Secondary terms in the asymptotics of moments of L-series

(Joint work with Henry Twiss)

Online Conference in Automorphic Forms, 1 - 5 June 2020

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Quadratic Dirichlet L-functions

For $d \in \mathbb{Z}$ non-zero and square-free, let

$$\chi_d(n) = \begin{cases} \left(\frac{d}{n}\right) & \text{if } d \equiv 1 \pmod 4 \\ \left(\frac{4d}{n}\right) & \text{if } d \equiv 2, 3 \pmod 4 \end{cases}$$

where $\left(\frac{a}{b}\right)$ is the Kronecker symbol. Let

$$L(s, \chi_d) = \sum_{n \geq 1} \chi_d(n) n^{-s} = \prod_{p-\mathrm{prime}} (1 - \chi_d(p) p^{-s})^{-1} \qquad (\text{for } \Re(s) > 1).$$

Throughout $\mathbb{F}_q$ will be a fixed finite field of odd characteristic. Over $\mathbb{F}_q[x]$ we have the analogous $\chi_d(m) = (d/m)$, for $d, m \in \mathbb{F}_q[x]$ with m monic. For $\Re(s) > 1$, let

$$L(s, \chi_d) = \sum_{m-\mathrm{monic}} \chi_d(m) |m|^{-s} = \prod_{\pi-\mathrm{monic}\,\&\,\mathrm{irred}} (1 - \chi_d(\pi) |\pi|^{-s})^{-1}$$

where $|a| = q^{\deg a}$.

Moments of L-series

The Conrey, Farmer, Keating, Rubinstein and Snaith conjecture

For integers $r \geq 1$, consider

$$\sum_{d<D}^{*} L\left(\tfrac{1}{2}, \chi_d\right)^r$$

the *star* indicating that the sum is over square-free integers.

The Conrey, Farmer, Keating, Rubinstein and Snaith conjecture

Conjecture (CFKRS): We have

$$\sum_{d<D}^{*} L\left(\tfrac{1}{2}, \chi_d\right)^r \sim D\mathcal{Q}_r(\log D) \qquad \text{as } D \to \infty$$

where $\mathcal{Q}_r(t)$ is some polynomial of degree $r(r+1)/2$.

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Over the rationals, asymptotic formulas are known only for the first three moments: Jutila (for the first two moments), Soundararajan, D-Goldfeld-Hoffstein, Young, and D-Whitehead for the third moment.

(Soundararajan and Young)/Shen proved an asymptotic formula for the fourth moment assuming GRH.

Over $\mathbb{F}_q(T)$ we know the corresponding asymptotic formulas for the first four moments: Hoffstein-Rosen and Andrade-Keating for the first moment, Florea for all four moments, and D. for the third moment.

The moment conjecture over $\mathbb{Q}$

Fix $r \geq 3$. Then, for every $N \geq 1$ and $(N+1)^{-1} < \Theta < N^{-1}$, we have

$$\sum_{d<D}^{*} L\big(\tfrac{1}{2}, \chi_d\big)^r = \sum_{n=1}^{N} D^{\frac{1}{2}+\frac{1}{2n}} \mathcal{Q}_{n,r}(\log D) + O\big(D^{(1+\Theta)/2}\big)$$

as $D \to \infty$, for computable polynomials $\mathcal{Q}_{n,r}(t)$.

This is in contrast with the moments of the Riemann-zeta function when we should have:

$$\int_0^T \big|\zeta\big(\tfrac{1}{2}+it\big)\big|^{2r} dt = T\mathcal{P}_r(\log T) + O\big(T^{\frac{1}{2}+\varepsilon}\big)$$

for all $r \geq 1$.

The function field case

The analogue of the conjecture is:

Conjecture: Fix $r \geq 3$. Then, for integers $D, N \geq 1$ and $(N+1)^{-1} < \Theta < N^{-1}$, we have

$$\boxed{\sum_{\substack{d-\text{monic \& sq. free} \\ \deg d = D}} L\left(\tfrac{1}{2}, \chi_d\right)^r = \sum_{n \leq N} q^{(\frac{1}{2}+\frac{1}{2n})D} Q_{n,r}(D,q) + O_{\Theta,q,r}\left(q^{D(1+\Theta)/2}\right)}$$

for computable $Q_{n,r}(D,q)$.

