The Weyl bound for Dirichlet L-functions

Ian Petrow

University College London

June 2, 2020

Joint work with Matthew P. Young

The subconvexity problem

Given π an automorphic form, let $C(\pi)$ be its analytic conductor.

The subconvexity problem

Given π an automorphic form, let $C(\pi)$ be its analytic conductor.
Example: χ a Dirichlet character modulo q and $|\cdot|^{it} : n \mapsto n^{it}$

$$C(\chi.|\cdot|^{it}) = (1+|t|)q.$$

The subconvexity problem

Given π an automorphic form, let $C(\pi)$ be its analytic conductor.
Example: χ a Dirichlet character modulo q and $|\cdot|^{it} : n \mapsto n^{it}$

$$C(\chi.|\cdot|^{it}) = (1+|t|)q.$$

Trivial "convexity" bound:

$$L(1/2, \pi) \ll C(\pi)^{1/4+\varepsilon}$$

The subconvexity problem

Given π an automorphic form, let $C(\pi)$ be its analytic conductor.
Example: χ a Dirichlet character modulo q and $|\cdot|^{it} : n \mapsto n^{it}$

$$C(\chi.|\cdot|^{it}) = (1 + |t|)q.$$

Trivial "convexity" bound:

$$L(1/2, \pi) \ll C(\pi)^{1/4+\varepsilon}$$

GRH $\Rightarrow$ Generalized Lindelöf hypothesis:

$$L(1/2, \pi) \ll C(\pi)^{\varepsilon}$$

The subconvexity problem

Given π an automorphic form, let $C(\pi)$ be its analytic conductor.
Example: χ a Dirichlet character modulo q and $|\cdot|^{it} : n \mapsto n^{it}$

$$C(\chi.|\cdot|^{it}) = (1+|t|)q.$$

Trivial "convexity" bound:

$$L(1/2,\pi) \ll C(\pi)^{1/4+\varepsilon}$$

GRH $\Rightarrow$ Generalized Lindelöf hypothesis:

$$L(1/2,\pi) \ll C(\pi)^{\varepsilon}$$

Subconvexity problem: show that there exists $\delta > 0$ so that

$$L(1/2,\pi) \ll C(\pi)^{1/4-\delta}$$

The subconvexity problem

Given π an automorphic form, let $C(\pi)$ be its analytic conductor.
Example: χ a Dirichlet character modulo q and $|\cdot|^{it} : n \mapsto n^{it}$

$$C(\chi.|\cdot|^{it}) = (1+|t|)q.$$

Trivial "convexity" bound:

$$L(1/2,\pi) \ll C(\pi)^{1/4+\varepsilon}$$

GRH $\Rightarrow$ Generalized Lindelöf hypothesis:

$$L(1/2,\pi) \ll C(\pi)^{\varepsilon}$$

Subconvexity problem: show that there exists $\delta > 0$ so that

$$L(1/2,\pi) \ll C(\pi)^{1/4-\delta}$$

Michel-Venkatesh (2010): π on GL_1 or GL_2 with unspecified $\delta > 0$.

Classical results

Classical results

First subconvexity result: Hardy-Littlewood-Weyl (1920s):

$$\zeta(\frac{1}{2} + it) \ll (1 + |t|)^{\frac{1}{6}+\varepsilon}.$$

Based on the method of Weyl differencing;
invariance of continuous functions under translation.

Classical results

First subconvexity result: Hardy-Littlewood-Weyl (1920s):

$$\zeta(\frac{1}{2} + it) \ll (1 + |t|)^{\frac{1}{6}+\varepsilon}.$$

Based on the method of Weyl differencing;
invariance of continuous functions under translation.

Burgess (1962) χ primitive modulo q:

$$L(1/2, \chi) \ll q^{\frac{3}{16}+\varepsilon}$$

Throws away correlation between character sums on many very short intervals, but uses Hölder and RH for *curves* over finite fields.

Classical results

First subconvexity result: Hardy-Littlewood-Weyl (1920s):

$$\zeta(\frac{1}{2} + it) \ll (1 + |t|)^{\frac{1}{6}+\varepsilon}.$$

Based on the method of Weyl differencing;
invariance of continuous functions under translation.

Burgess (1962) χ primitive modulo q:

$$L(1/2, \chi) \ll q^{\frac{3}{16}+\varepsilon}$$

Throws away correlation between character sums on many very short intervals, but uses Hölder and RH for *curves* over finite fields.

Hardy-Littlewood-Weyl refresher

It suffices to estimate

$$\sum_{n \ll t^{1/2}} n^{-it}.$$

Hardy-Littlewood-Weyl refresher

It suffices to estimate

$$\sum_{n \ll t^{1/2}} n^{-it}.$$

The function n^{-it} is quite oscillatory as a function of n when $n \ll \sqrt{t}$

Hardy-Littlewood-Weyl refresher

It suffices to estimate

$$\sum_{n \ll t^{1/2}} n^{-it}.$$

The function n^{-it} is quite oscillatory as a function of n when $n \ll \sqrt{t}$

Split the sum into many small intervals $n \in [a, b)$. For a large enough and $b - a$ small enough ($a \asymp t^{1/2}$, $b - a \asymp t^{1/6}$, say):

$$\left| \sum_{n=a+1}^{b} n^{-it} \right| = \left| \sum_{n=a+1}^{b} \exp(-it \log n) \right| \ll \left| \sum_{n=1}^{b-a} \exp(-it \left(\frac{n}{a} - \frac{n^2}{2a^2} \right)) \right|$$

Hardy-Littlewood-Weyl refresher

It suffices to estimate

$$\sum_{n \ll t^{1/2}} n^{-it}.$$

The function n^{-it} is quite oscillatory as a function of n when $n \ll \sqrt{t}$

