complex numbers $\boxed{\alpha = \{\alpha_1, \ldots, \alpha_n\}}$ which satisfy $\alpha_1 + \cdots + \alpha_n = 0$.

• *These parameters classify automorphic representations at the archimedean place.*

Power Function

Let $n \geq 2$. Let $\alpha = (\alpha_1, \ldots, \alpha_n)$ denote Langlands parameters and let $\rho = (\rho_1, \ldots, \rho_n)$, where $\rho_i = \frac{n+1}{2} - i$. We define a power function $\mathrm{I}(*, \alpha) : \mathfrak{h}^n \to \mathbb{C}$ by

$$\boxed{\mathrm{I}(xy, \alpha) = \prod_{i=1}^{n-1} y_i^{\alpha_1 + \cdots + \alpha_{n-i} + \rho_1 + \cdots + \rho_{n-i}}}$$

Power Function

Langlands Parameters

Let $n \geq 2$. The Langlands parameters are complex numbers $\boxed{\alpha = \{\alpha_1, \ldots, \alpha_n\}}$ which satisfy $\alpha_1 + \cdots + \alpha_n = 0$.

• *These parameters classify automorphic representations at the archimedean place.*

Power Function

Let $n \geq 2$. Let $\alpha = (\alpha_1, \ldots, \alpha_n)$ denote Langlands parameters and let $\rho = (\rho_1, \ldots, \rho_n)$, where $\rho_i = \frac{n+1}{2} - i$. We define a power function $\mathrm{I}(*, \alpha) : \mathfrak{h}^n \to \mathbb{C}$ by

$$\boxed{\mathrm{I}(xy, \alpha) = \prod_{i=1}^{n-1} y_i^{\alpha_1 + \cdots + \alpha_{n-i} + \rho_1 + \cdots + \rho_{n-i}}}$$

Key Fact: *The power function is a joint eigenfunction of $\mathcal{D}^n$.*

A Maass cusp form is a complex valued function $\phi : \mathfrak{h}^n \to \mathbb{C}$ satisfying the following conditions:

Maass Cusp forms for $SL(n, \mathbb{Z})\backslash\mathfrak{h}^n$

A Maass cusp form is a complex valued function $\phi : \mathfrak{h}^n \to \mathbb{C}$ satisfying the following conditions:

- $\boxed{\phi(\gamma g) \;=\; \phi(g)} \quad \forall\gamma \in SL(n, \mathbb{Z}),\; g \in \mathfrak{h}^n;$

Maass Cusp forms for $SL(n,\mathbb{Z})\backslash\mathfrak{h}^n$

A Maass cusp form is a complex valued function $\phi : \mathfrak{h}^n \to \mathbb{C}$ satisfying the following conditions:

- $\boxed{\phi(\gamma g) \;=\; \phi(g)}$ $\quad \forall \gamma \in SL(n,\mathbb{Z}),\ g \in \mathfrak{h}^n;$

- $\boxed{|\phi(xy)| \ll_B (y_1 \cdots y_{n-1})^{-B}}$ $\quad$ *for* $y_1, \ldots y_{n-1} \geq 1;$ *and* $B > 0;$

Maass Cusp forms for $SL(n,\mathbb{Z})\backslash\mathfrak{h}^n$

A Maass cusp form is a complex valued function $\phi : \mathfrak{h}^n \to \mathbb{C}$ satisfying the following conditions:

- $\boxed{\phi(\gamma g) \; = \; \phi(g)} \quad \forall \gamma \in SL(n,\mathbb{Z}),\; g \in \mathfrak{h}^n;$
- $\boxed{|\phi(xy)| \ll_B (y_1 \cdots y_{n-1})^{-B}} \quad$ *for* $y_1, \ldots y_{n-1} \geq 1;$ *and* $B > 0;$
- ϕ *is a joint eigenfunction of* $\mathcal{D}^n$.

Maass Cusp forms for $SL(n,\mathbb{Z})\backslash\mathfrak{h}^n$

A Maass cusp form is a complex valued function $\phi : \mathfrak{h}^n \to \mathbb{C}$ satisfying the following conditions:

- $\boxed{\phi(\gamma g) \ = \ \phi(g)} \quad \forall \gamma \in SL(n,\mathbb{Z}),\ g \in \mathfrak{h}^n;$
- $\boxed{|\phi(xy)| \ll_B (y_1 \cdots y_{n-1})^{-B}} \quad$ *for* $y_1, \dots y_{n-1} \geq 1;$ *and* $B > 0;$
- ϕ *is a joint eigenfunction of* $\mathcal{D}^n$.

Langlands parameters of a Maass cusp form

If the eigenvalues of ϕ agree with those of $\mathrm{I}(*,\alpha)$ we call α the Langlands parameters of ϕ.

A Maass cusp form is a complex valued function $\phi : \mathfrak{h}^n \to \mathbb{C}$ satisfying the following conditions:

- $\boxed{\phi(\gamma g) \;=\; \phi(g)}$ $\quad\forall\gamma\in SL(n,\mathbb{Z}),\; g\in\mathfrak{h}^n$;
- $\boxed{|\phi(xy)| \ll_B (y_1\cdots y_{n-1})^{-B}}$ *for* $y_1,\ldots y_{n-1}\geq 1$; *and* $B>0$;
- ϕ *is a joint eigenfunction of* $\mathcal{D}^n$.

Langlands parameters of a Maass cusp form

If the eigenvalues of ϕ agree with those of $\mathrm{I}(*,\alpha)$ we call α the Langlands parameters of ϕ.

Laplace eigenvalue of ϕ

$$\lambda_\Delta = \frac{n^3-n}{24} - \frac{\alpha_1^2+\alpha_2^2+\cdots+\alpha_n^2}{2}$$

Character of the unipotent group

Character of the unipotent group

Let $u \in U_n$ be given by

Character of the unipotent group

Let $u \in U_n$ be given by

$$u = \begin{pmatrix} 1 & u_{1,2} & u_{1,3} & \cdots & u_{1,n} \\ & 1 & u_{2,3} & \cdots & u_{2,n} \\ & & \ddots & \ddots & \vdots \\ & & & 1 & u_{n-1,n} \\ & & & & 1 \end{pmatrix}$$

Character of the unipotent group

Let $u \in U_n$ be given by

$$u = \begin{pmatrix} 1 & u_{1,2} & u_{1,3} & \cdots & u_{1,n} \\ & 1 & u_{2,3} & \cdots & u_{2,n} \\ & & \ddots & \ddots & \vdots \\ & & & 1 & u_{n-1,n} \\ & & & & 1 \end{pmatrix}$$

For $L = (\ell_1, \ell_2, \ldots, \ell_{n-1}) \in \mathbb{Z}^{m-1}$, we may define a character:

$$\boxed{\psi_L(u) := e^{2\pi i \left(\ell_1 u_{1,2} + \cdots + \ell_{n-1} u_{n-1,n}\right)}.}$$

Character of the unipotent group

Let $u \in U_n$ be given by

$$u = \begin{pmatrix} 1 & u_{1,2} & u_{1,3} & \cdots & u_{1,n} \\ & 1 & u_{2,3} & \cdots & u_{2,n} \\ & & \ddots & \ddots & \vdots \\ & & & 1 & u_{n-1,n} \\ & & & & 1 \end{pmatrix}$$

For $L = (\ell_1, \ell_2, \ldots, \ell_{n-1}) \in \mathbb{Z}^{m-1}$, we may define a character:

$$\boxed{\psi_L(u) := e^{2\pi i \left(\ell_1 u_{1,2} + \cdots + \ell_{n-1} u_{n-1,n}\right)}.}$$

Here

$$\psi_L(u \cdot u') = \psi_L(u)\psi_L(u'), \qquad (u, u' \in U_n).$$

Canonically Normalized Whittaker Function

Canonically Normalized Whittaker Function

Invariant measure on $U_n(\mathbb{R})$

In terms of the above coordinates, a bi-invariant Haar measure du on $U_n(\mathbb{R})$ is defined by $du := \prod_{1\le i<j\le n} du_{i,j}$.

