Quantum ergodicity in the Benjamini–Schramm limit for compact quotients of $\mathrm{SL}_n(\mathbb{R})$

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General question

If a sequence of spaces X_n "converges" toward X, what can one say of the convergence of certain "invariants" associated with X_n?

We'll take for X_n a sequence of locally symmetric spaces $\Gamma_n \backslash S$ which converge toward their "common geometry" S.

We are interested in *quantum invariants*: mass equidistribution of eigenfunctions.

Plan of the talk

1. Benjamini–Schramm convergence
2. Convergence of spectra
3. Quantum ergodicity

I. Benjamini–Schramm convergence

Three types of sequences X_n:

- dilated Riemannian manifolds,
- large finite regular graphs,
- locally symmetric spaces of large volume.

Benjamini–Schramm convergence unifies these three examples.

Dilated Riemannian manifolds

Let X be a compact connected complete d-dimension Riemannian manifold with metric g.

For all $t \geq 1$ we denote by X_t the same Riemannian manifold, equipped with the dilated metric $g_t = tg$.

Two points $x, y \in X$ of distance $1/t$ are of distance 1 in X_t.

Convergence toward $\mathbb{R}^d$

In the limit $t \to +\infty$ the sequence X_t 'converges' toward $\mathbb{R}^d$:

"An inhabitant of X_t for large t will have the impression of being in $\mathbb{R}^d$, equipped with the standard metric."

This is more or less the definition of a Riemannian manifold.

Benjamini–Schramm convergence

We would like to take the inhabitant *randomly*.

Definition

Let $\mathcal{M}$ be the space of locally compact pointed metric spaces (X, x), up to pointed isometry. We give $\mathcal{M}$ the Gromov–Hausdorff topology.

Thus $(X_n, x_n) \to (Y, y)$ if for all $R, \epsilon > 0$ there is N such that

$$\forall n \geq N : \qquad \mathrm{dist}(B_{X_n}(x_n, R), B_Y(y, R)) < \epsilon.$$

Probability space (X, μ): push-forward $\nu_X = \int_{\mathcal{M}} \delta_{(X,x)} d\mu(x)$.

Definition

A sequence (X_n, μ_n) converges in the sense of *Benjamini–Schramm* (BS) if the measures ν_{X_n} converge weak-* in $\mathcal{PM}$.

The case of finite graphs

Let $q \geq 2$. Let

- $\mathbb{T}_{q+1}$ be the $(q+1)$-regular tree, given the geodesic distance,
- Γ a cocompact discrete group of automorphisms,
- $X = \Gamma\backslash\mathbb{T}_{q+1}$, a connected finite $(q+1)$-regular graph.

We equip X with the uniform measure μ coming from $\mathbb{T}_{q+1}$.

Definition

For $x \in X$, the largest $\rho > 0$ such that $B(x,\rho)$ is a tree is called the **injectivity radius** about x, denoted $\mathrm{RayInj}_\Gamma(x)$.

Lemma

(X_n, μ_n) BS converges to $\mathbb{T}_{q+1}$ iff for all $R > 0$,

$$\lim_{n\to+\infty} \frac{|\{x \in X_n : \mathrm{RayInj}_{\Gamma_n}(x) < R\}|}{|X_n|} = 0.$$

"The probability that an R-ball is isometric to the tree is 1."

Example (Ramanujan graphs)

Let $B = B^{(2,\infty)}$ be the quaternion algebra over $\mathbb{Q}$ such that ramified at 2 and ∞. Let $\mathbf{G} = \mathrm{PB}^{\times}$. For $p > 2$ we have

$$\mathbb{T}_{p+1} = \mathbf{G}(\mathbb{Q}_p)/\mathbf{G}(\mathbb{Z}_p) = \mathrm{PGL}_2(\mathbb{Q}_p)/\mathrm{PGL}_2(\mathbb{Z}_p).$$

For $(N, p) = 1$ let $\Gamma(N)$ be the image in $\mathbf{G}(\mathbb{Q}_p) = \mathrm{PGL}_2(\mathbb{Q}_p)$ of

$$\ker\left(\mathbf{G}(\mathbb{Z}[1/p]) \to \mathbf{G}(\mathbb{Z}[1/p]/N\mathbb{Z}[1/p])\right) \subset \mathbf{G}(\mathbb{R}) \times \mathbf{G}(\mathbb{Q}_p).$$

The graph $X_N = \Gamma(N)\backslash\mathbb{T}_{p+1}$ is called a **Ramanujan graph**:

$$\mathrm{spec}_{L^2(X_N)}(A_p) \subset \mathrm{spec}_{L^2(\mathbb{T}_{p+1})}(A_p) = \left[-\frac{2\sqrt{p}}{p+1}, \frac{2\sqrt{p}}{p+1}\right],$$

where

$$(A_p f)(x) = \frac{1}{p+1}\sum_{y\sim x} f(y).$$

Locally symmetric spaces

Definition

A **globally Riemannian symmetric space** S is a connected Riemannian manifold for which geodesic inversion about each point is a global isometry.

