Weyl's law with remainder term and Hecke operators

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Weyl's law in the compact case (Hörmander 1968)

The number $N_X(T)$ of eigenfunctions of the Laplacian with eigenvalues $\leq T^2$ on a compact d-dimensional Riemannian manifold X satisfies

$$N_X(T) = \frac{\mathrm{vol}(X)}{(4\pi)^{d/2}\Gamma(\frac{d}{2}+1)} T^d + O(T^{d-1}).$$

Hyperbolic surfaces (Selberg 1956)

Locally symmetric spaces of non-compact type: $X = \Gamma\backslash G/K$, where Γ is a lattice in a semisimple Lie group G with a maximal compact subgroup K. (X can be compact or non-compact.)
For $G = \mathrm{SL}(2,\mathbb{R})$, the trace formula gives

$$N_{\Gamma\backslash G/K}(T) + M_\Gamma(T) = \frac{\mathrm{vol}(X)}{4\pi}T^2 - \frac{\kappa}{\pi}T\log\frac{2T}{e} + O(T/\log T).$$

- κ is the number of cusps of $\Gamma\backslash G/K$.
- $$M_\Gamma(T) = -\frac{1}{2\pi}\int_0^T \frac{\phi'}{\phi}(\frac{1}{2}+\mathrm{i}t)dt$$
  is the winding number of the determinant $\phi(s)$ of the scattering matrix.
- $\phi(s)$ is a meromorphic function that is holomorphic and has absolute value 1 on the line $\mathrm{Re}\, s = \frac{1}{2}$.

Selberg showed that

$$\liminf_{T\to\infty} \frac{M_\Gamma(T)}{T\log T} \geq \frac{\kappa}{\pi}.$$

If Γ is a congruence subgroup of $\mathrm{SL}(2,\mathbb{Z})$, then $\phi(s)$ is given in terms of Dirichlet L-functions, and

$$M_\Gamma(T) = \frac{\kappa}{\pi} T\log T + O(T).$$

Therefore, *for congruence subgroups* we have

$$N_\Gamma(T) = \frac{\mathrm{vol}(X)}{4\pi} T^2 - \frac{2\kappa}{\pi} T\log T + O(T).$$

One can refine the remainder term to $c'T + O(T/\log T)$.

Trivial upper bound $M_\Gamma(T) = O(T^2)$

Do we have $M_\Gamma(T) = o(T^2)$ in general?

Probably not in the non-arithmetic case (deformation theory, Phillips–Sarnak and others).

The discrete spectrum contains the cuspidal spectrum defined by the vanishing of all constant terms with respect to proper parabolic subgroups (decay at infinity). In higher rank, the residual (non-cuspidal discrete) spectrum can be infinite.

General upper bound (Donnelly 1982)

$$\limsup_{T\to\infty} \frac{N_{X,\mathrm{cusp}}(T)}{T^d} \leq \frac{\mathrm{vol}(X)}{(4\pi)^{d/2}\Gamma(\frac{d}{2}+1)}$$

Lindenstrauss-Venkatesh 2007: Weyl's law (without remainder estimate) for quotients $\Gamma\backslash G(\mathbb{R})/K$, where G is a split adjoint semisimple group over $\mathbb{Q}$ and Γ a congruence subgroup of $G(\mathbb{Q})$ (their method is completely general).

L–V use arithmeticity very directly, namely the existence of Hecke operators (Margulis showed that they do not exist in the non-arithmetic case).

The spectral parameters of Eisenstein series at different places (say, ∞ and a single prime p) satisfy simple relations.
One can use the Hecke algebra at p to construct a family of test functions f such that the convolution operators $R(f)$ map to the cuspidal spectrum.

Varying the test functions, one obtains

$$\liminf_{T\to\infty} \frac{N_{X,\mathrm{cusp}}(T)}{T^d} \geq (1-\varepsilon)\frac{\mathrm{vol}(X)}{(4\pi)^{d/2}\Gamma(\frac{d}{2}+1)}$$

for each $\varepsilon > 0$.

The main result (Weyl's law with remainder)

Theorem (F-Lapid 2019)

Let G be a simply connected, simple Chevalley group. Then, there exists $\delta > 0$ such that for any congruence subgroup Γ of $G(\mathbb{Q})$ we have

$$N_{X,\mathrm{cusp}}(T) = \frac{\mathrm{vol}(X)}{(4\pi)^{d/2}\Gamma(\frac{d}{2}+1)} T^d + O_\Gamma(T^{d-\delta}), \quad T \geq 1,$$

where $X = \Gamma \backslash G(\mathbb{R})/K$.

In the non-simply-connected case, our result is for manifolds which may be non-connected. The method is again general (but currently written down for simple Chevalley groups over $\mathbb{Q}$).

Open problem: Weyl's law for the entire discrete spectrum (it is known for classical groups)

A finger exercise

For $\Gamma = \mathrm{SL}(2,\mathbb{Z})$, we have

$$\phi(s) = \frac{\zeta^*(2s-1)}{\zeta^*(2s)}, \quad \zeta^*(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s).$$

De la Vallée Poussin's proof of the PNT yields

$$-\int_T^{T+1} \frac{\phi'(\frac{1}{2}+\mathrm{i}t)}{\phi(\frac{1}{2}+\mathrm{i}t)}\,dt = 2\log T + O(\log\log T).$$

The trace formula gives the trivial upper bound $O(T)$.

Note that $-\frac{\phi'(\frac{1}{2}+\mathrm{i}t)}{\phi(\frac{1}{2}+\mathrm{i}t)}$ is a priori bounded from below (it is negative for $t = 0$).

