Class numbers and representations by quadratic forms

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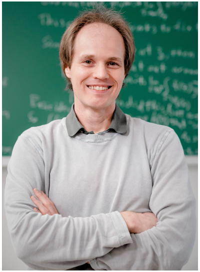

Joint work with Ben Kane (University of Hong Kong).

June 4, 2020

1. Representation numbers and modular forms

Outline

1. Representation numbers and modular forms
2. Class numbers and mock modular forms

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3. Generalizations

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4. 7-core partitions

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4. 7-core partitions

Representation numbers

Let for $n \in \mathbb{N}_0$ ($\boldsymbol{x} = (x_1, x_2, x_3)$)

$$r(n) := \#\left\{\boldsymbol{x} \in \mathbb{Z}^3 \,:\, x_1^2 + x_2^2 + x_3^2 = n\right\}.$$

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Example $n = 9$

$$(\pm 3)^2 = (\pm 2)^2 + (\pm 2)^2 + (\pm 1)^2 = 9$$

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Example $n = 9$

$$(\pm 3)^2 = (\pm 2)^2 + (\pm 2)^2 + (\pm 1)^2 = 9$$

$$\Rightarrow r(9) = 6 + 3 \cdot 2^3 = 30$$

Representation numbers

Generating function

$$\sum_{n\geq 0} r(n)q^n = \sum_{n_1,n_2,n_3\in\mathbb{Z}} q^{n_1^2+n_2^2+n_3^2} = \left(\sum_{n\in\mathbb{Z}} q^{n^2}\right)^3$$

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$$= 1 + 6q + 12q^2 + 8q^3 + 6q^4 + 24q^5 + 24q^6 + 12q^8 + 30q^9 + O\left(q^{10}\right)$$

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Confirms that $r(9)=30$.

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Confirms that $r(9) = 30$.

Goal Use symmetry properties.

Modularity

$f : \mathbb{H} \to \mathbb{C}$ holomorphic is modular of weight $k \in \mathbb{Z}$ if for all $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$

$$f\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^k f(\tau)$$

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plus growth condition

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Generalization include multiplier and half-integral weight

Examples

Fourier expansion ($q := e^{2\pi i\tau}$, $\tau \in \mathbb{H}$)

$$f(\tau) = \sum_{n\in\mathbb{Z}} c(n)q^n$$

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Examples

1. Dedekind η-function

$$\eta(\tau) := q^{\frac{1}{24}} \prod_{n\geq 1} (1 - q^n)$$

Modularity:

$$\eta(\tau + 1) = e^{\frac{\pi i}{12}}\eta(\tau), \qquad \eta\left(-\frac{1}{\tau}\right) = \sqrt{-i\tau}\eta(\tau).$$

Examples

2. Theta function

$$\Theta(\tau) := \sum_{n\in\mathbb{Z}} q^{n^2}$$

Θ is modular of weight $\frac{1}{2}$ for $\Gamma_0(4) := \{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z}) : 4 \mid c\}$.

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Θ is modular of weight $\frac{1}{2}$ for $\Gamma_0(4) := \{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z}) : 4 \mid c\}$.
Thus

$$\sum_{n\geq 0} r(n)q^n = \Theta^3(\tau)$$

is a modular form of weight $\frac{3}{2}$.

Valence formula

$f \neq 0$ modular of weight k for Γ satisfies

$$\sum_{\tau \in \Gamma\backslash\mathbb{H}} \frac{\mathrm{ord}_\tau(f)}{\omega_\tau} + \sum_{\varrho \in \Gamma\backslash(\mathbb{Q}\cup\{i\infty\})} \mathrm{ord}_\varrho(f) = [\mathrm{SL}_2(\mathbb{Z}) : \Gamma]\frac{k}{12}.$$

Identity of Gauss

Have

$$r(n) = \begin{cases} 12H(4n) & \text{if } n \equiv 1, 2 \pmod 4, \\ 24H(n) & \text{if } n \equiv 3 \pmod 8, \\ r\left(\frac{n}{4}\right) & \text{if } 4 \mid n, \\ 0 & \text{otherwise,} \end{cases}$$

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with

$$H(n) := \#\{\mathrm{SL}_2(\mathbb{Z})\text{-equivalence classes of integral binary quadratic forms of discriminant } n \text{ weighted by } \tfrac{1}{2} \text{ times the order of their automorphism group}\}.$$

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Question Why is the generating function of the right-hand side a modular form?

