Extreme Values of Twisted L-Functions

Online Conference in Automorphic Forms, June 2020
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(joint work with Blomer, Fouvry, Kowalski, Michel, Sawin)

Motivating example – $\zeta(\frac{1}{2}+it)$

$$\frac{1}{T}\int_T^{2T} |\zeta(\tfrac{1}{2}+it)|^2\,dt \sim \log T \qquad (T\to\infty).$$

so $|\zeta(\frac{1}{2}+it)| \approx \sqrt{\log t}$ "on average".

Lindelöf Hypothesis: $|\zeta(\frac{1}{2}+it)| \ll_\varepsilon (1+|t|)^\varepsilon$.

Theorem (Soundararajan)

$\exists\, t\in[T,2T]$ s.t.

$$|\zeta(\tfrac{1}{2}+it)| \geqslant \exp\left((1+o(1))\sqrt{\frac{\log T}{\log\log T}}\right).$$

Main Idea:

$$R(t) = \sum_{n\leq N}\frac{r(n)}{n^{it}} \qquad r(n) = \text{multiplicative, supp'd on sq.-free}$$

$$M_1(R,T) = \int_{-\infty}^{\infty} |R(t)|^2\,\Phi\left(\frac{t}{T}\right)dt \qquad \Phi\in C_c^\infty[1,2]$$

$$M_2(R,T) = \int_{-\infty}^{\infty} \zeta(\tfrac{1}{2}+it)\,|R(t)|^2\,\Phi\left(\frac{t}{T}\right)dt$$

Clearly $\displaystyle\max_{t\in[T,2T]}|\zeta(\tfrac{1}{2}+it)| \geqslant \frac{M_2(R,T)}{M_1(R,T)}$

$$M_1(R,T) = T\hat{\Phi}(0) \sum_{n \le N} |r(n)|^2 \quad + \text{negligible},$$

$$M_2(R,T) = T\hat{\Phi}(0) \cdot \sum_{mk = n \le N} \frac{r(m)\overline{r(n)}}{\sqrt{k}}$$

$$+ \text{Error Term} \quad + \text{negligible}.$$

Need to maximize $\dfrac{\displaystyle\sum_{mk=n\le N} \frac{r(m)\overline{r(mk)}}{\sqrt{k}}}{\displaystyle\sum_{n\le N} |r(n)|^2}$

$r(p) = f(p)$ supported on primes p in a specific range.

① $$\sum_{n \le N} |r(n)|^2 \le \prod_p (1 + f(p)^2)$$

② $$\sum_{mk \le N} \frac{r(m) r(mk)}{\sqrt{k}} =$$

$$= \sum_{k \le N} \frac{f(k)}{\sqrt{k}} \left(\prod_{p \nmid k} (1 + f(p)^2) + \text{Error} \right)$$

$$= \prod_p \left(\underbrace{1 + f(p)^2}_{} + \underbrace{\frac{f(p)}{\sqrt{p}}}_{} \right) + \text{Error}$$

$$\rightsquigarrow \prod_p \frac{1+f(p)^2+\frac{f(p)}{\sqrt{p}}}{1+f(p)^2} = \prod_p \left(1+\frac{f(p)}{\sqrt{p}\,(1+f(p)^2)}\right)$$

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f - Hecke (-Maaß) eigenform.

$\chi \longrightarrow L(s, f\times\chi)$

Theorems: f, g - fixed newforms of conductor $r \geq 1$, trivial central character, $I \subseteq \mathbb{R}/\pi\mathbb{Z}$ arbitrary, $q \geq q_0(f, g, I)$ prime.

Then:

* $\exists\ \chi \bmod q$ s.t.

$$|L(\tfrac{1}{2}, f\times\chi)| \geq \exp\left(c\sqrt{\frac{\log q}{\log\log q}}\right)$$

& $\arg L(\tfrac{1}{2}, f\times\chi) \in I$.

* $\exists\ \chi \bmod q$ s.t.

