Small Scale Equidistribution of Lattice Points on the Sphere

Peter Humphries (joint with Maksym Radziwiłł)

June 3, 2020

Sums of Three Squares: Classification

Question

When can a positive integer n be written as the sum of three squares?

Want to determine when the set

$$\mathcal{E}(n) := \left\{(x_1, x_2, x_3) \in \mathbb{Z}^3 : x_1^2 + x_2^2 + x_3^2 = n\right\}$$

is nonempty.

Geometric viewpoint: $\mathcal{E}(n)$ is the set of points on the lattice $\mathbb{Z}^3$ in $\mathbb{R}^3$ that lie on the sphere centred at the origin of radius $\sqrt{n}$.

Sums of Three Squares: Classification

Example

$x_1^2 + x_2^2 + x_3^2 = 11$ with $(x_1, x_2, x_3) \in \{(3, 1, 1), (1, 3, 1), \ldots\}$.

Example

$x_1^2 + x_2^2 + x_3^2 = 7$ has no integral solutions.

Sums of Three Squares: Classification

If $x \in \mathbb{Z}$, then $x^2 \equiv 0, 1, \text{ or } 4 \pmod{8}$. So if $(x_1, x_2, x_3) \in \mathbb{Z}^3$, then $x_1^2 + x_2^2 + x_3^2 \equiv 0, 1, 2, 3, 4, 5, \text{ or } 6 \pmod{8}$.

Theorem (Legendre (1798))

Any positive integer n can be written as a sum of three squares if and only if n is not of the form $n = 4^a(8b + 7)$ for some nonnegative integers a, b.

Question

Given a positive integer n, how many ways are there to write n as the sum of three squares?

Given n, want to determine

$$r_3(n) := \#\mathcal{E}(n) = \#\left\{(x_1, x_2, x_3) \in \mathbb{Z}^3 : x_1^2 + x_2^2 + x_3^2 = n\right\}.$$

Some obvious symmetries: permutations, multiplication of a coordinate by -1.

Sums of Three Squares: Number of Solutions

Interesting case is n squarefree.

Theorem (Gauss (1801))

For odd squarefree $n \not\equiv 7 \pmod 8$,

$$r_3(n) = \begin{cases} 12h(D) & \text{for } n \equiv 1, 2 \pmod 4 \text{ with } D = -4n, \\ 24h(D) & \text{for } n \equiv 3 \pmod 8 \text{ with } D = -n. \end{cases}$$

$h(D)$ is the class number of the imaginary quadratic number field $\mathbb{Q}(\sqrt{D})$.

Sums of Three Squares: Number of Solutions

Corollary

For all $\varepsilon > 0$, we have that

$$n^{\frac{1}{2}-\varepsilon} \ll_\varepsilon r_3(n) \ll \sqrt{n}\log n.$$

Upper bound is easy; not hard to show that $h(D) \ll \sqrt{|D|}\log|D|$.

Lower bound is nontrivial; Dirichlet class number formula (1839) plus Siegel ineffective bound (1935)

$$L(1,\chi_D) \gg_\varepsilon |D|^{-\varepsilon}.$$

Sums of Three Squares: Limiting Behaviour

Let

$$\widehat{\mathcal{E}}(n) := \left\{ \left(\frac{x_1}{\sqrt{n}}, \frac{x_2}{\sqrt{n}}, \frac{x_3}{\sqrt{n}} \right) \in \mathbb{R}^3 : (x_1, x_2, x_3) \in \mathcal{E}(n) \right\}$$

denote the projection of $\mathcal{E}(n)$ onto the unit sphere

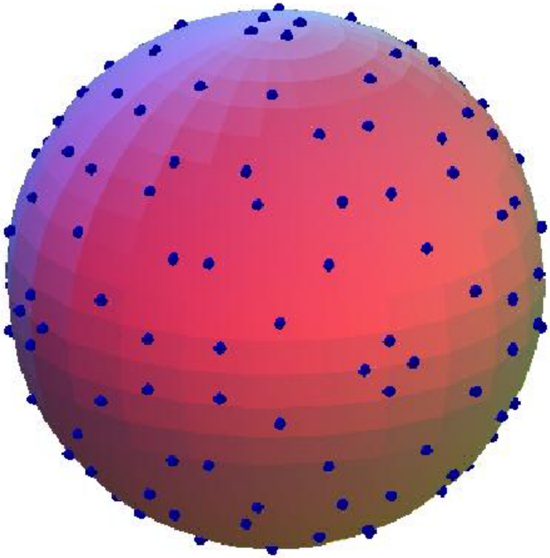
$$S^2 := \left\{ (x_1, x_2, x_3) \in \mathbb{R}^3 : x_1^2 + x_2^2 + x_3^2 = 1 \right\}.$$

Question

What are the limiting statistical properties of $\widehat{\mathcal{E}}(n) \subset S^2$ as $n \to \infty$?