The function field case

More generally, if $\mathbf{s}' = (s_1, \ldots, s_r)$, s_i distinct with $\Re(s_i) = \frac{1}{2}$, then

$$\boxed{\sum_{\substack{d-\text{monic \& sq. free} \\ \deg d = D}} \prod_{i=1}^{r} L(s_i, \chi_d) = \sum_{n \le N} Q_n(\mathbf{s}'; D, q) + O_{\Theta, q, r}\left(q^{D(1+\Theta)/2}\right)}$$

where

$$Q_n(\mathbf{s}'; D, q) = \frac{1}{n} \sum_{\alpha \in \Phi_n} \left\{ \sum_{\zeta^{2n}=1} \frac{\Gamma_{w_\alpha}(\mathbf{s}', \zeta)}{2^{l(w_\alpha)}} S_\alpha(\mathbf{s}', \zeta) \zeta^D \right\} q^{\frac{D(d(\alpha)+1-2\alpha(\mathbf{s}'))}{2n}},$$

1. $\Phi_n = \{\sum_{i \le r} k_i \alpha_i + n\alpha_{r+1}\}$ is a subset of positive real roots of a Kac-Moody Lie algebra of rank $r+1$, $\alpha(\mathbf{s}') := \sum_{i \le r} k_i s_i$, and w_α is a Weyl group element sending α to α_{r+1}.

2. Γ_{w_α} and $S_\alpha = \prod_\pi S_\pi^{w_\alpha}$ are computed in terms of M_w defined below.

Overview

For $\mathbf{s} = (s_1, \ldots, s_{r+1}) \in \mathbb{C}^{r+1}$, consider over $\mathbb{F}_q(T)$ (q odd):

$$Z(\mathbf{s}) = \sum_{\substack{m_1,\ldots,m_r,\, d-\text{monic} \\ d=d_0 d_1^2,\, d_0 \text{ square free}}} \frac{\chi_{d_0}(\widehat{m}_1 \cdots \widehat{m}_r) A(m_1, \ldots, m_r, d)}{|m_1|^{s_1} \cdots |m_r|^{s_r} |d|^{s_{r+1}}}$$

$\widehat{m}_i$ is the part of m_i coprime to d_0, and

$$A(m_1, \ldots, m_r, d) = \prod_{\substack{\pi^{k_i} \| m_i \\ \pi^{l} \| d}} A\left(\pi^{k_1}, \ldots, \pi^{k_r}, \pi^{l}\right).$$

Overview

$$\sum_{\substack{m_1,\ldots,m_r,\, d-\text{monic} \\ d=d_0 d_1^2,\, d_0 \text{ square free}}} \frac{\chi_{d_0}(\widehat{m}_1 \cdots \widehat{m}_r) A(m_1, \ldots, m_r, d)}{|m_1|^{s_1} \cdots |m_r|^{s_r} |d|^{s_{r+1}}}$$

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$$\sum_{\substack{m_1,\ldots,m_r,\, d-\text{monic} \\ d=d_0d_1^2,\; d_0 \text{ square free}}} \frac{\chi_{d_0}(\widehat{m}_1\cdots\widehat{m}_r)A(m_1,\ldots,m_r,d)}{|m_1|^{s_1}\cdots|m_r|^{s_r}|d|^{s_{r+1}}}$$

$$\sum_{\substack{m_1,\ldots,m_r,\, d-\text{monic} \\ d=d_0d_1^2,\; d_0 \text{ square free} \\ (m_1\cdots m_r d, c)=1}} \frac{\chi_{a_1c_1d_0}(\widehat{m}_1\cdots\widehat{m}_r)\chi_{a_2c_2}(d_0)A(m_1,\ldots,m_r,d)}{|m_1|^{s_1}\cdots|m_r|^{s_r}|d|^{s_{r+1}}}$$