Split the sum into many small intervals $n \in [a, b)$. For a large enough and $b - a$ small enough ($a \asymp t^{1/2}$, $b - a \asymp t^{1/6}$, say):

$$\left| \sum_{n=a+1}^{b} n^{-it} \right| = \left| \sum_{n=a+1}^{b} \exp(-it \log n) \right| \ll \left| \sum_{n=1}^{b-a} \exp(-it \left(\frac{n}{a} - \frac{n^2}{2a^2} \right)) \right|$$

Square out, geom. series $\Rightarrow$ $(\mathrm{RHS})^2$ is approx. invariant in a

Hardy-Littlewood-Weyl refresher

It suffices to estimate

$$\sum_{n \ll t^{1/2}} n^{-it}.$$

The function n^{-it} is quite oscillatory as a function of n when $n \ll \sqrt{t}$

Split the sum into many small intervals $n \in [a, b)$. For a large enough and $b - a$ small enough ($a \asymp t^{1/2}$, $b - a \asymp t^{1/6}$, say):

$$\left| \sum_{n=a+1}^{b} n^{-it} \right| = \left| \sum_{n=a+1}^{b} \exp(-it \log n) \right| \ll \left| \sum_{n=1}^{b-a} \exp(-it \left(\frac{n}{a} - \frac{n^2}{2a^2} \right)) \right|$$

Square out, geom. series $\Rightarrow$ (RHS)^2 is approx. invariant in a
So it is efficient to apply Cauchy-Schwarz to the sum over intervals.

The Burgess exponent

Burgess: Sum over many shifts $\chi(n + ab)$ instead. Hölder, exchange summation and use Weil's RH for curves on shift variable.

The Burgess exponent

Burgess: Sum over many shifts $\chi(n+ab)$ instead. Hölder, exchange summation and use Weil's RH for curves on shift variable.

Difference between HLW and Burgess: translation invariance of intervals is not available for $\chi \pmod{p}$.

The Burgess exponent

Burgess: Sum over many shifts $\chi(n + ab)$ instead. Hölder, exchange summation and use Weil's RH for curves on shift variable.

Difference between HLW and Burgess: translation invariance of intervals is not available for $\chi \pmod{p}$.

The exponent $3/16$ re-occurs often in modern incarnations of these problems (Blomer-Harcos-Michel, Blomer-Harcos, Han Wu).

The Weyl exponent

For a long time, the exponent $1/6$ only known for the t-aspect, and in special cases related to quadratic characters (Conrey-Iwaniec, Ivić, Young, P.-Young)

The Weyl exponent

For a long time, the exponent $1/6$ only known for the *t*-aspect, and in special cases related to quadratic characters (Conrey-Iwaniec, Ivić, Young, P.-Young)

Conrey-Iwaniec (2000): if $\chi^2 = 1$ with odd (sq.-free) conductor q

$$L(1/2, \chi) \ll q^{\frac{1}{6}+\varepsilon},$$

using input from automorphic forms and Deligne's RH for *varieties*.

The work of Conrey-Iwaniec

$H_{it}(m, \psi) = \{$Maass newforms of level m char. ψ and spec. par. $it\}$.

The work of Conrey-Iwaniec

$H_{it}(m,\psi) = \{$Maass newforms of level m char. ψ and spec. par. $it\}$.

Theorem (Conrey-Iwaniec)

We have for χ quadratic modulo q

$$\sum_{|t_j|\leq T} \sum_{m|q} \sum_{\pi \in H_{it_j}(m,1)} L(1/2, \pi \otimes \chi)^3 + \int_{-T}^{T} |L(1/2+it,\chi)|^6 \ell(t)\, dt$$

$$\ll T^B q^{1+\varepsilon}.$$

The work of Conrey-Iwaniec

$H_{it}(m,\psi) = \{$Maass newforms of level m char. ψ and spec. par. $it\}$.

Theorem (Conrey-Iwaniec)

We have for χ quadratic modulo q

$$\sum_{|t_j|\leq T} \sum_{m|q} \sum_{\pi \in H_{it_j}(m,1)} L(1/2, \pi\otimes\chi)^3 + \int_{-T}^{T} |L(1/2+it,\chi)|^6 \ell(t)\, dt \ll T^B q^{1+\varepsilon}.$$

$B < \infty$ unspecified, $\ell(t) = t^2(4+t^2)^{-1}$. $L(1/2,\pi\otimes\chi) \geq 0$ by Guo.

The work of Conrey-Iwaniec

$H_{it}(m,\psi) = \{$Maass newforms of level m char. ψ and spec. par. $it\}$.

Theorem (Conrey-Iwaniec)

We have for χ quadratic modulo q

$$\sum_{|t_j|\leq T} \sum_{m|q} \sum_{\pi \in H_{it_j}(m,1)} L(1/2, \pi\otimes\chi)^3 + \int_{-T}^{T} |L(1/2+it,\chi)|^6 \ell(t)\, dt \ll T^B q^{1+\varepsilon}.$$

$B < \infty$ unspecified, $\ell(t) = t^2(4+t^2)^{-1}$. $L(1/2,\pi\otimes\chi) \geq 0$ by Guo.

CI recover the Weyl bound by deforming through modular forms instead of deforming an interval by translation.

If $m \mid q$, χ conductor q, $\pi \in H_{it}(m, \chi^2)$, then $\pi \otimes \overline{\chi} \in H_{it}(q^2, 1)$.

If $m \mid q$, χ conductor q, $\pi \in H_{it}(m, \chi^2)$, then $\pi \otimes \overline{\chi} \in H_{it}(q^2, 1)$.

Theorem (P.-Young (2019))

Let χ be primitive of conductor q not quadratic.

$$\sum_{|t_j| \leq T} \sum_{m|q} \sum_{\pi \in H_{it_j}(m, \chi^2)} L(1/2, \pi \otimes \overline{\chi})^3 + \int_{-T}^{T} |L(1/2 + it, \chi)|^6 \, dt \ll T^B q^{1+\varepsilon}.$$

If $m \mid q$, χ conductor q, $\pi \in H_{it}(m, \chi^2)$, then $\pi \otimes \overline{\chi} \in H_{it}(q^2, 1)$.

