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Density conjecture for horizontal families

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G - semisimple Lie gr.

A - bounded closed subset of $\Pi(G)$

$(\Gamma_i)_{i \in \mathbb{N}}$ - family of lattices:

Problem: if $A \subseteq$ non temp. spectrum

$m(A, \Gamma) = \sum_{\pi \in A} m(\pi, \Gamma_i)$ in ~~term~~ terms of $\mathrm{vol}(\Gamma_i \backslash G)$

②

$m(\pi, \Gamma) := \dim_{\mathbb{C}} \mathrm{Hom}_G(\pi, L^2(\Gamma \backslash G))$.

Trivial bound

$$m(A, \Gamma_i) \ll_A \mathrm{vol}(\Gamma_i \backslash G).$$

since A ε non-temp.
we should do better

p. ex. $G = SL(2, \mathbb{R})$

Selberg conj:

if Γ is congruence arithmetic lattice then

$$m(A, \Gamma) \leq 1.$$

③

<u>In General</u>

if Γ is ~~on~~ a congruence arithmetic

Ramanujan Conjecture

$$m_{cusp}(A, \Gamma) = 0.$$

we know that sometimes it fails:

In these cases we expect that exceptions are functorial transfers from smaller group.

$$m(A, \Gamma) \ll_A vol(\Gamma \backslash G)^{1-\delta_A}$$

$$\delta_A > 0.$$

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Density conjecture

$\pi \in \Pi(G)$

$p(\pi) = \inf \{ p \geq 2 \mid \pi$ has non 0 matrix coeff. in $L^p(G) \}$.

$p(\pi) = 2$ iff π - is tempered

$A \subseteq \Pi(G)$, ~~bounded~~

$p(A) = \inf\limits_{\pi \in A} p(\pi)$.

<u>Conjecture</u> (Sarnak - Xue '92)

Γ - arithmetic, $\Gamma(N)$ - principal congruence subgroup of level N

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$$m(A, \Gamma(N)) \ll_{A,\varepsilon} \mathrm{vol}(\Gamma(N)\backslash G)^{2/p(A)+\varepsilon}$$

$$\ll_{A,\varepsilon} \mathrm{vol}(\Gamma(N)\backslash G)$$

Proved for $G = SL(2,\mathbb{R})$
$SL(2,\mathbb{C})$

by Sarnak–Xue '91.

(for $A = \{\pi\}$)

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conj is open of other G's

except if

- A consists of cohomological reps $G = U(n,1)$

(Marshall, Marshall–Shin)

- $G = GL(n)$,
  $\Gamma(N)$ replaced by subgroups
  $\begin{pmatrix} [\,*\,] & * \\ N\mathbb{Z} & \varepsilon \end{pmatrix}$ Blomer - Buttcane - Maga $n = 3$
  Blomer $n \geq 3$.
  $\Gamma = SL(n, \mathbb{Z})$ Jana $n \geq 3$.

- Density conjecture has applications in
  combinatorics (Golubev - Kamber)
  quantum computing
  (Evra - Parzanchevski - Kamber).

Two generalizations

I. G - is higher rank

non-tempered spectrum might not be bounded.

- what is the dependence on A.

(Golubev - Kamber '20)

(modified)

- fix some norm $\|\cdot\|$ on $(\mathfrak{h}_{\mathbb{C}})^*$ - dual complexified cartan.

we should have:

$$m(A, \Gamma(N)) \ll_{A,\varepsilon} V(\Gamma(N))^{2/p(A)+\varepsilon} (1 + \|A\|)^L$$

$$\|A\| = \sup_{\pi \in A} \|x_\pi\|.$$

L - some constant.

in fact $L = L_0/p(A) + \epsilon$.

II - look at bigger families of lattices of G.

Say Γ - any congruence arithmetic lattice of G. A-bounded

Can we expected

$$m(A,\Gamma) \ll_{A,\epsilon} \mathrm{vol}(\Gamma \backslash G)^{2/p+\epsilon} ?$$

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$\Gamma \subset G$ is arithmetic congruence

if $\exists$ H - lin alg $/k$

k - number field

$$\rho : H(k \otimes_{\mathbb{Q}} \mathbb{R}) \longrightarrow G$$

ρ - has compact kernel.

$\Gamma \subseteq \rho(H(k))$

and $\exists$ $U \subseteq H(\mathbb{A}_f)$

finite adeles of k

st $\Gamma = \rho(H(k) \cap U)$.

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if we ask for $o(\mathrm{vol}(\Gamma\backslash G))$ bound then there are positive results:

- $G = SL(2,\mathbb{R}), SL(2,\mathbb{C})$
  Γ - torsion free, cocompact
  $m(A,\Gamma) = o(\mathrm{vol}(\Gamma\backslash G))$.
  (F. 2016)

- G simple higher rank,
  Γ - cocompact, torsion free
  + either
  $[k:\mathbb{Q}]$ - bounded
  or
  assume Lehmer conjecture

$\Rightarrow$ $m(A,\Gamma) = o(\mathrm{vol}(\Gamma\backslash G))$.

