Multiple Dirichlet Series for affine Weyl groups

Alexandru A. Popa
(joint work in progress with Adrian Diaconu and Vicențiu Pașol)

Institute of mathematics "Simion Stoilow" of the Romanian Academy

Online Conference in Automorphic Forms
June 3, 2020

Outline

- Motivation: Moments of central values of quadratic Dirichlet L-functions
- Chinta-Gunnels MDS attached to Weyl-Coxeter groups
- Affine Coxeter groups
  - Residues of MDS for $\widetilde{A}_r$
  - Macdonald's formula
  - Extra functional equation

Moments of quadratic Dirichlet L-functions

For $d \neq 0$ square-free, let $\chi_d(n)$ the quadratic Dirichlet character (of modulus d or $4d$) associated with the quadratic field $\mathbb{Q}(\sqrt{d})$. Its L-function

$$L(s, \chi_d) = \sum_{n \geqslant 1} \frac{\chi_d(n)}{n^s}$$

has analytic continuation to $\mathbb{C}$ and satisfies a functional equation under $s \mapsto 1 - s$.

Moments of quadratic Dirichlet L-functions

For $d \neq 0$ square-free, let $\chi_d(n)$ the quadratic Dirichlet character (of modulus d or $4d$) associated with the quadratic field $\mathbb{Q}(\sqrt{d})$. Its L-function

$$L(s, \chi_d) = \sum_{n \geqslant 1} \frac{\chi_d(n)}{n^s}$$

has analytic continuation to $\mathbb{C}$ and satisfies a functional equation under $s \mapsto 1 - s$.

Conjecture (CFKRS Moment conjecture)

For $r \geqslant 1$ one has the asymptotic

$$\sum_{\substack{|d|<x \\ d \text{ square-free}}} L(1/2, \chi_d)^r \sim x P_r(\log x)$$

for an explicit polynomial P_r of degree $r(r+1)/2$.

Moments of quadratic Dirichlet L-functions

For $d \neq 0$ square-free, let $\chi_d(n)$ the quadratic Dirichlet character (of modulus d or $4d$) associated with the quadratic field $\mathbb{Q}(\sqrt{d})$. Its L-function

$$L(s, \chi_d) = \sum_{n \geqslant 1} \frac{\chi_d(n)}{n^s}$$

has analytic continuation to $\mathbb{C}$ and satisfies a functional equation under $s \mapsto 1 - s$.

Conjecture (CFKRS Moment conjecture)

For $r \geqslant 1$ one has the asymptotic

$$\sum_{\substack{|d|<x \\ d \text{ square-free}}} L(1/2, \chi_d)^r \sim x P_r(\log x)$$

for an explicit polynomial P_r of degree $r(r+1)/2$.

Known for $r = 1, 2$ (Jutila 1981) and $r = 3$ (Soundararajan 2000 and Diaconu-Goldfeld-Hoffstein 2001).

Beyond leading terms

Diaconu-Goldfeld-Hoffstein used multiple Dirichlet series, and conjectured the existence of another term in the asymptotics for the third moment, of order $x^{3/4}$. The existence of this lower order term was recently established:

Theorem (Diaconu-Whitehead 2019)

Let $W : (0,\infty) \to (0,1)$ be a "nice" function with Mellin transform $\widehat{W}$. Then

$$\sum_d L\left(\frac{1}{2}, \chi_{2d}\right)^3 W\left(\frac{d}{x}\right) = xQ_W(\log x) + R\widehat{W}\left(\frac{3}{4}\right)x^{\frac{3}{4}} + O\left(x^{\frac{2}{3}+\epsilon}\right)$$

where Q_W is a polynomial of degree 6, $R = -0.0034...$, and the sum is over square-free odd d.

Remarks

- A similar asymptotics holds over the rational function field $\mathbb{F}_q(T)$ (Diaconu 2019) for the moments

$$\sum_{\substack{d \in \mathbb{F}_q[T] \text{ monic, square-free} \\ \deg d = D}} L(1/2, \chi_d)^3.$$

The proof over $\mathbb{Q}$ is parallel to that over $\mathbb{F}_q(T)$. This is in agreement with the general philosophy, that results over $\mathbb{F}_q(T)$ have correspondents over $\mathbb{Q}$, but are usually easier to prove.