A non-trivial bound using Hecke operators

Let $\mathcal{H}$ be the Hecke algebra spanned by the normalized Hecke operators T_n. Associate to each $f \in \mathcal{H}$ a Dirichlet polynomial θf describing its action on the Eisenstein series, i.e.,

$$(\theta T_p)(z) = p^z + p^{-z}.$$

For every p and every $z \in \mathbb{R} \cup i\mathbb{R}$ we have $(\theta f)(z) > \frac{3}{4}$ for at least one of the following three operators

$$f_{p,1} = p^{-1} - T_{p^2}, \quad f_{p,2} = T_{p^2} + 3T_p - p^{-1}, \quad f_{p,3} = T_{p^2} - 3T_p - p^{-1}.$$

Indeed,

$$\max(1 + p^{-1} - x^2, x^2 + 3x - 1 - p^{-1}, x^2 - 3x - 1 - p^{-1}) > \frac{3}{4}.$$

Conclusion

For every p there exists $j_p \in \{1, 2, 3\}$ such that $(\theta f_{p,j_p})(it) \geq \frac{1}{2}$ for $|t - T| < c_1 / \log p$.

For $R > 0$ let $\mathcal{PW}_R(\mathbb{C})^{\text{even}}$ be the space of even Paley-Wiener functions with Fourier transform supported in $[-R, R]$.
Spectral side of the Selberg trace formula: the distribution on $\mathcal{PW}_R(\mathbb{C})^{\text{even}} \otimes \mathcal{H}$ given by

$$J(h \otimes f) = \sum_{\varphi} h(z_\varphi)\lambda_f(\varphi) - \frac{1}{4\pi}\int_{\mathbb{R}} \frac{\phi'(\frac{1}{2}+\mathrm{i}t)}{\phi(\frac{1}{2}+\mathrm{i}t)} h(\mathrm{i}t)(\theta f)(\mathrm{i}t)\, dt - \frac{1}{4}h(0)(\theta f)(0)$$

Here φ ranges over an orthonormal basis of even Maass eigenforms, the Laplace eigenvalue of φ is $\frac{1}{4} - z_\varphi^2$, and $\lambda_f(\varphi)$ is the eigenvalue of the Hecke operator f acting on φ.

Geometric side

$$J(h\otimes f)-\frac{\Delta(f)}{24}\int_{\mathbb{R}}h(\mathrm{i}t)t\tanh(\pi t)\ dt \\ \ll_R \|f\|_1\int_{\mathbb{R}}|h(\mathrm{i}t)|\log(2+|t|)\ dt,$$

where $\Delta(T_{m^2})=m^{-1}$, $\Delta(T_n)=0$ for non-square n (Plancherel measure). Note that $\Delta(f_{p,j}f_{p',j'})=0$ for $p\neq p'$ and $\Delta(f_{p,j}^2)\ll 1$. Fix $h\in\mathcal{PW}_R(\mathbb{C})^{\text{even}}$ such that $\overline{h(s)}=h(\overline{s})$ for all $s\in\mathbb{C}$ and $h(0)=1$ and let

$$h_T=\frac{1}{2}(h(\cdot+\mathrm{i}T)+h(\cdot-\mathrm{i}T))\in\mathcal{PW}_R(\mathbb{C})^{\text{even}}$$

Apply the trace formula to $h_T^2\otimes(\sum_{p\leq X}f_{p,j_p})^2$.

h_T^2 localizes around T. By "almost positivity," we get:

$$J((h_T)^2 \otimes (\sum_{p \le X} f_{p,j_p})^2) + O(\pi(X)^2) \gg \pi(X)^2 \left[-\int_T^{T+\frac{c_1}{\log X}} \frac{\phi'(\frac{1}{2}+\mathrm{i}t)}{\phi(\frac{1}{2}+\mathrm{i}t)} \, dt \right]$$

On the other hand,

$$J((h_T)^2 \otimes (\sum_{p \le X} f_{p,j_p})^2) \ll \pi(X)T + X^2\pi(X)^2 \log T.$$

Taking $X = T^{1/3}$ gives

$$-\int_T^{T+\frac{3c_1}{\log T}} \frac{\phi'(\frac{1}{2}+\mathrm{i}t)}{\phi(\frac{1}{2}+\mathrm{i}t)} \, dt \ll T^{2/3} \log T.$$

Let G be a Chevalley group over $\mathbb{Q}$ of rank r with a Chevalley model over $\mathbb{Z}$, T_0 a split maximal torus of G.
Let $\widehat{T_0}$ be the torus dual to T_0. The hermitian part of $\widehat{T_0}(\mathbb{C})$ is

$$\widehat{T_0}(\mathbb{C})^{\mathrm{hm}} = \cup_{w\in W}\{\gamma \in \widehat{T_0}(\mathbb{C}) : w(\bar{\gamma}) = \gamma^{-1}\}.$$

The maximal compact subgroup of $\widehat{T_0}(\mathbb{C})$ is

$$\widehat{T_0}(\mathbb{C})^1 = \{\gamma \in \widehat{T_0}(\mathbb{C}) : \bar{\gamma} = \gamma^{-1}\} \subset \widehat{T_0}(\mathbb{C})^{\mathrm{hm}}.$$

The orbit spaces

$$\widehat{T_0}(\mathbb{C})^1/W \subset \widehat{T_0}(\mathbb{C})_p^{\mathrm{unt}}/W \subset \widehat{T_0}(\mathbb{C})^{\mathrm{hm}}/W \subset \widehat{T_0}(\mathbb{C})/W$$

parametrize

tempered $\subset$ unitarizable $\subset$ hermitian $\subset$ admissible

unramified irreducible representations of $G(\mathbb{Q}_p)$.