Canonically Normalized Whittaker Function

Invariant measure on $U_n(\mathbb{R})$

In terms of the above coordinates, a bi-invariant Haar measure du on $U_n(\mathbb{R})$ is defined by $du := \prod_{1\le i<j\le n} du_{i,j}$.

Jacquet's Canonically Normalized Whittaker Function

Let $n \ge 2$, $w_n = \begin{pmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{pmatrix}$, and $\alpha = (\alpha_1, \ldots, \alpha_n)$ be Langlands parameters.

Then for $g \in \mathfrak{h}^n$

$$W_\alpha^{\pm}(g) := \prod_{1\le j<k\le n} \frac{\Gamma\left(\frac{1+\alpha_j-\alpha_k}{2}\right)}{\pi^{\frac{1+\alpha_j-\alpha_k}{2}}} \int_{U_n(\mathbb{R})} \mathrm{I}(w_n u g, \alpha)\, \overline{\psi_{1,\ldots,1,\pm 1}(u)}\, du.$$

Example 3 (GL(2) Canonically Normalized Whittaker function)

Example 3 (GL(2) Canonically Normalized Whittaker function)

$$g = \begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix} \in \mathfrak{h}^2, \text{ Langlands parameter} = (\alpha, -\alpha).$$

Example 3 (GL(2) Canonically Normalized Whittaker function)

$$g = \begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix} \in \mathfrak{h}^2, \;\; \text{Langlands parameter} = (\alpha, -\alpha).$$

$$W_{(\alpha,-\alpha)}(g) = \frac{\Gamma\left(\frac{1}{2}+\alpha\right)}{\pi^{\frac{1}{2}+\alpha}} \int\limits_{-\infty}^{\infty} \left(\frac{y}{(x+u)^2+y^2}\right)^{\frac{1}{2}+\alpha} e^{-2\pi i u}\, du$$

$$= 2\sqrt{y}\, K_\alpha(2\pi y) \cdot e^{2\pi i x}$$

Fourier-Whittaker expansion of cusp forms (Shalika-Piatetski-Shapiro (1973))

$$\phi(g) = \sum_{\gamma \in U_{n-1} \backslash SL_{n-1}} \sum_{\substack{M \\ m_1 \cdots m_{n-1} \neq 0}} \frac{A_\phi(M)}{\prod_{k=1}^{n-1} |m_k|^{\frac{k(n-k)}{2}}} \, W_\alpha\left(M^* \begin{pmatrix} \gamma & 0 \\ 0 & 1 \end{pmatrix} g \right)$$

Fourier-Whittaker expansion of cusp forms (Shalika-Piatetski-Shapiro (1973))

$$\phi(g) = \sum_{\gamma \in U_{n-1}\backslash SL_{n-1}} \sum_{\substack{M \\ m_1 \cdots m_{n-1} \neq 0}} \frac{A_\phi(M)}{\prod\limits_{k=1}^{n-1} |m_k|^{\frac{k(n-k)}{2}}} W_\alpha\left(M^* \begin{pmatrix} \gamma & 0 \\ 0 & 1 \end{pmatrix} g \right)$$

where $g \in \mathfrak{h}^n$ and

$$M = (m_1, m_2, \ldots, m_{n-1}), \qquad M^* = \begin{pmatrix} m_1 \cdots m_{n-2}|m_{n-1}| & & & \\ & \ddots & & \\ & & m_1 & \\ & & & 1 \end{pmatrix},$$

Fourier-Whittaker expansion of cusp forms (Shalika-Piatetski-Shapiro (1973))

$$\phi(g) = \sum_{\gamma \in U_{n-1}\backslash SL_{n-1}} \sum_{\substack{M \\ m_1\cdots m_{n-1}\neq 0}} \frac{A_\phi(M)}{\prod_{k=1}^{n-1} |m_k|^{\frac{k(n-k)}{2}}} W_\alpha\left(M^* \begin{pmatrix} \gamma & 0 \\ 0 & 1 \end{pmatrix} g\right)$$

where $g \in \mathfrak{h}^n$ and

$$M = (m_1, m_2, \ldots, m_{n-1}), \qquad M^* = \begin{pmatrix} m_1\cdots m_{n-2}|m_{n-1}| & & & \\ & \ddots & & \\ & & m_1 & \\ & & & 1 \end{pmatrix},$$

- *$A_\phi(M)$ is called the M^{th} Fourier coefficient of ϕ,*

Fourier-Whittaker expansion of cusp forms (Shalika-Piatetski-Shapiro (1973))

$$\phi(g) = \sum_{\gamma \in U_{n-1}\backslash SL_{n-1}} \sum_{\substack{M \\ m_1 \cdots m_{n-1} \neq 0}} \frac{A_\phi(M)}{\prod_{k=1}^{n-1} |m_k|^{\frac{k(n-k)}{2}}} W_\alpha\left(M^* \begin{pmatrix} \gamma & 0 \\ 0 & 1 \end{pmatrix} g \right)$$

where $g \in \mathfrak{h}^n$ and

$$M = (m_1, m_2, \ldots, m_{n-1}), \qquad M^* = \begin{pmatrix} m_1 \cdots m_{n-2}|m_{n-1}| & & & \\ & \ddots & & \\ & & m_1 & \\ & & & 1 \end{pmatrix},$$

- $A_\phi(M)$ *is called the* M^{th} *Fourier coefficient of* ϕ,
- $\alpha = (\alpha_1, \ldots, \alpha_n)$ *is the Langlands parameter of* ϕ.

Fourier coefficients of a Hecke cusp form

Let $M = (m_1, \ldots, m_{n-1})$ and $\mathbf{1} = (1, \ldots, 1)$. Then

$$A_\phi(M) = \underbrace{A_\phi(\mathbf{1})}_{\text{first coeff.}} \cdot \underbrace{\lambda_\phi(M)}_{M^{th}\text{ Hecke coeff.}}$$

where $\lambda_\phi(\mathbf{1}) = 1$.

Fourier coefficients of a Hecke cusp form

Let $M = (m_1, \ldots, m_{n-1})$ and $\mathbf{1} = (1, \ldots, 1)$. Then

$$A_\phi(M) = \underbrace{A_\phi(\mathbf{1})}_{\text{first coeff.}} \cdot \underbrace{\lambda_\phi(M)}_{M^{th}\text{ Hecke coeff.}}$$

where $\lambda_\phi(\mathbf{1}) = 1$.

First Coefficient of a cusp form ϕ

$$\left|A_\phi(\mathbf{1})\right|^2 = \frac{\langle \phi,\ \phi \rangle}{\mathrm{Vol}\left(\Gamma \backslash \mathfrak{h}^n\right) \cdot L(1, \mathrm{Ad}\,\phi) \prod\limits_{1 \le j \ne k \le n} \Gamma\left(\frac{1+\alpha_j - \alpha_k}{2}\right)}$$

Fourier coefficients of a Hecke cusp form

Let $M = (m_1, \ldots, m_{n-1})$ and $\mathbf{1} = (1, \ldots, 1)$. Then

$$A_\phi(M) = \underbrace{A_\phi(\mathbf{1})}_{\text{first coeff.}} \cdot \underbrace{\lambda_\phi(M)}_{M^{th}\text{ Hecke coeff.}}$$

where $\lambda_\phi(\mathbf{1}) = 1$.