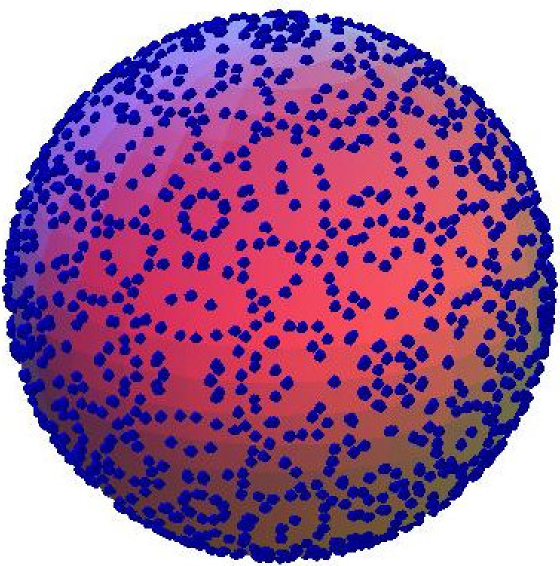
Examples: $n = 101$

(Image: Ellenberg–Michel–Venkatesh)



Examples: $n = 104851$

(Image: Ellenberg–Michel–Venkatesh)



Sums of Three Squares: Limiting Behaviour

The normalised lattice points $\widehat{\mathcal{E}}(n) \subset S^2$ appear to behave just like *random* points on the sphere.

Goal

Quantify the limiting behaviour of $\widehat{\mathcal{E}}(n)$ as $n \to \infty$ in ways that are shared by *randomly chosen* points.

Duke's Theorem

Theorem (Duke (1988), Duke–Schulze-Pillot (1990), Golubeva–Fomenko (1990))

As $n \to \infty$ along squarefree integers with $n \not\equiv 7 \pmod 8$, the lattice points on the sphere $\widehat{\mathcal{E}}(n)$ equidistribute on S^2.

Informally, the points $\widehat{\mathcal{E}}(n)$ spread out randomly on S^2.

Proved earlier by Linnik (1968) for certain subsequences of n.

Equidistribution

Let M be a topological space and μ a probability measure on M. Let μ_n be a sequence of probability measures on M.

Definition

The probability measures μ_n equidistribute on M w.r.t. μ if

$$\lim_{n \to \infty} \mu_n(B) = \mu(B)$$

for every continuity set $B \subset M$ (boundary has μ-measure zero).

Equidistribution

Let M be a topological space and μ a probability measure on M. Let μ_n be a sequence of probability measures on M.

Definition

The probability measures μ_n equidistribute on M w.r.t. μ if

$$\lim_{n\to\infty} \int_M f(x)\, d\mu_n(x) = \int_M f(x)\, d\mu(x)$$

for all $f \in C_b(M)$ (continuous bounded).

Equidistribution

Definition

We define a probability measure μ_n on S^2 by

$$\mu_n := \frac{1}{\#\widehat{\mathcal{E}}(n)} \sum_{x \in \widehat{\mathcal{E}}(n)} \delta_x.$$

So for $B \subset S^2$ and $f : S^2 \to \mathbb{C}$,

$$\mu_n(B) := \frac{\#(\widehat{\mathcal{E}}(n) \cap B)}{\#\widehat{\mathcal{E}}(n)},$$

$$\int_{S^2} f(y)\, d\mu_n(y) := \frac{1}{\#\widehat{\mathcal{E}}(n)} \sum_{x \in \widehat{\mathcal{E}}(n)} f(x).$$

Duke's Theorem

Theorem (Duke (1988), Duke–Schulze-Pillot (1990), Golubeva–Fomenko (1990))

As $n \to \infty$ along squarefree integers with $n \not\equiv 7 \pmod 8$, the probability measures μ_n equidistribute on S^2 with respect to the normalised surface measure μ on S^2.

Duke's Theorem

Theorem (Duke (1988), Duke–Schulze-Pillot (1990), Golubeva–Fomenko (1990))

As $n \to \infty$ along squarefree integers with $n \not\equiv 7 \pmod 8$,

$$\frac{\#(\widehat{\mathcal{E}}(n) \cap B)}{\#\widehat{\mathcal{E}}(n)} \to \mathrm{vol}(B)$$

for every continuity set $B \subset S^2$.

Duke's Theorem

Theorem (Duke (1988), Duke–Schulze-Pillot (1990), Golubeva–Fomenko (1990))

As $n \to \infty$ along squarefree integers with $n \not\equiv 7 \pmod 8$,

$$\frac{1}{\#\widehat{\mathcal{E}}(n)} \sum_{x \in \widehat{\mathcal{E}}(n)} f(x) \to \int_{S^2} f(y)\, dy$$

for every continuous function f on S^2.