For example, in the case of $G = \mathrm{PGL}(2)$,

$$\begin{aligned}
\widehat{T_0}(\mathbb{C})^1 &= \{z : |z| = 1\}, \\
\widehat{T_0}(\mathbb{C})_p^{\mathrm{unt}} &= \{z : |z| = 1\} \cup \pm[p^{-\frac{1}{2}}, p^{\frac{1}{2}}], \\
\widehat{T_0}(\mathbb{C})^{\mathrm{hm}} &= \{z : |z| = 1\} \cup \mathbb{R}^*, \\
\widehat{T_0}(\mathbb{C}) &= \mathbb{C}^*.
\end{aligned}$$

Explicit parametrization: unramified characters χ_γ of $T_0(\mathbb{Q}_p)$ are parametrized by $\gamma \in \widehat{T_0}(\mathbb{C})$:

$$\chi_\gamma(\nu(p)) = \nu(\gamma) \quad \forall \nu \in X_*(T_0).$$

The representation $\mathrm{Ind}_{P_0(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)} \chi_\gamma$ has a unique unramified irreducible subquotient π_γ depending only on the W-orbit of γ.
$\mathcal{H}_p$ Hecke algebra of bi-$G(\mathbb{Z}_p)$-invariant, compactly supported functions on $G(\mathbb{Q}_p)$ with convolution, $*$-operation.
The Satake transform

$$\mathcal{S} = \mathcal{S}_p : \mathcal{H}_p \to \mathbb{C}[\widehat{T_0}]^W$$

is an isomorphism of commutative $*$-algebras. $f \in \mathcal{H}_p$ acts on $\pi_\gamma^{G(\mathbb{Z}_p)}$ by the scalar $(\mathcal{S}_p f)(\gamma)$.

Denote by $(\mathcal{H}_p)_{\mathbb{R}}$ the real subalgebra of self-adjoint elements of $\mathcal{H}_p$ and by $(\mathcal{H}_p)_{\geq 0} \subset (\mathcal{H}_p)_{\mathbb{R}}$ the convex cone generated by x^*x, $x \in \mathcal{H}_p$.

Non-archimedean Separation Lemma

Lemma

Let $\emptyset \neq U \subset \widehat{T_0}(\mathbb{C})^1$ be open and W-invariant. There exist constants $A, B, a > 0$, and for every prime p an element

$$f_{U,p} \in (\mathcal{H}_p)_{\mathbb{R}}$$

such that:

1. $f_{U,p}(e) = 0$.
2. $\|f_{U,p}\|_1 \leq Bp^A$.
3. $\|f_{U,p}\|_2 \leq B$.
4. $\mathcal{S}_p f_{U,p} \geq 1$ *on* $\widehat{T_0}(\mathbb{C})^{\mathrm{hm}} \setminus U$.
5. $\operatorname{supp} f_{U,p} \subset \{x \in G(\mathbb{Q}_p) \,:\, \|x\|_p \leq p^a\}$.

Let $T_0 \subset M \neq G$ be a Levi subgroup, T_M the split part of the center of M. Identify the dual torus of T_0/T_M with a subtorus $\widehat{T_0^M} \subset \widehat{T_0}$. We have

$$\widehat{T_0^M}(\mathbb{C}) = \{\gamma \in \widehat{T_0}(\mathbb{C}) : \chi_\gamma|_{T_M(\mathbb{Q}_p)} \equiv 1\}.$$

Claim

Given $\eta \in i\mathfrak{a}_0^*$, we can find $f_{\eta,p} \in (\mathcal{H}_p)_\mathbb{R}$ as in the Separation Lemma with $\widehat{f_{\eta,p}}(I_P(\pi_p, \lambda)) \geq 1$ for all unramified irreducible representations π_p of $M(\mathbb{Q}_p)$ trivial on $T_M(\mathbb{Q}_p)$, and all $\lambda \in i\mathfrak{a}_M^*$ with $\|\lambda - \eta_M\| \leq \delta_1(\log p)^{-1}$.

For this we need the open sets U_i of the following

Observation

For a Levi subgroup $T_0 \subset M \neq G$, there exist

- $\emptyset \neq U_1, U_2 \subset \widehat{T_0}(\mathbb{C})^1$, $U_1 \cap U_2 = \emptyset$, *open and W-invariant,*
- *an open neighborhood V of e in $\widehat{T_0}(\mathbb{C})^1$,*

with the property:

For every $\gamma \in \widehat{T_0}(\mathbb{C})^1$, the set $\gamma V \widehat{T_0^M}(\mathbb{C})$ is disjoint from U_1 or U_2.

Proof of the Separation Lemma

- Wlog U open in $\widehat{T_0}(\mathbb{C})^{\mathrm{hm}}$, $V \neq \emptyset$ open, W-invariant with $V^{cl} \subset U$.
- Cruical point: $\delta := \inf_p \mu_{\mathrm{pl},p}(V) > 0$ (the weak-$*$-limit of the $\mu_{\mathrm{pl},p}$, the Sato-Tate measure, is supported on all of $\widehat{T_0}(\mathbb{C})^1$).
- Let $X > 0$ be a parameter.
- Find $h^{\mathrm{prop}} \in \mathbb{C}[\widehat{T_0}]^W_{\geq 0}$ inducing a proper map $\widehat{T_0}(\mathbb{C})^{\mathrm{hm}} \to \mathbb{R}^{\geq 0}$. Normalize $\max_{\widehat{T_0}(\mathbb{C})^1} h^{\mathrm{prop}} = 1$, then $\mu_{\mathrm{pl},p}(h^{\mathrm{prop}}) \leq 1$.

