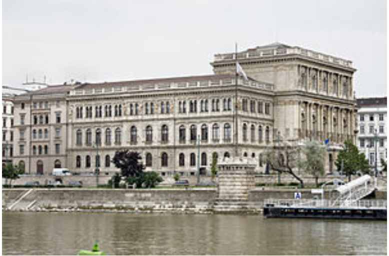
On Bernstein's proof of the meromorphic continuation of Eisenstein series

Erez Lapid

Weizmann Institute of Science

June 2020



Eisenstein series are the fulcrum of the spectral theory of automorphic forms for non-uniform lattices. They furnish the continuous spectrum of $L^2(\Gamma\backslash G)$. But they first have to be meromorphically continued. In the rank-one case this was first taken up by Maaß and Roelcke and resolved by Selberg in the 1950s. In the higher rank case it was completed by Langlands in the 1960s in one his major works.

Most proofs treat Eisenstein series emanating from cusp forms. Langlands uses these Eisenstein series to construct the discrete spectrum by residue calculus. He then considers Eisenstein series from square-integrable automorphic forms as residues of cuspidal Eisenstein series.

Many variants of Selberg's second proof exist in the literature (Cohen–Sarnak, Wong, Mœglin–Waldspurger, Garrett and others). For SL_2 there is also a proof by Lax–Phillips, revisited by Colin de Verdière and extended to cuspidal Eisenstein series in higher rank by Werner Müller, who also showed finiteness of order.

These proofs fall short of general (non-cuspidal) Eisenstein series.

One can deduce the meromorphic continuation of general Eisenstein series from the cuspidal case using Franke's theorem (1990s) that describes any automorphic form in terms of cuspidal Eisenstein series.

In the 1980s Joseph Bernstein penned a letter to Ilya Piatetski-Shapiro where he outlined a simple strategy for proving the meromorphic continuation of Eisenstein series using the so-called "Principle of Meromorphic Continuation".

The goal here is to present this proof, which finally appeared in `arXiv:1911.02342`. It simplifies all previously published proofs. In particular, it doesn't use any complicated spectral theory (only rudimentary Fredholm theory, in the number field case). At the same time, the proof works uniformly for any automorphic form.

Besides the principle of meromorphic continuation (which is "soft" and has nothing to do with automorphic forms), the linchpin of the argument is a new uniqueness result for automorphic forms.

Another important ingredient is the (classical) computation of the constant term of Eisenstein series.

Classical Eisenstein series for $\Gamma = SL_2(\mathbb{Z})$

$$E^*_{2k}(z) = \sum_{(m,n)\in\mathbb{Z}^2\setminus\{(0,0)\}} \frac{1}{(mz+n)^{2k}}$$

$$= \zeta(2k) \sum_{(m,n):\gcd(m,n)=1} \frac{1}{(mz+n)^{2k}}, \quad z \in \mathbb{H}, k > 1$$

a modular form with respect to Γ of weight $2k$.
For spectral theory we consider a related object, which is only real analytic in z but which interpolates k holomorphically

$$\boxed{E(z;s) = \sum_{(m,n):\gcd(m,n)=1} \frac{y^s}{|mz+n|^{2s}}}, \quad z \in \mathbb{H}.$$

The series converges for $\mathrm{Re}\, s > 1$.

The normalized Eisenstein series

$$E^*(z;s) = \sum_{(m,n)\in\mathbb{Z}^2\setminus\{(0,0)\}} \frac{y^s}{|mz+n|^{2s}} = \zeta(2s)E(z;s)$$

can be analytically continued exactly as Riemann did for $\zeta(s)$ (using Poisson summation formula). It has a simple pole at $s=1$ and a functional equation

$$\pi^{-s}\Gamma(s)E^*(z;s) = \pi^{s-1}\Gamma(1-s)E^*(z;1-s).$$

In contrast, the unnormalized Eisenstein series $E(z;s)$ (which will be our main concern) has infinitely many poles (corresponding to zeros of $\zeta(2s)$).
Our (immediate) goal is to meromorphically continue $E(z;s)$ by a different method (thereby giving an alternative proof of the meromorphic continuation of $\zeta(s)$).
Remark One can prove PNT using $E(z;s)$.
RH is equivalent to holomorphy of $E(z;s)$ for $\frac{1}{4} < \operatorname{Re} s < \frac{1}{2}$.

Some notation

- $G = \mathrm{SL}_2(\mathbb{R})$
- $\mathbb{H}$ the upper half-plane, with the left G-action by Möbius transformations and the G-invariant metric $ds^2 = \frac{dx^2+dy^2}{y^2}$.
- We identify $\mathbb{H}$ with G/K where $K = \mathrm{SO}(2)$.
- $\mathrm{y} : \mathbb{H} \to \mathbb{R}_{>0}$, $\mathrm{y}(x + \mathrm{i}y) = y$, considered also as a right K-invariant function on G.
- $\boxed{\mathcal{X} = \Gamma\backslash\mathbb{H}}$. We view functions on $\mathcal{X}$ as Γ-invariant functions on $\mathbb{H}$.
- $\mathfrak{F}_{\mathrm{umg}}(\mathcal{X})$ – the space of smooth functions on $\mathcal{X}$ of uniform moderate growth. This means that there exists $N \geq 1$ such that the lift f to G satisfies

$$|(\delta(X)f)(g)| \ll \mathrm{y}(g)^N \text{ when } \mathrm{y}(g) \geq 1$$

for any $X \in \mathcal{U}(G)$ (acting on the right).

