The fourth moment of Dirichlet L-functions along a coset

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Motivation

- Let $\mathcal{H}_k(q, \psi)$ (resp. $\mathcal{H}_{it_j}(q, \psi)$) be the set of holomorphic newforms of weight k (resp. Maass newforms of spectral parameter t_j), level q, and central character ψ.

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- Let $\mathcal{H}_k(q,\psi)$ (resp. $\mathcal{H}_{it_j}(q,\psi)$) be the set of holomorphic newforms of weight k (resp. Maass newforms of spectral parameter t_j), level q, and central character ψ.
- For χ of conductor q, consider

$$\sum_{k \leq T} \sum_{f \in \mathcal{H}_k(q,\overline{\chi}^2)} L(f \otimes \chi, 1/2)^3$$

and

$$\sum_{|t_j| \leq T} \sum_{f \in \mathcal{H}_{it_j}(q,\overline{\chi}^2)} L(f \otimes \chi, 1/2)^3 + \int_{|t| \leq T} |L(1/2 + it, \chi)|^6 dt.$$

- Call either of these expressions $\mathcal{M}_{\leq T}(q)$.

The cubic moment

Theorem (Petrow, Y. 2018-2019)

We have

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Corollary (Weyl bound)

We have

$$L(f \otimes \chi, 1/2) \ll q^{1/3+\varepsilon},$$

and

$$L(1/2 + it, \chi) \ll q^{1/6+\varepsilon}.$$

Motivation

Roughly speaking, the proof gives a kind of spectral identity

$$\mathcal{M}_{\leq T}(q) \approx \frac{1}{q} \sum_{\psi \pmod{q}} |L(1/2, \psi)|^4 g(\chi, \psi)$$

where

$$g(\chi, \psi) = \sum_{u,t \pmod{q}} \chi\Big(\frac{t(u+1)}{u(t+1)}\Big) \psi(ut - 1).$$

Motivation

If for all ψ

$$|g(\chi, \psi)| \ll q^{1+\varepsilon}, \tag{1}$$

then we can use an easy bound

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to complete the proof of the theorem.

- If q is cube-free then (1) is true.
- For more general q, (1) is not true.

Higher prime powers

If $q = p^k$, $k \geq 3$, and $p \equiv 1 \pmod 4$, then there exist characters ψ so that $|g(\chi, \psi)| \geq qp^{\alpha(k)}$ for some $\alpha(k) > 0$, $\alpha(k) \in \frac{1}{2}\mathbb{Z}$.

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Strategy to treat higher prime powers:

1. Understand the possible values of $\alpha(k)$.
2. Understand the structure of the set of ψ so that $\alpha(k) > 0$ is attained.
3. Bound
$$\sum_{\substack{\psi \pmod q \\ \alpha(k) \text{ attained}}} |L(1/2, \psi)|^4 \ll p^{-\alpha(k)} q^{1+\varepsilon}.$$

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It turns out that the set of "singular" ψ so that $\alpha(k) > 0$ is a *coset* of the subgroup of characters modulo d, for some d with $p \leq d \leq p^{k-1}$.

Example

Say $q = p^3$. If $p \equiv 1 \pmod 4$, then there exist $2(p-1)$ singular characters ψ so that $|g(\chi, \psi)| = qp^{1/2}$. The set of singular characters is a union of two cosets of the form

$$\{\eta.\alpha^{\pm 1} : \eta \pmod p\}.$$

Fourth moment of Dirichlet L-functions along a coset

Theorem (Petrow, Y. 2019)

Let $d|q$ and let q^* be the least integer so that $q^2|(q^*)^3$ (so if $q = \prod_p p^{\beta_p}$, then $q^* = \prod_p p^{\lceil 2\beta_p/3 \rceil}$). Let χ have conductor q. Then

$$\sum_{\eta \pmod{d}} |L(1/2, \chi \cdot \eta)|^4 \ll [d, q^*] q^{\varepsilon}.$$

This bound turns out to be strong enough for item 3.