- $C = \{\gamma \in \widehat{T_0}(\mathbb{C})^{\mathrm{hm}} : h^{\mathrm{prop}}(\gamma) \leq X + 2\} \supset \widehat{T_0}(\mathbb{C})^1$ is compact and W-invariant.
- By Stone-Weierstrass on C, we can find $h^{\mathrm{SW}} \in \mathbb{C}[\widehat{T_0}]^W_{\geq 0}$ with $h^{\mathrm{SW}}\big|_C \leq 1$, $h^{\mathrm{SW}}\big|_{V^{cl}} \leq \frac{1}{4}\delta$, $h^{\mathrm{SW}}\big|_{C \setminus U} \geq 1 - \frac{1}{4}\delta$. We have $\mu_{\mathrm{pl},p}(h^{\mathrm{SW}}) \leq 1 - \frac{3}{4}\delta$.
- Set

$$h_p = X(h^{\mathrm{SW}} - \mu_{\mathrm{pl},p}(h^{\mathrm{SW}})) + h^{\mathrm{prop}} - \mu_{\mathrm{pl},p}(h^{\mathrm{prop}}) \in \mathbb{C}[\widehat{T_0}]^W_{\mathbb{R}}$$

and $f_p = \mathcal{S}_p^{-1}(h_p) \in (\mathcal{H}_p)_{\mathbb{R}}$. $f_p(e) = \mu_{\mathrm{pl},p}(h_p) = 0$.

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$$h_p(\gamma) \geq \begin{cases} X(\frac{3}{4}\delta - \frac{1}{4}\delta) - 1 = \frac{1}{2}X\delta - 1, & \gamma \in C \setminus U, \\ -X + X + 2 - 1 = 1, & \gamma \in \widehat{T_0}(\mathbb{C})^{\mathrm{hm}} \setminus C. \end{cases}$$

Taking $X = 4\delta^{-1}$ we get $h_p \geq 1$ on $\widehat{T_0}(\mathbb{C})^{\mathrm{hm}} \setminus U$.

- $\|f_p\|_2^2 = \mu_{\mathrm{pl},p}(h_p^2) \le \max_{\widehat{T_0}(\mathbb{C})^1} h_p^2 \le (X+1)^2 \le B$.
- $\|f_p\|_1 \le \|\mathcal{S}_p^{-1}(Xh^{\mathrm{SW}} + h^{\mathrm{prop}})\|_1 + X + 1 \le Bp^A$.

Recall

$$h_p = X(h^{\mathrm{SW}} - \mu_{\mathrm{pl},p}(h^{\mathrm{SW}})) + h^{\mathrm{prop}} - \mu_{\mathrm{pl},p}(h^{\mathrm{prop}}), \quad f_p = \mathcal{S}_p^{-1}(h_p).$$

The method of Duistermaat-Kolk-Varadarajan (1979)

- $\mathbf{K}_\infty$ fixed maximal compact subgroup of $G(\mathbb{R})$, K arbitrary open compact subgroup of $G(\mathbb{A}_{\mathrm{fin}})$
- $\mathfrak{a}_0$ is the Lie algebra of $T_0(\mathbb{R})$, $\mathfrak{a}_0^*$ the dual space
- $\Pi_2(G)$ the set of all irreducible unitary representations π of $G(\mathbb{A})$ appearing discretely in $L^2(G(\mathbb{Q})\backslash G(\mathbb{A}))$
- $m_{\mathrm{cusp}}(\pi) = \dim \mathrm{Hom}(\pi, L^2_{\mathrm{cusp}}(G(\mathbb{Q})\backslash G(\mathbb{A})))$ for $\pi \in \Pi_2(G)$
- $\Lambda(B) = \frac{\mathrm{vol}(G(\mathbb{Q})\backslash G(\mathbb{A}))}{|W|} \int_B \beta(\lambda)\, d\lambda$, where β is the spherical Plancherel measure of $G(\mathbb{R})$

- For compactly supported $f : G(\mathbb{A}_{\mathrm{fin}})//K \longrightarrow \mathbb{C}$ and $\lambda \in (\mathfrak{a}_0)^*_{\mathbb{C}}$ set

$$m_{\mathrm{cusp}}(\lambda, f) = \frac{1}{|W\lambda|} \sum_{\substack{\pi \in \Pi_2(G),\ \pi_\infty^{\mathbf{K}_\infty} \neq 0, \\ W\lambda_{\pi_\infty} = W\lambda}} m_{\mathrm{cusp}}(\pi)\, \mathrm{tr}\, \pi_{\mathrm{fin}}(f).$$

- For any bounded subset B of $(\mathfrak{a}_0)^*_{\mathbb{C}}$ set

$$m_{\mathrm{cusp}}(B, f) = \sum_{\lambda \in B} m_{\mathrm{cusp}}(\lambda, f).$$

- Important special case: $f = \mathbf{e}_K$, the idempotent associated to K. $\mathrm{tr}\, \pi_{\mathrm{fin}}(\mathbf{e}_K)$ is the dimension of the space of K-fixed vectors in π_{fin}.
- ν_G is the weight function $(1 + \log\|\cdot\|)^r$ on $G(\mathbb{A}_{\mathrm{fin}})$, where $\|x\| = \prod_{p \leq \infty} \|x_p\|_p$ is a height on $G(\mathbb{A})$.