Proof of Duke's Theorem

Idea of proof.

Approximate $f \in C(S^2)$ by spherical harmonics.

Reduces problem to showing that for every spherical harmonic ϕ,

$$\frac{1}{\#\widehat{\mathcal{E}}(n)} \sum_{x \in \widehat{\mathcal{E}}(n)} \phi(x) \to \int_{S^2} \phi(y)\, dy.$$

Trivial if ϕ is constant. RHS is zero if ϕ is nonconstant.

Since $\#\widehat{\mathcal{E}}(n) \gg_{\varepsilon} n^{1/2-\varepsilon}$, suffices to show that there exists $\delta > 0$ such that

$$\sum_{x \in \widehat{\mathcal{E}}(n)} \phi(x) \ll_{\phi} n^{\frac{1}{2}-\delta}.$$

□

Waldspurger's Identity

Theorem (Waldspurger (1981))

Given a spherical harmonic ϕ of degree $m_\phi \geq 1$, there exists a holomorphic modular form f of weight $2 + 2m_\phi$ such that

$$\left| \sum_{x \in \widehat{\mathcal{E}}(n)} \phi(x) \right|^2 \approx \sqrt{n} L\left(\frac{1}{2}, f\right) L\left(\frac{1}{2}, f \otimes \chi_{-n}\right).$$

Waldspurger's Identity: Step 1

Waldspurger's identity proceeds in two steps.

Proposition

Given a spherical harmonic ϕ of degree $m_\phi \geq 1$, there exists a half-integral weight modular form g of weight $m_\phi + 1/2$ and level 4 *lying in the Kohnen minus space such that*

- *the ratio of Petersson norms is $\approx L(1/2, f)$, and*
- *the n-th Fourier coefficient $\rho_g(n)$ of g satisfies*

$$\rho_g(n) = \sum_{x \in \widehat{\mathcal{E}}(n)} \phi(x).$$

This is a special case of the Rallis inner product formula.

Waldspurger's Identity: Step 2

Proposition

Given a half-integral weight modular form g of weight $m + 1/2$ and level 4 lying in the Kohnen minus space, there exists a holomorphic modular form f of level $2 + 2m$ and level 2 such that

$$|\rho_g(n)|^2 \approx \sqrt{n} L\left(\frac{1}{2}, f \otimes \chi_{-n}\right).$$

Remark

Alternatively, one can circumvent the need for half-integral weight modular forms via work of Martin–Whitehouse (following Waldspurger, Gross, Böcherer–Schulze-Pillot, Zhang, Jacquet–Nan, Popa, . . .)

Iwaniec's Subconvex Bound

Theorem (Iwaniec (1987))

There exists $\delta > 0$ such that

$$L\left(\frac{1}{2}, f \otimes \chi_{-n}\right) \ll_f n^{\frac{1}{2}-\delta}.$$

This is a case of *subconvexity*. Trivial bound is

$$L\left(\frac{1}{2}, f \otimes \chi_{-n}\right) \ll_{f,\varepsilon} n^{\frac{1}{2}+\varepsilon}.$$

Consequence of the Phragmén–Lindelöf convexity principle.
Generalisation of the bound

$$\zeta\left(\frac{1}{2} + it\right) \ll_{\varepsilon} (|t| + 1)^{\frac{1}{4}+\varepsilon}.$$

Rate of Equidistribution: Decay of Error Term

What is the rate of equidistribution of μ_n on S^2 w.r.t. μ?

Goal

Find the most rapidly decreasing function $\alpha(n)$ for which

$$\mu_n(B) = \frac{\#(\widehat{\mathcal{E}}(n) \cap B)}{\#\widehat{\mathcal{E}}(n)}$$

is equal to

$$\mathrm{vol}(B) + O_B(\alpha(n))$$

for a fixed continuity set $B \subset S^2$.

Informally, determine how quickly the points $\widehat{\mathcal{E}}(n)$ spread out randomly on S^2.

Rate of Equidistribution: Decay of Error Term

Heuristic

Like *random* points, we should expect square-root cancellation: since $\#\widehat{\mathcal{E}}(n) \approx \sqrt{n}$, we should hope for $\alpha(n) \approx n^{-1/4}$.

Rate of Equidistribution: Decay of Error Term

Theorem (Conrey–Iwaniec (2000))

For a fixed continuity set $B \subset S^2$,

$$\frac{\#(\widehat{\mathcal{E}}(n) \cap B)}{\#\widehat{\mathcal{E}}(n)} = \mathrm{vol}(B) + O_{B,\varepsilon}\left(n^{-\frac{1}{12}+\varepsilon}\right)$$

for all $\varepsilon > 0$.