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Example. Let $q = p^3$, so $q^* = p^2$. Then

$$\sum_{\eta \pmod{p^2}} |L(1/2, \chi \cdot \eta)|^4 \ll p^{2+\varepsilon}.$$

This implies $|L(1/2, \chi)| \ll p^{1/2+\varepsilon} = q^{1/6+\varepsilon}$, which happens to have been first shown by Heath-Brown in 1978.

Related works

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The previous theorem is a mod q analog of a result of Iwaniec from 1980:

$$\int_T^{T+\Delta} |\zeta(1/2+it)|^4 dt \ll \max(\Delta, T^{2/3}) T^{\varepsilon}.$$

In this analogy, Δ corresponds to d and $T^{2/3}$ corresponds to q^*.

Related works

Some work on the second moment of Dirichlet L-functions along cosets appears in:

- Nunes (*The twelfth moment of Dirichlet L-functions with smooth moduli*, IMRN, 2019)
- Milićević-White (*Twelfth moment of Dirichlet L-functions to prime power moduli*, arXiv:1908.04833)

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- Milićević-White (*Twelfth moment of Dirichlet L-functions to prime power moduli*, arXiv:1908.04833)
- Bradford Garcia (Texas A&M PhD Student) has further work in progress on this family.

Related works

Some variants on the fourth moment problem for Dirichlet L-functions include:

- Blomer, Humphries, Khan, and Milinovich. *Motohashi's fourth moment identity for non-archimedean test functions and applications*, arXiv:1902.07042, 2019.
- Young. *The fourth moment of Dirichlet L-functions.* Ann. of Math. (2), 173(1):1–50, 2011.

Cuspidal analogs

An analogous family is the collection of twists $\{L(f \otimes \chi, 1/2) : \chi \pmod{q}\}$ for f a GL_2 cusp form.

- Blomer and Milićević. *The second moment of twisted modular L-functions.* Geom. Funct. Anal., 25(2):453–516, 2015.
- Kowalski, Michel, and Sawin. *Bilinear forms with Kloosterman sums and applications.* Ann. of Math. (2), 186(2):413–500, 2017.
- Work in progress by Texas A&M PhD student Agniva Dasgupta for cuspidal analog of the coset family.

The fourth moment along a coset

Recall: We want to show

$$\sum_{\eta \pmod{d}} |L(1/2, \chi \cdot \eta)|^4 \ll dq^{\varepsilon},$$

when $d|q$ and $q^3|d^2$. Take the case $q = p^3$, $d = p^2$.

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By an approximate functional equation and orthogonality of characters, this reduces to showing

$$\sum_{\substack{m \equiv n \pmod{p^2} \\ m,n \approx p^3}} \frac{d(m)\chi(m)d(n)\overline{\chi}(n)}{\sqrt{mn}} \ll q^{\varepsilon}.$$

So the problem boils down to a shifted divisor sum twisted by characters of large conductor.

Shifted divisor problems

In general it is difficult to solve a shifted convolution problem with arithmetic coefficients of large conductor. The QUE problem can be reduced to a shifted convolution problem of this type, and a robust power-saving bound is still unknown.

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Specifically, one desires a bound of the form (for $h \neq 0$)

$$\sum_{n \approx k} \lambda_f(n)\lambda_f(n+h) \ll k^{1-\delta},$$

for a holomorphic form f of weight k, and similarly for Maass forms.

Zeta in a short interval

For comparison, the fourth moment of the zeta function on a short interval reduces to showing

$$\sum_{\substack{m=n+O(\frac{T}{\Delta}) \\ m,n\approx T}} \frac{d(m)m^{iT}d(n)n^{-iT}}{\sqrt{mn}} \ll T^{\varepsilon}.$$

Shifted divisor problems, cont.

Note that if $\Delta \approx T^{2/3}$ then

$$(m/n)^{iT} = e^{iT \log(1+\frac{m-n}{n})} \approx e^{iT\frac{h}{n}}, \quad \text{with} \quad h = m - n.$$

Shifted divisor problems, cont.

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The magnitude of the phase is roughly of size $h \ll \frac{T}{\Delta} \approx T^{1/3}$ which is much smaller than T.