- $\Gamma_\infty = \{\left(\begin{smallmatrix} 1 & n \\ 0 & 1 \end{smallmatrix}\right) : n \in \mathbb{Z}\} \subset \Gamma$.
- The constant term of a function f on $\Gamma_\infty \backslash \mathbb{H}$ is given by

$$Cf(y) = \int_{\mathbb{Z}\backslash\mathbb{R}} f(x + \mathrm{i}y)\ dx = \int_0^1 f(x + \mathrm{i}y)\ dx, \quad y > 0.$$

- The growth of $f \in \mathfrak{F}_{\mathrm{umg}}(\mathcal{X})$ is controlled by Cf:

$$|f(z) - Cf(\mathrm{y}(z))| \ll \mathrm{y}(z)^{-n} \ \ \forall n \geq 1 \ \ (\mathrm{y}(z) \geq \tfrac{1}{2}).$$

- $f \in \mathfrak{F}_{\mathrm{umg}}(\mathcal{X})$ is a cusp form if $Cf \equiv 0$, in which case f is rapidly decreasing.
- We can write $\boxed{E(z; s) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} \mathrm{y}(\gamma z)^s}$.
- For $\mathrm{Re}\, s > 1$, $E(\cdot; s) \in \mathfrak{F}_{\mathrm{umg}}(\mathcal{X})$ (the growth rate depends on $\mathrm{Re}\, s$).

Two elementary properties (in the region $\mathrm{Re}\, s > 1$)

1. For any cusp form f on $\mathcal{X}$ we have $(E(\cdot; s), f)_{\mathcal{X}} = 0$ with respect to the Petersson inner product

$$(f_1, f_2)_{\mathcal{X}} = \int_{\mathcal{X}} f_1(z)\overline{f_2(z)}\ \mu(z), \quad \text{where } \mu = \frac{dx\, dy}{y^2}.$$

2. There exists a holomorphic function $m(s)$ such that

$$CE(y; s) = y^s + m(s) y^{1-s}, \quad y > 0.$$

Remark In reality, $m(s) = \pi^{\frac{1}{2}} \frac{\Gamma(s-\frac{1}{2})\zeta(2s-1)}{\Gamma(s)\zeta(2s)}$

$$\text{where } \zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \text{ and } \Gamma(s) = \int_0^{\infty} e^{-t} t^s \frac{dt}{t}.$$

However, we will not use the meromorphic continuation of $\zeta(s)$ (which is in fact equivalent to the meromorphic continuation of $E(z; s)$).

Uniqueness

For any $s \in \mathbb{C}$ and $a > 0$ denote by D_a^s the difference operator on functions on $\mathbb{R}_{>0}$ given by

$$\mathrm{D}_a^s f(y) = f(ay) - a^s f(y).$$

Thus $\bigcap_{a>0} \operatorname{Ker} D_a^s$ is spanned by y^s.
These operators pairwise commute.

Claim

In the region $\operatorname{Re} s > 1$, *the Eisenstein series* $E(z; s)$ *is the unique* $\psi \in \mathfrak{F}_{\mathrm{umg}}(\mathcal{X})$ *satisfying the following two properties:*

$$(\psi, f)_{\mathcal{X}} = 0 \quad \textit{for every cusp form } f \textit{ on } \mathcal{X},$$
$$\mathrm{D}_a^{1-s}(C\psi(y) - y^s) \equiv 0 \quad \forall a > 0.$$

Claim In the region $\mathrm{Re}\, s > 1$, the Eisenstein series $E(z; s)$ is the unique $\psi \in \mathfrak{F}_{\mathrm{umg}}(\mathcal{X})$ satisfying the following two properties:

$$(\psi, f)_{\mathcal{X}} = 0 \text{ for every cusp form } f \text{ on } \mathcal{X}, \quad (1)$$

$$\mathrm{D}_a^{1-s}(C\psi(y) - y^s) \equiv 0 \quad \forall a > 0. \quad (2)$$

Proof.