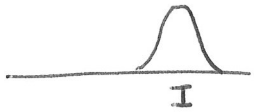
$$|L(\tfrac{1}{2}, f\times\chi)\, L(\tfrac{1}{2}, g\times\chi)| \geq \exp\left(c\sqrt{\frac{\log q}{\log\log q}}\right).$$

$$R(\chi) = \sum_{n \leq N} r(n)\, \lambda_f(n)\, \chi(n)$$

$$N \leq q^{1-\varepsilon}.$$

Compare

$$\sum_{\chi \bmod q}^{*} |R(\chi)|^2 \qquad \text{vs.}$$



$$\sum_{\chi \bmod q}^{*} |R(\chi)|^2 \, |L(\tfrac{1}{2}, f \times \chi)| \, \psi\big(\theta(f \times \chi)\big)$$

$$\text{or} \quad \sum_{\chi \bmod q}^{*} |R(\chi)|^2 \, L(\tfrac{1}{2}, f \times \chi) \, \overline{L(\tfrac{1}{2}, g \times \chi)}$$

<u>Moment Evaluations:</u>

1) $$\frac{1}{\varphi^*(q)} \sum_{\chi \bmod q}^{*} L(\tfrac{1}{2}, f \times \chi)\, \varepsilon_\chi^{k}\, \chi(\ell)$$

$$= \delta_{k=0} \frac{\lambda_f(\bar{\ell}_q)}{\bar{\ell}_q^{1/2}} + \delta_{k=-2} \cdots + \text{Error}.$$

(Fouvry - Kowalski - Michel)

$$2)\quad \frac{1}{\varphi^*(q)} \sum_{\chi \bmod q}^{*} L(f\times\chi,\tfrac{1}{2})\,\overline{L(g\times\chi,\tfrac{1}{2})}\,\chi(\ell)\overline{\chi(\ell')}$$

$$= \frac{1}{2} L^*(f\times g,1)\left(\frac{\lambda_f^*(\ell')\lambda_g^*(\ell)}{\sqrt{\ell\ell'}} + \text{dual}\right)$$

$$+ O\left(L^{3/2} q^{-\frac{1}{144}} + q^{-\frac{1}{2}+\varepsilon}\right).$$

(Blomer - Fouvry - Kowalski - Michel - M. - Sawin)

$\rightsquigarrow$ After moments / resonator, leads to

$$\prod_p \left(1 + \lambda_f(p)^2 r(p)^2\right)$$

$$\text{vs}\quad \prod_p \left(1 + \underbrace{\lambda_f(p)^2}\, r(p)^2 + \frac{\lambda_f(p)^2 r(p)}{\sqrt{p}}\right).$$

Proposition:

$$\sum_{n \le N} r(n)^2 \omega(n) \stackrel{?}{=} \prod_p \quad + \text{Error}$$

$$\sum_{\substack{nm \le N \\ (n,m)=1}} \frac{r(n)^2 r(m) \omega(n) \omega'(m)}{\sqrt{m}} \stackrel{?}{=} \prod_p \quad + \text{Error}$$

$$\prod_p \left(1 + \frac{r(p) \omega'(p)}{\sqrt{p}\,(1 + r(p)^2 \omega(p))} \right) \stackrel{?}{\gg} \cdots$$

Above can be done if $\forall\, Y \geqslant 2X \geqslant 4$,

$$\sum_{X \le p \le Y} \frac{\omega(p)}{p \log p} \le a_\omega \left(\frac{1}{\log X} - \frac{1}{\log Y} \right) + O_\omega\left(\frac{1}{\log^2 X} \right)$$

$$\sum_{X \le p \le Y} \frac{\omega'(p)}{p \log p} \geqslant a'_{\omega'} \left(\frac{1}{\log X} - \frac{1}{\log Y} \right) + O_{\omega'}\left(\frac{1}{\log^2 X} \right)$$

$$\sum_{p \le X} \omega(p)^\delta \omega'(p) \ll_{\omega, \omega', \delta} X (\log X)^\delta$$

$$\sum_{p \le X} \omega'(p)^{1+\delta'} \ll_{\omega', \delta'} X^{1+\delta''/2}$$