The spectral approach

If f and g are $GL_2/\mathbb{Q}$ automorphic newforms of levels q_f and q_g, then there is a somewhat direct spectral approach (originally put forward by Selberg and greatly extended by many people) as follows:

$$\sum_n \lambda_f(n)\lambda_g(n+h) \approx \langle fg, P_h \rangle,$$

where P_h is a Poincare series, and the inner product is on $\Gamma_0(Q)$, with $Q = [q_f, q_g]$.

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One then applies the spectral decomposition

$$\langle fg, P_h \rangle = \sum_{u \text{ on } \Gamma_0(Q)} \langle fg, u \rangle \langle u, P_h \rangle,$$

where $\langle u, P_h \rangle \simeq \lambda_u(h)$

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We will use a different approach that eventually uses a spectral decomposition on $\Gamma_1(p)$.

The shifted convolution problem with a congruence

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Say $m = n + p^2 k$. Then

$$\chi(m)\overline{\chi}(n) = \chi(1 + p^2 k\overline{n}) = e_p(\ell_\chi k\overline{n}).$$

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Say $m = n + p^2 k$. Then

$$\chi(m)\overline{\chi}(n) = \chi(1 + p^2 k\overline{n}) = e_p(\ell_\chi k\overline{n}).$$

Compare this with

$$(m/n)^{iT} \approx e^{iT\frac{h}{n}}.$$

This first step has a significant conductor-reducing effect.

Separation of variables

Typically one of the first steps in a solution of a shifted convolution problem is a way to separate the variables n and h in $\lambda(n+h)$. The circle method or the delta symbol method (of Duke, Friedlander, and Iwaniec) could be used for this purpose.

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Lemma

Let χ be a primitive Dirichlet character modulo q. Let $G(s)$ be an even entire holomorphic function with rapid decay in vertical strips, satisfying $G(0)=1$ (e.g. $G(s)=\exp(s^2)$). Then

$$d(n)\chi(n) = \frac{2}{\tau(\overline{\chi})}\sum_{c=1}^{\infty}\frac{\chi(c)}{c}f\left(\frac{c}{\sqrt{n}}\right)\sum_{r \pmod{cq}}^{*}\overline{\chi}(r)e_{cq}(nr),$$

where

$$f(x) = \frac{1}{2\pi i}\int_{(1)} x^{-2s}L(1+2s,\chi_{0,q})\frac{G(s)}{s}ds,$$

and where $\chi_{0,q}$ denotes the trivial character modulo q.

Separation of variables, cont.

- The origin of the previous lemma is that there exists an Eisenstein series $E(z, s, \chi)$ with Fourier coefficients related to $\chi(n) \sum_{ab=n} (b/a)^{s-\frac{1}{2}}$.
- The Eisenstein series has a functional equation under $s \to 1 - s$, and the calculation of its Fourier expansion naturally uses the identity (for $\mathrm{Re}(s)$ large enough)

$$\frac{n^{s-\frac{1}{2}}}{\tau(\overline{\chi})} \sum_{c=1}^{\infty} \frac{\chi(c)}{c^{2s}} \sum_{r \pmod{cq}} \overline{\chi}(r) e_{cq}(nr) = \chi(n) \sum_{ab=n} (b/a)^{s-\frac{1}{2}}.$$

- This is then an approximate functional equation for each Fourier coefficient
- The fact that c runs over integers coprime to q is highly convenient.

Applying the Lemma

$$\sum_{\substack{m\equiv n \pmod{p^2} \\ m,n\approx p^3}} d(m)\chi(m)d(n)\overline{\chi}(n)$$

$$\approx \sum_{n\approx p^3}\sum_{k\approx p} d(n+p^2k)\chi(n+p^2k)d(n)\overline{\chi}(n)$$

$$\approx \sum_{n\approx p^3}\sum_{k\approx p} \frac{d(n)\overline{\chi}(n)}{\tau(\overline{\chi})} \sum_{c\approx p^{3/2}} \frac{\chi(c)}{c} \sideset{}{^*}\sum_{r \pmod{cq}} \overline{\chi}(r)e_{cq}((n+p^2k)r).$$

This decomposition into additive characters allows the variables n and k to be separated, arithmetically.