Examples: $\mathbb{R}^d$, S^d, $\mathbb{H}^d$, Pos_d^1

Classification of irreducible symmetric spaces (Cartan)

We have $S = G/K$, where G is a connected simple Lie group with finite center, and K is a compact subgroup of G.

$$\mathbb{R}^d = \mathrm{E}_d/\mathrm{O}(d), \qquad S^d = \mathrm{SO}(d)/\mathrm{SO}(d-1),$$
$$\mathbb{H}^d = \mathrm{SO}_0(d,1)/\mathrm{SO}(d), \qquad \mathrm{Pos}_d^1 = \mathrm{SL}_d(\mathbb{R})/\mathrm{SO}(d).$$

Locally symmetric spaces

Let $X = \Gamma \backslash S$ **locally symmetric**, where

- $S = G/K$, and
- Γ is a cocompact discrete subgroup of G.

We equip S (and thus X) with its Riemannian measure.

Example: $\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}$ and its congruence covers $\Gamma(N)\backslash\mathbb{H}$.

Lemma (7 samuraï[1])

A sequence $X_n = \Gamma_n \backslash S$ converges BS to S iff for all $R > 0$

$$\lim_{n \to +\infty} \frac{\mathrm{vol}(\{x \in X_n : \mathrm{RayInj}_{\Gamma_n}(x) < R\})}{\mathrm{vol}(X_n)} = 0.$$

"The probability that an R-ball is isometric to a ball in S is 1."

[1] Abert–Bergeron–Biringer–Gelander–Nikolov–Raimbault–Samet

Recent progress

Let G be semisimple real or p-adic. Let

- $\Gamma < G$ be discrete cocompact,
- $K < G$ a maximal compact (open, for p-adic) subgroup,
- $S = G/K$.

Theorem

Every $X_n = \Gamma_n \backslash S$, with Γ_n (torsion free) *arithmetic* and such that $\mathrm{vol}(X_n) \to \infty$, BS converges to S.

Due to

- 7 samuraï ($\Gamma < \Gamma_0$ fixed)
- Fraczyk (Γ arbitrary)
- Levit (G p-adic)

In particular, in higher rank one needs only $\mathrm{vol}(X_n) \to \infty$.

II. Convergence of spectra

Let

- G be a real or p-adic Lie group,
- Γ a discrete cocompact subgroup in G,
- μ a Haar measure on G.

The regular representation of G on $L^2(\Gamma\backslash G)$ decomposes discretely

$$L^2(\Gamma\backslash G) = \bigoplus_{\pi\in\widehat{G}} m(\pi,\Gamma)\pi,$$

where $\widehat{G}$ is the unitary dual of G and

$$m(\pi,\Gamma) = \dim \mathrm{Hom}_G(\pi, L^2(\Gamma\backslash G)) \in \mathbb{N}.$$

This defines a measure

$$\hat{\nu}_\Gamma = \frac{1}{\mathrm{vol}(\Gamma\backslash G)} \sum_{\pi\in\widehat{G}} m(\pi,\Gamma)\,\delta_\pi,$$

on $\hat{G}$, called the **Plancherel measure of** G **relative to** Γ.

Limit Multiplicity

On the other hand, $L^2(G)$ decomposes (under $G \times G$) as

$$L^2(G) = \int_{\hat{G}} H_\pi \, \mathrm{d}\hat{\nu}_G(\pi), \qquad H_\pi = \pi \otimes \tilde{\pi},$$

where $\hat{\nu}_G$ is the **Plancherel measure** on $\hat{G}$ (which depends on μ).

Limit Multiplicity Problem

Given a sequence of uniform lattices Γ_n in G, under what conditions on Γ_n does one have $\hat{\nu}_{\Gamma_n} \to \hat{\nu}_G$?

Remark. The support of $\hat{\nu}_G$ is the **tempered** unitary dual (H-Ch). If $\hat{\nu}_{\Gamma_n} \to \hat{\nu}_G$ the non-tempered π are negligible in the limit.

A first result

For $\pi \in \hat{G}$ we have $\hat{\nu}_G(\pi) > 0$ iff π is square-integrable.

Theoreme: De George and Wallach, 1978

Let G be real. Take the Γ_n (cocompact) in a *tower*:

$$\Gamma_n \text{ normal}, \qquad \Gamma_n \supset \Gamma_{n+1}, \quad \text{and} \quad \bigcap_n \Gamma_n = \{1\}.$$

Then for all $\pi \in \hat{G}$ we have

$$\lim_{n \to +\infty} \hat{\nu}_{\Gamma_n}(\pi) = \begin{cases} \hat{\nu}_G(\pi), & \pi \text{ square integrable}, \\ 0, & \text{else}. \end{cases}$$

For example, the principal congruence subgroups

$$\Gamma_n = \Gamma(p^n) = \ker\left(\mathrm{SL}_d(\mathbb{Z}) \to \mathrm{SL}_d(\mathbb{Z}/p^n\mathbb{Z})\right)$$

form a tower of (non-uniform) lattices in $\mathrm{SL}_d(\mathbb{Z})$.

Uniformly discrete

Definition

A sequence Γ_n is **uniformly discrete** if $\mathrm{RayInd}_{\Gamma_n}$ is bounded away from zero uniformly in n.