Main result (asymptotics of Hecke operators)

Set $d = r + |\Phi^+| = \dim G(\mathbb{R})/\mathbf{K}_\infty$.

Theorem

Let $\Omega \subset i(\mathfrak{a}_0)^*$ be a bounded domain with rectifiable boundary. For all bi-K-invariant compactly supported $f : G(\mathbb{A}_f) \longrightarrow \mathbb{C}$ and all $t \geq 1$ we have:

$$m_{\text{cusp}}(t\Omega, f) - \Lambda(t\Omega) \sum_{\gamma \in Z(\mathbb{Q})} f(\gamma) \ll_\Omega \|f\nu_G\|_1 t^{d-\delta}.$$

Note that asymptotically $\Lambda(t\Omega) = C_\Omega t^d + O(t^{d-1})$, $t \to \infty$, with $C_\Omega > 0$.

Spherical Fourier transform at ∞: an isomorphism

$$\mathcal{S}_\infty : C_c^\infty(G(\mathbb{R})//\mathbf{K}_\infty) \to \mathcal{PW}(\mathfrak{a}_{0,\mathbb{C}}^*)^W$$

For any $f \in C_c^\infty(G(\mathbb{R})//\mathbf{K}_\infty)$ we have

$$f(e) = \frac{1}{|W|} \int_{\mathrm{i}\mathfrak{a}_0^*} \mathcal{S}_\infty f(\lambda)\beta(\lambda)\, d\lambda$$

with the Plancherel density

$$\beta(\lambda) \sim \prod_{\alpha\in\Phi^+} \tfrac{\langle\lambda,\alpha^\vee\rangle}{2\mathrm{i}} \tanh\left(\tfrac{\pi\langle\lambda,\alpha^\vee\rangle}{2\mathrm{i}}\right).$$

Simple upper bound for β on $B_1(\lambda)$, $\lambda \in \mathrm{i}\mathfrak{a}_0^*$:

$$\tilde{\beta}(\lambda) = \prod_{\alpha\in\Phi^+} (1 + |\langle\lambda, \alpha^\vee\rangle|).$$

For $h \in \mathcal{PW}(\mathfrak{a}^*_{0,\mathbb{C}})^W$ and $\eta \in \mathrm{i}\mathfrak{a}^*_0$ define $h^\eta \in \mathcal{PW}(\mathfrak{a}^*_{0,\mathbb{C}})^W$ by

$$h^\eta(\lambda) = \frac{1}{|W|} \sum_{w \in W} h(\lambda - w\eta).$$

For $\mathcal{S}_\infty f = h$ set $f^\eta = \mathcal{S}_\infty^{-1}(h^\eta) \in C_c^\infty(G(\mathbb{R})//\mathbf{K}_\infty)$.
The main feature of h^η is that it localizes near $W\eta$.

Following DKV, one can deduce the eigenvalue asymptotics from the following inequality for test functions (and local upper bounds on the cuspidal spectrum):

Theorem

There exists $\delta > 0$ *such that for any* $f_\infty \in C_c^\infty(G(\mathbb{R})//\mathbf{K}_\infty)$ *and* $\eta \in i\mathfrak{a}_0^*$ *we have*

$$|\operatorname{tr} R_{\mathrm{cusp}}(f_\infty^\eta \otimes \mathbf{e}_K) - v_K f_\infty^\eta(e)| \ll_{f_\infty, K} \frac{\tilde{\beta}(\eta)}{(1 + \|\eta\|)^\delta},$$

where

$$v_K = |Z(\mathbb{Q}) \cap K| \frac{\operatorname{vol}(G(\mathbb{Q})\backslash G(\mathbb{A}))}{\operatorname{vol} K}.$$

Arthur's trace formula

The right regular representation gives rise to a representation R of $C_c^\infty(G(\mathbb{A})^1)$ on $L^2(G(\mathbb{Q})\backslash G(\mathbb{A}))$. $R(f)$ is an integral operator with kernel

$$K_f(x,y) = \sum_{\gamma \in G(\mathbb{Q})} f(x^{-1}\gamma y), \quad x, y \in G(\mathbb{Q})\backslash G(\mathbb{A}).$$

Fix a Borel subgroup P_0 containing T_0. Truncation parameters are elements $T \in \mathfrak{a}_0^{++} := \{T \in \mathfrak{a}_0 : d(T) > C_0\}$, where $d(T) = \min_{\alpha \in \Delta_0} \langle \alpha, T \rangle$. For such a T let Λ^T be Arthur's truncation operator (Arthur 1980), an orthogonal projection on $L^2(G(\mathbb{Q})\backslash G(\mathbb{A}))$. Let

$$J^T(f) = \mathrm{tr}(\Lambda^T \circ R(f) \circ \Lambda^T) = \int_{G(\mathbb{Q})\backslash G(\mathbb{A})} (\Lambda^{(T,T)}_{G\times G} K_f)(x,x)\, dx.$$

Spectral expansion

For a standard parabolic P (i.e., $P \supset P_0$) and $\varphi \in \mathcal{A}_P^2$ the Eisenstein series

$$E_P(g,\varphi,\lambda) = \sum_{\gamma \in P(\mathbb{Q})\backslash G(\mathbb{Q})} \varphi_\lambda(\gamma g), \quad g \in G(\mathbb{A}),$$

converges if $\operatorname{Re}\langle \lambda, \alpha^\vee \rangle \gg 1 \ \forall \alpha \in \Delta_P$, admits a meromorphic continuation to $\mathfrak{a}^*_{M,\mathbb{C}}$ and is analytic for $\lambda \in i\mathfrak{a}^*_M$. E_P and E_Q are connected by a functional equation if P and Q are associate. We decompose

$$J^T(f) = \sum_{P \in \mathcal{P}/\sim} J_P^T(f)$$

according to associate classes of standard parabolic subgroups.