Compute class numbers

Gives efficient ways to compute class numbers!

Recall

$$\sum_{n\in\mathbb{Z}} r(n)q^n = 1+6q+12q^2+8q^3+6q^4+24q^5+24q^6+12q^8+O\left(q^9\right)$$

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$$H(4) = \frac{1}{12}r(1) = \frac{1}{12},$$
$$H(3) = \frac{1}{24}r(3) = \frac{1}{3},$$
$$H(8) = \frac{1}{12}r(2) = 1.$$

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Definition

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J. Bruinier

J. Funke

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$$\Delta_k(F) = 0$$

with $(\tau = \tau_1 + i\tau_2)$

$$\Delta_k := -\tau_2^2 \left(\frac{\partial^2}{\partial \tau_1^2} + \frac{\partial^2}{\partial \tau_2^2} \right) + ik\tau_2 \left(\frac{\partial}{\partial \tau_1} + i \frac{\partial}{\partial \tau_2} \right).$$

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plus growth condition

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Examples

- weight 2 Eisenstein series

$$\widehat{E}_2(\tau) := \underset{\substack{\uparrow \\ \text{quasimodular}}}{E_2}(\tau) - \frac{3}{\pi\tau_2}$$

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where

$$E_2(\tau) := 1 - 24 \sum_{n\geq 1} \underset{\substack{\downarrow \\ \sum_{d|n} d}}{\sigma(n)}\, q^n$$

Examples

- Class number generating function

$$\widehat{\mathcal{H}}(\tau) := \overbrace{\sum_{\substack{n\geq 0\\ n\equiv 0,3 \pmod 4}} \underset{\substack{\uparrow\\ \text{mock modular}}}{H(n)}q^n}^{\mathcal{H}(\tau)} + \frac{i}{8\sqrt{2}\pi}\int_{-\overline{\tau}}^{i\infty} \frac{\overset{\substack{\text{shadow}\\ \downarrow}}{\Theta(w)}}{(-i(\tau+w))^{\frac{3}{2}}}dw.$$

Natural splitting

$\mathcal{F}$ harmonic Maass form

$$\mathcal{F} \quad = \quad \mathcal{F}^+ \quad + \quad \mathcal{F}^-$$

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with

$$\mathcal{F}^+(\tau) := \sum_{n \gg -\infty} c^+(n) q^n,$$

$$\mathcal{F}^-(\tau) := \sum_{n \geq 1} c^-(n) \Gamma(k-1; 4\pi |n| \tau_2) q^n.$$

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incomplete gamma function

Ramanujan's last letter

"I am extremely sorry for not writing you a single letter up to now. I recently discovered very interesting functions which I call "Mock" ϑ-functions. Unlike the "False" ϑ-functions they enter into mathematics as beautifully as the theta functions. I am sending you with this letter some examples."

S. Ramanujan

Mock theta functions

These mock theta functions are 22 peculiar q-series.

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Example

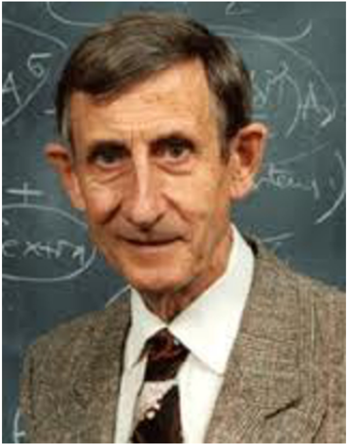
$$f(q) := \sum_{n \geq 0} \frac{q^{n^2}}{(-q;q)_n^2}$$

with

$$(a;q)_n := \prod_{m=0}^{n-1} (1 - aq^m).$$

Dyson's challenge for the future

"The mock theta-functions give us tantalizing hints of a grand synthesis still to be discovered. Somehow it should be possible to build them into a coherent group-theoretical structure, analogous to the structure of modular forms which Hecke built around the old theta functions of Jacobi. This remains a challenge for the future..."