Voronoi summation

Voronoi summation (in n modulo cp) converts to a sum of the shape

$$\sum_{\substack{c \approx p^{3/2} \\ (c,p)=1}} \sum_{\substack{k \approx p \\ (k,p)=1}} \sum_{n \ll p^2} d(n) S(pk, -\overline{p}n; c) \mathrm{Kl}_3(\ell_\chi k, n, \overline{c}^2; p).$$

Here

$$Kl_3(a,b,c;p) = \sum_{xyz \equiv 1 \pmod{p}} e_p(ax + by + cz).$$

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Using Deligne's bound for Kl_3 and Weil's bound for the Kloosterman sum gives only a weak result.

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We have

$$\sum_{c} \overline{\chi}(c) S(\overline{p}m, n; c) \to \sum_{t_j} \sum_{f \in \mathcal{H}_{it_j}(p,\chi)} \overline{\lambda_f}(m) \overline{\lambda_f}(n)$$

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We will use this with $\chi = \eta^2$.

The spectral decomposition

The resulting formula roughly looks like

$$\sum_{t_j \ll 1} \sum_{\eta \pmod{p}} \frac{\tau(\overline{\eta})^3}{p^{3/2}} \sum_{f \in \mathcal{H}_{it_j}(p,\eta^2)} \lambda_f(p) L(\overline{f} \otimes \eta, 1/2)^3.$$

The normalizations are such that Lindelöf-on-average suffices to prove the main theorem.

The final move

Since $\overline{f} \otimes \eta \in \mathcal{H}_{it_j}(p^2)$ (with trivial central character after the twist), we can use the simple bound

$$\sum_{t_j \ll 1} \sum_{g \in \mathcal{H}_{it_j}(N)} |L(g, 1/2)|^4 \ll N^{1+\varepsilon}$$

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Subtlety: given a $g \in \mathcal{H}_{it_j}(p^2)$, for how many pairs f, η do we have $\overline{f} \otimes \eta = g$? Answer: at most 2 (via a quadratic twist).

This final step was deceptively simple because of the focus on the case $q = p^3$.

A generalization

Theorem (Petrow, Y.)

Suppose $q = q_1 q_2$ *with* $(q_1, q_2) = 1$. *Then*

$$\sum_{\eta \pmod{q_2}} \sum_{|t_j| \ll 1} \sum_{m|q} \sum_{f \in \mathcal{H}_{it_j}(m,\eta^2)} |L(1/2, \overline{f} \otimes \eta)|^4 \ll q_2 q^{1+\varepsilon}.$$

A similar bound holds true for holomorphic forms, as well as the Eisenstein series.

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Example. Consider $q_1 = 1$, $q_2 = p^2$. If $m = p^2$ and η is primitive modulo p^2, then $\overline{f} \otimes \eta \in \mathcal{H}_{it_j}(p^4, 1)$. The multiplicity of the map $(f, \eta) \to \overline{f} \otimes \eta$ is bounded, and the standard level p^4 fourth moment bound suffices to estimate these terms.

A generalization

$$\sum_{\eta \pmod{q_2}} \sum_{|t_j| \ll 1} \sum_{m|q} \sum_{f \in \mathcal{H}_{it_j}(m,\eta^2)} |L(1/2, \overline{f} \otimes \eta)|^4 \ll q_2 q^{1+\varepsilon}.$$

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On the other hand, the map $(f, \eta) \to \overline{f} \otimes \eta$ has multiplicity $\approx p$ (since for $\chi \pmod{p}$),

$$(f \otimes \chi, \eta \otimes \chi) \mapsto \overline{f \otimes \chi} \otimes (\eta \otimes \chi) = \overline{f} \otimes \eta,$$

and $f \otimes \chi$ still has level p^2, and $\eta \otimes \chi$ still has modulus p).

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and $f \otimes \chi$ still has level p^2, and $\eta \otimes \chi$ still has modulus p).

Luckily, the extra multiplicity is compensated by the saving in the number of forms of level p^2 compared to those of level p^4.

Thank you for listening!