Remarks

- A similar asymptotics holds over the rational function field $\mathbb{F}_q(T)$ (Diaconu 2019) for the moments

$$\sum_{\substack{d \in \mathbb{F}_q[T] \text{ monic, square-free} \\ \deg d = D}} L(1/2, \chi_d)^3.$$

  The proof over $\mathbb{Q}$ is parallel to that over $\mathbb{F}_q(T)$. This is in agreement with the general philosophy, that results over $\mathbb{F}_q(T)$ have correspondents over $\mathbb{Q}$, but are usually easier to prove.
- Over $\mathbb{F}_q(T)$ asymptotics for the 3rd and 4th moments were proven by A. Florea (2017).

Multiple Dirichlet series

Consider multiple Dirichlet series (MDS) of the type:

$$Z_{\mathrm{arithm}}(s_1, s_2, \ldots, s_{r+1}) = \sum_d \frac{L(s_1, \chi_d)L(s_2, \chi_d)\ldots L(s_r, \chi_d)}{|d|^{s_{r+1}}}$$

where $s_i \in \mathbb{C}$ and the sum is over discriminants d in the number field setting (over $\mathbb{Q}$), or over monic polynomials d in the rational function field setting (over $\mathbb{F}_q(T)$). In the latter case, $|d| = q^{\deg d}$.

Multiple Dirichlet series

Consider multiple Dirichlet series (MDS) of the type:

$$Z_{\mathrm{arithm}}(s_1, s_2, \ldots, s_{r+1}) = \sum_d \frac{L(s_1, \chi_d)L(s_2, \chi_d) \ldots L(s_r, \chi_d)}{|d|^{s_{r+1}}}$$

where $s_i \in \mathbb{C}$ and the sum is over discriminants d in the number field setting (over $\mathbb{Q}$), or over monic polynomials d in the rational function field setting (over $\mathbb{F}_q(T)$). In the latter case, $|d| = q^{\deg d}$.

The term in the summation for d not square-free has to be adjusted by suitable correction factors, so that Z_{arithm} has a large group of functional equations. The correction factors are unique for $r = 1, 2, 3$, but not so for $r \geqslant 4$.

The most general construction of Z_{arithm} is due to Diaconu-Pașol (2018), who give a set of axioms that determine uniquely the correction factors for every r (using some of the geometric machinery used by Deligne in the proof of the Weil conjectures).

The most general construction of Z_{arithm} is due to Diaconu-Pașol (2018), who give a set of axioms that determine uniquely the correction factors for every r (using some of the geometric machinery used by Deligne in the proof of the Weil conjectures).

Strategy of MDS method: Obtain meromorphic continuation of Z_{arithm} to a region containing $(1/2, \ldots, 1/2, 1)$. Tauberian theorems then give the desired asymptotics from knowledge of the residues of Z_{arithm} at poles.

The most general construction of Z_{arithm} is due to Diaconu-Pașol (2018), who give a set of axioms that determine uniquely the correction factors for every r (using some of the geometric machinery used by Deligne in the proof of the Weil conjectures).

Strategy of MDS method: Obtain meromorphic continuation of Z_{arithm} to a region containing $(1/2, \ldots, 1/2, 1)$. Tauberian theorems then give the desired asymptotics from knowledge of the residues of Z_{arithm} at poles.

Remark: When $r \leqslant 3$, the functional equations form a finite group and give meromorphic continuation to $\mathbb{C}^{r+1}$ by means of Böchner's theorem, but for $r \geqslant 4$ the group is infinite, and there is a natural boundary of singularities.