Overview

Fix $\theta_0 \in \mathbb{F}_q^\times \setminus (\mathbb{F}_q^\times)^2$, $c \in \mathbb{F}_q[T]$ monic and square free, and write $c = c_1 c_2 c_3$, with c_i monic. For $a_1, a_2 \in \{1, \theta_0\}$, let

$$Z^{(c)}_{\mathrm{arithm}}(\mathbf{s};\chi_{a_2c_2}, \chi_{a_1c_1}) = \sum_{\substack{m_1,\ldots,\, d \\ d=d_0d_1^2 \\ (m_1\cdots d,c)=1}} \frac{\chi_{a_1c_1d_0}(\widehat{m}_1\cdots\widehat{m}_r)\chi_{a_2c_2}(d_0)A(m_1,\ldots,m_r,d)}{|m_1|^{s_1}\cdots|m_r|^{s_r}|d|^{s_{r+1}}}.$$

Overview

We can write

$$Z^{(c)}_{\mathrm{arithm}}(\mathbf{s};\chi_{a_2c_2},\chi_{a_1c_1}) = \sum_{\substack{(d,c)=1\\ d=d_0d_1^2}} \frac{\prod_{i=1}^{r} L^{(c_2c_3)}(s_i,\chi_{a_1c_1d_0})\chi_{a_2c_2}(d_0)P_d(\mathbf{s}';\chi_{a_1c_1d_0})}{|d|^{s_{r+1}}}$$

$$= \sum_{\substack{(m_1\cdots m_r,c)=1\\ m_1\cdots m_r=n_0n_1^2}} \frac{L^{(c_1c_3)}(s_{r+1},\chi_{a_2c_2n_0})\,\chi_{a_1c_1}(n_0)Q_{\underline{m}}(s_{r+1};\chi_{a_2c_2n_0})}{|m_1|^{s_1}\cdots|m_r|^{s_r}}.$$

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$$= \sum_{\substack{(m_1\cdots m_r,c)=1\\ m_1\cdots m_r=n_0n_1^2}} \frac{L^{(c_1c_3)}(s_{r+1},\chi_{a_2c_2n_0})\chi_{a_1c_1}(n_0)Q_{\underline{m}}(s_{r+1};\chi_{a_2c_2n_0})}{|m_1|^{s_1}\cdots|m_r|^{s_r}}.$$

If $a_2 = c_2 = 1$, it has a simple pole at $s_{r+1} = 1$ with residue

$$\zeta_c(1)^{-1} \cdot \sum_{\substack{(m_1\cdots m_r,c)=1\\ m_1\cdots m_r=\square}} \frac{Q_{\underline{m}}(1;1)}{|m_1|^{s_1}\cdots|m_r|^{s_r}}$$

$$= \prod_{\pi|c}(1-|\pi|^{-1}) \cdot \prod_{\pi\nmid c}\left(\sum_{|\underline{k}|-\mathrm{even}} Q_{\underline{k}}(|\pi|^{-1};|\pi|)|\pi|^{-k_1s_1-\cdots-k_rs_r}\right).$$

Overview

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Generalized Cartan matrix:

$$A = \begin{pmatrix} 2 & & & & -1 \\ & 2 & & & -1 \\ & & & & \vdots \\ & & & 2 & -1 \\ -1 & -1 & \cdots & -1 & 2 \end{pmatrix} \qquad ((r+1)\times(r+1) \text{ matrix})$$

We have: $\det A = -\,2^{r-1}(r-4)$.

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$$A=(A_{ij})=\begin{pmatrix} 2 & & & & -1 \\ & 2 & & & -1 \\ & & & & \vdots \\ & & & 2 & -1 \\ -1 & -1 & \cdots & -1 & 2 \end{pmatrix}$$

Let $(\mathfrak{h},\Pi,\Pi^\vee)$ be a realization of A, that is:

(i). $\mathfrak{h}$ is a complex vector space of dimension $2r+2-\operatorname{rank}A$.

(ii). $\Pi=\{\alpha_i : 1\le i\le r+1\}\subset\mathfrak{h}^*,\Pi^\vee=\{\alpha_i^\vee : 1\le i\le r+1\}\subset\mathfrak{h}$ are linearly independent sets such that $\alpha_j(\alpha_i^\vee)=A_{ij}$.