We already showed that $E(z; s)$ satisfies these properties. Conversely, suppose that ψ satisfies the above and $\mathrm{Re}\, s > 1$. We need to show that $\psi = E(z; s)$. Consider $\psi' = \psi - E(z; s)$. By (2), the constant term $C\psi'$ is proportional to y^{1-s}. In particular, it is bounded for $\mathrm{y} \geq \frac{1}{2}$. On the other hand, the function $\psi' - C\psi'$ is bounded (and in fact, rapidly decreasing in y) for $\mathrm{y} \geq \frac{1}{2}$. Therefore, ψ' is bounded on $\mathrm{y} \geq \frac{1}{2}$, and hence on $\mathcal{X}$ (since it is Γ-invariant). This implies that $C\psi'$ is bounded (not just for $\mathrm{y} \geq \frac{1}{2}$). Since $C\psi'$ is proportional to y^{1-s} and $\mathrm{Re}\, s > 1$ this means that $C\psi' \equiv 0$. Thus, ψ' is a cusp form and by (1), $\psi' \equiv 0$, i.e., $\psi = E(z; s)$. □

Analytic functions in HLCTVSs

Let $\mathfrak{E}$ be a complex, Hausdorff, locally convex topological vector space (HLCTVS) and U an open subset $\mathbb{C}^n$. A function $f : U \to \mathfrak{E}$ is analytic (or holomorphic) if for every continuous linear form μ of $\mathfrak{E}$, the scalar function $\langle \mu, f(s) \rangle : U \to \mathbb{C}$ is analytic.
For reasonable (e.g., Fréchet) spaces this is equivalent to strong analyticity (existence of partial derivatives) or to having a convergent power series expansion in $\mathfrak{E}$ around any point of U.
Let $\mathfrak{F}$ be another HLCTVS and consider the space $\mathfrak{L}(\mathfrak{E}, \mathfrak{F})$ of operators from $\mathfrak{E}$ to $\mathfrak{F}$. By an *operator* we will always mean a continuous linear map. An analytic family of operators is an analytic function $A : U \to \mathfrak{L}(\mathfrak{E}, \mathfrak{F})$ with respect to either the pointwise convergence topology (i.e., for every $v \in \mathfrak{E}$ the function $s \mapsto A(s)(v) \in \mathfrak{F}$ is holomorphic) or the topology of uniform convergence on bounded subsets. For reasonable spaces, the two notions coincide. The composition of two analytic families of operators is analytic by Hartogs's Theorem.

Analytic systems of linear equations

Let $\mathfrak{E}$ be a HLCTVS. An analytic family of systems $\Xi(s)$, $s \in \mathbb{C}^n$ of linear equations (on $v \in \mathfrak{E}$) consists of

$$\mu_i(s)(v) = c_i(s), \;\; i \in I$$

(possibly with an infinite index set I) where for every i

- $c_i : \mathbb{C}^n \to \mathfrak{E}_i$ is an analytic function to a HLCTVS $\mathfrak{E}_i$.
- $\mu_i(s)$, $s \in \mathbb{C}^n$ is analytic family of operators from $\mathfrak{E}$ to $\mathfrak{E}_i$.

WLOG $\mathfrak{E}_i = \mathbb{C}$ for all i.
Let $\mathrm{Sol}(\Xi(s))$ be the set of solutions of the system $\Xi(s)$ in $\mathfrak{E}$.
We say that Ξ is locally of finite type if $\forall s_0 \in \mathbb{C}^n$, $\exists$ an open neighborhood U, a finite-dimensional vector space L and an analytic family $\lambda(s)$, $s \in U$ of operators $L \to \mathfrak{E}$ such that $\mathrm{Sol}(\Xi(s)) \subset \mathrm{Im}\,\lambda(s)$ for all $s \in U$.

Principle of meromorphic continuation

Theorem

Let $\Xi = (\Xi(s))_{s\in\mathbb{C}^n}$ be an analytic family of systems of linear equations on a HLCTVS $\mathfrak{E}$. Assume that Ξ is locally of finite type. Let

$$S = \{s \in \mathbb{C}^n : \mathrm{Sol}(\Xi(s)) = \{v(s)\}\}$$

be the set of $s \in \mathbb{C}^n$ for which the system $\Xi(s)$ has a unique solution $v(s)$. Suppose that the interior S° of S is nonempty. Then, S contains an open dense subset U of $\mathbb{C}^n$ such that v is holomorphic on U and meromorphic on $\mathbb{C}^n$, i.e., for every $s_0 \in \mathbb{C}^n$ there exists a polydisc D around s_0 and a holomorphic function $0 \not\equiv g : D \to \mathbb{C}$ such that the function $g(s)v(s) : U \cap D \to \mathfrak{E}$ extends holomorphically to D.

The proof is a simple application of Cramer's rule.

Remark It is not claimed that v is holomorphic on S°.

Application of Fredholm theory

(For any compact operator K on a Banach space, $I - K$ is left invertible modulo operators of finite rank.)

Lemma (Fredholm's criterion)

Let $\mathfrak{B}, \mathfrak{C}$ be Banach spaces and let $\mu(s)$, $s \in \mathbb{C}^n$ be an analytic family of operators from $\mathfrak{B}$ to $\mathfrak{C}$. Suppose that for some $s_0 \in \mathbb{C}^n$, $\mu(s_0)$ is left-invertible modulo compact operators, i.e., there exists an operator $D : \mathfrak{C} \to \mathfrak{B}$ such that $D\mu(s_0) - \mathrm{Id}_{\mathfrak{B}} : \mathfrak{B} \to \mathfrak{B}$ is compact. Then, the homogeneous system $\Xi(s)$ (in $\mathfrak{B}$) given by

$$\mu(s)v = 0$$

is locally finite near s_0.
Slightly more generally, suppose that $\mathfrak{B} = \mathfrak{B}_1 \oplus \mathfrak{B}_2$, $\mu(s_0)|_{\mathfrak{B}_1}$ is left-invertible modulo compact operators and that the family $p_2(\mathrm{Sol}(\Xi(s)))$ of subsets of $\mathfrak{B}_2$ is locally finite where $p_2 : \mathfrak{B} \to \mathfrak{B}_2$ is the projection. Then, $\Xi(s)$ is locally finite near s_0.