Follows from Waldspurger's identity together with the *Weyl-strength* subconvex bound

$$L\left(\frac{1}{2}, f \otimes \chi_{-n}\right) \ll_{f,\varepsilon} n^{\frac{1}{3}+\varepsilon}.$$

Rate of Equidistribution: Decay of Error Term

Assuming the generalised Lindelöf hypothesis, we instead have

$$L\left(\frac{1}{2}, f \otimes \chi_{-n}\right) \ll_{f,\varepsilon} n^{\varepsilon}.$$

Theorem

For a fixed continuity set $B \subset S^2$,

$$\frac{\#(\widehat{\mathcal{E}}(n) \cap B)}{\#\widehat{\mathcal{E}}(n)} = \mathrm{vol}(B) + O_{B,\varepsilon}\left(n^{-\frac{1}{4}+\varepsilon}\right)$$

for all $\varepsilon > 0$ under the assumption of the generalised Lindelöf hypothesis.

Optimal.

Rate of Equidistribution: Small Scale Equidistribution

What is the rate of equidistribution of μ_n on S^2 w.r.t. μ?

Goal

Find the most rapidly decreasing function $\alpha(n)$ for which

$$\lim_{n \to \infty} \frac{1}{\mathrm{vol}(B_n)} \frac{\#(\widehat{\mathcal{E}}(n) \cap B_n)}{\#\widehat{\mathcal{E}}(n)} = 1$$

for a family of sets $B = B_n$ with $\mathrm{vol}(B_n) = \alpha(n)$.

Informally, determine the scale at which the points $\widehat{\mathcal{E}}(n)$ no longer look random.

How small does a set B_n have to be to **not** contain the expected number of points?

Rate of Equidistribution: Small Scale Equidistribution

Heuristic

Like *random* points, we should expect small scale equidistribution provided we are at a scale for which $\#(\widehat{\mathcal{E}}(n) \cap B_n) \to \infty$. Since $\#\widehat{\mathcal{E}}(n) \approx \sqrt{n}$, the optimal scale should be $\alpha(n) \approx n^{-1/2}$.

Optimal Small Scale Equidistribution

Proposition

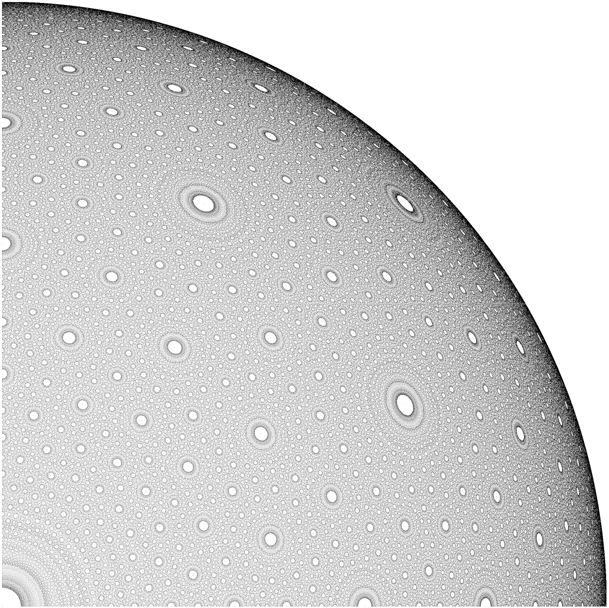
Generically, $\widehat{\mathcal{E}}(n)$ cannot equidistribute on shrinking sets B_n for which $\mathrm{vol}(B_n) \leq n^{-\frac{1}{2}-\delta}$ for some $\delta > 0$.

Sketch of Proof.

There are $\approx \sqrt{n}$ points in $\widehat{\mathcal{E}}(n)$, so if $\mathrm{vol}(B_n) \leq n^{-\frac{1}{2}-\delta}$, then generically $\widehat{\mathcal{E}}(n) \cap B_n = \emptyset$ by the pigeonhole principle. $\square$

Example: $n \leq 2048$

(Image: Stefan Kohl)



Optimal Small Scale Equidistribution

Conjecture

Lattice points $\widehat{\mathcal{E}}(n)$ equidistribute on shrinking sets B_n for which $\mathrm{vol}(B_n) \gg n^{-\frac{1}{2}+\delta}$ for some $\delta > 0$.

Optimal scale.

Remark

Conjecture does *not* follow from the generalised Lindelöf hypothesis!

Small Scale Equidistribution

Conjecture looks very hard, especially for balls $B_n = B_R(w)$.