Overview

For $1 \leq i \leq r+1$, we define the fundamental reflection σ_i of $\mathfrak{h}^*$ by

$$\sigma_i(\lambda) = \lambda - \lambda(\alpha_i^\vee)\alpha_i \qquad (\text{for } \lambda \in \mathfrak{h}^*).$$

The subgroup $W := \langle \sigma_i : 1 \leq i \leq r+1 \rangle$ of $\mathrm{GL}(\mathfrak{h}^*)$ is the Weyl group of

$$\mathfrak{g}(A) = \mathfrak{h} \oplus \Big(\bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha \Big);$$

$\Phi \subset \sum_i \mathbb{Z}\alpha_i \setminus \{\mathbf{0}\}$ is the set of all roots of $\mathfrak{g}(A)$, and let $m_\alpha = \dim \mathfrak{g}_\alpha \geq 1$ denote the multiplicity of α.

Lemma: Let $\alpha = \sum k_i \alpha_i$ be a positive root with $k_{r+1} \geq 1$. Then $k_i \leq k_{r+1}$ for all $i = 1, \ldots, r$.

Thus the set $\Phi_n := \{\alpha \text{ positive real root with } k_{r+1} = n\}$ is finite.

Overview

The Weyl-Kac denominator

$$\prod_{\alpha>0}(1-e^{-\alpha})^{m_\alpha} \qquad (e^{\lambda}(h)=e^{\lambda(h)}, \text{ for } \lambda\in\mathfrak{h}^*, h\in\mathfrak{h})$$

is holomorphic on the interior of the complexified Tits cone $\mathrm{X}_{\mathbb{C}}$.

Setting $e^{-\alpha_i(h)} \mapsto q^{-2\mathfrak{s}_i}$, the Weyl-Kac denominator becomes

$$\Delta = \prod_{\alpha>0}\left(1-q^{-2\alpha(\mathfrak{s})}\right)^{m_\alpha}$$

where, for $\alpha=\sum k_i\alpha_i$ and $\mathfrak{s}=(\mathfrak{s}_1,\mathfrak{s}_2,\ldots)$, we set

$$\alpha(\mathfrak{s}) := \sum k_i\mathfrak{s}_i.$$

Δ is holomorphic on X_0^*: the interior of the corresponding complexified convex cone.

Overview

Let $\mathrm{Q}_+^{\mathrm{ev}} \subset \sum \mathbb{N}\alpha_i$ consist of $\alpha = \sum k_i\alpha_i$ with both $k_1 + \cdots + k_r$ and k_{r+1} *non-zero even* natural numbers. For $t > 1$, let $\mathcal{S}_W$ consist of the absolutely convergent series on X_0^*,

$$1 \ + \sum_{\alpha \in \mathrm{Q}_+^{\mathrm{ev}} \setminus \{\mathbf{0}\}} f_\alpha(t) t^{-\alpha(\mathfrak{s})}$$

with $t^{d(\alpha)/2} f_\alpha(t) \in \mathbb{Q}[t]$, which are W-invariant.

Let

$$D^{\mathrm{re}}(\mathfrak{s}) \ := \prod_{\substack{\alpha > 0 \\ \alpha \text{ real}}} \left(1 - q^{1 - 2\alpha(\mathfrak{s})}\right).$$

Conjecture 1. There exists $\mathrm{I}(\mathfrak{s}, t) \in \mathcal{S}_W$, non-vanishing on X_0^* for all $t > 1$, such that

$$\prod_{\pi \nmid c} \mathrm{I}(\mathfrak{s}, |\pi|) \cdot D^{\mathrm{re}}(\mathfrak{s}) Z_{\mathrm{arithm}}^{(c)}\left(\mathfrak{s} + \tfrac{\mathbf{1}}{\mathbf{2}}; \chi_{a_2c_2}, \chi_{a_1c_1}\right), \ \tfrac{\mathbf{1}}{\mathbf{2}} := \left(\tfrac{1}{2}, \ldots, \tfrac{1}{2}\right) \in \mathbb{C}^{r+1}$$

is holomorphic on X_0^*; the product $\prod_\pi \mathrm{I}(\mathfrak{s}, |\pi|)$ converges absolutely to a holomorphic function on X_0^*.