Theorem (P.-Young (2019))

Let χ be primitive of conductor q not quadratic.

$$\sum_{|t_j| \leq T} \sum_{m|q} \sum_{\pi \in H_{it_j}(m, \chi^2)} L(1/2, \pi \otimes \overline{\chi})^3 + \int_{-T}^{T} |L(1/2 + it, \chi)|^6 \, dt \ll T^B q^{1+\varepsilon}.$$

Theorem (P.-Young (2019))

Let χ be primitive of conductor q and $T \gg q^{\varepsilon}$.

$$\sum_{T < |t_j| \leq T+1} \sum_{m|q} \sum_{\pi \in H_{it_j}(m, \chi^2)} L(1/2, \pi \otimes \overline{\chi})^3 + \int_{T}^{T+1} |L(1/2 + it, \chi)|^6 \, dt \ll (Tq)^{1+\varepsilon}.$$

The Weyl Bound

Corollary (P.-Young 2019)

For all primitive χ modulo q we have

$$L(1/2+it,\chi) \ll ((1+|t|)q)^{\frac{1}{6}+\varepsilon}.$$

The Weyl Bound

Corollary (P.-Young 2019)

For all primitive χ modulo q we have

$$L(1/2 + it, \chi) \ll ((1 + |t|)q)^{\frac{1}{6}+\varepsilon}.$$

In other language: For any Hecke character χ on GL_1 over $\mathbb{Q}$ we have

$$L(1/2, \chi) \ll C(\chi)^{\frac{1}{6}+\varepsilon}.$$

Another Weyl Bound

Corollary (P.-Young 2019)

For all primitive χ modulo q, $m \mid q$, and $\pi \in H_{it_j}(m, \chi^2)$ we have

$$L(1/2, \pi \otimes \overline{\chi}) \ll_\varepsilon ((1 + |t_j|)q)^{\frac{1}{3}+\varepsilon}$$

(and similarly for holomorphic forms $k \geq 2$).

Another Weyl Bound

Corollary (P.-Young 2019)

For all primitive χ modulo q, $m \mid q$, and $\pi \in H_{it_j}(m, \chi^2)$ we have

$$L(1/2, \pi \otimes \overline{\chi}) \ll_\varepsilon ((1 + |t_j|)q)^{\frac{1}{3}+\varepsilon}$$

(and similarly for holomorphic forms $k \geq 2$).

In other language: For $\pi = \bigotimes_v \pi_v$ a cuspidal automorphic rep. on PGL_2 over $\mathbb{Q}$ with π_p principal series for all $p < \infty$ we have

$$L(1/2, \pi) \ll C(\pi)^{\frac{1}{6}+\varepsilon}.$$

Another Weyl Bound

Corollary (P.-Young 2019)

For all primitive χ modulo q, $m \mid q$, and $\pi \in H_{it_j}(m, \chi^2)$ we have

$$L(1/2, \pi \otimes \overline{\chi}) \ll_\varepsilon ((1 + |t_j|)q)^{\frac{1}{3}+\varepsilon}$$

(and similarly for holomorphic forms $k \geq 2$).

In other language: For $\pi = \bigotimes_v \pi_v$ a cuspidal automorphic rep. on PGL_2 over $\mathbb{Q}$ with π_p principal series for all $p < \infty$ we have

$$L(1/2, \pi) \ll C(\pi)^{\frac{1}{6}+\varepsilon}.$$

Complementary case: π has specified supercuspidal local component at a single p: future work of Hu-P.-Young.

Summary of proof

Summary of proof

Key is a reciprocal "dual moment" (P. 2014 in quadratic case)

$$\sum_{t_j} \sum_{m|q} \sum_{\pi \in H_{it_j}(m,\chi^2)} L(1/2, \pi \otimes \overline{\chi})^3 \leftrightarrow \sideset{}{^*}\sum_{\psi \pmod q} |L(1/2,\psi)|^4 g(\chi,\psi)$$

$$g(\chi,\psi) := \sum_{u,v \pmod q} \chi(u)\overline{\chi(u+1)\chi(v)}\chi(v+1)\psi(uv-1).$$

Summary of proof

Key is a reciprocal "dual moment" (P. 2014 in quadratic case)

$$\sum_{t_j} \sum_{m|q} \sum_{\pi \in H_{it_j}(m,\chi^2)} L(1/2, \pi \otimes \overline{\chi})^3 \leftrightarrow \sum_{\psi \pmod q}^{*} |L(1/2, \psi)|^4 g(\chi, \psi)$$

$$g(\chi, \psi) := \sum_{u,v \pmod q} \chi(u)\overline{\chi(u+1)\chi(v)}\chi(v+1)\psi(uv-1).$$

To get this we apply:

1. Approximate functional equation to expand $L(1/2, \pi \otimes \overline{\chi})$

Summary of proof

Key is a reciprocal "dual moment" (P. 2014 in quadratic case)

$$\sum_{t_j} \sum_{m|q} \sum_{\pi \in H_{it_j}(m,\chi^2)} L(1/2, \pi \otimes \overline{\chi})^3 \leftrightarrow \sum_{\psi \pmod q}^{*} |L(1/2,\psi)|^4 g(\chi,\psi)$$

$$g(\chi,\psi) := \sum_{u,v \pmod q} \chi(u)\overline{\chi(u+1)\chi(v)}\chi(v+1)\psi(uv-1).$$

To get this we apply:

1. Approximate functional equation to expand $L(1/2, \pi \otimes \overline{\chi})$
2. Bruggeman-Kuznetsov formula
   (for newforms, using explicit orthonormal basis of $S(q,\chi^2)$)