F. Dyson

Mock modularity of $f(q)$

Theorem (Zwegers)

The function $f(q)$ is a mock modular form.

S. Zwegers

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Generalization

Let for $\boldsymbol{a}, \boldsymbol{h} \in \mathbb{N}^3$

$$r_{\boldsymbol{a},\boldsymbol{h},N}(n) := \#\left\{\boldsymbol{x} \in \mathbb{Z}^3 : a_1x_1^2 + a_2x_2^2 + a_3x_3^2 = n,\ x_j \equiv h_j \pmod{N}\right\}.$$

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Motivation Question of Petersson

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Motivation Question of Petersson

Example We have for $n \equiv 2 \pmod 8$

$$r_{\boldsymbol{1},\boldsymbol{h},N}(n) = H(4n).$$

Main Theorem

Let

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$S_{\boldsymbol{a}}$ certain set of $(\boldsymbol{h}, N)$ (explicit),

$d_{\boldsymbol{a},\boldsymbol{h},N}$ explicit constant only depending on $n \pmod{N}$.

Class number relations

Theorem

For each $\boldsymbol{a} \in \mathcal{C}$, $(h, N) \in S_{\boldsymbol{a}}$, $n \in \mathbb{N}$

$$r_{\boldsymbol{a},\boldsymbol{h},N}(n) = d_{\boldsymbol{a},\boldsymbol{h},N}(n) r_{\boldsymbol{a}}(n).$$

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Corollary

We have many relations to class numbers.

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Corollary

We have many relations to class numbers.

Gives an efficient way to compute class numbers!

Modularity

Have

$$\Theta_{\boldsymbol{a},\boldsymbol{h},N}(\tau) := \prod_{j=1}^{3} \vartheta_{h_j,N}(2Na_j\tau) = \sum_{n\geq 0} r_{\boldsymbol{a},\boldsymbol{h},N}(n)q^n,$$

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Let

$$\Theta_{\boldsymbol{a}} := \Theta_{\boldsymbol{a},\boldsymbol{1},1}.$$

Idea of proof

Step 1: Take generating functions

$$\Theta_{\boldsymbol{a},\boldsymbol{h},N} = \sum_{m \pmod{N}} d_{\boldsymbol{a},\boldsymbol{h},N}(m)\Theta_{\boldsymbol{a}}\big|S_{M,m},$$

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where for $f(\tau) = \sum_n c(n)q^n$,

$$f\big|S_{M,m}(\tau) := \sum_{n\equiv m \pmod M} c(n)q^n.$$

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A key lemma

Let for $N, M \in \mathbb{N}$

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Lemma

$\Theta_{\boldsymbol{a},\boldsymbol{h},N}$ is modular of weight $\frac{3}{2}$ on $\Gamma_{4\ell N^2,N}$ with character $\left(\frac{d}{\cdot}\right)$.

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$\Theta_{\boldsymbol{a}}\big|S_{M,n}$ has weight $\frac{3}{2}$ on $\Gamma_{\operatorname{lcm}(4\ell,M^2,MN_{\left(\frac{d}{\cdot}\right)}),M}$, with N_χ the conductor of character χ.

Idea of proof cont.

Step 3: Use valence formula.

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Lemma

For $M \mid N$

$$[\mathrm{SL}_2(\mathbb{Z}) : \Gamma_{N,M}] = N \prod_{p \mid N} \left(1 + \frac{1}{p}\right) \varphi(M).$$

Idea of proof cont.

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Problem: Bounds too big in some cases, new ideas required.