Theorem (H.–Radziwiłł (2019))

Fix $w \in S^2$. If $R \geq n^{-\delta}$ for some fixed $\delta < \frac{1}{24}$,

$$\lim_{n \to \infty} \frac{1}{\mathrm{vol}(B_R(w))} \frac{\#(\widehat{\mathcal{E}}(n) \cap B_R(w))}{\#\widehat{\mathcal{E}}(n)} = 1.$$

Assuming the generalised Lindelöf hypothesis, this holds for $\delta < \frac{1}{8}$.

Under Lindelöf, implies small scale equidistribution at scales down to $\mathrm{vol}(B_n) \approx n^{-1/4}$; far shy of the optimal scale $\mathrm{vol}(B_n) \approx n^{-1/2}$.

What about for annuli?

Linnik's conjecture

Conjecture (Linnik (1968))

Fix $\delta > 0$. For all sufficiently large squarefree $n \not\equiv 7 \pmod 8$,

$$x_1^2 + x_2^2 + x_3^2 = n$$

has an integral solution $(x_1, x_2, x_3) \in \mathbb{Z}^3$ with $|x_3| < n^\delta$.

Special case of optimal small scale equidistribution:
B_n the annulus (belt about the equator) of optimally shrinking width.

Linnik's conjecture

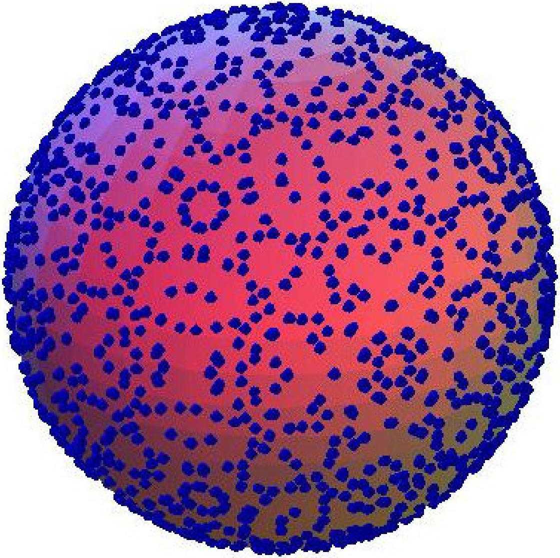
Conjecture (Linnik (1968))

Fix $\delta > 0$. For all sufficiently large squarefree $n \not\equiv 7 \pmod 8$, there exists $(x_1, x_2, x_3) \in \widehat{\mathcal{E}}(n)$ with $|x_3| < n^{-\frac{1}{2}+\delta}$.

Special case of optimal small scale equidistribution:
B_n the annulus (belt about the equator) of optimally shrinking width.

Example: $n = 104851$

(Image: Ellenberg–Michel–Venkatesh)



Linnik's Conjecture

Theorem (H.–Radziwiłł (2019))

Fix $\delta > 0$. For all sufficiently large squarefree $n \not\equiv 7 \pmod 8$,

$$x_1^2 + x_2^2 + x_3^2 = n$$

has an integral solution $(x_1, x_2, x_3) \in \mathbb{Z}^3$ with $|x_3| < n^{\frac{4}{9}+\delta}$.

Assuming the generalised Lindelöf hypothesis, the same result is true with $|x_3| < n^{\frac{1}{4}+\delta}$.

Still fall well short of Linnik's conjecture $|x_3| < n^{\delta}$.

Proof shows small scale equidistribution when $\mathrm{vol}(B_n) \gg n^{-\frac{1}{18}+\delta}$.

Rotated Linnik's Conjecture

Linnik's conjecture is small scale equidistribution on thin annuli around the equator, with respect to the north pole $(0, 0, 1) \in S^2$.

Nothing special about this choice of north pole; could also choose any other equator with respect to a point $w = (w_1, w_2, w_3) \in S^2$.

Conjecture (Rotated Linnik's Conjecture)

Fix $\delta > 0$. For all sufficiently large squarefree $n \not\equiv 7 \pmod 8$,

$$x_1^2 + x_2^2 + x_3^2 = n$$

has an integral solution $x = (x_1, x_2, x_3) \in \mathbb{Z}^3$ with $|x \cdot w| < n^{\delta}$.

Rotated Linnik's Conjecture

Theorem (H.–Radziwiłł (2019))

Fix $\delta > 0$ and $w \in S^2$. For all sufficiently large squarefree $n \not\equiv 7 \pmod 8$,

$$x_1^2 + x_2^2 + x_3^2 = n$$

has an integral solution $x = (x_1, x_2, x_3) \in \mathbb{Z}^3$ with $|x \cdot w| < n^{\frac{4}{9}+\delta}$.