The contribution of the associate class of $P = MU$ is

$$J_P^T(f) = |W(M)|^{-1} \sum_{\pi \in \Pi_2(M)} \int_{i\mathfrak{a}_M^*} \sum_{\varphi \in \mathcal{OB}_P(\pi)} \int_{G(\mathbb{Q})\backslash G(\mathbb{A})} \Lambda^T E_P(x, I_P(f,\pi,\lambda)\varphi, \lambda) \overline{\Lambda^T E_P(x,\varphi,\lambda)}\, dx\, d\lambda.$$

- $\Pi_2(M)$ is the set of equivalence classes of irreducible representations of $M(\mathbb{A})$ occuring discretely in $L^2(A_M M(\mathbb{Q})\backslash M(\mathbb{A}))$.
- For $\pi \in \Pi_2(M)$ fix an orthonormal basis $\mathcal{OB}_P(\pi)$ of the space $\mathcal{A}^2_{P,\pi}$ of automorphic forms on $U(\mathbb{A})M(\mathbb{Q})\backslash G(\mathbb{A})$ of type π.
- $I_P(f,\pi,\lambda)$ is the action of f on the (normalized) parabolic induction from $P(\mathbb{A})$ to $G(\mathbb{A})$ of $\pi \otimes e^{\langle \lambda, H_P(\cdot)\rangle}$.

Positivity:

$$J_P^T(f * f^*) \geq 0.$$

Note that

$$J_G^T(f) = \operatorname{tr} R_{\text{cusp}}(f) + \operatorname{tr}(\Lambda^T \circ R_{\text{res}}(f)),$$

where $R_{\text{cusp}}(f)$ (resp., $R_{\text{res}}(f)$) is the restriction of $R(f)$ to the cuspidal (resp. residual) part of $L^2_{\text{disc}}(G(\mathbb{Q})\backslash G(\mathbb{A}))$.
Clearly

$$\lim_{d(T)\to\infty} J_G^T(f) = \operatorname{tr} R_{\text{disc}}(f),$$

once we have shown that the right-hand side exists (the trace class conjecture, Müller 1989, 1998, Ji 1998) – but in general we do not know how to control the rate of convergence.

On the other hand, for $d(T) > C_1$ and f supported in $\{x \in G(\mathbb{A}) : \log\|x\| < C_1^{-1} d(T)\}$ we have

$$J^T(f) = \int_{G(\mathbb{Q})\backslash G(\mathbb{A})} k_f^T(x)\, dx$$

with Arthur's modified kernel k_f^T (Arthur 1978). This is a polynomial function of T (Arthur 1981). Moreover (F-Lapid 2016),

$$J^T(f) - \int_{G(\mathbb{Q})\backslash G(\mathbb{A})_{\leq T}} K_f(x,x)\, dx \ll_K (1+\|T\|)^r e^{-d(T)} \sum_i \|X_i * f\|_1.$$

Let $S = \{p : G(\mathbb{Z}_p) \not\subset K\}$. For $f_\infty \in C_c^\infty(G(\mathbb{R})//\mathbf{K}_\infty)$, $f^S \in \mathcal{H}^S = \bigotimes_{p \notin S} \mathcal{H}_p$ and $T \in \mathfrak{a}_0^{++}$ we have

$$J_P^T(f_\infty \otimes \mathbf{e}_K \otimes f^S) = |W(M)|^{-1} \sum_{\pi \in \Pi_2(M)^{\mathbf{K}_\infty K}} \int_{\mathrm{i}\mathfrak{a}_M^*} (\mathcal{S}_\infty f_\infty)(\lambda_{\pi_\infty} + \lambda) \hat{f}^S(I_P(\pi^S, \lambda)) \sum_{\varphi \in \mathcal{OB}_P(\pi)^{\mathbf{K}_\infty K}} \|\Lambda^T E_P(\cdot, \varphi, \lambda)\|_2^2 \, d\lambda.$$

- $\lambda_{\pi_\infty} \in (\mathfrak{a}_0^M)^*_{\mathbb{C}}/W_M$ is the archimedean parameter of π
- $\mathcal{OB}_P(\pi)^{\mathbf{K}_\infty K}$ is an orthonormal basis of the finite-dimensional space $(\mathcal{A}^2_{P,\pi})^{\mathbf{K}_\infty K}$, the $\mathbf{K}_\infty K$-invariant part of $\mathcal{A}^2_{P,\pi}$

Geometric estimates

Theorem (F-Matz 2019)

For $f_\infty \in C_R^\infty(G(\mathbb{R})//\mathbf{K}_\infty)$, $f^S \in \mathcal{H}^S$ *we have*

$$J^T(f_\infty \otimes \mathbf{e}_K \otimes f^S) - v_K f^S(e) f_\infty(e) \ll_{K,R}$$
$$\|f^S\|_1 (1 + \|T\|)^r \int_{i\mathfrak{a}_0^*} |\mathcal{S}_\infty f_\infty(\lambda)| \frac{\beta(\lambda)}{D(\lambda)} \log(2 + \|\lambda\|)^r d\lambda$$

with

$$D(\lambda) \gg (1 + \|\lambda\|)^{\frac{1}{2}}$$

for all $T \in \mathfrak{a}_0^{++}$ *with*
$d(T) > C_1(1 + R + \max_{x \in K} \log\|x\| + \mathfrak{ms}(f^S))$.