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is holomorphic on X_0^*; the product $\prod_p \mathrm{I}(\mathfrak{s}, p)$ converges absolutely to a holomorphic function on X_0^*.

Remark

For $Z_{\mathrm{arithm}}(\mathbf{s};\chi_{a_2},\chi_{a_1})$ (that is, $c=1$) over $\mathbb{F}_q(T)$, the conjecture is true when $r=4$, with $\mathrm{I}(\mathfrak{s},t)\equiv 1$ (forthcoming joint work with V. Pașol and A. Popa).

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The proof uses in a crucial way the new functional equation:

$$(Z_{\mathrm{arithm}}(\mathfrak{s}+\tfrac{\mathbf{1}}{\mathbf{2}};\chi_{a_2},\chi_{a_1};q))_{a_1,a_2}=(Z_{\mathrm{arithm}}(\mathfrak{s}+\tfrac{\mathbf{1}}{\mathbf{2}};\chi_{a_2},\chi_{a_1};q^{1-2\delta(\mathfrak{s})}))_{a_1,a_2}B(\mathfrak{s})$$

where $\delta=\alpha_1+\alpha_2+\alpha_3+\alpha_4+2\alpha_5$ is the minimal positive imaginary root, and $B(\mathfrak{s})$ is a 4 by 4 matrix with rational entries in q^{-s_i} $(1\le i\le 5)$.

Overview

For $h \in \mathbb{F}_q[T]$ monic and square free, put

$$Z(\mathbf{s}, h) \quad = \sum_{\substack{m_1,\ldots,m_r,\, d-\mathrm{monic} \\ d=d_0 d_1^2 \\ d_1 \equiv 0 \pmod{h}}} \frac{\chi_{d_0}(\widehat{m}_1 \cdots \widehat{m}_r) A(m_1, \ldots, m_r, d)}{|m_1|^{s_1} \cdots |m_r|^{s_r} |d|^{s_{r+1}}};$$

it can be expressed in terms of the functions $Z^{(h)}_{\mathrm{arithm}}(\mathbf{s}; \chi_{c_2}, \chi_{c_1})$.

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Under a reasonable assumption,

$$\prod_{\pi \nmid h} \mathrm{I}(\mathfrak{s}, |\pi|) \cdot D^{\mathrm{re}}(\mathfrak{s}) Z\left(\mathfrak{s} + \tfrac{1}{2}, h\right)$$

is also holomorphic on X_0^*.

Overview

We have

$$\sum_{d-\text{square free}} \frac{\prod_{i=1}^{r} L(s_i, \chi_d)}{|d|^{s_{r+1}}} = \sum_{h-\text{monic}} \mu(h) Z(\mathbf{s}, h).$$

Conjecture 2. The right-hand side is absolutely convergent for $\Re(s_i) \geq \frac{1}{2}$ $(i = 1, \ldots, r)$ and $\Re(s_{r+1}) > \frac{1}{2}$, away from the zero-set of $D^{\mathrm{re}}\big(\mathbf{s} - \mathbf{\frac{1}{2}}\big)$.

MDS (D-Pașol)

Fix:

- $\mathbb{F}_q$ a finite field of odd characteristic
- $\overline{\mathbb{F}}_q$ an algebraic closure of $\mathbb{F}_q$
- $r \geq 1$ integer

Consider formal power series:

$$\sum_{\mathbf{k}} \Big\{ \sum_{j} \mathbf{P}_j^{(\mathbf{k})}(q)\lambda_j \Big\} \mathbf{z}^{\mathbf{k}} \qquad (\mathbf{k} \in \mathbb{N}^{r+1},\ \mathbf{z}^{\mathbf{k}} = z_1^{k_1} z_2^{k_2} \cdots) \qquad (*)$$

with coefficients finite sums, $\mathbf{P}_j^{(\mathbf{k})}(x) \in \mathbb{Q}[x]$, and λ_j (q-Weil) algebraic integers of weights $\nu_j \in \mathbb{N}$.