Assuming the generalised Lindelöf hypothesis, the same result is true with $|x \cdot w| < n^{\frac{1}{4}+\delta}$.

Question

Can we do better for "most" $w \in S^2$?

Averaged Rotated Linnik's Conjecture

Theorem (H.–Radziwiłł (2019))

Fix $\delta > 0$. For squarefree $n \not\equiv 7 \pmod 8$, the volume of the set of $w \in S^2$ for which

$$x_1^2 + x_2^2 + x_3^2 = n$$

has no integral solutions $x = (x_1, x_2, x_3) \in \mathbb{Z}^3$ with $|x \cdot w| < n^\delta$ is $o(1)$ as $n \to \infty$.

Unconditionally resolves the rotated Linnik's conjecture for *almost every* pole $w \in S^2$.

Optimal. Fails if instead one demands $|x \cdot w| < 1000$.

Optimal Small Scale Equidistribution on Annuli

Theorem follows from the following result on the equidistribution of lattice points in the annulus $B_n = B_n(w)$ around the equator with respect to the north pole $w = (w_1, w_2, w_3) \in S^2$ of volume $n^{-\frac{1}{2}+\delta}$.

Theorem

For any fixed $\varepsilon > 0$,

$$\lim_{n\to\infty} \mathrm{vol}\left(\left\{w \in S^2 : \left|\frac{1}{\mathrm{vol}(B_n)} \frac{\#(\widehat{\mathcal{E}}(n) \cap B_n(w))}{\#\mathcal{E}(n)} - 1\right| > \varepsilon\right\}\right) = 0.$$

In particular, the normalised lattice points $\widehat{\mathcal{E}}(n)$ *equidistribute on the shrinking annulus* $B_n(w)$ *of volume* $n^{-\frac{1}{2}+\delta}$ *for almost every* $w \in S^2$.

Rate of shrinking is **optimal**.

Optimal Small Scale Equidistribution on Annuli

Method of proof.

By Chebyshev's inequality, this result follows upon showing that

$$\mathrm{Var}(\widehat{\mathcal{E}}(n); B_n) := \int_{S^2} \left(\#(\widehat{\mathcal{E}}(n) \cap B_n(w)) - \mathrm{vol}(B_n)\#\widehat{\mathcal{E}}(n)\right)^2 \, dw$$

is $O(\mathrm{vol}(B_n)^2 n^{1-\delta})$ as $n \to \infty$. $\square$

Can ask for more refined results about this variance.

Bourgain–Rudnick–Sarnak Conjecture

Conjecture (Bourgain–Rudnick–Sarnak (2017))

Let $B_n(w)$ be a sequence of balls (spherical caps) or annuli on S^2 of shrinking volume as $n \to \infty$. Then as $n \to \infty$,

$$\mathrm{Var}(\widehat{\mathcal{E}}(n); B_n) \sim \mathrm{vol}(B_n) \# \widehat{\mathcal{E}}(n).$$

Motivation

Such an asymptotic holds for *random* points.

Highly refined quantification of randomness of lattice points on the sphere; far beyond equidistribution!

Bourgain–Rudnick–Sarnak Conjecture

Theorem (H.–Radziwiłł (2019))

Let $B_n(w)$ be a sequence of annuli on S^2 with fixed inner radius for which $\mathrm{vol}(B_n) \ll n^{-\frac{5}{12}-\delta}$ for some $\delta > 0$. Then as $n \to \infty$,

$$\mathrm{Var}(\widehat{\mathcal{E}}(n); B_n) \sim \mathrm{vol}(B_n) \# \widehat{\mathcal{E}}(n).$$

Resolves the Bourgain–Rudnick–Sarnak conjecture for *small* annuli, namely $\mathrm{vol}(B_n) \ll n^{-\frac{5}{12}-\delta}$.

For less small annuli, namely $n^{-\frac{5}{12}-\delta} \ll \mathrm{vol}(B_n) \ll 1$, we still get nontrivial upper bounds in place of asymptotics for the variance.

Idea of Proof

First step of proof to bound the variance: spectral expansion on $L^2(S^2)$ plus Waldspurger's formula.

Lemma

We have that

$$\mathrm{Var}(\widehat{\mathcal{E}}(n); B_n) \approx \mathrm{vol}(B_n)^2 \sqrt{n} \sum_{f} L\left(\frac{1}{2}, f\right) L\left(\frac{1}{2}, f \otimes \chi_{-n}\right) |h(k_f)|^2$$

where the sum is over modular forms of even weight $k_f \in 2\mathbb{N}$, and

$$h(k) \ll \begin{cases} \dfrac{1}{\sqrt{k}} & \text{for } k \leq \dfrac{1}{\mathrm{vol}(B_n)}, \\ \dfrac{1}{\mathrm{vol}(B_n) k^{3/2}} & \text{for } k \geq \dfrac{1}{\mathrm{vol}(B_n)}. \end{cases}$$

The function $h : 2\mathbb{N} \to \mathbb{C}$ is the Selberg–Harish-Chandra transform of the indicator function of the annulus; can be explicitly written in terms of integrals of Legendre polynomials.