Back to Eisenstein series

Let $C_c^\infty(G//K)$ be the algebra of smooth, compactly supported, bi-K-invariant functions on G. This algebra acts on the right on locally L^1 functions on $\mathbb{H}$. We denote this action by $f \mapsto \delta(h)f$.
We have

$$\delta(h)\mathrm{y}^s = \hat{h}(s)\mathrm{y}^s$$

where $\hat{h}(s)$ is an entire function, which can be computed explicitly.
All we need to know is the elementary fact that for every $s \in \mathbb{C}$ there exists $h \in C_c^\infty(G//K)$ such that $\hat{h}(s) \neq 0$.
Consider the following system $\Xi(s)$, $s \in \mathbb{C}$ on $\psi \in \mathfrak{F}_{\mathrm{umg}}(\mathcal{X})$ (a countable union of Fréchet spaces):

$$(\psi, f)_{\mathcal{X}} = 0 \quad \text{for every cusp form } f \text{ on } \mathcal{X},$$
$$\mathrm{D}_a^{1-s}(C\psi(y) - y^s) \equiv 0 \quad \forall a > 0,$$
$$\delta(h)\psi = \hat{h}(s)\psi \quad \forall h \in C_c^\infty(G//K).$$

This is a holomorphic system in $s \in \mathbb{C}$. We already know that $E(z;s)$ is the unique solution for $\mathrm{Re}\, s > 1$.

In order to apply the principle of meromorphic continuation to Eisenstein series it remains to prove the following.

Claim

The holomorphic system (parameterized by $s \in \mathbb{C}$)

$$\mathrm{D}_{a_2}^{1-s}\mathrm{D}_{a_1}^{s} C\psi(y) \equiv 0 \quad \forall a_1, a_2 > 0,$$
$$\delta(h)\psi = \hat{h}(s)\psi \quad \forall h \in C_c^\infty(G//K).$$

on $\psi \in \mathfrak{F}_{\mathrm{umg}}(\mathcal{X})$ is locally of finite type.

We sketch the proof.

Step 1: Replace $\mathfrak{F}_{\mathrm{umg}}(\mathcal{X})$ by a suitable Hilbert space

Since the problem is local, we may assume that s is in a small disc U and $\hat{h}$ is non-vanishing on U for some fixed $h \in C_c^\infty(G//K)$. Let $N = 1 + \sup_{s\in U} |\mathrm{Re}\, s|$. By the condition on $C\psi$, any $\psi \in \mathrm{Sol}(\Xi(s))$, $s \in U$ satisfies $|\psi(x + \mathrm{i}y)| \ll y^N$ for $y \geq \frac{1}{2}$. Let

$$w_1(z) = \max_{\gamma\in\Gamma} \mathrm{y}(\gamma z), \quad z \in \mathcal{X}$$

and $\mathfrak{H}_N(\mathcal{X}) := L^2(\mathcal{X}; w_1^{-2N}\mu)$. (This is a non-unitary representation of G.) Then $\mathrm{Sol}(\Xi(s)) \subset \mathfrak{H}_N(\mathcal{X})$, $s \in U$. It is enough to prove the local finiteness for the system in $\mathfrak{H}_N(\mathcal{X})$. The reason is that $\delta(h)(\mathrm{Sol}(\Xi(s))) = \mathrm{Sol}(\Xi(s))$ and $\delta(h)$ is continuous from $\mathfrak{H}_N(\mathcal{X})$ to $\mathfrak{F}_{\mathrm{umg}}(\mathcal{X})$.

Step 2: Replace $\mathcal{X}$ by a simpler space

Let $\mathcal{Z} = \Gamma_\infty \backslash \mathbb{H}$ and for any $c > 0$ consider the space

$$\boxed{\mathcal{Z}_c = \Gamma_\infty \backslash \{z \in \mathbb{H} : \mathrm{y}(z) > c\}} \subset \mathcal{Z}.$$

The restriction p_c of the projection map $\mathcal{Z} \to \mathcal{X}$ to $\mathcal{Z}_c$ is finite-to-one. It is surjective if c is sufficiently small (say $c \leq c_0 = \frac{1}{2}$). For any $0 < c \leq c_0$ let $\boxed{\mathfrak{H}_N(\mathcal{Z}_c) := L^2(\mathcal{Z}_c; \mathrm{y}^{-2N}\mu)}$. The pullback via p_c defines a closed embedding of Banach spaces

$$\boxed{\iota_{c,N} : \mathfrak{H}_N(\mathcal{X}) \to \mathfrak{H}_N(\mathcal{Z}_c)}.$$