First Coefficient of a cusp form ϕ

$$\left|A_\phi(\mathbf{1})\right|^2 = \frac{\langle \phi,\ \phi \rangle}{\mathrm{Vol}\left(\Gamma \backslash \mathfrak{h}^n\right) \cdot L(1, \mathrm{Ad}\,\phi) \prod\limits_{1 \le j \ne k \le n} \Gamma\left(\frac{1+\alpha_j-\alpha_k}{2}\right)}$$

Adjoint L-function

$$L(s, \mathrm{Ad}\,\phi) := \frac{L\left(s,\ \phi \times \overline{\phi}\right)}{\zeta(s)}.$$

L-function associated to a Hecke cusp form ϕ

L-function associated to a Hecke cusp form ϕ

Godement-Jacquet L-function

$$L(s,\phi) = \sum_{m=1}^{\infty} \frac{\lambda_\phi(m,1,\ldots,1)}{m^s}$$

L-function associated to a Hecke cusp form ϕ

Godement-Jacquet L-function

$$L(s,\phi) = \sum_{m=1}^{\infty} \frac{\lambda_\phi(m,1,\ldots,1)}{m^s}$$

Euler Product

$$L(s,\phi) = \prod_p \left(1 - \frac{\lambda_\phi(p,1,\ldots,1)}{p^s} + \frac{\lambda_\phi(1,p,1,\ldots,1)}{p^{2s}} - \frac{\lambda_\phi(1,1,p,\ldots,1)}{p^{3s}} + \cdots + (-1)^{n-1}\frac{\lambda_\phi(1,,\ldots,1,p)}{p^{(n-1)s}} + \frac{(-1)^n}{p^{ns}}\right)^{-1}$$

Functional Equation of $L(s, \phi)$

$L(s, \phi)$ is a degree n L-function. This means the completed L-function has n Gamma factors and satisfies the functional equation

$$\boxed{L^*(s, \phi) := \pi^{-\frac{ns}{2}} \prod_{i=1}^{n} \Gamma\left(\frac{s - \alpha_i}{2}\right) L(s, \phi) = L^*(1 - s, \widetilde{\phi})}$$

where $\widetilde{\phi}$ denotes the dual form which has M^{th} Fourier coefficient (for $M = (m_1, m_2, \ldots, m_{n-1})$) given by $A(m_{n-1}, m_{n-2}, \ldots, m_1)$.

Conjectured orthogonality relation for GL(n)

Conjectured orthogonality relation for GL(n)

Good Test Function

Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbb{C}^n$ be Langlands parameters.
A function $h_T(\alpha)$ is a *"good test function"* if it satisfies:

Conjectured orthogonality relation for GL(n)

Good Test Function

Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbb{C}^n$ be Langlands parameters.
A function $h_T(\alpha)$ is a *"good test function"* if it satisfies:

- *h_T is smooth, real valued and positive.*

Conjectured orthogonality relation for GL(n)

Good Test Function

Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbb{C}^n$ be Langlands parameters.
A function $h_T(\alpha)$ is a *"good test function"* if it satisfies:

- *h_T is smooth, real valued and positive.*
- $h_T(\alpha) \gg 1$ *if* $\sum_{k=1}^{n} |\alpha_k|^2 \le T^b$; *for fixed $b > 0$ as $T \to \infty$.*

Conjectured orthogonality relation for GL(n)

Good Test Function

Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbb{C}^n$ be Langlands parameters.
A function $h_T(\alpha)$ is a *"good test function"* if it satisfies:

- *h_T is smooth, real valued and positive.*
- $h_T(\alpha) \gg 1$ *if* $\sum_{k=1}^{n} |\alpha_k|^2 \le T^b$; *for fixed* $b > 0$ *as* $T \to \infty$.
- $h_T(\alpha)$ *has exponential decay as* $\sum_{k=1}^{n} |\alpha_k|^2 \to \infty$ *and* T *fixed;*

Conjectured orthogonality relation for GL(n)

Good Test Function

Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbb{C}^n$ be Langlands parameters.
A function $h_T(\alpha)$ is a *"good test function"* if it satisfies:

- *h_T is smooth, real valued and positive.*
- *$h_T(\alpha) \gg 1$ if $\sum_{k=1}^{n} |\alpha_k|^2 \le T^b$; for fixed $b > 0$ as $T \to \infty$.*
- *$h_T(\alpha)$ has exponential decay as $\sum_{k=1}^{n} |\alpha_k|^2 \to \infty$ and T fixed;*
- *h_T is invariant under permutation of Langlands parameters.*

Conjectured orthogonality relation for GL(n)

Good Test Function

Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbb{C}^n$ be Langlands parameters.
A function $h_T(\alpha)$ is a *"good test function"* if it satisfies:

- *h_T is smooth, real valued and positive.*
- $h_T(\alpha) \gg 1$ *if* $\sum_{k=1}^{n} |\alpha_k|^2 \leq T^b$; *for fixed* $b > 0$ *as* $T \to \infty$.
- $h_T(\alpha)$ *has exponential decay as* $\sum_{k=1}^{n} |\alpha_k|^2 \to \infty$ *and* T *fixed;*
- *h_T is invariant under permutation of Langlands parameters.*

Conjectured orthogonality relation for GL(n)

Conjectured orthogonality relation for GL(n)

Orthogonal Basis of Maass Forms

Let ϕ_1, ϕ_2, ϕ_3, ... be an orthogonal basis of Maass cusp forms for $SL(n,\mathbb{Z})$ where ϕ_j has Langlands parameter α_j.

Conjectured orthogonality relation for GL(n)

Orthogonal Basis of Maass Forms

Let ϕ_1, ϕ_2, ϕ_3, ... be an orthogonal basis of Maass cusp forms for $SL(n,\mathbb{Z})$ where ϕ_j has Langlands parameter α_j.

Two Fourier coefficients of a cusp form ϕ_j

Fix $n \geq 2$. For $j = 1, 2, 3, \ldots$ let

$$\boxed{A_j(M) = A_{\phi_j}(m_1, \ldots, m_{n-1}), \qquad A_j(L) = A_{\phi_j}(\ell_1, \ldots, \ell_{n-1}),}$$

denote two Fourier coefficients of a cusp form ϕ_j for $SL(n,\mathbb{Z})$.

Conjectured orthogonality relation for GL(n)

Orthogonal Basis of Maass Forms

Let $\phi_1, \phi_2, \phi_3, \ldots$ be an orthogonal basis of Maass cusp forms for $SL(n,\mathbb{Z})$ where ϕ_j has Langlands parameter α_j.

Two Fourier coefficients of a cusp form ϕ_j

Fix $n \geq 2$. For $j = 1, 2, 3, \ldots$ let

$$\boxed{A_j(M) = A_{\phi_j}(m_1, \ldots, m_{n-1}), \qquad A_j(L) = A_{\phi_j}(\ell_1, \ldots, \ell_{n-1}),}$$

denote two Fourier coefficients of a cusp form ϕ_j for $SL(n,\mathbb{Z})$.

Conjecture (Fan Zhou 2013)

$$\boxed{\lim_{T\to\infty} \frac{\sum\limits_{j=1}^{\infty} A_j(M)\overline{A_j(L)} \cdot \frac{H_T\left(\alpha^{(j)}\right)}{L(1, \mathrm{Ad}\,\phi_j)}}{\sum\limits_{j=1}^{\infty} \frac{H_T\left(\alpha^{(j)}\right)}{L(1, \mathrm{Ad}\,\phi_j)}} = \begin{cases} 1, & \text{if } M = L, \\ 0, & \text{otherwise.} \end{cases}}$$

Orthogonality relation for GL(2): History

Orthogonality relation for GL(2): History

1978:

R. Bruggeman: *Fourier coefficients of cusp forms*, Inventiones.

Orthogonality relation for GL(2): History

1978:

R. Bruggeman: *Fourier coefficients of cusp forms*, Inventiones.

Orthogonal basis of Maass cusp forms for $SL(2,\mathbb{Z})$

$$\boxed{\phi_j(z) = \sum_{n\neq 0} a_j(n)\sqrt{2\pi y}\, K_{it_j}(2\pi|n|y)\cdot e^{2\pi i n x},} \qquad (z = x+iy \in \mathfrak{h}^2).$$

Orthogonality relation for GL(2): History

1978:

R. Bruggeman: *Fourier coefficients of cusp forms*, Inventiones.

Orthogonal basis of Maass cusp forms for $SL(2,\mathbb{Z})$

$$\boxed{\phi_j(z) = \sum_{n\neq 0} a_j(n)\sqrt{2\pi y}\; K_{it_j}(2\pi|n|y)\cdot e^{2\pi inx},} \qquad (z = x+iy \in \mathfrak{h}^2).$$

Each ϕ_j has Langlands parameter $(it_j, -it_j)$ and Laplace eigenvalue

$$\lambda_j = 1/4 + t_j^2.$$

Orthogonality relation for GL(2): History

1978:

R. Bruggeman: *Fourier coefficients of cusp forms*, Inventiones.

Orthogonal basis of Maass cusp forms for $SL(2,\mathbb{Z})$

$$\boxed{\phi_j(z) = \sum_{n\neq 0} a_j(n)\sqrt{2\pi y}\; K_{it_j}(2\pi|n|y)\cdot e^{2\pi i n x},} \qquad (z = x+iy \in \mathfrak{h}^2).$$

Each ϕ_j has Langlands parameter $(it_j, -it_j)$ and Laplace eigenvalue

$$\lambda_j = 1/4 + t_j^2.$$

Bruggeman Orthogonality Relation

$$\lim_{T\to\infty} \frac{4\pi^2}{T} \sum_{j=1}^{\infty} \frac{a_j(m)\,\overline{a_j(n)}}{\cosh(\pi t_j)} \cdot e^{-\lambda_j/T} = \begin{cases} 1 & \text{if } m = n, \\ 0 & \text{if } m \neq n. \end{cases}$$

Orthogonality relation for GL(2): History

1999: W. Luo (*Values of symmetric square L-functions at 1,*) J. Reine Ang. Math. Found removal technique for adjoint L-function.