Reduction to Bounds for Moments of L-Functions

Break up sum into dyadic ranges; reduces problem to bounding moments of L-functions.

Corollary

Good bounds for $\mathrm{Var}(\widehat{\mathcal{E}}(n); B_n)$ *follow from good bounds for the moment of L-functions*

$$\sum_{T \leq k_f \leq 2T} L\left(\frac{1}{2}, f\right) L\left(\frac{1}{2}, f \otimes \chi_{-n}\right)$$

associated to modular forms f *of even weight* $k_f \in [T, 2T] \cap 2\mathbb{N}$.

Need uniformity in T and n; hybrid problem.

Bounds for Moments of L-Functions

Lemma

Assuming the generalised Lindelöf hypothesis,

$$\sum_{T \leq k_f \leq 2T} L\left(\frac{1}{2}, f\right) L\left(\frac{1}{2}, f \otimes \chi_{-n}\right) \ll_\varepsilon n^\varepsilon T^{2+\varepsilon}.$$

Would like results of this strength *unconditionally*.

Lemma

Unconditionally, the moment above is

$$\ll_\varepsilon \begin{cases} n^{\frac{1}{3}+\varepsilon} T^{2+\varepsilon} & \text{for } T \ll n^{\frac{1}{12}}, \\ n^{\frac{1}{2}+\varepsilon} & \text{for } n^{\frac{1}{12}} \ll T \ll n^{\frac{1}{4}}, \\ n^\varepsilon T^{2+\varepsilon} & \text{for } T \gg n^{\frac{1}{4}}. \end{cases}$$

"Lindelöf on average" for T sufficiently large.

Dropping all but one term yields subconvexity.

Asymptotics for Moments of L-Functions

For the Bourgain–Rudnick–Sarnak conjecture on the variance, we need *asymptotics* instead of upper bounds for this moment.

Lemma

The moment

$$\sum_{T \leq k_f \leq 2T} L\left(\frac{1}{2}, f\right) L\left(\frac{1}{2}, f \otimes \chi_{-n}\right)$$

is exactly *equal to the sum of the main term*

$$L(1, \chi_{-n}) T^2$$

and an error term

$$\frac{T}{\sqrt{n}} \sum_{m=1}^{n} \lambda_{\chi,1}(m) \lambda_{\chi,1}(n-m) \sum_{T \leq k \leq 2T} P_{k-1}\left(1 - \frac{2m}{n}\right),$$

a shifted convolution sum weighted by a (smoothed) sum of Legendre polynomials.

Asymptotics for Moments of L-Functions

Idea of proof.

- Dirichlet series for $L(s, f)L(s, f \otimes \chi_{-n})$ with $\Re(s) \gg 1$,
- Petersson trace formula,
- open up Kloosterman sums and use the Poisson summation formula twice/Voronoĭ summation formula once,
- analytically continue to $s = 1/2$.

Diagonal term and zero frequency from Poisson gives main term. Off-diagonal gives error term. □

Well trodden road: Bykovskiĭ, Goldfeld–Zhang, Nelson.

Could instead use approximate functional equations: Holowinsky–Templier, H.–Khan.

Could also use relative trace formula: Ramakrishnan–Rogawski, Feigon–Whitehouse, Michel–Ramakrishnan.

Bounding the Error Term

Error term is

$$\frac{T}{\sqrt{n}} \sum_{m=1}^{n} \lambda_{\chi,1}(m)\lambda_{\chi,1}(n-m) \sum_{T \leq k \leq 2T} P_{k-1}\left(1 - \frac{2m}{n}\right).$$

Insert the trivial bound for the Hecke eigenvalues and *nontrivial* bounds for the sum over $k \in [T, 2T]$.

Major issue: special functions behave differently in various regimes, so many separate cases to deal with.

- Bound Legendre polynomials by 1 when $m \ll n/T^2$ (no oscillation),
- Mellin inversion plus Poisson summation when $m \gg n/T^2$ (oscillatory), yielding integrals of Bessel functions, then stationary phase. Many subcases to deal with due to uniformity in T, m, n.

Sketch of Proof

End up with

$$\sum_{T \leq k_f \leq 2T} L\left(\frac{1}{2}, f\right) L\left(\frac{1}{2}, f \otimes \chi_{-n}\right) = L(1, \chi_{-n}) T^2 + O_\varepsilon\left(n^{\frac{1}{2}+\varepsilon}\right).$$

Main term dominates when $T \gg n^{\frac{1}{4}}$.