Summary of proof

Key is a reciprocal "dual moment" (P. 2014 in quadratic case)

$$\sum_{t_j}\sum_{m|q}\sum_{\pi\in H_{it_j}(m,\chi^2)} L(1/2,\pi\otimes\overline{\chi})^3 \leftrightarrow \sideset{}{^*}\sum_{\psi \pmod{q}} |L(1/2,\psi)|^4 g(\chi,\psi)$$

$$g(\chi,\psi) := \sum_{u,v \pmod{q}} \chi(u)\overline{\chi(u+1)\chi(v)}\chi(v+1)\psi(uv-1).$$

To get this we apply:

1. Approximate functional equation to expand $L(1/2,\pi\otimes\overline{\chi})$
2. Bruggeman-Kuznetsov formula
   (for newforms, using explicit orthonormal basis of $S(q,\chi^2)$)
3. Poisson summation (Voronoi formula for Eis. series on GL_3)

Summary of proof

Key is a reciprocal "dual moment" (P. 2014 in quadratic case)

$$\sum_{t_j}\sum_{m|q}\sum_{\pi\in H_{it_j}(m,\chi^2)} L(1/2,\pi\otimes\overline{\chi})^3 \leftrightarrow \sum_{\psi \pmod q}^{*} |L(1/2,\psi)|^4 g(\chi,\psi)$$

$$g(\chi,\psi) := \sum_{u,v \pmod q} \chi(u)\overline{\chi(u+1)\chi(v)}\chi(v+1)\psi(uv-1).$$

To get this we apply:

1. Approximate functional equation to expand $L(1/2,\pi\otimes\overline{\chi})$
2. Bruggeman-Kuznetsov formula (for newforms, using explicit orthonormal basis of $S(q,\chi^2)$)
3. Poisson summation (Voronoi formula for Eis. series on GL_3)
4. Stationary phase, explicit computation of complete character sums, Mellin inversion.

Other examples of dual moments

Motohashi (c. 1995):

$$\int w(t)|\zeta(1/2+it)|^4\,dt \leftrightarrow \sum_{t_j} \sum_{\pi\in H_{it_j}(1,1)} \check{w}(t_j)L(1/2,\pi)^3$$

Other examples of dual moments

Motohashi (c. 1995):

$$\int w(t)|\zeta(1/2+it)|^4\,dt \leftrightarrow \sum_{t_j}\sum_{\pi\in H_{it_j}(1,1)} \check{w}(t_j)L(1/2,\pi)^3$$

Michel-Venkatesh (2010): $\int_{y>0} E^*(\frac{1}{2},iy)E^*(\frac{1}{2},iy)\,d^\times y$ in two ways

Other examples of dual moments

Motohashi (c. 1995):

$$\int w(t)|\zeta(1/2+it)|^4\,dt \leftrightarrow \sum_{t_j} \sum_{\pi\in H_{it_j}(1,1)} \check{w}(t_j)L(1/2,\pi)^3$$

Michel-Venkatesh (2010): $\int_{y>0} E^*(\frac{1}{2}, iy)E^*(\frac{1}{2}, iy)\,d^\times y$ in two ways

Young (2007):

$$\sum_{\chi \pmod p} |L(1/2,\chi)|^4 \leftrightarrow \sum_{t_j} \sum_{\pi\in H_{it_j}(1,1)} \lambda_\pi(p)L(1/2,\pi)^3$$

Other examples of dual moments

Motohashi (c. 1995):

$$\int w(t)|\zeta(1/2+it)|^4\,dt \leftrightarrow \sum_{t_j}\sum_{\pi\in H_{it_j}(1,1)} \check{w}(t_j)L(1/2,\pi)^3$$

Michel-Venkatesh (2010): $\int_{y>0} E^*(\frac{1}{2},iy)E^*(\frac{1}{2},iy)\,d^\times y$ in two ways

Young (2007):

$$\sum_{\chi \,(\mathrm{mod}\ p)} |L(1/2,\chi)|^4 \leftrightarrow \sum_{t_j}\sum_{\pi\in H_{it_j}(1,1)} \lambda_\pi(p)L(1/2,\pi)^3$$

See also recent work of Blomer-Khan (2017) and Zacharias (2018).

Other examples of dual moments

Motohashi (c. 1995):

$$\int w(t)|\zeta(1/2+it)|^4\,dt \leftrightarrow \sum_{t_j}\sum_{\pi\in H_{it_j}(1,1)} \check{w}(t_j)L(1/2,\pi)^3$$

Michel-Venkatesh (2010): $\int_{y>0} E^*(\frac{1}{2},iy)E^*(\frac{1}{2},iy)\,d^\times y$ in two ways

Young (2007):

$$\sum_{\chi\ (\mathrm{mod}\ p)} |L(1/2,\chi)|^4 \leftrightarrow \sum_{t_j}\sum_{\pi\in H_{it_j}(1,1)} \lambda_\pi(p)L(1/2,\pi)^3$$

See also recent work of Blomer-Khan (2017) and Zacharias (2018).
Nelson (2020) explicit and useful transforms for test functions in the Michel-Venkatesh method.

Other examples of dual moments

Motohashi (c. 1995):

$$\int w(t)|\zeta(1/2+it)|^4\,dt \leftrightarrow \sum_{t_j} \sum_{\pi\in H_{it_j}(1,1)} \check{w}(t_j)L(1/2,\pi)^3$$

Michel-Venkatesh (2010): $\int_{y>0} E^*(\frac{1}{2}, iy)E^*(\frac{1}{2}, iy)\, d^\times y$ in two ways
Young (2007):

$$\sum_{\chi \pmod p} |L(1/2,\chi)|^4 \leftrightarrow \sum_{t_j} \sum_{\pi\in H_{it_j}(1,1)} \lambda_\pi(p)L(1/2,\pi)^3$$

See also recent work of Blomer-Khan (2017) and Zacharias (2018).
Nelson (2020) explicit and useful transforms for test functions in the Michel-Venkatesh method. Recovers our dual moment (with $g(\chi,\psi)$), and gives a generalization of our results to number fields.