Functional equations

From now on assume the function field setting. The functional equations

$$L(s,\chi_d) \mapsto |d|^{1/2-s} L(1-s,\chi_d)$$

imply that Z_{arithm} has a group of functional equations under the transformations :

$$\sigma_i : (s_1,\ldots,s_i,\ldots,s_{r+1}) \mapsto (s_1,\ldots,1-s_i,\ldots,s_{r+1}+s_i-1/2)$$

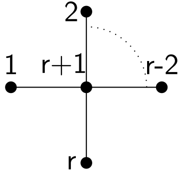
for $i=1,\ldots,r$ and

$$\sigma_{r+1} : (s_1,\ldots,s_{r+1}) \mapsto (s_1+s_{r+1}-1/2, s_2+s_{r+1}-1/2,\ldots,1-s_{r+1})$$

(for this latter equation one needs the correction factors in definition of Z_{arithm}).

Coxeter groups and their root systems

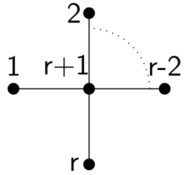
These transformations form a Weyl-Coxeter group W_r with Coxeter graph:



$$W_r = \langle \sigma_i \mid \sigma_i^2 = 1,\ (\sigma_i \sigma_j)^3 = 1 \text{ for } i \sim j,\ (\sigma_i \sigma_j)^2 = 1 \text{ for } i \not\sim j \rangle.$$

Key fact: W_r is finite for $r = 1, 2, 3$ and infinite if $r \geqslant 4$.

These transformations form a Weyl-Coxeter group W_r with Coxeter graph:



$$W_r = \langle \sigma_i \mid \sigma_i^2 = 1,\ (\sigma_i\sigma_j)^3 = 1 \text{ for } i \sim j,\ (\sigma_i\sigma_j)^2 = 1 \text{ for } i \not\sim j \rangle.$$

Key fact: W_r is finite for $r = 1, 2, 3$ and infinite if $r \geqslant 4$.

Remark: The case $r = 4$ is special in that the Weyl group is *affine*. The corresponding root system is denoted $\widetilde{D}_4$.

Another construction of MDS

Our goal is to study another construction of MDS, obtained as a Chinta-Gunnels average over the Weyl group, and to relate it to Z_{arithm}.

Another construction of MDS

Our goal is to study another construction of MDS, obtained as a Chinta-Gunnels average over the Weyl group, and to relate it to Z_{arithm}.

What is known beyond the case of finite Weyl group:

- Diaconu-Bucur (2009) studied such an MDS associated to $W_4 = \widetilde{D}_4$, proved its meromorphic continuation up to the natural boundary, and related it to Z_{arithm} up to a "diagonal factor";

Our goal is to study another construction of MDS, obtained as a Chinta-Gunnels average over the Weyl group, and to relate it to Z_{arithm}.

What is known beyond the case of finite Weyl group:

- Diaconu-Bucur (2009) studied such an MDS associated to $W_4 = \widetilde{D}_4$, proved its meromorphic continuation up to the natural boundary, and related it to Z_{arithm} up to a "diagonal factor";
- Lee-Zhang (2012) established meromorphic continuation for such a Chinta-Gunnels average, attached to symmetrizable Kac-Moody root systems.

Another construction of MDS

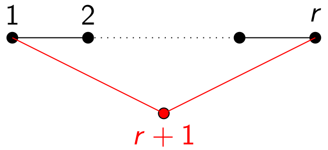
Our goal is to study another construction of MDS, obtained as a Chinta-Gunnels average over the Weyl group, and to relate it to Z_{arithm}.

What is known beyond the case of finite Weyl group:

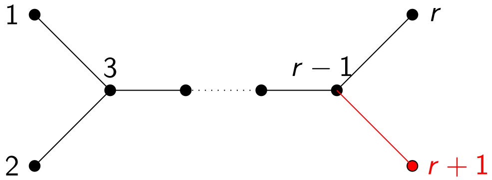
- Diaconu-Bucur (2009) studied such an MDS associated to $W_4 = \widetilde{D}_4$, proved its meromorphic continuation up to the natural boundary, and related it to Z_{arithm} up to a "diagonal factor";
- Lee-Zhang (2012) established meromorphic continuation for such a Chinta-Gunnels average, attached to symmetrizable Kac-Moody root systems.