Orthogonality relation for GL(2): History

1999: W. Luo (*Values of symmetric square L-functions at 1,*) J. Reine Ang. Math. Found removal technique for adjoint L-function.

The following papers derive orthogonality relations for holomorphic modular forms on $GL(2)$ and do not include the adjoint L-function.

1984: P. Sarnak (*Statistical properties of eigenvalues of the Hecke operators,* Proc. Conf. at Oklahoma State Univ.

1997: B. Conrey, W. Duke, D. Farmer (*The distribution of the eigenvalues of Hecke operators,*) Acta Arith.

1997: J.P. Serre (*Répartition asymptotique des valeurs propres de l'operateur de Hecke* T_p, J. Amer. Math. Soc.

Orthogonality relation for GL(3)

Let $\phi_1, \phi_2, \phi_3, \ldots$ be a Hecke basis of cusp forms for $SL(3, \mathbb{Z})$ with Langlands parameters $\alpha^{(1)}, \alpha^{(2)}, \alpha^{(3)}, \ldots$ Let $\mathcal{L}_j = L(1, \mathrm{Ad}\, \phi_j)$.

Orthogonality relation for GL(3)

Let $\phi_1, \phi_2, \phi_3, \ldots$ be a Hecke basis of cusp forms for $SL(3, \mathbb{Z})$ with Langlands parameters $\alpha^{(1)}, \alpha^{(2)}, \alpha^{(3)}, \ldots$ Let $\mathcal{L}_j = L(1, \mathrm{Ad}\, \phi_j)$.

Theorem: (Kontorovich-G (2012))

Let $T \to \infty$, and $H_T : \mathbb{C}^2 \to \mathbb{R}$ be a family of good test functions essentially supported on cusp forms ϕ_j with eigenvalue $\leq T^2$. Fix $M = (m_1, m_2)$ and $L = (\ell_1, \ell_2)$. Then for all $\epsilon > 0$,

$$\frac{\sum\limits_{j=1}^{\infty} A_j(M)\overline{A_j(L)} \cdot \frac{H_T(\alpha^{(j)})}{\mathcal{L}_j}}{\sum\limits_{j=1}^{\infty} \frac{H_T(\alpha^{(j)})}{\mathcal{L}_j}} = \delta_{M,L} + \mathcal{O}\left(|m_1 m_2 \ell_1 \ell_2|^2 \, T^{-2+\epsilon} \right).$$

Orthogonality relation for GL(3)

Let $\phi_1, \phi_2, \phi_3, \ldots$ be a Hecke basis of cusp forms for $SL(3, \mathbb{Z})$ with Langlands parameters $\alpha^{(1)}, \alpha^{(2)}, \alpha^{(3)}, \ldots$ Let $\mathcal{L}_j = L(1, \mathrm{Ad}\, \phi_j)$.

Theorem: (Kontorovich-G (2012))

Let $T \to \infty$, and $H_T : \mathbb{C}^2 \to \mathbb{R}$ be a family of good test functions essentially supported on cusp forms ϕ_j with eigenvalue $\leq T^2$. Fix $M = (m_1, m_2)$ and $L = (\ell_1, \ell_2)$. Then for all $\epsilon > 0$,

$$\frac{\sum\limits_{j=1}^{\infty} A_j(M)\overline{A_j(L)} \cdot \frac{H_T(\alpha^{(j)})}{\mathcal{L}_j}}{\sum\limits_{j=1}^{\infty} \frac{H_T(\alpha^{(j)})}{\mathcal{L}_j}} = \delta_{M,L} + \mathcal{O}\left(|m_1 m_2 \ell_1 \ell_2|^2 \, T^{-2+\epsilon} \right).$$

Remark: *A result of this type was proved independently by V. Blomer, Inventiones Math. (2012).*

Choice of Test Function for $GL(n)$

Ramanujan conjecture at ∞

We want to study cusp forms on $GL(n)$ with Langlands parameter $\alpha := (\alpha_1, \alpha_2, \ldots \alpha_n)$. Then $\frac{-\alpha_1^2 - \alpha_2^2 - \cdots - \alpha_n^2}{2}$ is essentially the Laplace eigenvalue of the cusp form. The Selberg eigenvalue conjecture predicts $\alpha_j \in i\,\mathbb{R}$ (the cusp form is tempered at ∞).

Choice of Test Function for $GL(n)$

Ramanujan conjecture at ∞

We want to study cusp forms on $GL(n)$ with Langlands parameter $\alpha := (\alpha_1, \alpha_2, \ldots \alpha_n)$. Then $\frac{-\alpha_1^2 - \alpha_2^2 - \cdots - \alpha_n^2}{2}$ is essentially the Laplace eigenvalue of the cusp form. The Selberg eigenvalue conjecture predicts $\alpha_j \in i\,\mathbb{R}$ (the cusp form is tempered at ∞).

Test Function

Let $T \to +\infty$ and $R \gg 1$ be fixed. Define

$$H_{T,R}(\alpha) = e^{\frac{\alpha_1^2+\alpha_2^2+\cdots+\alpha_n^2}{2T^2}} \cdot \frac{\prod\limits_{1 \le j \ne k \le n} \Gamma\left(\frac{2+R+\alpha_j-\alpha_k}{4}\right)^2}{\prod\limits_{1 \le j \ne k \le n} \Gamma\left(\frac{1+\alpha_j-\alpha_k}{2}\right)}.$$

Then $H_{T,R}$ has support on eigenvalues $\ll T^2$. This choice is motivated by the fact that we need $H_{T,R}$ to be invariant under the action of the Weyl group.

Orthogonality relation for GL(4)

Let $\{\phi_j \mid j = 1, 2, 3, \ldots\}$ be a Hecke basis of cusp forms for $SL(4, \mathbb{Z})$ with Langlands parameters $\alpha^{(j)} = (\alpha_1^{(j)}, \alpha_2^{(j)}, \alpha_3^{(j)}, \alpha_4^{(j)})$. We assume each Maass cusp form ϕ_j is normalized so that its first Fourier coefficient $A_j(1, 1, 1) = 1$.

Orthogonality relation for GL(4)

Let $\{\phi_j \mid j = 1, 2, 3, \ldots\}$ be a Hecke basis of cusp forms for $SL(4, \mathbb{Z})$ with Langlands parameters $\alpha^{(j)} = (\alpha_1^{(j)}, \alpha_2^{(j)}, \alpha_3^{(j)}, \alpha_4^{(j)})$. We assume each Maass cusp form ϕ_j is normalized so that its first Fourier coefficient $A_j(1,1,1) = 1$.