For any $c \leq c_0$ we have a commutative diagram

$$\begin{array}{ccc}
\mathfrak{H}_N(\mathcal{X}) & \xrightarrow{\iota_{c,N}} & \mathfrak{H}_N(\mathcal{Z}_c) \\
& \searrow^{\iota_{c_0,N}} & \downarrow \textit{restriction} \\
& & \mathfrak{H}_N(\mathcal{Z}_{c_0})
\end{array}$$

Express system on $\mathfrak{H}_N(\mathcal{Z}_c)$

- Denote by $\pi_{c,N} : \mathfrak{H}_N(\mathcal{Z}_c) \to \mathfrak{H}_N(\mathcal{X})$ the surjective operator such that $\iota_{c,N}\pi_{c,N}$ is the orthogonal projection onto the image of $\iota_{c,N} : \mathfrak{H}_N(\mathcal{X}) \to \mathfrak{H}_N(\mathcal{Z}_c)$.
- For any given $h \in C_c^\infty(G//K)$ there exists $0 < c < c_0$ (depending on the support of h) and an operator $\delta_{c,c_0}(h) : \mathfrak{H}_N(\mathcal{Z}_c) \to \mathfrak{H}_N(\mathcal{Z}_{c_0})$ such that the following diagram commutes

$$\begin{array}{ccc} \mathfrak{H}_N(\mathcal{X}) & \xrightarrow{\iota_{c,N}} & \mathfrak{H}_N(\mathcal{Z}_c) \\ \downarrow \delta(h) & & \downarrow \delta_{c,c_0}(h) \\ \mathfrak{H}_N(\mathcal{X}) & \xrightarrow{\iota_{c_0,N}} & \mathfrak{H}_N(\mathcal{Z}_{c_0}) \end{array}$$

- The constant term map defines an orthogonal projection C_N of $\mathfrak{H}_N(\mathcal{Z}_c)$ onto the closed subspace consisting of the functions that factor through y.

An auxiliary system

Lemma

Suppose that $U \subset \mathbb{C}$ *is bounded and* $h \in C_c^\infty(G//K)$ *is such that* $\hat{h}$ *is non-vanishing on* U. *Let* $N = 1 + \sup_{s \in U} |\mathrm{Re}\, s|$ *and let* $0 < c < c_0$ *be such that* $\delta_{c,c_0}(h) : \mathfrak{H}_N(\mathcal{Z}_c) \to \mathfrak{H}_N(\mathcal{Z}_{c_0})$ *is defined. Let*

$$\alpha_1(s) = \mathrm{y}^s|_{\mathcal{Z}_c}, \quad \alpha_2(s) = \frac{\mathrm{y}^s - \mathrm{y}^{1-s}}{2s-1}|_{\mathcal{Z}_c} \in \mathfrak{H}_N(\mathcal{Z}_c).$$

Consider the family of systems $\tilde{\Xi}(s)$, $s \in U$ *of linear equations on* $f \in \mathfrak{H}_N(\mathcal{Z}_c)$:

$$\iota_{c,N}\pi_{c_0,N}(f|_{\mathcal{Z}_{c_0}}) = f,$$
$$\delta_{c,c_0}(h)f = \hat{h}(s)f|_{\mathcal{Z}_{c_0}},$$
$$C_N f \in \textit{linear span of } \{\alpha_1(s), \alpha_2(s)\}.$$

Then $\tilde{\Xi}(s)$ *is holomorphic and locally of finite type on* U.

Proof.

The last condition can be written equivalently as the holomorphic system of linear equations

$$\begin{vmatrix} (f,\xi_1) & (\alpha_1(s),\xi_1) & (\alpha_2(s),\xi_1) \\ (f,\xi_2) & (\alpha_1(s),\xi_2) & (\alpha_2(s),\xi_2) \\ (f,\xi_3) & (\alpha_1(s),\xi_3) & (\alpha_2(s),\xi_3) \end{vmatrix} = 0 \quad \forall \xi_1,\xi_2,\xi_3 \in \mathfrak{H}_N(\mathcal{Z}_c)$$

where the inner products are taken in $\mathfrak{H}_N(\mathcal{Z}_c)$.
By Gelfand–Piatetski-Shapiro, the restriction of $\delta_{c,c_0}(h)$ to $\mathfrak{H}_N^{\mathrm{cusp}}(\mathcal{Z}_c) := \operatorname{Ker} C_N$ is a compact operator (in fact Hilbert–Schmidt). Combining the first two equations we get $\hat{h}(s)^{-1}\iota_{c,N}\pi_{c_0,N}\delta_{c,c_0}(h)f = f$. Therefore, the lemma follows from Fredholm's criterion. □

Finally, the local finiteness of the original system $\Xi(s)$, $s \in U$ on $\mathfrak{H}_N(\mathcal{X})$ follows since for any $\psi \in \mathrm{Sol}(\Xi(s))$, $\iota_{c,N}\psi \in \mathrm{Sol}(\tilde{\Xi}(s))$ and $\psi = \pi_{c,N}\iota_{c,N}\psi$, so that $\mathrm{Sol}(\Xi(s)) \subset \pi_{c,N}(\mathrm{Sol}(\tilde{\Xi}(s)))$.