We consider simply laced Coxeter root systems (all the roots have the same length), which are of **affine** type. The irreducible ones are $\widetilde{A}_r$, $\widetilde{D}_r$ and those associated with the finite exceptional root systems of type E_r, $6 \leqslant r \leqslant 8$.

$\widetilde{A}_r, r \geqslant 2$:



$\widetilde{D}_r, r \geqslant 4$:



Weyl group and root system associated to a diagram

- Let $\alpha_1, \ldots, \alpha_{r+1}$ be a basis of $\mathbb{R}^{r+1}$ (called simple roots), and define an inner product: all the α_i have the same length, α_i and α_j are orthogonal to each other if $i \not\sim j$ and they make a $2\pi/3$ angle if $i \sim j$. The inner-product is positive **semi-definite** in the affine case.

- Let $\alpha_1, \ldots, \alpha_{r+1}$ be a basis of $\mathbb{R}^{r+1}$ (called simple roots), and define an inner product: all the α_i have the same length, α_i and α_j are orthogonal to each other if $i \not\sim j$ and they make a $2\pi/3$ angle if $i \sim j$. The inner-product is positive **semi-definite** in the affine case.
- **The Weyl-Coxeter group** W: generated by the reflections σ_i in the plane orthogonal to α_i (Tits representation).

Weyl group and root system associated to a diagram

- Let $\alpha_1, \ldots, \alpha_{r+1}$ be a basis of $\mathbb{R}^{r+1}$ (called simple roots), and define an inner product: all the α_i have the same length, α_i and α_j are orthogonal to each other if $i \not\sim j$ and they make a $2\pi/3$ angle if $i \sim j$. The inner-product is positive **semi-definite** in the affine case.
- **The Weyl-Coxeter group** W: generated by the reflections σ_i in the plane orthogonal to α_i (Tits representation).
- **Root system** $\Phi \subset \mathbb{R}^{r+1}$: generated by the images of simple roots α_i under the Weyl group W.

Weyl group and root system associated to a diagram

- Let $\alpha_1, \ldots, \alpha_{r+1}$ be a basis of $\mathbb{R}^{r+1}$ (called simple roots), and define an inner product: all the α_i have the same length, α_i and α_j are orthogonal to each other if $i \not\sim j$ and they make a $2\pi/3$ angle if $i \sim j$. The inner-product is positive **semi-definite** in the affine case.
- **The Weyl-Coxeter group** W: generated by the reflections σ_i in the plane orthogonal to α_i (Tits representation).
- **Root system** $\Phi \subset \mathbb{R}^{r+1}$: generated by the images of simple roots α_i under the Weyl group W.
- Φ is contained in the root lattice $\bigoplus_{i=1}^{r+1} \mathbb{Z}\alpha_i$, and it decomposes $\Phi = \Phi^+ \cup \Phi^-$, into positive and negative roots.

Structure of affine Weyl groups

For affine root systems there is a special element in the root lattice ("imaginary root") fixed by the entire Weyl group. For $\widetilde{A}_r$ it is:

$$\delta = \alpha_1 + \ldots + \alpha_{r+1}.$$

Structure of affine Weyl groups

For affine root systems there is a special element in the root lattice ("imaginary root") fixed by the entire Weyl group. For $\widetilde{A}_r$ it is:

$$\delta = \alpha_1 + \ldots + \alpha_{r+1}.$$

Assume W comes from a finite Weyl group W_0 (e.g $\widetilde{A}_r$). We have

$$W = W_0 \ltimes T$$

with $T \simeq \mathbb{Z}^r$ a normal subgroup of translations.