Theorem: (Woodbury-Stade-G 2019)

Let $T \to +\infty$ and let $R \gg 1$ be fixed. Let $\ell, m \in \mathbb{Z}$ with $\ell m \neq 0$. Then for all $\varepsilon > 0$,

$$\sum_{j=1}^{\infty} A_j(\ell,1,1)\,\overline{A_j(m,1,1)}\,\frac{H_{T,R}\left(\alpha^{(j)}\right)}{\mathcal{L}_j}$$

$$= \delta_{\ell,m} \cdot \left(\mathfrak{c}_1\,T^{\,9+6R} + \mathfrak{c}_2\,T^{\,8+6R} + \mathfrak{c}_3\,T^{\,7+6R}\right)$$

$$+ \mathcal{O}_{\varepsilon,R}\Big(|\ell m|^7 \cdot T^{\,6+6R+\varepsilon}\Big).$$

Orthogonality relation for GL(4)

Let $\{\phi_j \mid j = 1, 2, 3, \ldots\}$ be a Hecke basis of cusp forms for $SL(4, \mathbb{Z})$ with Langlands parameters $\alpha^{(j)} = (\alpha_1^{(j)}, \alpha_2^{(j)}, \alpha_3^{(j)}, \alpha_4^{(j)})$. We assume each Maass cusp form ϕ_j is normalized so that its first Fourier coefficient $A_j(1,1,1) = 1$.

Theorem: (Woodbury-Stade-G 2019)

Let $T \to +\infty$ and let $R \gg 1$ be fixed. Let $\ell, m \in \mathbb{Z}$ with $\ell m \neq 0$. Then for all $\varepsilon > 0$,

$$\sum_{j=1}^{\infty} A_j(\ell,1,1)\,\overline{A_j(m,1,1)}\,\frac{H_{T,R}\left(\alpha^{(j)}\right)}{\mathcal{L}_j}$$

$$= \delta_{\ell,m} \cdot \left(\mathfrak{c}_1\, T^{\,9+6R} + \mathfrak{c}_2\, T^{\,8+6R} + \mathfrak{c}_3\, T^{\,7+6R}\right) + \mathcal{O}_{\varepsilon,R}\left(|\ell m|^7 \cdot T^{\,6+6R+\varepsilon}\right).$$

This result is new and proves Fan Zhou's conjecture for $GL(4)$ with a power savings error term.

Whittaker Transform

Let $f : \mathbb{R}_+^{n-1} \to \mathbb{C}$ be an integrable function and

$$y := (y_1, y_2, \ldots y_{n-1}), \qquad y^* := \begin{pmatrix} y_1 y_2 \cdots y_{n-1} & & & \\ & \ddots & & \\ & & y_1 & \\ & & & 1 \end{pmatrix}.$$

Whittaker Transform

Let $f : \mathbb{R}_+^{n-1} \to \mathbb{C}$ be an integrable function and

$$y := (y_1, y_2, \ldots y_{n-1}), \qquad y^* := \begin{pmatrix} y_1 y_2 \cdots y_{n-1} & & & \\ & \ddots & & \\ & & y_1 & \\ & & & 1 \end{pmatrix}.$$

Whittaker Transform

$$\boxed{f^{\#}(\alpha) := \int\limits_{y_1=0}^{\infty} \cdots \int\limits_{y_{n-1}=0}^{\infty} f(y)\, W_\alpha(y^*) \prod_{k=1}^{n-1} \frac{dy_k}{y_k^{k(n-k)+1}}.}$$

Whittaker Transform

Let $f : \mathbb{R}_+^{n-1} \to \mathbb{C}$ be an integrable function and

$$y := (y_1, y_2, \ldots y_{n-1}), \qquad y^* := \begin{pmatrix} y_1 y_2 \cdots y_{n-1} & & & \\ & \ddots & & \\ & & y_1 & \\ & & & 1 \end{pmatrix}.$$

Whittaker Transform

$$f^{\#}(\alpha) := \int_{y_1=0}^{\infty} \cdots \int_{y_{n-1}=0}^{\infty} f(y)\, W_\alpha(y^*) \prod_{k=1}^{n-1} \frac{dy_k}{y_k^{k(n-k)+1}}.$$

Inverse Whittaker Transform (kontorovich-G 2012)

$$f(y) = \frac{1}{\pi^{n-1}} \int_{\mathrm{Re}(\alpha_1)=0} \cdots \int_{\mathrm{Re}(\alpha_{n-1})=0} f^{\#}(\alpha) W_{-\alpha}(t(y)) \frac{d\alpha_1 d\alpha_2 \cdots d\alpha_{n-1}}{\prod\limits_{1\le k\ne \ell \le n} \Gamma\left(\frac{\alpha_k - \alpha_\ell}{2}\right)}.$$

Poincaré Series for $GL(n, \mathbb{R})$

The $p^{\#}_{T,R}$ function

$$p^{\sharp}_{T,R}(\alpha) \;:=\; e^{\frac{\alpha_1^2+\alpha_2^2+\,\cdots\,+\alpha_n^2}{2T^2}} \prod_{1\le j\ne k\le n} \Gamma\left(\frac{2+R+\alpha_j-\alpha_k}{4}\right)$$

Poincaré Series for $GL(n, \mathbb{R})$

The $p^{\#}_{T,R}$ function

$$p^{\sharp}_{T,R}(\alpha) := e^{\frac{\alpha_1^2+\alpha_2^2+\cdots+\alpha_n^2}{2T^2}} \prod_{1 \le j \ne k \le n} \Gamma\left(\frac{2+R+\alpha_j-\alpha_k}{4}\right)$$

(Poincaré Series)

Let $g \in \mathfrak{h}^n$, $M = (m_1, m_2, \ldots, m_{n-1}) \in \mathbb{Z}^{n-1}$. The $SL(n, \mathbb{Z})$ Poincaré series is defined by

$$\boxed{P^M(g) := \sum_{\gamma \in U_n(\mathbb{Z}) \backslash SL(n,\mathbb{Z})} \psi_M(\gamma g)\, p_{T,R}(\gamma g).}$$

Set $M = (m_1, m_2, \ldots, m_{n-1})$ and $L = (\ell_1, \ell_2, \ldots, \ell_{n-1})$.

Let $P^M(g), P^L(g)$ be two Poincaré series. The Kuznetsov trace formula is an evaluation of the Petersson inner product

$$\langle P^M, P^L \rangle = \int\limits_{SL(n,\mathbb{Z})\backslash\mathfrak{h}^n} P^M(g) \cdot \overline{P^L(g)}\, d^*g$$

in two different ways.

Set $M = (m_1, m_2, \ldots, m_{n-1})$ and $L = (\ell_1, \ell_2, \ldots, \ell_{n-1})$.

Let $P^M(g), P^L(g)$ be two Poincaré series. The Kuznetsov trace formula is an evaluation of the Petersson inner product

$$\langle P^M, P^L \rangle = \int\limits_{SL(n,\mathbb{Z})\backslash \mathfrak{h}^n} P^M(g) \cdot \overline{P^L(g)}\, d^*g$$

in two different ways.

Using SPECTRAL THEORY.

Set $M = (m_1, m_2, \ldots, m_{n-1})$ and $L = (\ell_1, \ell_2, \ldots, \ell_{n-1})$.

Let $P^M(g), P^L(g)$ be two Poincaré series. The Kuznetsov trace formula is an evaluation of the Petersson inner product

$$\langle P^M,\, P^L \rangle = \int\limits_{SL(n,\mathbb{Z})\backslash\mathfrak{h}^n} P^M(g)\cdot \overline{P^L(g)}\, d^*g$$

in two different ways.

<u>Using SPECTRAL THEORY.</u>

<u>Using GEOMETRY.</u>

Trace formula

$$\boxed{\underbrace{\mathcal{C} + \mathcal{E}}_{\textit{spectral side}} = \underbrace{\mathcal{M} + \mathcal{K}}_{\textit{geometric side}}.}$$

$GL(n, \mathbb{R})$ Kuznetsov trace formula

Trace formula

$$\boxed{\underbrace{\mathcal{C} + \mathcal{E}}_{\textit{spectral side}} = \underbrace{\mathcal{M} + \mathcal{K}}_{\textit{geometric side}}.}$$

$$\mathcal{C} = \text{cuspidal contribution} = \sum_{j=1}^{\infty} A_j(M)\overline{A_j(L)} \cdot \frac{H_{T,R}(\alpha^{(j)})}{\mathcal{L}_J}$$

Trace formula

$$\boxed{\underbrace{\mathcal{C} + \mathcal{E}}_{\textit{spectral side}} = \underbrace{\mathcal{M} + \mathcal{K}}_{\textit{geometric side}}.}$$

$$\mathcal{C} = \text{cuspidal contribution} = \sum_{j=1}^{\infty} A_j(M)\overline{A_j(L)} \cdot \frac{H_{T,R}(\alpha^{(j)})}{\mathcal{L}_J}$$

$$\mathcal{M} = \text{Main Term} \sim \delta_{M,L} \cdot c\, T^{\frac{n(n-1)R}{2} + \frac{(n+2)(n-1)}{2}}$$

$\mathcal{E}$ = Eisenstein contribution, $\mathcal{K}$ = Kloosterman sum contribution. These should be small for suitable choice of the test function $H_{T,R}$.