$$\boxed{\lambda_{j'} = q^n \lambda_j, \text{ for some } n \in \mathbb{N}, \text{ iff } j = j'}$$

MDS (D-Pașol)

We are assuming that:

1. Each λ_j occurs together with all its complex conjugates.
2. If λ_j and $\lambda_{j'}$ are conjugates over $\mathbb{Q}$, then $\mathbf{P}_j^{(\mathbf{k})}(x) = \mathbf{P}_{j'}^{(\mathbf{k})}(x)$.
3. For each j, $\deg \mathbf{P}_j^{(\mathbf{k})} + \nu_j \leq |\mathbf{k}|$, and $\mathbf{P}_j^{(\mathbf{k})}(x)^2 \equiv 0 \pmod{x^{|\mathbf{k}|-\nu_j+2}}$ when $|\mathbf{k}| = k_1 + \cdots + k_{r+1} \geq 2$.

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For $\mathbf{s} = (s_1, \ldots, s_{r+1}) \in \mathbb{C}^{r+1}$, consider

$$Z(\mathbf{s}) = \sum_{\substack{m_1,\ldots,m_r,\, d-\text{monic} \\ d=d_0 d_1^2,\, d_0 \text{ square free}}} \frac{\chi_{d_0}(\widehat{m}_1 \cdots \widehat{m}_r) A(m_1, \ldots, m_r, d)}{|m_1|^{s_1} \cdots |m_r|^{s_r} |d|^{s_{r+1}}}. \qquad (**)$$

If $x_i := q^{-s_i}$, for $1 \leq i \leq r+1$, we can write $(**)$ as

$$\sum_{k_1,\ldots,k_r,l \geq 0} b(k_1, \ldots, k_r, l; q) x_1^{k_1} \cdots x_r^{k_r} x_{r+1}^{l}.$$

MDS (D-Pașol)

Theorem (D-Pașol): There exists a unique series $(*)$ giving a $Z(\mathbf{s})$ such that:

a). For $\mathbb{F}_{q^n} \subset \overline{\mathbb{F}}_q$,

$$b(k_1, \ldots, l; q^n) = \sum_j \mathbf{P}_j^{(k_1, \ldots, l)}(q^n)\lambda_j^n;$$

the polynomials $\mathbf{P}_j^{(\mathbf{k})}$ are independent of q.

b). For every monic irreducible π of degree $e \geq 1$,

$$A\big(\pi^{k_1}, \ldots, \pi^l\big) = q^{e(k_1 + \cdots + l)} \sum_j \mathbf{P}_j^{(k_1, \ldots, l)}(q^{-e})\lambda_j^{-e}.$$

MDS (D-Paşol)

c). The subseries

$$\sum_{k_1,\ldots,k_r\geq 0} b(k_1,\ldots,k_r,0;q)x_1^{k_1}\cdots x_r^{k_r} = \prod_{i=1}^{r}\frac{1}{1-qx_i}$$

i.e., a product of r zeta functions. In addition,

$$\sum_{k_1,\ldots,k_r\geq 0} b(k_1,\ldots,k_r,1;q)x_1^{k_1}\cdots x_r^{k_r} = q$$

and

$$\sum_{l\geq 0} b(0,\ldots,0,l;q)x_{r+1}^{l} = \frac{1}{1-qx_{r+1}}.$$

In particular, $A(1,\ldots,1,1) = b(0,\ldots,0,0;q) = 1$.

MDS over $\mathbb{Q}$

Over $\mathbb{Q}$ this series is defined similarly, with $A(m_1, \ldots, m_r, d)$ multiplicative and

$$A\left(p^{k_1}, \ldots, p^{k_r}, p^l\right) = p^{k_1 + \cdots + k_r + l} \sum_j \mathbf{P}_j^{(k_1, \ldots, k_r, l)}(p^{-1}) \lambda_j^{-1}$$

for any prime $p \geq 3$.