Structure of affine Weyl groups

For affine root systems there is a special element in the root lattice ("imaginary root") fixed by the entire Weyl group. For $\widetilde{A}_r$ it is:

$$\delta = \alpha_1 + \ldots + \alpha_{r+1}.$$

Assume W comes from a finite Weyl group W_0 (e.g $\widetilde{A}_r$). We have

$$W = W_0 \ltimes T$$

with $T \simeq \mathbb{Z}^r$ a normal subgroup of translations.
Elements $t \in T$ act on simple roots by

$$t\alpha_i = \alpha_i + n_i\delta, \quad n_i \in \mathbb{Z}.$$

The Chinta-Gunnels action

Assume Φ is an irreducible, simply laced Coxeter root system, with Weyl group W generated by simple reflections $\sigma_1, \ldots, \sigma_{r+1}$.

The Chinta-Gunnels action

Assume Φ is an irreducible, simply laced Coxeter root system, with Weyl group W generated by simple reflections $\sigma_1, \ldots, \sigma_{r+1}$.

For $\mathbf{x} = (x_1, \ldots, x_{r+1})$, and $\alpha = \sum_{i=1}^{r+1} k_i \alpha_i$ in the root lattice:

$$\mathbf{x}^{\alpha} := \prod_{i=1}^{r+1} x_i^{k_i}.$$

The Weyl group W acts on variables $\mathbf{x}$ by

$$(w\mathbf{x})_i = \mathbf{x}^{w^{-1}\alpha_i}.$$

The action of generators σ_i correspond precisely to the functional equations of the MDS for $x_i = q^{1/2-s_i}$.

The Chinta-Gunnels action

Also define the sign function on multivariables $\mathbf{x}$:

$$(\varepsilon_i(\mathbf{x}))_j = \begin{cases} -x_j & \text{if } j \sim i \\ x_j & \text{otherwise} \end{cases}$$

The Chinta-Gunnels action

Also define the sign function on multivariables $\mathbf{x}$:

$$(\varepsilon_i(\mathbf{x}))_j = \begin{cases} -x_j & \text{if } j \sim i \\ x_j & \text{otherwise} \end{cases}$$

Chinta-Gunnels action (2006): for a rational function $f(\mathbf{x})$

$$f|\sigma_i(\mathbf{x}) = f(\sigma_i \mathbf{x}) J(x_i, 0) + f(\varepsilon_i \sigma_i \mathbf{x}) J(x_i, 1)$$

$$J(x, \delta) = \frac{x}{2}\left(\frac{\sqrt{q} - x}{1 - \sqrt{q}x} - (-1)^\delta\right),$$

and the action extends to W by $f|w_1|w_2 = f|w_1 w_2$.

Zeta function

Form the zeta function (Chinta-Gunnels average)

$$Z(\mathbf{x}) = \sum_{w \in W} 1|w(\mathbf{x}).$$

In the **affine** case, we want to:

- compute its residues at poles;
- relate it to Z_{arithm} that encodes moments.

The case $\widetilde{A}_r$

We restrict to $\widetilde{A}_r$ for simplicity. Let

$$I = \begin{cases} \{2, 4, \ldots, r+1\} & \text{for } r \text{ odd} \\ \{3, 5, \ldots, r+1\} & \text{for } r \text{ even.} \end{cases}$$

and let J be the complement of I in the set of indices $\{1, \ldots, r+1\}$. Let $\mathbf{x}_I$, $\mathbf{x}_J$ the corresponding multivariables.

The case $\widetilde{A}_r$

We restrict to $\widetilde{A}_r$ for simplicity. Let

$$I = \begin{cases} \{2, 4, \dots, r+1\} & \text{for } r \text{ odd} \\ \{3, 5, \dots, r+1\} & \text{for } r \text{ even.} \end{cases}$$

and let J be the complement of I in the set of indices $\{1, \dots, r+1\}$. Let $\mathbf{x}_I$, $\mathbf{x}_J$ the corresponding multivariables.

Set $u = \sqrt{q}$. The zeta function has poles at $x_i = 1/\sqrt{u}$ and it is completely determined by its functional equations, plus knowledge of the iterated residue:

$$R(\mathbf{x}_J; u) = \prod_{i \in I} \frac{1 - u x_i}{1 - x_i^2} Z(\mathbf{x}) \bigg|_{\mathbf{x}_i = 1/u,\, i \in I} .$$

A conjecture

Let $P = \prod_{j \in J} x_j / u^{|I|}$ (specialization of $\mathbf{x}^\delta$ for $x_i = 1/u$, $i \in I$).