A more careful analysis of the proof yields that Eisenstein series are of finite order (as meromorphic functions) (Musicantov). This is technically important for applications to *L*-functions and functoriality (Cogdell–Kim–Piatetski-Shapiro–Shahidi).
Grosso modo, the higher rank case follows the same pattern, except that there are more parabolic subgroups and the constant terms are more complicated.
We will explain it in the adelic setup although it should be emphasized that the theory has nothing to do with number theory. (In the rank one case all what matters is the structure of the manifold at ∞.)

Some notation for the higher rank case

- G – a split, semisimple group over $\mathbb{Q}$ (to fix ideas).
- $\mathbb{A}$ – the ring of adeles.
- $\mathcal{P}$ – the finite set of standard parabolic subgroups of G.
- Define $\mathcal{X}_G = G(\mathbb{Q})\backslash G(\mathbb{A})$ and for any $P = M \ltimes U \in \mathcal{P}$ $\mathcal{X}_P = P(\mathbb{Q})U(\mathbb{A})\backslash G(\mathbb{A}) = M(\mathbb{Q})U(\mathbb{A})\backslash G(\mathbb{A})$.
- Fixing a "good" maximal compact subgroup $\boldsymbol{K}$ of $G(\mathbb{A})$ we can identify $\mathcal{X}_P$ with $M(\mathbb{Q})\backslash M(\mathbb{A}) \times_{\boldsymbol{K}\cap M(\mathbb{A})} \boldsymbol{K}$.
- $\mathcal{A}_P$ – space of automorphic forms on $\mathcal{X}_P$.
- We have a constant term map

$$C_P : \mathcal{A}_G \to \mathcal{A}_P, \quad C_P\phi(g) = \int_{U(\mathbb{Q})\backslash U(\mathbb{A})} \phi(ug)\, du.$$

- Goal: define a dual map

$$\mathcal{A}_P \to \mathcal{A}_G \quad \text{“by”} \ \varphi \mapsto \sum_{\gamma\in P(\mathbb{Q})\backslash G(\mathbb{Q})} \varphi(\gamma\cdot).$$

Unfortunately, this does not converge in general.

Eigenvalues and cuspidality

Fix $P = M \ltimes U \in \mathcal{P}$. The group $Z_M(\mathbb{Q})\backslash Z_M(\mathbb{A})$ acts on $\mathcal{X}_P$ on the left. Let $A_P = Z_M(\mathbb{R})^\circ$, a vector space/$\mathbb{R}$ of dimension $d_P =$ corank P. Let $M(\mathbb{A})^1 = \cap_{\chi \in X^*(M)} \operatorname{Ker} |\chi|$. We have a splitting

$$M(\mathbb{A}) = A_P \times M(\mathbb{A})^1 \quad \text{and} \quad \operatorname{vol}(M(\mathbb{Q})\backslash M(\mathbb{A})^1) < \infty.$$

Let $X_P \simeq \mathbb{C}^{d_P}$ be the group of quasi-characters of A_P.
We decompose $\mathcal{A}_P$ according to A_P-generalized eigenvalues (with respect to left action).

$$\boxed{\mathcal{A}_P = \oplus_{\lambda \in X_P} \mathcal{A}_{P,\lambda}.}$$

The exponents of $\varphi \in \mathcal{A}_P$ (a subset of X_P) are the non-zero coordinates in the decomposition of φ.
Also, define the cuspidal part

$$\boxed{\mathcal{A}_P^{\text{cusp}} = \cap_{Q \subsetneq P} \operatorname{Ker}(C_Q : \mathcal{A}_P \to \mathcal{A}_Q)}.$$

We have a similar decomposition $\mathcal{A}_P^{\text{cusp}} = \oplus_{\lambda \in X_P} \mathcal{A}_{P,\lambda}^{\text{cusp}}$.

Let $\phi \in \mathcal{A}_G$ and $P = M \ltimes U \in \mathcal{P}$. We say that $\lambda \in X_P$ is a cuspidal exponent of ϕ along P if there exist $g \in G(\mathbb{A})$ and a cusp form ψ on $M(\mathbb{Q})\backslash M(\mathbb{A})$ such that

$$(\phi'(\cdot g), \psi)_{M(\mathbb{Q})\backslash M(\mathbb{A})^1} \neq 0$$

where ϕ' is the component of $C_P\phi$ in $\mathcal{A}_{P,\lambda}$.
It is a basic (and relatively easy) fact due to Langlands that every $0 \not\equiv \phi \in \mathcal{A}_G$ admits a cuspidal exponent along some $P \in \mathcal{P}$.

Theorem

For any $0 \not\equiv \phi \in \mathcal{A}_G$ there exist a parabolic subgroup $P \in \mathcal{P}$ and a cuspidal exponent $\lambda \in X_P$ of ϕ along P such that $\operatorname{Re}\lambda$ lies in the closure of the positive Weyl chamber.