Example

$$r_{(1,1,1),(0,0,1),3}(n) = \frac{1}{6}\delta_{n\equiv 1 \pmod 3} r_{(1,1,1)}(n)$$

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Generating functions

$$\Theta_{(1,1,1),(0,1,1),3} = \frac{1}{6}\Theta_{(1,1,1)}\big| S_{3,1}$$

$\ell = d = 1$,

Example cont.

Modularity

left-hand side: modular on $\Gamma_{4\cdot 3^2,3}$,

right-hand side: $\Gamma_{\underbrace{\mathrm{lcm}(4,3^2,3)}_{4\cdot 3^2},3}$.

Example cont.

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Valence formula

$$\begin{aligned}\#\text{coefficients} &= \frac{1}{8}\left[\mathrm{SL}_2(\mathbb{Z}) : \Gamma_{4\cdot 3^2,3}\right]\\ &= \frac{1}{8} 4\cdot 3^2\left(1+\frac{1}{2}\right)\left(1+\frac{1}{3}\right)2 = 18\end{aligned}$$

Idea of proof : Class number identities

Step 1 Corresponding linear combination of class numbers is modular.

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Define for $f : \mathbb{H} \to \mathbb{C}$

$$f\big|V_d(\tau) := f(d\tau).$$

Key lemma

Lemma

For $\ell_1, \ell_2 \in \mathbb{N}$ *with* $\gcd(\ell_1, \ell_2) = 1$ *and* ℓ_2 *square-free*

$$\mathcal{H}_{\ell_1,\ell_2} := \mathcal{H}\Big|(U_{\ell_1\ell_2} - \ell_2 U_{\ell_1} V_{\ell_2})$$

is a modular form of weight $\frac{3}{2}$ *on* $\Gamma_0(4\ell_2 \prod_{p|\ell_1} p)$ *with character* $(\frac{\ell_1\ell_2}{\cdot})$.

Idea of proof : Class number identities

Proof (of key lemma).

1. Show modularity of

$$\widehat{\mathcal{H}}_{\ell_1,\ell_2} := \widehat{\mathcal{H}}\Big|(U_{\ell_1\ell_2} - \ell_2 U_{\ell_1} V_{\ell_2}).$$

Proof (of key lemma).

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where for $f(\tau) = \sum_n c_{\tau_2}(n)q^n$,

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Proof (of key lemma).

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$$\widehat{\mathcal{H}}_{\ell_1,\ell_2} := \widehat{\mathcal{H}}\Big|\left(U_{\ell_1\ell_2} - \ell_2 U_{\ell_1} V_{\ell_2}\right).$$

where for $f(\tau) = \sum_n c_{\tau_2}(n) q^n$,

$$f\Big|U_d(\tau) = \sum_n c_{\frac{\tau_2}{d}}(dn) q^n.$$

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<u>Step 2</u> Valence formula.

Example

$$r_{(1,1,1)}(n) = 12\left(H(4n) - 2H(n)\right)$$

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Modularity

$\Theta_{(1,1,1)}$ is a modular form of weight $\frac{3}{2}$ on $\Gamma_0(4)$,
$\mathcal{H}_{1,2}$ is a modular form of weight $\frac{3}{2}$ on $\Gamma_0(8)$, U_2 keeps that property

Valence formula

$$\#\text{coefficients} = \frac{1}{8}\left[\mathrm{SL}_2(\mathbb{Z}) : \Gamma_0(8)\right] = \frac{1}{8}4 = \frac{1}{2}$$

Check 1 coefficient!