Alternative strategy: Hölder's inequality plus bounds for cubic moments of L-functions (Conrey–Iwaniec (2000), Young (2017), Petrow–Young (2019)) yields

$$\sum_{T \leq k_f \leq 2T} L\left(\frac{1}{2}, f\right) L\left(\frac{1}{2}, f \otimes \chi_{-n}\right) \ll_\varepsilon n^{\frac{1}{3}+\varepsilon} T^{2+\varepsilon}.$$

Better when $T \ll n^{\frac{1}{12}}$.

Optimal Small Scale Equidistribution on Balls

Question

What about small scale equidistribution on balls (spherical caps) instead of annuli?

Equidistribution implied by

$$\mathrm{Var}(\widehat{\mathcal{E}}(n); B_n) = o(\mathrm{vol}(B_n)^2 n)$$

for $\mathrm{vol}(B_n) \gg n^{-\frac{1}{2}+\delta}$. For $T \ll n^{\frac{1}{4}-\delta}$, we need

$$\sum_{T \leq k_f \leq 2T} L\left(\frac{1}{2}, f\right) L\left(\frac{1}{2}, f \otimes \chi_{-n}\right) = o(\sqrt{n}).$$

Unfortunately, can only prove $O_\varepsilon\left(n^{\frac{1}{2}+\varepsilon}\right)$ for $n^{\frac{1}{12}} \ll T \ll n^{\frac{1}{4}}$. Need to find additional cancellation from error term.

Related Problems: Heegner Points and Closed Geodesics

Method also works for ternary quadratic forms other than just

$$x_1^2 + x_2^2 + x_3^2 = n.$$

Can instead work with the indefinite ternary quadratic form

$$x_2^2 - 4x_1x_3 = D.$$

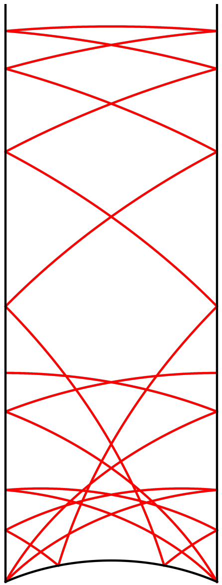
Involves different geometric objects in place of normalised lattice points $\widehat{\mathcal{E}}(n)$ on the sphere S^2:

- Heegner points on the modular surface $\Gamma\backslash\mathbb{H}$ when $D < 0$,
- Closed geodesics on $\Gamma\backslash\mathbb{H}$ when $D > 0$.

Equidistribution as $|D| \to \infty$ along fundamental discriminants: Duke's theorem.

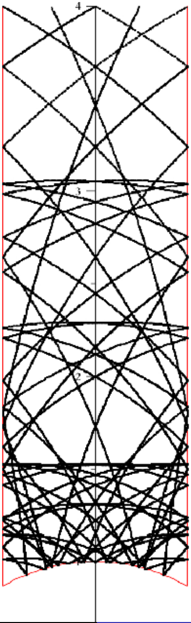
Example: $D = 19$

(Image: Constantin Kogler)



Example: $D = 377$

(Image: Einsiedler–Lindenstrauss–Michel–Venkatesh)



More Optimal Small Scale Equidistribution

Theorem (H.–Radziwiłł (2019))

(1) *Exact same results hold for Heegner points as for lattice points on the sphere.*

(2) *For closed geodesics, we obtain stronger results: small scale equidistribution on almost every shrinking* ball *down to the optimal scale.*

Geometric difference between closed geodesics compared to Heegner points and lattice points on the sphere: codimension 1 instead of 2.

Analytic difference for closed geodesics: gamma factors arising from Waldspurger's formula are different; Stirling's formula implies *better* decay as Laplacian eigenvalue increases.

More Optimal Small Scale Equidistribution

Method of proof is very similar. Main differences:

- spectral expansion of the variance involves Maaß forms instead of modular forms, so use the Kuznetsov formula instead of the Petersson formula,
- error term has integrals of associated Legendre functions $P_{-\frac{1}{2}+it}(1+\frac{2m}{|D|})$ instead of sums of Legendre polynomials $P_{k-1}(1-\frac{2m}{n})$.

For closed geodesics on shrinking balls, we end up needing to show that for $T \ll D^{\frac{1}{2}-\delta}$,

$$\frac{1}{T}\sum_{T \leq t_f \leq 2T} L\left(\frac{1}{2}, f\right) L\left(\frac{1}{2}, f \otimes \chi_D\right) = o(\sqrt{D}).$$

Presence of $\frac{1}{T}$ comes from gamma factors and is why we win.

Thank you!