This is proved by reducing to the case where all the cuspidal components are along maximal parabolic subgroups. As in the SL_2 case, negative cuspidal exponents would yield that ϕ is bounded, and hence so are the constant terms, resulting in a contradiction.

An equivalent formulation

Let

$$\mathcal{L} : \mathcal{A}_G \to \oplus_{P\in\mathcal{P}} \oplus_{\lambda\in X_P:\mathrm{Re}\,\lambda \text{ dominant}} \mathcal{A}_{P,\lambda}^{\text{cusp}}$$

be the linear map obtained by taking for each $P \in \mathcal{P}$ the image of $C_P\phi$ under the projection

$$\mathcal{A}_P \to \oplus_{\lambda\in X_P:\mathrm{Re}\,\lambda \text{ dominant}}\mathcal{A}_{P,\lambda}$$

and then taking the "cuspidal projection".
Roughly speaking, we keep the cuspidal components along leading exponents.

Theorem

$\mathcal{L}$ is injective.

Characterization of Eisenstein series

Let $\varphi \in \mathcal{A}_P$. For any $\lambda \in X_P$ define

$$\varphi_\lambda(g) = \mathrm{m}_P(g)^\lambda \varphi(g)$$

where $\mathrm{m}_P(g)$ is the $M(\mathbb{A})$-part in the Iwasawa decomposition of g. We have $\varphi_\lambda \in \mathcal{A}_P$.

Proposition

For $\mathrm{Re}\,\lambda$ *sufficiently dominant (depending on* φ*) there exists a unique automorphic form* $E(\varphi, \lambda) \in \mathcal{A}_G$ *such that*

$$\mathcal{L}(E(\varphi, \lambda)) = (C_Q^{\mathrm{cusp}} \varphi_\lambda)_{Q \subset P}.$$

It is given by

$$E(\varphi, \lambda) = \sum_{\gamma \in P(\mathbb{Q}) \backslash G(\mathbb{Q})} \varphi_\lambda(\gamma g).$$

Geometric Lemma

Let ${}_Q\Omega_P$ be the set of double cosets of the Weyl group Ω of G by the Weyl groups of M_P and M_Q. By Bruhat decomposition $Q(F)\backslash G(F)/P(F) \longleftrightarrow {}_Q\Omega_P$.

Lemma

Let $\varphi \in \mathcal{A}_P$ and $\lambda \in X_P$ be such that $\mathrm{Re}\,\lambda$ *is sufficiently dominant. Then for any $Q \in \mathcal{P}$ we have*

$$C_Q(E(\varphi,\lambda)) = \sum_{w \in {}_Q\Omega_P} E^Q(M(w,\lambda)(C_{P_w}\varphi), w\lambda).$$

Here $P_w, Q_w \in \mathcal{P}$ are s.t. $M_{P_w} = M_P \cap w^{-1} M_Q w = w^{-1} M_{Q_w} w$. Each summand on the right-hand side is a composition of three operations: taking a constant term (from $\mathcal{A}_P$ to $\mathcal{A}_{P_w}$), intertwining operator (from $\mathcal{A}_{P_w}$ to $\mathcal{A}_{Q_w}$) and Eisenstein series (from $\mathcal{A}_{Q_w}$ to $\mathcal{A}_Q$). The last two operations are taken in their range of convergence.

Cuspidal components of Eisenstein series

Corollary

$$C_Q^{\text{cusp}} E(\varphi, \lambda) = \sum_{w \in \Omega(P;Q)} [M(w, \lambda)(C_{P_w}^{\text{cusp}} \varphi)]_{w\lambda}$$

where

$$\Omega(P; Q) = \{w \in {}_Q\Omega_P : wM_P w^{-1} \supset M_Q\} = \{w \in {}_Q\Omega_P : Q_w = Q\}.$$

This implies the proposition because of the following fact.

Lemma

If $\mu \in X_P$ with $\operatorname{Re} \mu$ sufficiently dominant wrt P then $w \operatorname{Re} \mu$ is "far" from the positive Weyl chamber for any $w \in \Omega(P; Q) \setminus \{e\}$.

Characterization of automorphic forms

For any $a \in A_Q$ and $\lambda \in X_Q$ define the difference operator

$$\boxed{\mathrm{D}^{\lambda}_{a,Q} f(g) = f(ag) - a^{\lambda} f(g)} \qquad \text{on functions on } \mathcal{X}_Q.$$

Theorem (Harish-Chandra)

Let $\phi \in \mathfrak{F}_{\mathrm{umg}}(\mathcal{X}_G)$. Then ϕ is an automorphic form if and only if the following two conditions are satisfied.

1. *There exists a smooth, compactly supported, bi-$\boldsymbol{K}$-finite function h on $G(\mathbb{A})$ such that $\delta(h)\phi = \phi$.*
2. *For every $Q \in \mathcal{P}$ there exists $\lambda_1, \ldots, \lambda_n \in X_Q$ such that*

$$\mathrm{D}^{\lambda_1}_{a_1,Q} \ldots \mathrm{D}^{\lambda_n}_{a_n,Q} C_Q \phi \equiv 0$$

   for all $a_1, \ldots, a_n \in A_Q$.