The assumption

For $\mathbf{k} \in \mathbb{N}^{r+1} \setminus \{\mathbf{0}\}$, let

$$a(\mathbf{k};q) = \sum_j \mathbf{P}_j^{(\mathbf{k})}(q^{-1}) q^{|\mathbf{k}|-\nu_j} \lambda_j.$$

We have $\mathbf{P}_j^{(\mathbf{k})}(x^{-1}) x^{|\mathbf{k}|-\nu_j} \in \mathbb{Q}[x]$, and

$$\max_j \left\{ \tfrac{\nu_j}{2} + \deg \mathbf{P}_j^{(\mathbf{k})}(x^{-1}) x^{|\mathbf{k}|-\nu_j} \right\} \leq \tfrac{|\mathbf{k}|}{2} - 1 \quad (\text{if } |\mathbf{k}| > 1).$$

Assumption. For $\varepsilon > 0$ and $|\mathbf{k}| \geq 2$, $a(\mathbf{k};q) \ll_\varepsilon q^{\left(\frac{1}{2}+\varepsilon\right)|\mathbf{k}|-1}$.

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Assumption. For $\varepsilon > 0$ and $|\mathbf{k}| \geq 2$, $a(\mathbf{k};q) \ll_\varepsilon q^{\left(\frac{1}{2}+\varepsilon\right)|\mathbf{k}|-1}$.

Define

$$f(\mathbf{x};q) = 1 + \sum_{\mathbf{k} \neq \mathbf{0}} a(\mathbf{k};q) \mathbf{x}^{\mathbf{k}};$$

it converges absolutely when $|x_i| < q^{-\frac{1}{2}}$ $(i = 1, \ldots, r+1)$. Note that

$$\boxed{Z(\mathbf{s}) = f(q^{1-\mathbf{s}}; 1/q)}$$

Functional equations

For $i = 1, \ldots, r+1$, define $\sigma_i \mathbf{x} = \mathbf{x}'$ by

$$x_j' := \begin{cases} x_j & \text{if } i, j \le r,\ i \ne j \\ 1/(qx_i) & \text{if } i = j \\ \sqrt{q} x_i x_j & \text{if } i \ne j \text{ and either } i = r+1 \text{ or } j = r+1 \end{cases}$$

and set $W := \langle \sigma_1, \ldots, \sigma_{r+1} \rangle$. If we define

$$\mathbf{f}(\mathbf{x};q) := \frac{1}{2} \begin{pmatrix} f(\underline{x}, x_{r+1}; q) + f(\underline{x}, -x_{r+1}; q) \\ f(\underline{x}, x_{r+1}; q) - f(\underline{x}, -x_{r+1}; q) \\ f(-\underline{x}, x_{r+1}; q) + f(-\underline{x}, -x_{r+1}; q) \end{pmatrix}$$

where $\underline{x} = (x_1, \ldots, x_r)$, then

$$\boxed{\mathbf{f}(\mathbf{x};q) = \mathrm{M}_w(\mathbf{x};q)\mathbf{f}(w\mathbf{x};q)}$$

for $w \in W$.

The cocycle $\mathrm{M}_w(\mathbf{x};q)$

Define

$$M(u,q) = \begin{pmatrix} -\frac{1-qu}{qu(1-u)} & & \\ & \frac{1}{\sqrt{q}u} & \\ & & \frac{1+qu}{qu(1+u)} \end{pmatrix}$$

and

$$U = \begin{pmatrix} 1/2 & 1 & 1/2 \\ 1/2 & 0 & -1/2 \\ 1/2 & -1 & 1/2 \end{pmatrix}.$$

Then

$$\mathrm{M}_{\sigma_i}(x_i;q) = \begin{cases} M(x_i,q) & \text{if } 1 \le i \le r \\ UM(x_{r+1},q)U & \text{if } i = r+1, \end{cases}$$

and $\mathrm{M}_{ww'}(\mathbf{x};q) = \mathrm{M}_{w'}(\mathbf{x};q)\mathrm{M}_w(w'\mathbf{x};q)$, for $w, w' \in W$.

Thank you!