Dual moment estimates

To win, need

$$\sideset{}{^*}\sum_{\psi \pmod{q}} |L(1/2, \psi)|^4 g(\chi, \psi) \ll q^{2+\varepsilon}.$$

Dual moment estimates

To win, need

$$\sideset{}{^*}\sum_{\psi \,(\mathrm{mod}\ q)} |L(1/2, \psi)|^4 g(\chi, \psi) \ll q^{2+\varepsilon}.$$

To estimate $g(\chi, \psi)$, it suffices to consider q a prime power.

Dual moment estimates

To win, need

$$\sideset{}{^*}\sum_{\psi \pmod q} |L(1/2,\psi)|^4 g(\chi,\psi) \ll q^{2+\varepsilon}.$$

To estimate $g(\chi,\psi)$, it suffices to consider q a prime power.

- If $q = p$ then $g(\chi,\psi) \ll p$ follows the RH of Deligne.

Dual moment estimates

To win, need

$$\sum_{\psi \,(\mathrm{mod}\ q)}^{*} |L(1/2,\psi)|^4 g(\chi,\psi) \ll q^{2+\varepsilon}.$$

To estimate $g(\chi,\psi)$, it suffices to consider q a prime power.

- If $q = p$ then $g(\chi,\psi) \ll p$ follows the RH of Deligne.
- If $q = p^2$, then $g(\chi,\psi) \ll p^2$ by an elementary computation.

Dual moment estimates

To win, need

$$\sum_{\psi \pmod q}^{*} |L(1/2,\psi)|^4 g(\chi,\psi) \ll q^{2+\varepsilon}.$$

To estimate $g(\chi,\psi)$, it suffices to consider q a prime power.

- If $q = p$ then $g(\chi,\psi) \ll p$ follows the RH of Deligne.
- If $q = p^2$, then $g(\chi,\psi) \ll p^2$ by an elementary computation.

In these cases we have by a standard large-sieve type inequality:

$$\sum_{\psi \pmod q}^{*} |L(1/2,\psi)|^4 g(\chi,\psi) \ll q^{1+\varepsilon} \sum_{\psi \pmod q}^{*} |L(1/2,\psi)|^4 \ll q^{2+\varepsilon}.$$

Dual moment estimates

To win, need

$$\sideset{}{^*}\sum_{\psi \pmod q} |L(1/2,\psi)|^4 g(\chi,\psi) \ll q^{2+\varepsilon}.$$

To estimate $g(\chi,\psi)$, it suffices to consider q a prime power.

- If $q = p$ then $g(\chi,\psi) \ll p$ follows the RH of Deligne.
- If $q = p^2$, then $g(\chi,\psi) \ll p^2$ by an elementary computation.

In these cases we have by a standard large-sieve type inequality:

$$\sideset{}{^*}\sum_{\psi \pmod q} |L(1/2,\psi)|^4 g(\chi,\psi) \ll q^{1+\varepsilon} \sideset{}{^*}\sum_{\psi \pmod q} |L(1/2,\psi)|^4 \ll q^{2+\varepsilon}.$$

Pointwise bound on $g(\chi,\psi)$ finish the proof if q is cube-free.

Prime cubes

If $q = p^3$ with $p \equiv 1(4)$, then

Prime cubes

If $q = p^3$ with $p \equiv 1(4)$, then (SUPRISE!) there exist $2(p-1)$ characters ψ mod q such that

$$|g(\chi,\psi)| = p^{\frac{1}{2}} q.$$

Prime cubes

If $q = p^3$ with $p \equiv 1(4)$, then (SUPRISE!) there exist $2(p-1)$ characters $\psi \mod q$ such that

$$|g(\chi, \psi)| = p^{\frac{1}{2}} q.$$

The "bad" ψ are in two cosets of the subgroup of characters mod p:

$$\widehat{(\mathbb{Z}/p\mathbb{Z})^\times} \hookrightarrow \widehat{(\mathbb{Z}/p^3\mathbb{Z})^\times}.$$

Prime cubes

If $q = p^3$ with $p \equiv 1(4)$, then (SUPRISE!) there exist $2(p-1)$ characters $\psi \mod q$ such that

$$|g(\chi, \psi)| = p^{\frac{1}{2}} q.$$

The "bad" ψ are in two cosets of the subgroup of characters mod p:

$$(\widehat{\mathbb{Z}/p\mathbb{Z}})^\times \hookrightarrow (\widehat{\mathbb{Z}/p^3\mathbb{Z}})^\times.$$

Stay tuned for Matt's talk!

Bound on $g(\chi, \psi)$ for q prime, χ quadratic

Refresher on C-I's method to bound $g(\chi, \psi)$ when χ quadratic:

Bound on $g(\chi, \psi)$ for q prime, χ quadratic

Refresher on C-I's method to bound $g(\chi, \psi)$ when χ quadratic:
Dwork, Deligne:
$\alpha_{i,+}, \alpha_{i,-} \in \overline{\mathbb{Q}}$ with $|\alpha_{i,+}| = p^{k_i/2}$, $|\alpha_{i,-}| = p^{\ell_i/2}$, $k_i, \ell_i \in \mathbb{Z}$ s.t.

$$g(\chi, \psi) = \sum_{i=1}^{N_+} \alpha_{i,+} + \sum_{i=1}^{N_-} \alpha_{i,-}.$$

Bound on $g(\chi, \psi)$ for q prime, χ quadratic

Refresher on C-I's method to bound $g(\chi, \psi)$ when χ quadratic:
Dwork, Deligne:
$\alpha_{i,+}, \alpha_{i,-} \in \overline{\mathbb{Q}}$ with $|\alpha_{i,+}| = p^{k_i/2}$, $|\alpha_{i,-}| = p^{\ell_i/2}$, $k_i, \ell_i \in \mathbb{Z}$ s.t.

$$g(\chi, \psi) = \sum_{i=1}^{N_+} \alpha_{i,+} + \sum_{i=1}^{N_-} \alpha_{i,-}.$$

Adolphson-Sperber, Katz: $N_+, N_- \ll 1$.