Reformulation

For any $Q \in \mathcal{P}$ define

$$\mathfrak{F}^{\text{noncusp}}_{\text{umg}}(\mathcal{X}_Q) = \{\phi \in \mathfrak{F}_{\text{umg}}(\mathcal{X}_Q) : (\phi(\cdot g), \theta)_{M_Q(\mathbb{Q})\backslash M_Q(\mathbb{A})^1} = 0$$
$$\text{for all cusp forms } \theta \text{ on } M_Q(\mathbb{Q})\backslash M_Q(\mathbb{A}) \text{ and all } g \in G(\mathbb{A})\}.$$

We can reformulate the previous theorem as follows.

Theorem (Harish-Chandra)

Let $\phi \in \mathfrak{F}_{\text{umg}}(\mathcal{X}_G)$. Then ϕ is an automorphic form if and only if the following two conditions are satisfied.

1. *There exists a smooth, compactly supported, bi-**K**-finite function h on $G(\mathbb{A})$ such that $\delta(h)\phi = \phi$.*
2. *For every $Q \in \mathcal{P}$ there exists $\lambda_1, \ldots, \lambda_n \in X_Q$ such that*

$$\mathrm{D}^{\lambda_1}_{a_1,Q} \ldots \mathrm{D}^{\lambda_n}_{a_n,Q} C_Q \phi \in \mathfrak{F}^{\text{noncusp}}_{\text{umg}}(\mathcal{X}_Q)$$

for all $a_1, \ldots, a_n \in A_Q$.

Refined characterization of Eisenstein series

Now fix $P \in \mathcal{P}$ and $\varphi \in \mathcal{A}_P$. For any $P' \subset P$ let $\mu_1, \ldots, \mu_{n_{P'}}$ be the cuspidal exponents of φ along P' (including multiplicities). Consider the system of linear equations on $\psi \in \mathfrak{F}_{\mathrm{umg}}(\mathcal{X}_G)$:

$$\prod_{w \in \Omega(P;Q) \setminus \{e\}} \prod_{i=1}^{n_{P_w}} D_{a_{w,i},Q}^{w \cdot (\mu_{P_w,i} + \lambda)} [C_Q(\psi) - C_Q(\varphi_\lambda)] \in \mathfrak{F}_{\mathrm{umg}}^{\mathrm{noncusp}}(\mathcal{X}_Q)$$

for any $Q \in \mathcal{P}$ and elements $a_{w,i} \in A_Q$ for all $w \in \Omega(P;Q) \setminus \{e\}$, $i = 1, \ldots, n_{P_w}$ where by convention, $C_Q(\varphi_\lambda) = 0$ if $Q \not\subset P$. Combining all the above, we get

Theorem

The system is holomorphic in $\lambda \in X_P$ and admits $E(\varphi, \lambda)$ as its unique solution when $\mathrm{Re}\,\lambda$ is sufficiently dominant.

End of proof

In order to use the principle of meromorphic continuation we need to add equations of the form

$$\delta(h_i(\lambda))\psi = c_i(\lambda)\psi, \quad i \in I$$

where for each $i \in I$

1. $c_i : X_P \to \mathbb{C}$ is a holomorphic function.
2. $\lambda \in X_P \mapsto h_i(\lambda)$ is a holomorphic family of bi-$\boldsymbol{K}$-finite, compactly supported, smooth functions on $G(\mathbb{A})$.

Moreover, for each $\lambda \in X_P$ there exists $i \in I$ such that $c_i(\lambda) \neq 0$.
Such a family exists by a result of Harish-Chandra.
The local finiteness is an extension of the classical finiteness result of Harish-Chandra, and is proved along the same lines, namely using reduction theory and the compactness of convolution operators on the space of cusp forms as explained before for SL_2.

Complements

- The meromorphic continuation of Eisenstein series implies the same for the intertwining operators $M(w, \lambda)$.
- The functional equations

$$E(M(w, \lambda)\varphi, w\lambda) = E(\varphi, \lambda) \quad \lambda \in X_P$$

for any $w \in \Omega(P, Q)$ and

$$M(w'w, \lambda) = M(w', w\lambda) \circ M(w, \lambda) \quad \lambda \in X_P$$

for any $w \in \Omega(P, Q)$ and $w' \in \Omega(Q, Q')$ follow from the characterization of Eisenstein series.
- In the function field case (where X_P is complex torus) there exists a polynomial p on X_P s.t. $\lambda \mapsto p(\lambda)E(g, \varphi, \lambda)$ is a polynomial on X_P for all $g \in G(\mathbb{A})$.
  Similarly $M(w, \lambda)$ is a rational function.
- Finally, the singularities of $E(\varphi, \lambda)$ and $M(w, \lambda)\varphi$ are along root hyperplanes.

Odds and ends

As it stands, the proof does not give any information about the location of the poles of Eisenstein series.
Also, it doesn't directly give the result for smooth (but not K-finite) automorphic forms. However, this can be deduced from uniform Casselman–Wallach type results due to Bernstein–Krötz and (using different methods) Wallach.