Here,

$$\mathfrak{ms}(f^S) = \max_{x \in G(\mathbb{A}^S) : f^S(x) \neq 0} \log\|x\|, \quad f^S \in \mathcal{H}^S.$$

Consequence: for $\eta \in \mathrm{i}\mathfrak{a}_0^*$ we have

$$J^T(f_\infty^\eta * (f_\infty^\eta)^* \otimes \mathbf{e}_K \otimes f^S) \ll_{f_\infty, K} \tilde{\beta}(\eta)\left[f^S(e) + \|f^S\|_1 \frac{(1+\|T\|)^r \log(2+\|\eta\|)^r}{D(\eta)}\right].$$

Let $\nu_P^{T,K}$ be the Radon measure on $\mathrm{i}\mathfrak{a}_0^*/W$ given by

$$\nu_P^{T,K}(g) = |W(M)|^{-1}$$
$$\sum_{\pi \in \Pi_2(M)^{\mathbf{K}_\infty K}} \int_{\mathrm{i}\mathfrak{a}_M^*} g(\operatorname{Im} \lambda_{\pi_\infty} + \lambda) \sum_{\varphi \in \mathcal{OB}_P(\pi)^{\mathbf{K}_\infty K}} \|\Lambda^T E_P(\cdot, \varphi, \lambda)\|_2^2 \, d\lambda$$

for $g \in C_c(\mathrm{i}\mathfrak{a}_0^*)^W$. In addition, decompose

$$\nu_G^{T,K} = \nu_{\mathrm{cusp}}^K + \nu_{\mathrm{res}}^{T,K}.$$

The key technical step

Proposition

There exists $\delta > 0$ (depending only on G) such that

$$\nu_P^{T,K}(WB_1(\eta)) \ll_K \frac{\tilde{\beta}(\eta)(1+\|T\|)^r}{(1+\|\eta\|)^\delta}$$

for all $P \neq G$, $\eta \in i\mathfrak{a}_0^*$ and $T \in \mathfrak{a}_0^{++}$. The same bound holds for $\nu_{\mathrm{res}}^{T,K}$.

Sketch of the proof

For simplicity let $K = \mathbf{K}_{\text{fin}} = \prod_p G(\mathbb{Z}_p)$, $S = \emptyset$.

Observation

Any automorphic representation of $M(\mathbb{A})/A_M$ with a $\mathbf{K}_{\text{fin}}$-fixed vector is trivial on $T_M(\mathbb{A})$ (because only the trivial Dirichlet character has conductor 1).

Let $\eta \in i\mathfrak{a}_0^*$ and let $X \geq 2$ be a parameter. Recall:

Claim

Given $\eta \in i\mathfrak{a}_0^*$, we can find $f_{\eta,p} \in (\mathcal{H}_p)_{\mathbb{R}}$ as in the Separation Lemma with $\widehat{f_{\eta,p}}(I_P(\pi_p, \lambda)) \geq 1$ for all unramified irreducible representations π_p of $M(\mathbb{Q}_p)$ trivial on $T_M(\mathbb{Q}_p)$, and all $\lambda \in i\mathfrak{a}_M^*$ with $\|\lambda - \eta_M\| \leq \delta_1 (\log p)^{-1}$.

Set

$$\phi = \sum_{p \le X} f_{\eta,p} \in \mathcal{H}_{\mathbb{R}}.$$

1. We have $\hat{\phi}(I_P(\pi_{\text{fin}}, \lambda)) \ge \pi(X)$ for $\pi \in \Pi_2(M)^{\mathbf{K}_\infty \mathbf{K}_{\text{fin}}}$ and $\|\lambda - \eta_M\| \le \delta_1 (\log p)^{-1}$.
2. For any $p_1 \ne p_2$, the functions f_{η,p_1} and f_{η,p_2} have disjoint support in $G(\mathbb{A}_{\text{fin}})$.
3. $(\phi * \phi)(e) = \|\phi\|_2^2 = \sum_{p \le X} \|f_{\eta,p}\|_2^2 \ll \pi(X)$.
4. $\|\phi\|_1 \le \sum_{p \le X} \|f_{\eta,p}\|_1 \ll X^A \pi(X)$.
5. $\mathfrak{ms}(\phi) \ll \log X$.

For suitable $f_\infty \in C_c^\infty(G(\mathbb{R})//\mathbf{K}_\infty)_{\geq 0}$ we have $|h^\eta(\lambda)| \geq 1$ for $\lambda \in \mathfrak{a}_{0,\mathbb{C}}^*$ with $\|\mathrm{Re}\,\lambda\| \leq \|\rho\|$ and $\|\mathrm{Im}(\lambda - \eta)\| \leq 1$. Let

$$F_{\eta,X} = f_\infty^\eta * f_\infty^\eta \otimes \phi * \phi \in C_c^\infty(G(\mathbb{A})^1).$$

$$\pi(X)^2 \nu_P^{T,K}(W B_{\frac{\delta_1}{\log X}}(\eta)) \leq J_P^T(F_{\eta,X}) \leq J^T(F_{\eta,X})$$

for all $T \in \mathfrak{a}_0^{++}$. On the other hand,

$$J^T(F_{\eta,X}) \ll \tilde{\beta}(\eta) \left[\pi(X) + \frac{X^{2A}\pi(X)^2(1 + \|T\|)^r \log(2 + \|\eta\|)^r}{D(\eta)} \right]$$

for $d(T) > C_2 \log X$. Take $X = D(\eta)^{(2A+1)^{-1}}$.