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Piecing together

$$r_{(1,1,1),(0,0,1),3}(n) = \frac{1}{6}\delta_{n\equiv 1 \pmod 3} r_{(1,1,1)}(n)$$

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Piecing together

$$\begin{aligned} r_{(1,1,1),(0,0,1),3}(n) &= \frac{1}{6}\delta_{n\equiv 1 \pmod 3} r_{(1,1,1)}(n) \\ &= 2\delta_{n\equiv 1 \pmod 3}\left(H(4n) - 2H(n)\right) \end{aligned}$$

Outline

1. Representation numbers and modular forms
2. Class numbers and mock modular forms
3. Generalizations
4. 7-core partitions

Partitions

A partition of $n \in \mathbb{N}_0$ is a non-increasing sequence of positive integers whose sum is n. Denote

$$p(n) := \# \text{ of partitions of } n.$$

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Example

$n = 4$ $\qquad$ $p(4) = 5$

$4 \quad 3+1, \quad 2+2, \quad 2+1+1 \quad 1+1+1+1$

Generating function

Euler

$$P(q) := \sum_{n\geq 0} p(n)q^n = \prod_{n\geq 1} \frac{1}{1-q^n}$$

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$$P(q) := \sum_{n\geq 0} p(n)q^n = \prod_{n\geq 1} \frac{1}{1-q^n} = \frac{q^{\frac{1}{24}}}{\eta(\tau)}$$

a modular form of weight $-\frac{1}{2}$.

Ferrers-Young diagram

Let $n_1 + n_2 + \ldots + n_\ell$ be a partition of n.

Ferrers-Young diagram

$$\begin{array}{cccccccc}
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \quad n_1 & \text{nodes} \\
\bullet & \bullet & \bullet & \bullet & \bullet & & \quad n_2 & \text{nodes} \\
\vdots & & & & & & & \\
\bullet & \bullet & & & & & \quad n_\ell & \text{nodes}
\end{array}$$

Example

The partition $3+3+2+1$ has Ferrers-Young diagram

$$\begin{array}{ccc} \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet \\ \bullet & \bullet & \\ \bullet & & \end{array}$$

t-cores

Hook numbers

$$H(j, \ell) = n_j + n'_\ell - j - \ell + 1$$

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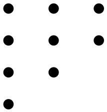
$$H(j,\ell) = n_j + \underset{\substack{\uparrow \\ \text{\#of nodes in column } \ell}}{n'_\ell} - j - \ell + 1$$

Example



Example

$$\begin{array}{lll}
\bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet \\
\bullet & \bullet & \\
\bullet & &
\end{array}$$

$$H(1,1) = 3 + 4 - 1 - 1 + 1 = 6$$
$$H(1,2) = 3 + 3 - 1 - 2 + 1 = 4$$
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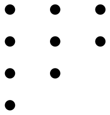

$$
\begin{aligned}
H(1,1) &= 3+4-1-1+1 = 6 & \qquad H(2,1) &= 3+4-2-1+1 = 5\\
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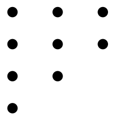
H(1,3) &= 3+2-1-3+1 = 2 & \qquad H(2,3) &= 3+2-2-3+1 = 1
\end{aligned}
$$

Example



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$$H(3,1) = 2 + 4 - 3 - 1 + 1 = 3$$
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$$H(3,3) = 2 + 2 - 3 - 3 + 1 = -1$$

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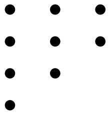

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t-core of n if $t \nmid H(j,\ell)\ \forall j,\ell$

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$3+3+2+1$ is a 7-core partition of 9.

Importance

Arise in:

- Combinatorics
  e.g. combinatorial proof of Ramanujan congruences

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$$\sum_{n\geq 0} c_t(n)q^n = \prod_{n\geq 1} \frac{\left(1-q^{tn}\right)^t}{1-q^n} = q^{\frac{1-t^2}{24}} \frac{\eta(t\tau)^t}{\eta(\tau)}$$

A modular form of weight $\frac{1}{2}(t^2-1)$.

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Known facts:

- For $t \in \{2,3\}$, $c_t(n) = 0$ for almost all $n \in \mathbb{N}$.

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A modular form of weight $\frac{1}{2}(t^2-1)$.

Known facts:

- For $t \in \{2,3\}$, $c_t(n) = 0$ for almost all $n \in \mathbb{N}$.
- For $t \geq 4$, $c_t(n) > 0$ for all $n \in \mathbb{N}$.