Bound on $g(\chi, \psi)$ for q prime, χ quadratic

Refresher on C-I's method to bound $g(\chi, \psi)$ when χ quadratic:
Dwork, Deligne:
$\alpha_{i,+}, \alpha_{i,-} \in \overline{\mathbb{Q}}$ with $|\alpha_{i,+}| = p^{k_i/2}$, $|\alpha_{i,-}| = p^{\ell_i/2}$, $k_i, \ell_i \in \mathbb{Z}$ s.t.

$$g(\chi, \psi) = \sum_{i=1}^{N_+} \alpha_{i,+} + \sum_{i=1}^{N_-} \alpha_{i,-}.$$

Adolphson-Sperber, Katz: $N_+, N_- \ll 1$.
Conrey-Iwaniec:

$$\frac{1}{p-1} \sum_{\psi(p)} |g(\chi, \psi)|^2 = p^2 + O(p).$$

Bound on $g(\chi,\psi)$ for q prime, χ quadratic

Refresher on C-I's method to bound $g(\chi,\psi)$ when χ quadratic:
Dwork, Deligne:
$\alpha_{i,+},\alpha_{i,-} \in \overline{\mathbb{Q}}$ with $|\alpha_{i,+}| = p^{k_i/2}$, $|\alpha_{i,-}| = p^{\ell_i/2}$, $k_i,\ell_i \in \mathbb{Z}$ s.t.

$$g(\chi,\psi) = \sum_{i=1}^{N_+} \alpha_{i,+} + \sum_{i=1}^{N_-} \alpha_{i,-}.$$

Adolphson-Sperber, Katz: $N_+, N_- \ll 1$.
Conrey-Iwaniec:

$$\frac{1}{p-1} \sum_{\psi(p)} |g(\chi,\psi)|^2 = p^2 + O(p).$$

Thus $g(\chi,\psi) \ll p$ with at most one exception, which must have $g(\chi,\psi) \asymp p^{3/2}$.

Bound on $g(\chi,\psi)$ for q prime, χ quadratic

Refresher on C-I's method to bound $g(\chi,\psi)$ when χ quadratic:
Dwork, Deligne:
$\alpha_{i,+}, \alpha_{i,-} \in \overline{\mathbb{Q}}$ with $|\alpha_{i,+}| = p^{k_i/2}$, $|\alpha_{i,-}| = p^{\ell_i/2}$, $k_i, \ell_i \in \mathbb{Z}$ s.t.

$$g(\chi,\psi) = \sum_{i=1}^{N_+} \alpha_{i,+} + \sum_{i=1}^{N_-} \alpha_{i,-}.$$

Adolphson-Sperber, Katz: $N_+, N_- \ll 1$.
Conrey-Iwaniec:

$$\frac{1}{p-1} \sum_{\psi(p)} |g(\chi,\psi)|^2 = p^2 + O(p).$$

Thus $g(\chi,\psi) \ll p$ with at most one exception, which must have $g(\chi,\psi) \asymp p^{3/2}$. Since χ quadratic, the exception can only be $g(\chi,\chi)$, which has a special structure.

Bound on $g(\chi, \psi)$ for q prime, χ quadratic

Refresher on C-I's method to bound $g(\chi, \psi)$ when χ quadratic:
Dwork, Deligne:
$\alpha_{i,+}, \alpha_{i,-} \in \overline{\mathbb{Q}}$ with $|\alpha_{i,+}| = p^{k_i/2}$, $|\alpha_{i,-}| = p^{\ell_i/2}$, $k_i, \ell_i \in \mathbb{Z}$ s.t.

$$g(\chi, \psi) = \sum_{i=1}^{N_+} \alpha_{i,+} + \sum_{i=1}^{N_-} \alpha_{i,-}.$$

Adolphson-Sperber, Katz: $N_+, N_- \ll 1$.
Conrey-Iwaniec:

$$\frac{1}{p-1} \sum_{\psi(p)} |g(\chi, \psi)|^2 = p^2 + O(p).$$

Thus $g(\chi, \psi) \ll p$ with at most one exception, which must have $g(\chi, \psi) \asymp p^{3/2}$. Since χ quadratic, the exception can only be $g(\chi, \chi)$, which has a special structure. FAILS if $\chi^2 \neq 1$!

Bound on $g(\chi, \psi)$ for q prime, χ general

Use Deligne's RH directly; theory of ℓ-adic trace functions following Deligne, Laumon, Katz, Fouvry-Kowalski-Michel.