Conjecture (Diaconu-Pașol-P.)

For $\widetilde{A}_r$, we have

$$R(\mathbf{x}_J; u) = f(P; u) \cdot R_0(\mathbf{x}_J; u)$$

where $f(P; u)$ is an explicit power series in P, and R_0 is an explicit residue, common to all zeta functions satisfying the same group of functional equations. Both R_0 and P are products of Pochhammer symbols.

Partial result

Denote by $(a;b)_\infty = \prod_{k=0}^{\infty}(1-ab^k)$, the b-Pochhammer symbol.

Theorem (Diaconu-Pașol-P.)

The conjecture holds for $\widetilde{A}_2, \widetilde{A}_3$. For example, for $\widetilde{A}_2$ we have

$$R(x_1,x_2;u) = \left(P^2;P^2\right)_\infty \left(P^2/u^2;P^2\right)_\infty \prod_{i=1}^{2} \left(x_i^2/u^2;P^2\right)_\infty$$

where $P = x_1x_2/u$ (recall $u=\sqrt{q}$).

Remarks

$$R(\mathbf{x}_J; u) = f(P; u) \cdot R_0(\mathbf{x}_J; u)$$

- We expect a similar result to hold in all affine cases. We proved it for $\widetilde{D}_4$.
- The factor R_0 is computed using the functional equations of Z under translations in affine Weyl group.
- The determination of the "diagonal factor" $f(P; u)$ is more subtle, and it requires a new type of functional equation.
- The multiple residue of Z_{arithm} for $\widetilde{A}_r$, in the same variables, was computed by Whitehead (2014). Both Z and Z_{arithm} satisfy the same affine Weyl group of functional equations, and proving the conjecture would also determine the proportionality factor between the two.

Connection with Macdonald's formula

The length function on a Coxeter group W has the property:

$$\ell(w) = \#\Phi(w), \quad \Phi(w) := \{\alpha \in \Phi^+ : w\alpha \in \Phi^-\}.$$

Set $s(w) = \sum_{\beta \in \Phi(w)} \beta$, an element in the root lattice.

$$Z(\mathbf{x}; q = 1) = \sum_{w \in W} (-1)^{\ell(w)} (-\mathbf{x})^{s(w)} =: F_{MD}(-\mathbf{x}).$$

Connection with Macdonald's formula

The length function on a Coxeter group W has the property:

$$\ell(w) = \#\Phi(w), \quad \Phi(w) := \{\alpha \in \Phi^+ : w\alpha \in \Phi^-\}.$$

Set $s(w) = \sum_{\beta \in \Phi(w)} \beta$, an element in the root lattice.

$$Z(\mathbf{x}; q = 1) = \sum_{w \in W} (-1)^{\ell(w)} (-\mathbf{x})^{s(w)} =: F_{MD}(-\mathbf{x}).$$

Key fact: For $q = 1$, the residue conjecture specializes to *Macdonald's formula* for $F_{MD}(-\mathbf{x})$ in the case $\widetilde{A}_r$ (specialized at $x_i = 1$ for $i \in I$). It can be regarded as a q-deformation of it.

Macdonald's formula

For **finite** W, we have the well-known Weyl denominator formula:

$$F_{MD}(\mathbf{x}) := \sum_{w \in W} (-1)^{\ell(w)} \mathbf{x}^{s(w)} = \prod_{\beta > 0} (1 - \mathbf{x}^{\beta})$$

where the product is over all positive roots in Φ.

Macdonald's formula

For **finite** W, we have the well-known Weyl denominator formula:

$$F_{MD}(\mathbf{x}) := \sum_{w \in W} (-1)^{\ell(w)} \mathbf{x}^{s(w)} = \prod_{\beta > 0} (1 - \mathbf{x}^{\beta})$$

where the product is over all positive roots in Φ.