Let $M = (m_1, m_2, \ldots, m_{n-1})$ and $L = (\ell_1, \ell_2, \ldots, \ell_{n-1})$.

The Kloosterman contribution to the the trace formula is the most difficult to estimate. It arises from the L^{th} Fourier coefficient of the Poincaré series $P^M(g,\, s)$ and is given by

$$\int\limits_{U_n(\mathbb{Z})\backslash U_n(\mathbb{R})} P^M(ug,\, s) \cdot \overline{\psi_L(u)}\, d^*u \;=\; \sum_{w\in W_n} \sum_{v\in V_n} \sum_{c_1=1}^{\infty} \cdots \sum_{c_{n-1}=1}^{\infty} \cdot S_w(\psi_M, \psi_L^v, c)\, J_w(g; s, \psi_M, \psi_L^v, c)$$

where S_w is the Kloosterman sum and J_w is the Kloosterman integral.

Key Idea for bounding the Kloosterman terms on $GL(n)$

• *When computing the inner product of two Poincaré series formed with a test function, the Whittaker transform of the test function arises along with Kloosterman sums. We take the Mellin transform of the Whittaker function and then take the inverse Mellin transform to recover the Whittaker function. The final result is a multiple integral over a ratio of many Gamma functions.*

Key Ideas for bounding the Kloosterman terms on $GL(n)$

• *A theorem of Friedberg (1987) gives methods for explicitly evaluating Kloosterman sums arising in the trace formula and showing many of them are identically zero.*

- *A theorem of Friedberg (1987) gives methods for explicitly evaluating Kloosterman sums arising in the trace formula and showing many of them are identically zero.*

Example 4

The **non-zero** (non-identity) Kloosterman sums on $GL(n)$ are associated to the following Weyl group elements:

$$n = 2, \quad \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}$$

$$n = 3, \quad \begin{pmatrix} & & 1 \\ 1 & & \\ & 1 & \end{pmatrix}, \begin{pmatrix} & 1 & \\ & & 1 \\ 1 & & \end{pmatrix}, \begin{pmatrix} & & 1 \\ & 1 & \\ 1 & & \end{pmatrix}$$

$$n = 4, \quad \begin{pmatrix} & & & 1 \\ 1 & & & \\ & 1 & & \\ & & 1 & \end{pmatrix}, \begin{pmatrix} & 1 & & \\ & & 1 & \\ & & & 1 \\ 1 & & & \end{pmatrix}, \begin{pmatrix} & & 1 & \\ & & & 1 \\ 1 & & & \\ & 1 & & \end{pmatrix}, \begin{pmatrix} & & & 1 \\ & 1 & & \\ & & 1 & \\ 1 & & & \end{pmatrix},$$

$$\begin{pmatrix} & & 1 & \\ & & & 1 \\ & 1 & & \\ 1 & & & \end{pmatrix}, \begin{pmatrix} & & & 1 \\ & & 1 & \\ 1 & & & \\ & 1 & & \end{pmatrix}, \begin{pmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{pmatrix}.$$

Key Ideas for bounding the Kloosterman terms on $GL(n)$

• *Some of the $GL(n)$ Kloosterman sums can be bounded using Deligne's (1974) estimates obtained from his proof of the Riemann hypothesis for varieties over finite fields. In addition to this other methods have been developed.*

We may mention:
Bump-Friedberg-G-Larsen (1987), Stevens (1987),
Friedberg (1987), Dabrowski-Reeder (1998), Huang (2016).

Key Ideas for bounding the Kloosterman terms on $GL(n)$

• *Another important ingredient in bounding Kloosterman contributions to the trace formula is* <u>*Stade's formula*</u> *(2001) for Mellin transforms of Whittaker functions.*

Key Ideas for bounding the Kloosterman terms on $GL(n)$

- *Another important ingredient in bounding Kloosterman contributions to the trace formula is* <u>*Stade's formula*</u> *(2001) for Mellin transforms of Whittaker functions.*

Example 5

Here are examples of Mellin transforms of Whittaker functions

$$W_\alpha^{(n)}(s) := \int_0^\infty \cdots \int_0^\infty W_\alpha^{(n)}(y) \prod_{j=1}^{n-1} y_j^{s_j - \frac{j(n-j)}{2}} \frac{dy_j}{y_j},$$

$\alpha = (\alpha_1, \ldots, \alpha_n) =$ spectral parameter and $\alpha_1 + \cdots + \alpha_n = 0$.

$$W_\alpha^{(2)}(s) = \Gamma\left(\tfrac{s+\alpha_1}{2}\right)\Gamma\left(\tfrac{s+\alpha_2}{2}\right)$$

$$W_\alpha^{(3)}(s) = \frac{\Gamma(\frac{s_1+\alpha_1}{2})\Gamma(\frac{s_1+\alpha_2}{2})\Gamma(\frac{s_1+\alpha_3}{2})\Gamma(\frac{s_2-\alpha_1}{2})\Gamma(\frac{s_2-\alpha_2}{2})\Gamma(\frac{s_2-\alpha_3}{2})}{\Gamma(\frac{s_1+s_2}{2})}$$

The last example for $n = 3$ is due to Bump (1984).

Example 6

Stade's formula for $GL(4)$ takes the form:

$$\widetilde{W}_{\alpha}^{(4)}(s) = \frac{\Gamma(\frac{s_1+\alpha_1}{2})\Gamma(\frac{s_1+\alpha_2}{2})\Gamma(\frac{s_2-\alpha_1-\alpha_2}{2})\Gamma(\frac{s_2+\alpha_1+\alpha_2}{2})\Gamma(\frac{s_3-\alpha_1}{2})\Gamma(\frac{s_3-\alpha_2}{2})}{8\pi^{s_1+s_2+s_3}}$$

$$\cdot \frac{1}{2\pi i} \int\limits_{\mathrm{Re}(s_0)=-\epsilon} \frac{\Gamma(\frac{-2s_0+\alpha_3}{2})\Gamma(\frac{-2s_0+\alpha_4}{2})\Gamma(\frac{2s_0+s_1}{2})\Gamma(\frac{2s_0+s_2+\alpha_1}{2})}{\Gamma(\frac{2s_0+s_1+s_2+\alpha_1+\alpha_2}{2})}$$

$$\cdot \frac{\Gamma(\frac{2s_0+s_2+\alpha_2}{2})\Gamma(\frac{2s_0+s_3+\alpha_1+\alpha_2}{2})}{\Gamma(\frac{2s_0+s_2+s_3}{2})}\, ds_0.$$

Key Ideas for bounding the Kloosterman terms on $GL(n)$

• *Stade (2001), (2018) explicitly obtained the poles and residues of the Mellin transform of the Whittaker function* $\widetilde{W}_{\alpha}^{(n)}(s)$*. This is important because there are Kloosterman terms in the trace formula for every pole.*

Key Ideas for bounding the Kloosterman terms on $GL(n)$

• *Shift relations for the Mellin transform* $\widetilde{W}_\alpha^{(n)}(s)$ *of the Whittaker function play a major role in estimating the Kloosterman terms in the trace formula. The shift equations were first worked out in Friedberg-Goldfeld (1993) and more recently by Stade (2018).*

When $\widetilde{W}_\alpha^{(n)}(s)$ is just a ratio of Gamma functions then the shift equations come from

$$\Gamma(s+1) = s\,\Gamma(s).$$

When $n \geq 4$ the situation changes drastically.