How to prove the necessary geometric estimate? We need to bound

$$\int_{G(\mathbb{Q})\backslash G(\mathbb{A})_{\leq T}} \sum_{\gamma \in G(\mathbb{Q})\backslash Z(\mathbb{Q})} (f_\infty \otimes \mathbf{e}_K \otimes f^S)(x^{-1}\gamma x)\, dx.$$

Use estimates for the decay of spherical functions to reduce to the estimation of

$$\int_{G(\mathbb{Q})\backslash G(\mathbb{A})_{\leq T}} \sum_{\gamma \in G(\mathbb{Q})\backslash Z(\mathbb{Q})} (\phi(\varepsilon^{-1} q(\cdot)) \otimes \mathbf{e}_K \otimes f^S)(x^{-1}\gamma x)\, dx,$$

where ϕ is a fixed non-negative compactly supported function on $\mathbb{R}^{\geq 0}$, and $q(g) = \operatorname{tr} \rho(g)^t \rho(g) - n$ for a fixed faithful representation ρ of G of dimension n.
For this, we use reduction theory and the Bruhat decomposition. It is crucial to understand the contributions of Bruhat cells that are not contained in proper standard parabolics (we call them "regular").

We need to estimate truncated adelic integrals over these Bruhat cells. This can be done "spectrally" by invoking intertwining operators for the spherical principal series. Let $w \in W$ be regular. For $f \in C_c^\infty(G(\mathbb{A})^1/\mathbf{K})$ non-negative, $f = \phi(\varepsilon^{-1}q(\cdot)) \otimes \mathbf{e}_K \otimes f^S$, we need to estimate

$$I_w^{G,\mathrm{sm}}(f) = \int_{U_0(\mathbb{A})/U_w(\mathbb{A})} \int_{A_0} \int_{U_0(\mathbb{A})} \int_{T_0(\mathbb{A})^1} f(a^{-1}uwau_1t)\kappa(H_0(a))\, dt\, du_1\, da\, du,$$

where κ is a suitable smooth cutoff function, $U_w = U_0 \cap wU_0w^{-1}$.

Mellin inversion yields

$$I_w^{G,\mathrm{sm}}(f) = \int_{\lambda_0 + i(\mathfrak{a}_0)^*} \hat{\kappa}((1-w^{-1})\lambda) m(w^{-1},\lambda)\Phi(f,\lambda)\, d\lambda,$$

where $\lambda_0 \in (\mathfrak{a}_0)^*$ is such that $\lambda_0 - \rho$ lies in the positive Weyl chamber. Here,

$$m(w,\lambda) = \prod_{\alpha \in \Phi_w^+} \frac{\zeta^*(\langle \lambda, \alpha^\vee \rangle)}{\zeta^*(\langle \lambda, \alpha^\vee \rangle + 1)},$$

where Φ_w^+ is the set of $\alpha \in \Phi^+$ with $w\alpha < 0$, is the scalar-valued spherical intertwining operator, and

$$\Phi(f,\lambda) = \int_{A_0} \int_{P_0(\mathbb{A})^1} f(pa) a^{-\lambda-\rho}\, dp\, da$$

the adelic Harish-Chandra transform of f.

Now do a contour shift in the integral crossing the singular hyperplanes associated to the simple roots (and no others). One obtains integrals

$$\int_{\lambda_{1,L}+i(\mathfrak{a}_L^G)^*} \hat{\kappa}((1-w^{-1})\lambda) \prod_{\alpha\in\Phi^+_{w^{-1}}\setminus\Delta^L} \frac{\zeta^*(\langle\lambda,\alpha^\vee\rangle)}{\zeta^*(\langle\lambda,\alpha^\vee\rangle+1)}\Phi(f,\lambda)\,d\lambda,$$

where L is a standard Levi subgroup with $\Delta^L \subset \Phi^+_{w^{-1}}$, and $\lambda_{1,L} = \rho - \delta\sum_{\alpha\notin\Delta^L}\varpi_\alpha$ with $0<\delta<1/2$.
By Stirling's formula, the factors $\frac{\zeta^*(s)}{\zeta^*(s+1)}$ decay like $\frac{|\zeta(s)|}{(1+|t|)^{1/2}}$ for $s=\sigma+it\neq 1$, $\sigma\geq c>0$. The factor $\zeta(s)$ is bounded "on average" (by the standard second moment estimate).

Lemma

If w is regular, then the roots in $\Phi^+_{w^{-1}}$ span the space $(\mathfrak{a}_0)^*$. Moreover, for any $L \in \mathcal{L}$ there exists a subset $\Psi_L \subset \Phi^+_{w^{-1}}$ such that the images of the co-roots $\alpha^\vee$, $\alpha \in \Psi_L$, under the projection map $\mathfrak{a}_0 \longrightarrow \mathfrak{a}^w_L$ are of the form $b_1, -b_1, b_2, \dots, b_d, -b_d$ for a basis $b_1, \dots, b_d$ of $\mathfrak{a}^w_L$.

We obtain the estimate

$$I^{G,\mathrm{sm}}_w(\phi(\varepsilon^{-1}q(\cdot)) \otimes \mathbf{e}_K \otimes f^S) \ll \epsilon^{\frac{r+|\Phi^+|}{2}} |\log \epsilon|^{\dim(\mathfrak{a}_0)^w} \|f^S\|_1.$$