Bound on $g(\chi, \psi)$ for q prime, χ general

Use Deligne's RH directly; theory of ℓ-adic trace functions following Deligne, Laumon, Katz, Fouvry-Kowalski-Michel.

$$g(\chi,\psi) = \sum_{u(q)} \chi(u)\overline{\chi(u+1)} \underbrace{\sum_{v(q)} \overline{\chi(v)}\chi(v+1)\psi(uv-1)}_{\text{a trace function}}$$

Bound on $g(\chi, \psi)$ for q prime, χ general

Use Deligne's RH directly; theory of ℓ-adic trace functions following Deligne, Laumon, Katz, Fouvry-Kowalski-Michel.

$$g(\chi, \psi) = \sum_{u(q)} \chi(u)\overline{\chi(u+1)} \underbrace{\sum_{v(q)} \overline{\chi(v)}\chi(v+1)\psi(uv-1)}_{\text{a trace function}}$$

Let $\ell \neq p$, $\iota : \overline{\mathbb{Q}}_\ell \to \mathbb{C}$ fixed isom.,

Bound on $g(\chi, \psi)$ for q prime, χ general

Use Deligne's RH directly; theory of ℓ-adic trace functions following Deligne, Laumon, Katz, Fouvry-Kowalski-Michel.

$$g(\chi, \psi) = \sum_{u(q)} \chi(u)\overline{\chi(u+1)} \underbrace{\sum_{v(q)} \overline{\chi(v)}\chi(v+1)\psi(uv-1)}_{\text{a trace function}}$$

Let $\ell \neq p$, $\iota : \overline{\mathbb{Q}}_\ell \to \mathbb{C}$ fixed isom., X a variety $/\mathbb{F}_p$, $\mathcal{F}$ a (constructible ℓ-adic) sheaf on X,

Bound on $g(\chi, \psi)$ for q prime, χ general

Use Deligne's RH directly; theory of ℓ-adic trace functions following Deligne, Laumon, Katz, Fouvry-Kowalski-Michel.

$$g(\chi,\psi) = \sum_{u(q)} \chi(u)\overline{\chi(u+1)} \underbrace{\sum_{v(q)} \overline{\chi(v)}\chi(v+1)\psi(uv-1)}_{\text{a trace function}}$$

Let $\ell \neq p$, $\iota : \overline{\mathbb{Q}}_\ell \to \mathbb{C}$ fixed isom., X a variety $/\mathbb{F}_p$, $\mathcal{F}$ a (constructible ℓ-adic) sheaf on X,x a geometric point, $\mathcal{F}_x$ stalk at x.

Bound on $g(\chi, \psi)$ for q prime, χ general

Use Deligne's RH directly; theory of ℓ-adic trace functions following Deligne, Laumon, Katz, Fouvry-Kowalski-Michel.

$$g(\chi,\psi) = \sum_{u(q)} \chi(u)\overline{\chi(u+1)} \underbrace{\sum_{v(q)} \overline{\chi(v)}\chi(v+1)\psi(uv-1)}_{\text{a trace function}}$$

Let $\ell \neq p$, $\iota : \overline{\mathbb{Q}}_\ell \to \mathbb{C}$ fixed isom., X a variety $/\mathbb{F}_p$, $\mathcal{F}$ a (constructible ℓ-adic) sheaf on X,x a geometric point, $\mathcal{F}_x$ stalk at x.

$$t_{\mathcal{F}}(x) := \iota(\mathrm{Tr}(\mathrm{Fr}_p|\mathcal{F}_x))).$$

Bound on $g(\chi, \psi)$ for q prime, χ general

Use Deligne's RH directly; theory of ℓ-adic trace functions following Deligne, Laumon, Katz, Fouvry-Kowalski-Michel.

$$g(\chi,\psi) = \sum_{u(q)} \chi(u)\overline{\chi(u+1)} \underbrace{\sum_{v(q)} \overline{\chi(v)}\chi(v+1)\psi(uv-1)}_{\text{a trace function}}$$

Let $\ell \neq p$, $\iota : \overline{\mathbb{Q}}_\ell \to \mathbb{C}$ fixed isom., X a variety $/\mathbb{F}_p$, $\mathcal{F}$ a (constructible ℓ-adic) sheaf on X,x a geometric point, $\mathcal{F}_x$ stalk at x.

$$t_{\mathcal{F}}(x) := \iota(\mathrm{Tr}(\mathrm{Fr}_p|\mathcal{F}_x))).$$

Let $\mathcal{H}$ be the sheaf on $\mathbb{A}^2/\mathbb{F}_p$ with

$$t_{\mathcal{H}}(u,v) = \overline{\chi(v)}\chi(v+1)\psi(uv-1).$$

Bound on $g(\chi, \psi)$ (continued)

Grothendieck-Lefshetz trace formula (with $U = \mathbb{A}^1 - \{0, -1, 1/u\}$):

$$\sum_{v \in U(\mathbb{F}_p)} t_{\mathcal{H}}(u, v) = \mathrm{Tr}(\mathrm{Fr}_p | H_c^0(U, \mathcal{H})) - \mathrm{Tr}(\mathrm{Fr}_p | H_c^1(U, \mathcal{H})) + \mathrm{Tr}(\mathrm{Fr}_p | H_c^2(U, \mathcal{H}))$$

Bound on $g(\chi, \psi)$ (continued)

Grothendieck-Lefshetz trace formula (with $U = \mathbb{A}^1 - \{0, -1, 1/u\}$):

$$\sum_{v \in U(\mathbb{F}_p)} t_{\mathcal{H}}(u, v) = \mathrm{Tr}(\mathrm{Fr}_p | H^0_c(U, \mathcal{H})) - \mathrm{Tr}(\mathrm{Fr}_p | H^1_c(U, \mathcal{H})) + \mathrm{Tr}(\mathrm{Fr}_p | H^2_c(U, \mathcal{H}))$$

Easy verifications (excision, co-invariants formula):

$$H^0_c(U, \mathcal{H}) = \{0\}, \quad \text{and} \quad H^2_c(U, \mathcal{H}) = \{0\}$$

Bound on $g(\chi, \psi)$ (continued)

Grothendieck-Lefshetz trace formula (with $U = \mathbb{A}^1 - \{0, -1, 1/u\}$):

$$\sum_{v \in U(\mathbb{F}_p)} t_{\mathcal{H}}(u, v) = \mathrm{Tr}(\mathrm{Fr}_p | H^0_c(U, \mathcal{H})) - \mathrm{Tr}(\mathrm{Fr}_p | H^1_c(U, \mathcal{H})) + \mathrm{Tr}(\mathrm{Fr}_p | H^2_c(U, \mathcal{H}))$$