• *Shift relations for the Mellin transform* $\widetilde{W}_{\alpha}^{(n)}(s)$ *of the Whittaker function play a major role in estimating the Kloosterman terms in the trace formula. The shift equations were first worked out in Friedberg-Goldfeld (1993) and more recently by Stade (2018).*

When $\widetilde{W}_{\alpha}^{(n)}(s)$ is just a ratio of Gamma functions then the shift equations come from

$$\Gamma(s+1) = s\,\Gamma(s).$$

When $n \geq 4$ the situation changes drastically.

Next we turn to the Eisenstein series appearing in the $\mathcal{E}$ *the Eisenstein term in the Kuznetsov trace formula.*

The Template Method (S.D. Miller, M. Woodbury, G)

Template

According to the Merriam-Webster dictionary a *template* is:
"*a gauge, pattern, or mold used as a guide*" for producing other similar things.

The Template Method (S.D. Miller, M. Woodbury, G)

Template

According to the Merriam-Webster dictionary a *template* is:
"*a gauge, pattern, or mold used as a guide*" for producing other similar things.

In the theory of automorphic forms, Eisenstein series, in particular those associated to the Borel subgroup, can sometimes serve as a template to deduce properties of other automorphic forms.

x, y Coordinates for $\mathfrak{h}^3$

$$x = \begin{pmatrix} 1 & x_{12} & x_{13} \\ & 1 & x_{23} \\ & & 1 \end{pmatrix}, \qquad y = \begin{pmatrix} y_1 y_2 & & \\ & y_1 & \\ & & 1 \end{pmatrix}.$$

The Template Method for $GL(3)$ Eisenstein Series

x, y Coordinates for $\mathfrak{h}^3$

$$x = \begin{pmatrix} 1 & x_{12} & x_{13} \\ & 1 & x_{23} \\ & & 1 \end{pmatrix}, \qquad y = \begin{pmatrix} y_1 y_2 & & \\ & y_1 & \\ & & 1 \end{pmatrix}.$$

Power function for Langlands parameter $\alpha = (\alpha_1, \alpha_2, \alpha_3)$

$$\mathrm{I}(g, \alpha) = y_1^{1-\alpha_3} y_2^{1+\alpha_1}, \qquad (g = xy \in \mathfrak{h}^3).$$

The Template Method for $GL(3)$ Eisenstein Series

x, y Coordinates for $\mathfrak{h}^3$

$$x = \begin{pmatrix} 1 & x_{12} & x_{13} \\ & 1 & x_{23} \\ & & 1 \end{pmatrix}, \qquad y = \begin{pmatrix} y_1 y_2 & & \\ & y_1 & \\ & & 1 \end{pmatrix}.$$

Power function for Langlands parameter $\alpha = (\alpha_1, \alpha_2, \alpha_3)$

$$\mathrm{I}(g, \alpha) = y_1^{1-\alpha_3} y_2^{1+\alpha_1}, \qquad (g = xy \in \mathfrak{h}^3).$$

Borel Eisenstein Series

$$\mathcal{B} := \left(\begin{smallmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{smallmatrix}\right), \qquad \boxed{E_{\mathcal{B}}(g, \alpha) := \sum_{\gamma \in U_3(\mathbb{Z}) \backslash SL(3,\mathbb{Z})} \mathrm{I}(\gamma g, \alpha).}$$

The Template Method for $GL(3)$ Eisenstein Series

Character of $U_3(\mathbb{R})$

$$M = (m_1, m_2) \in \mathbb{Z}^2, \qquad x = \begin{pmatrix} 1 & x_{12} & x_{13} \\ & 1 & x_{23} \\ & & 1 \end{pmatrix}$$

$$\boxed{\psi_M(x) = e^{2\pi i \left(m_1 x_{12} + m_2 x_{23} \right)}.}$$

The Template Method for $GL(3)$ Eisenstein Series

Character of $U_3(\mathbb{R})$

$$M = (m_1, m_2) \in \mathbb{Z}^2, \qquad x = \begin{pmatrix} 1 & x_{12} & x_{13} \\ & 1 & x_{23} \\ & & 1 \end{pmatrix}$$

$$\boxed{\psi_M(x) = e^{2\pi i \left(m_1 x_{12} + m_2 x_{23} \right)}.}$$

Fourier Coefficient of the Borel Eisenstein series

$$\boxed{\int\limits_{U_3(\mathbb{Z})\backslash U_3(\mathbb{R})} E_{\mathcal{B}}(ug, \alpha)\psi_M(u)du = A_{E_{\mathcal{B}}}(M, \alpha) W_\alpha(M^* g)}$$

for

$$M = (m_1, m_2), \qquad M^* = \begin{pmatrix} m_1 m_2 & & \\ & m_1 & \\ & & 1 \end{pmatrix}$$

The Template Method for $GL(3)$ Eisenstein Series

The first coefficient of the Borel Eisenstein series

Theorem(Selberg, Imai/Terras, Vinogradov/Tahktadshyan, Bump): *Let* $\alpha = (\alpha_1, \alpha_2, \alpha_3) \in \mathbb{C}^3$ *with* $\alpha_1 + \alpha_2 + \alpha_3 = 0$. *Then*

$$A_{E_{\mathcal{B}}}((1,1),\,\alpha) = \frac{1}{\zeta^*(1+\alpha_1-\alpha_2)\zeta^*(1+\alpha_2-\alpha_3)\zeta^*(1+\alpha_1-\alpha_3)}.$$

There is another Eisenstein series for $SL(3,\mathbb{Z})$ associated to the parabolic subgroup $\mathcal{P}_{2,1} = \left(\begin{smallmatrix} * & * & * \\ * & * & * \\ 0 & 0 & * \end{smallmatrix}\right)$ by inducing an $SL(2,\mathbb{Z})$ cusp form ϕ with Laplace eigenvalue $1/4+\nu^2$ on the 2×2 Levi in red:

$$\begin{pmatrix} {\color{red}*} & {\color{red}*} & * \\ {\color{red}*} & {\color{red}*} & * \\ 0 & 0 & * \end{pmatrix}.$$

The Template Method for $GL(3)$ Eisenstein Series

There is another Eisenstein series for $SL(3,\mathbb{Z})$ associated to the parabolic subgroup $\mathcal{P}_{2,1} = \begin{pmatrix} * & * & * \\ * & * & * \\ 0 & 0 & * \end{pmatrix}$ by inducing an $SL(2,\mathbb{Z})$ cusp form ϕ with Laplace eigenvalue $1/4+\nu^2$ on the 2×2 Levi in red:

$$\begin{pmatrix} * & * & * \\ * & * & * \\ 0 & 0 & * \end{pmatrix}.$$

The Eisenstein series $E_{\mathcal{P}_{2,1},\phi}$ for the parabolic $\mathcal{P}_{2,1}$

Let $y = \begin{pmatrix} y_1y_2 & & \\ & y_1 & \\ & & 1 \end{pmatrix}$.

$$E_{\mathcal{P}_{2,1},\phi}(y,s) := \sum_{\gamma\in\mathcal{P}_{2,1}\backslash SL(3,\mathbb{Z})} \underbrace{\phi\left(\left(\begin{smallmatrix} y_1y_2 & \\ & y_1 \end{smallmatrix}\right)\right)}_{=\phi\left(\left(\begin{smallmatrix} y_2 & \\ & 1 \end{smallmatrix}\right)\right)} \left(y_1^2y_2\right)^{\frac{1}{2}+s} \Bigg|_{\gamma}$$

Here $F(g)\Big|_{\gamma} := F(\gamma g)$ for all $g\in\mathfrak{h}^3$.