Generating function

Relation to class numbers (Ono-Sze)
For $8n+5$ square-free

$$c_4(n) = \frac{1}{2} H(32n+20).$$

Self-conjugate t-cores

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Example

$$\begin{array}{ccc} \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet \\ \bullet & \bullet & \\ \bullet & & \end{array}$$

$3+3+2+1$ is not self-conjugate.

Theorem (Ono-Raji)

If $n \in \mathbb{N}$, $n \not\equiv 5 \pmod 7$, n odd, then

$$\mathrm{sc}_7(n) = \begin{cases} \frac{1}{4}H(28n+56) & \text{if } n \equiv 1 \pmod 4, \\ \frac{1}{2}H(7n+14) & \text{if } n \equiv 3 \pmod 4, \\ 0 & \text{otherwise.} \end{cases}$$

Extending the result of Ono/ Raji

Let $H_7(D)$ denote the number of *7-primitive quadratic forms* (for $[a, b, c]$, $7 \nmid \gcd(a, b, c)$) with discriminant $-D$ and the same weighting as $H(D)$.

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Lemma

$$H_7(D) = H(D) - H\left(\frac{D}{7^2}\right).$$

Generalization

Let

$$D_n := \begin{cases} 28n+56 & \text{if } n \equiv 0,1 \pmod 4, \\ 7n+14 & \text{if } n \equiv 3 \pmod 4, \\ D_{\frac{n+2}{2^{2\ell}}-2} & \text{if } n \equiv 2 \pmod 4, \end{cases}$$

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$$\nu_n := \begin{cases} \frac{1}{4} & \text{if } n \equiv 0,1 \pmod{4}, \\ \frac{1}{2} & \text{if } n \equiv 3 \pmod{8}, \\ \nu_{\frac{n+2}{2^{2\ell}}-2} & \text{if } n \equiv 2 \pmod{4}, \\ 0 & \text{otherwise,} \end{cases}$$

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where $\ell \in \mathbb{N}$ is maximal s.t. $n \equiv -2 \pmod{2^{2\ell}}$.

Generalization

Theorem (B.-Kane)

We have for $n \in \mathbb{N}$

$$\mathrm{sc}_7(n) = \nu_n H_7(D_n).$$

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Remark

Includes result by Ono/ Raji since $H(D_n) = H_7(D_n)$ for $n \not\equiv -2 \pmod 7$.

Generalization

Key theorem

Theorem (B.-Kane)

For every $n \in \mathbb{N}$, we have

$$\mathrm{sc}_7(n) = \frac{1}{4}\left(H(28n+56) - H\left(\frac{4n+8}{7}\right) - 2H(7n+14) \right.$$
$$\left. +2H\left(\frac{n+2}{7}\right)\right).$$

Consequences

Corollary

For $n \in \mathbb{N}$ with $n+2$ square-free

$$\mathrm{sc}_7(n) = -\frac{\nu_n}{D_n} \begin{cases} \sum_{m=1}^{D_n-1} \left(\frac{-D_n}{m}\right) m & \text{if } n \not\equiv -2 \pmod 7, \\ 7^2 \left(7 + \left(\frac{\frac{D_n}{7^2}}{7}\right)\right) \sum_{m=1}^{\frac{D_n}{7^2}-1} \left(\frac{-\frac{D_n}{7^2}}{m}\right) m & \text{if } n \equiv -2 \pmod 7. \end{cases}$$

Consequences

Corollary

For $n \in \mathbb{N}$ with $n+2$ square-free, $\ell, r \in \mathbb{N}_0$, $f \in \mathbb{N}$ with $\gcd(f,14)=1$

$$\mathrm{sc}_7\left((n+2)2^{2\ell}f^2 7^{2r}-2\right) = 7^r\,\mathrm{sc}_7(n)\sum_{1\le d\mid f}\mu(d)\left(\frac{-D_n}{d}\right)\sigma_1\left(\frac{f}{d}\right).$$

Open questions

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2. Find bijective proofs for the t-core identities (work in progress with Males).

Thank you for your attention