Easy verifications (excision, co-invariants formula):

$$H^0_c(U, \mathcal{H}) = \{0\}, \quad \text{and} \quad H^2_c(U, \mathcal{H}) = \{0\}$$

There is a sheaf on $\mathbb{P}^1$ with stalks $H^1_c(U, \mathcal{H})$ (proper base change):

$$\mathcal{G} := R^1_{p_{1,!}} \mathcal{H}$$

Bound on $g(\chi, \psi)$ (continued)

Grothendieck-Lefshetz trace formula (with $U = \mathbb{A}^1 - \{0, -1, 1/u\}$):

$$\sum_{v \in U(\mathbb{F}_p)} t_{\mathcal{H}}(u, v) = \mathrm{Tr}(\mathrm{Fr}_p | H^0_c(U, \mathcal{H})) - \mathrm{Tr}(\mathrm{Fr}_p | H^1_c(U, \mathcal{H})) + \mathrm{Tr}(\mathrm{Fr}_p | H^2_c(U, \mathcal{H}))$$

Easy verifications (excision, co-invariants formula):

$$H^0_c(U, \mathcal{H}) = \{0\}, \quad \text{and} \quad H^2_c(U, \mathcal{H}) = \{0\}$$

There is a sheaf on $\mathbb{P}^1$ with stalks $H^1_c(U, \mathcal{H})$ (proper base change):

$$\mathcal{G} := R^1_{p_{1,!}} \mathcal{H} = T^1_{\mathcal{K}}(\mathcal{F})$$

a "Cohomological transform sheaf" in sense of Katz. Upshot:

$$t_{\mathcal{G}}(u) = \sum_{v(q)} \underbrace{\overline{\chi(v)}\chi(v+1)}_{t_{\mathcal{F}}(v)} \underbrace{\psi(uv-1)}_{t_{\mathcal{K}}(u,v)}.$$

Bound on $g(\chi, \psi)$ (continued $\times 2$)

Key difficult fact:

Lemma

The sheaf $\mathcal{G}$ is geometrically irreducible and pure of weight 1.

Bound on $g(\chi, \psi)$ (continued $\times 2$)

Key difficult fact:

Lemma

The sheaf $\mathcal{G}$ is geometrically irreducible and pure of weight 1.

Uses the full strength of RH from Deligne's Weil II,

Bound on $g(\chi, \psi)$ (continued $\times 2$)

Key difficult fact:

Lemma

The sheaf $\mathcal{G}$ is geometrically irreducible and pure of weight 1.

Uses the full strength of RH from Deligne's Weil II,
as well as Deligne's semi-continuity theorem (Laumon) to identify the set on which $\mathcal{G}$ is lisse.

Bound on $g(\chi, \psi)$ (continued $\times 2$)

Key difficult fact:

Lemma

The sheaf $\mathcal{G}$ is geometrically irreducible and pure of weight 1.

Uses the full strength of RH from Deligne's Weil II,
as well as Deligne's semi-continuity theorem (Laumon) to identify the set on which $\mathcal{G}$ is lisse.

$$g(\chi, \psi) = \sum_{u(q)} \chi(u)\overline{\chi(u+1)} t_{\mathcal{G}}(u)$$

Bound on $g(\chi, \psi)$ (continued $\times 2$)

Key difficult fact:

Lemma

The sheaf $\mathcal{G}$ is geometrically irreducible and pure of weight 1.

Uses the full strength of RH from Deligne's Weil II,
as well as Deligne's semi-continuity theorem (Laumon) to identify the set on which $\mathcal{G}$ is lisse.

$$g(\chi, \psi) = \sum_{u(q)} \chi(u)\overline{\chi(u+1)}t_{\mathcal{G}}(u)$$

Recall $\mathcal{F}$ is the sheaf on $\mathbb{A}^1$ s.t. $t_{\mathcal{F}}(u) = \overline{\chi(u)}\chi(u+1)$.

Bound on $g(\chi, \psi)$ (continued $\times 2$)

Key difficult fact:

Lemma

The sheaf $\mathcal{G}$ is geometrically irreducible and pure of weight 1.

Uses the full strength of RH from Deligne's Weil II,
as well as Deligne's semi-continuity theorem (Laumon) to identify the set on which $\mathcal{G}$ is lisse.

$$g(\chi, \psi) = \sum_{u(q)} \chi(u)\overline{\chi(u+1)} t_{\mathcal{G}}(u)$$

Recall $\mathcal{F}$ is the sheaf on $\mathbb{A}^1$ s.t. $t_{\mathcal{F}}(u) = \overline{\chi(u)}\chi(u+1)$.
By Deligne's RH a second time, conclude $g(\chi, \psi) \ll q$ so long as $\mathcal{G} \not\simeq \mathcal{F}$.

Bound on $g(\chi, \psi)$ (continued $\times 2$)

Key difficult fact:

Lemma

The sheaf $\mathcal{G}$ is geometrically irreducible and pure of weight 1.

Uses the full strength of RH from Deligne's Weil II,
as well as Deligne's semi-continuity theorem (Laumon) to identify the set on which $\mathcal{G}$ is lisse.

$$g(\chi, \psi) = \sum_{u(q)} \chi(u)\overline{\chi(u+1)} t_{\mathcal{G}}(u)$$

Recall $\mathcal{F}$ is the sheaf on $\mathbb{A}^1$ s.t. $t_{\mathcal{F}}(u) = \overline{\chi(u)}\chi(u+1)$.
By Deligne's RH a second time, conclude $g(\chi, \psi) \ll q$ so long as $\mathcal{G} \not\simeq \mathcal{F}$. Not hard: $\mathrm{rank}(\mathcal{G}) = 2$, $\mathrm{rank}(\mathcal{F}) = 1$, and $\mathcal{G}$ irred. so clear.