Theorem (Macdonald 1972)

For **affine** W *associated to a finite, irreducible root system of rank* r*:*

$$F_{MD}(\mathbf{x}) = \prod_{k \geqslant 1} (1 - \mathbf{x}^{k\delta})^r \prod_{\beta > 0} (1 - \mathbf{x}^{\beta})$$

where δ *is the minimal positive imaginary root, fixed by all of* W*.*

Macdonald's formula

For **finite** W, we have the well-known Weyl denominator formula:

$$F_{MD}(\mathbf{x}) := \sum_{w \in W} (-1)^{\ell(w)} \mathbf{x}^{s(w)} = \prod_{\beta > 0} (1 - \mathbf{x}^{\beta})$$

where the product is over all positive roots in Φ.

Theorem (Macdonald 1972)

For **affine** W *associated to a finite, irreducible root system of rank* r*:*

$$F_{MD}(\mathbf{x}) = \prod_{k \geqslant 1} (1 - \mathbf{x}^{k\delta})^r \prod_{\beta > 0} (1 - \mathbf{x}^{\beta})$$

where δ *is the minimal positive imaginary root, fixed by all of* W*.*

Ideea of proof: both sides have the same functional equations under suitable Weyl group elements, and they agree when variables are appropriately specialized.

A cocycle associated to the Chinta-Gunnels action

Assume Φ is of type $\widetilde{A}_2$, $\widetilde{A}_3$ or $\widetilde{D}_4$. For a function $F(\mathbf{x})$ we denote

$$\overrightarrow{F}(\mathbf{x}) := \begin{pmatrix} F(\mathbf{x}) \\ F(\varepsilon_{r+1}\mathbf{x}) \\ F(\varepsilon_1\mathbf{x}) \\ F(\varepsilon_1\varepsilon_{r+1}\mathbf{x}) \end{pmatrix}$$

and we consider a matrix valued cocycle $M_w(\mathbf{x})$ such that

$$\overrightarrow{F|w}(\mathbf{x}) = M_w(\mathbf{x})\overrightarrow{F}(w\mathbf{x}).$$

The function M_w is defined initially on the generators of W, and then extended to the whole group by the cocycle relation

$$M_{w'w}(\mathbf{x}) = M_w(\mathbf{x})M_{w'}(w\mathbf{x}), \quad \text{for } w, w' \in W.$$

Functional equations

For the zeta function $Z = \sum_{w \in W} 1|w$, we have

$$\vec{Z}(\mathbf{x}) = \sum_{w \in W} M_w(\mathbf{x})\vec{1}$$

The cocycle relation for M_w implies the functional equations

$$\boxed{M_w(\mathbf{x})\vec{Z}(w\mathbf{x}) = \vec{Z}(\mathbf{x})}$$

(when $x = q^{1/2-s_i}$ these are of the same type as before in s_i).

Main result: an additional symmetry

The vector $\vec{Z}$ has an extra functional equation, when viewing $u = \sqrt{q}$ as an additional variable and making the transformation $u \mapsto u/\mathbf{x}^{\delta}$.

Main result: an additional symmetry

The vector $\vec{Z}$ has an extra functional equation, when viewing $u = \sqrt{q}$ as an additional variable and making the transformation $u \mapsto u/\mathbf{x}^\delta$.

Theorem (Diaconu-Pașol-P.)

Assume Φ is of type $\widetilde{A}_2$, $\widetilde{A}_3$ or $\widetilde{D}_4$. Then there exists a 4×4 matrix with **rational function** *entries $A(\mathbf{x})$ such that*

$$A(\mathbf{x})\vec{Z}(\mathbf{x}; u/\mathbf{x}^\delta) = \vec{Z}(\mathbf{x}; u).$$

Applications

- Our explicit formulas for the residue (including for $\widetilde{D}_4$) follow from the functional equations, together with Macdonald's formula.

Applications

- Our explicit formulas for the residue (including for $\widetilde{D}_4$) follow from the functional equations, together with Macdonald's formula.
- All the analytic information of the Chinta-Gunnels average Z transfers to Z_{arithm}.