(Abert, Bergeron + 5, 2015)

(11) our main result:

$G = SL(2,\mathbb{R})^a \times SL(2,\mathbb{C})^b$

Choose $S_1 \subset \{1, \ldots, a+b\}$

$(\lambda_i)_{i \in S_1} \in (0, 1/2)^{S_1}$

$(t_i)_{i \notin S_1} \in [0, \infty)^{\{1, \ldots, a+b\} \setminus S_1}$

$A = A((\lambda_i), (t_i)) = \{ \pi_s = \pi_{s_1} \times \ldots \pi_{s_{a+b}} \in \Pi(G) \mid$

- π_{s_i} -spherical of norm. Casimir eigenvalue $1/4 - s_i^2$.
- $s_i \in (\lambda_i, 1/2)$ $i \in S_1$
- $s_i \in i[t_i - 1, t_i + 1]$.

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let $\rho_i = 1$ if $i \leq a$

$\rho_i = 2$ if $a < i \leq a+b$.

<u>Thm</u> (F., Mavros, Mitiga, Milicevic)

if Γ is a congruence*

arithmetic lattice with trace

field k then

$$m(A, \Gamma) \ll_{\varepsilon, [k:\mathbb{Q}]} \left(\mathrm{vol}(\Gamma \backslash G) \cdot \prod_{i=1}^{a+b} (1+t_i)^{\rho_i} \right)^{2/p(A)+\varepsilon}$$

$$2/p(A) = 1 - \max_{i \in S_\infty} 2\lambda_i .$$

Idea of the proof

assume $a=1$, $b=0$.

A = quaternion alg / $\mathbb{Q}$
split at every finite place

and

$$A \otimes_{\mathbb{Q}} \mathbb{R} = M(2,\mathbb{R}) \times \mathcal{H}^c$$

Hamilton

$H = SL(1, A)$.

$$H(\mathbb{A}) \cong SL(2,\mathbb{R}) \times SU(2)^c \cdot \underbrace{\prod_p SL(2, \mathbb{Q}_p)}_{= H(\mathbb{A}_f)}$$

$U \in H(\mathbb{A}_f)$

$U = \prod SL(2, \mathcal{O}_p)$

(14)

$\rho : \mathrm{SL}_1(A) \to G = \mathrm{SL}(2,\mathbb{R})$

$\Gamma := \rho(H(k) \cap U)$.

(classically Γ is the ~~unit~~ gp. of norm 1 elements in $\mathcal{O} \subseteq A$, $\mathcal{O}$-maximal order).

$S_2 = \{2\}$, λ_2.

we want to prove

$$m(A,\Gamma) \ll_\varepsilon \mathrm{vol}(\Gamma|G)^{2/p(\alpha)+\varepsilon}$$

$$= \mathrm{vol}(\Gamma|G)^{1-2\lambda_2+\varepsilon}$$

Borel volume formula

$$\mathrm{vol}(\Gamma|G) \asymp \Delta_k^{3/2}.$$

(15)

$$c^{2.1} \quad m(A,\Gamma) \ll \sum_{[\gamma] \subseteq H(k)}{}' \mathrm{vol}(E_\gamma(k)^1 \backslash E_\gamma(A)) \cdot O(f,\gamma) \cdot O(1_u,\gamma).$$

$H = SL(2,A)$

- $\mathrm{vol}(E_\gamma(k)^1 \backslash E_\gamma(A)) \ll \Delta_k^{1/2} \left|N_{k/\mathbb{Q}}(\Delta_{k(\gamma)/k})\right|^{1/2}$

- $O(f,\gamma) \ll \left|N_{k/\mathbb{Q}}(\Delta(\gamma))\right|^{-1/2}$

- $O(1_u,\gamma) \ll \left|N_{k/\mathbb{Q}}(\Delta(\gamma))\right|^{1/2} \cdot \left|N_{k/\mathbb{Q}}(\Delta_{k(\gamma)/k})\right|^{-1/2}$

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We find a test function $f \in C_c(G)$ with the following properties: (parameter $R > 0$)

$f(1) \ll 1$. $\cancel{f(\pm 1) \gg e^{-2R\lambda_1}}$.

- $O(f,\gamma) = \int_{G_\gamma \backslash G} f(g^{-1}\gamma g)\,dg$

  $\ll |\Delta(\gamma)|^{-1/2}$

  $\Delta(\gamma) = \det(\mathrm{Ad}\,\gamma|_{\mathfrak{g}/\mathfrak{g}_\gamma})$.

- $\mathrm{tr}\,\pi(f) \gg e^{2R\lambda_1}$

  $\forall\, \pi \in A$

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$$e^{2R\lambda} m(A,\Gamma) \ll \sum_{\substack{[\gamma]\subset H(u) \\ O(\gamma,t)\neq 0 \\ O(\gamma,2u)\neq 0}} \Delta_u^{1/2}$$

$$\ll \Delta_u^{1/2} \cdot \frac{V}{\Delta_u^{1/2}} = V.$$

for $R = \log V$

we get the theorem.

$\square$