The key idea in the template method is to replace the cusp form ϕ with a Borel Eisenstein series E^* on $GL(2)$ with the same Langlands parameters as ϕ. We get a new Eisenstein series $E_{\text{new}}(*, s)$ given by

The Template Method for $GL(3)$ Eisenstein Series

The key idea in the template method is to replace the cusp form ϕ with a Borel Eisenstein series E^* on $GL(2)$ with the same Langlands parameters as ϕ. We get a new Eisenstein series $E_{\text{new}}(*, s)$ given by

The Borel Eisenstein series E_{new}

$$E_{\text{new}}(y, s) = \sum_{\gamma \in \mathcal{P}_{2,1} \backslash SL(3,\mathbb{Z})} E^*\left(\left(\begin{smallmatrix} y_1 y_2 & \\ & y_1 \end{smallmatrix}\right), \nu\right) \left(y_1^2 y_2\right)^{\frac{1}{2}+s} \Big|_{\gamma}$$

The key idea in the template method is to replace the cusp form ϕ with a Borel Eisenstein series E^* on $GL(2)$ with the same Langlands parameters as ϕ. We get a new Eisenstein series $E_{\rm new}(*, s)$ given by

The Borel Eisenstein series $E_{\rm new}$

$$E_{\rm new}(y, s) = \sum_{\gamma \in \mathcal{P}_{2,1} \backslash SL(3,\mathbb{Z})} E^* \left(\begin{pmatrix} y_1 y_2 & \\ & y_1 \end{pmatrix}, \nu \right) \left(y_1^2 y_2 \right)^{\frac{1}{2}+s} \Bigg|_{\gamma}$$

Theorem

$$A_{E_{\rm new}}((1,1), s) = \Big(\zeta^*(1+3s+v)\zeta^*(1+3s-v) \Big)^{-1}.$$

The Template Method for $GL(3)$ Eisenstein Series

The key idea in the template method is to replace the cusp form ϕ with a Borel Eisenstein series E^* on $GL(2)$ with the same Langlands parameters as ϕ. We get a new Eisenstein series $E_{\mathrm{new}}(*, s)$ given by

The Borel Eisenstein series E_{new}

$$E_{\mathrm{new}}(y, s) = \sum_{\gamma \in \mathcal{P}_{2,1} \backslash SL(3,\mathbb{Z})} E^*\left(\begin{pmatrix} y_1 y_2 & \\ & y_1 \end{pmatrix}, \nu\right) \left(y_1^2 y_2\right)^{\frac{1}{2}+s} \Bigg|_{\gamma}$$

Theorem

$$A_{E_{\mathrm{new}}}((1,1), s) = \Big(\zeta^*(1+3s+v)\zeta^*(1+3s-v)\Big)^{-1}.$$

$$\boxed{A_{E_{\mathcal{P}_{2,1},\phi}}((1,1), s) = \Big(L^*(1, \mathrm{Ad}\ \phi)^{1/2} \cdot L^*(\phi, 1+3s)\Big)^{-1}.}$$

$$\mathcal{P}_{2,2} = \begin{pmatrix} * & * & * & * \\ * & * & * & * \\ & & * & * \\ & & * & * \end{pmatrix}$$

$$\mathcal{P}_{2,2} = \begin{pmatrix} * & * & * & * \\ * & * & * & * \\ & & * & * \\ & & * & * \end{pmatrix}$$

$SL(2,\mathbb{Z})$ cusp forms ϕ_1, ϕ_2 with Laplace eigenvalues $1/4+v_1^2$, $1/4+v_2^2$.

$$\mathcal{P}_{2,2} = \begin{pmatrix} * & * & * & * \\ * & * & * & * \\ & & * & * \\ & & * & * \end{pmatrix}$$

$SL(2,\mathbb{Z})$ cusp forms ϕ_1, ϕ_2 with Laplace eigenvalues $1/4+v_1^2$, $1/4+v_2^2$.

$SL(2,\mathbb{Z})$ Borel Eisenstein series E_1^*, E_2^* matching ϕ_1, ϕ_2.

The Template Method for $GL(4)$ Eisenstein Series

$$\mathcal{P}_{2,2} = \begin{pmatrix} * & * & * & * \\ * & * & * & * \\ & & * & * \\ & & * & * \end{pmatrix}$$

$SL(2,\mathbb{Z})$ cusp forms ϕ_1, ϕ_2 with Laplace eigenvalues $1/4+v_1^2$, $1/4+v_2^2$.

$SL(2,\mathbb{Z})$ Borel Eisenstein series E_1^*, E_2^* matching ϕ_1, ϕ_2.

The Borel Eisenstein series E_{new}

$s = (1+s_1,\ -1+s_2$ where $2s_1 + 2s_2 = 0$.

$$E_{\mathrm{new}}(y,s) = \sum_{\gamma \in \mathcal{P}_{2,2}\backslash SL(4,\mathbb{Z})} E_1^*\left(\left(\begin{smallmatrix} y_3 & \\ & 1 \end{smallmatrix}\right), \nu_1\right) E_2^*\left(\left(\begin{smallmatrix} y_1 & \\ & 1 \end{smallmatrix}\right), \nu_2\right) \left(y_1^2 y_2^2 y_3\right)^{1+s_1} y_1^{-1+s_2} \Bigg|_{\gamma}$$

$$\mathcal{P}_{2,2} = \begin{pmatrix} * & * & * & * \\ * & * & * & * \\ & & * & * \\ & & * & * \end{pmatrix}$$

$SL(2,\mathbb{Z})$ cusp forms ϕ_1, ϕ_2 with Laplace eigenvalues $1/4+v_1^2$, $1/4+v_2^2$.

$SL(2,\mathbb{Z})$ Borel Eisenstein series E_1^*, E_2^* matching ϕ_1, ϕ_2.

The Borel Eisenstein series E_{new}

$s = (1+s_1, \ -1+s_2$ where $2s_1 + 2s_2 = 0$.

$$E_{\mathrm{new}}(y,s) = \sum_{\gamma \in \mathcal{P}_{2,2}\backslash SL(4,\mathbb{Z})} E_1^*\left(\left(\begin{smallmatrix} y_3 & \\ & 1 \end{smallmatrix}\right), \nu_1\right) E_2^*\left(\left(\begin{smallmatrix} y_1 & \\ & 1 \end{smallmatrix}\right), \nu_2\right) \left(y_1^2 y_2^2 y_3\right)^{1+s_1} y_1^{-1+s_2} \Bigg|_{\gamma}$$

Theorem

$$A_{E_{\mathrm{new}}}(\mathbf{1}) = \Big(\zeta^*(1+2s_1-v-v')\zeta^*(1+2s_1+v-v')\zeta^*(1+2s_1-v+v')\zeta^*(1+2s_1+v+v')\Big)^{-1}$$

The Template Method for $GL(4)$ Eisenstein Series

$$\mathcal{P}_{2,2} = \begin{pmatrix} * & * & * & * \\ * & * & * & * \\ & & * & * \\ & & * & * \end{pmatrix}$$

$SL(2,\mathbb{Z})$ cusp forms ϕ_1, ϕ_2 with Laplace eigenvalues $1/4+\nu_1^2$, $1/4+\nu_2^2$.

$SL(2,\mathbb{Z})$ Borel Eisenstein series E_1^*, E_2^* matching ϕ_1, ϕ_2.

The Borel Eisenstein series E_{new}

$s = (1+s_1,\ -1+s_2$ where $2s_1+2s_2=0$.

$$E_{\text{new}}(y,s) = \sum_{\gamma\in\mathcal{P}_{2,2}\backslash SL(4,\mathbb{Z})} E_1^*\left(\left(\begin{smallmatrix} y_3 & \\ & 1\end{smallmatrix}\right), \nu_1\right) E_2^*\left(\left(\begin{smallmatrix} y_1 & \\ & 1\end{smallmatrix}\right), \nu_2\right) \left(y_1^2 y_2^2 y_3\right)^{1+s_1} y_1^{-1+s_2} \Bigg|_{\gamma}$$

Theorem

$$A_{E_{\text{new}}}(\mathbf{1}) = \left(\zeta^*(1+2s_1-v-v')\zeta^*(1+2s_1+v-v')\zeta^*(1+2s_1-v+v')\zeta^*(1+2s_1+v+v')\right)^{-1}$$

$$\boxed{E_{\mathcal{P}_{2,2}}(\mathbf{1}) = \left(\sqrt{L(1,\mathrm{Ad}\,\phi_1)L(1,\mathrm{Ad}\,\phi_2)}\cdot\Gamma(1/2+v_1)\Gamma(1/2+\nu_2)L^*(1+2s_1,\phi_1\times\phi_2)\right)^{-1}}$$