Examples:

- a sequence $\Gamma_n < \Gamma_0$ (Γ_n cocompact) is uniformly discrete;
- every sequence $\{\Gamma_n\}$ in a p-adic G is uniformly discrete.

Conjecture (Margulis)

For real G, every arithmetic $\{\Gamma_n\}$ is uniformly discrete.

This is a weak form of Lehmer's conjecture (and implies Salem's conjecture).

Recent breakthroughs (compact case)

Theoreme (7 samuraï, Gelander–Levit)

The Limit Multiplicity property is true as soon as

1. the sequence Γ_n is uniformly discrete,
2. the $X_n = \Gamma_n \backslash X$ converges in the BS sense toward X.

Remark: For Maass forms f on $X = \mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}$ such that

$$\Delta f = \lambda f \quad \text{and} \quad T_p f = \lambda_f(p) f,$$

we can **fix** p and vary λ: the $\lambda_f(p)$ equidistribute as $\widehat{\nu}^{\mathrm{nr}}_{\mathrm{PGL}_2(\mathbb{Q}_p)}$. If X_t is the dilation of X and Δ_t is the Laplacian on X_t, then

$$\Delta_t f = \tilde{\lambda} f, \qquad \text{with} \quad \tilde{\lambda} \approx 1.$$

The “high frequency” limit becomes the BS limit.

III. Quantum ergodicity

Let X be a compact Riemannian manifold of *negative curvature*.

The geodesic flow γ_t on $S^*(X)$ is *chaotic*:

(Top) almost all orbit is dense,

(Erg) ergodic relative to the Liouvillle measure μ_L on $S^*(X)$,

(Mix) mixing: $\mu_L(\gamma_t X \cap Y) \to \mu_L(X)\mu_L(Y)$.

A typical geodesic is *delocalised* in phase space $S^*(X)$.

Quantum physics: the state of a particle in $(S^*(X), \gamma_t)$ is **replaced** by the prob measure $|f|^2 \mathrm{d}Vol$, where $\Delta f = \lambda f$ and $\|f\|_2 = 1$.

Question (quantum chaos)

Let $\{f_j\}$ be a sequence in $L^2(X)$ with $\Delta f_j = \lambda_j f_j$ and $\|f\|_2 = 1$. Understand the weak-* limits of $|f_j|^2 \mathrm{d}Vol$ as $\lambda_j \to \infty$.

In particular, are there 'quantum limits' other than $\frac{\mathrm{d}Vol}{\mathrm{Vol}(X)}$?

Schnirelmann's theorem

Quantum ergodicity (Schirelmann, Colin de Verdière, Zelditch)

Assume that the geodesic flow is ergodic on $S^*(X)$. Let $\mathcal{B} = \{f_j\}$ be an o.n.b. of $L^2(X)$, consisting of Laplacian eigenfunctions $\Delta f_j = \lambda_j f_j$. Then $\forall a \in C(X)$

$$\lim_{T \to +\infty} \frac{1}{N(T)} \sum_{\lambda_j \leq T} \left| \langle a f_j, f_j \rangle - \int_X a \, \mathrm{d}Vol \right|^2 = 0,$$

where $N(T) = |\{j : \lambda_j \leq T\}|$.

Corollary

There is a density one subsequence $\{f_{i_j}\}$ such that $\forall a \in C(X)$

$$\langle a f_{i_j}, f_{i_j} \rangle \longrightarrow \int_X a \, \mathrm{d}Vol,$$

as $j \to +\infty$.

Replacement for negative curvature

For hyperbolic surfaces X we have

- $X = \Gamma\backslash\mathbb{H}$ of negative curvature,
- $S^*(\Gamma\backslash\mathbb{H}) = \Gamma\backslash\mathrm{PSL}_2(\mathbb{R})$,
- the geodesic flow γ_t is the action by $A \simeq \mathbb{R}$.

The exponential rate of mixing of A, as a measure preserving flow on $L^2_0(X)$, is given by the first non-zero Laplacian eigenvalue

$$\lambda_1(X) = \sup_{\int f=0} \frac{\int_X |\nabla f|^2}{\int_X |f|^2} > 0.$$

Note that $\lambda_0(X) = 0$.

We call $\lambda_1(X)$ the **spectral gap**: it will replace negative curvature in what follows.

Delocalization in the large spacial regime: rank 1

We concentrate on the (chronologically first) case of **graphs**.

Fix p prime. Let X be a $(p+1)$-regular finite connected graph.

Write $\lambda \in \operatorname{spec}_{L^2(X)}(A_p) \subset [-1,1]$ as

$$\lambda = \frac{2\sqrt{p}}{p+1} \cos(s \log p) \qquad (s \in \mathbb{C}/\{\pm 1\}).$$

Thus

$$\lambda \text{ is } \textbf{tempered} \text{ (i.e., } \lambda \in \operatorname{spec}_{L^2(\mathbb{T}_{p+1})}(A_p)) \quad \text{iff} \quad s \in \mathbb{R}/\{\pm 1\}.$$

Write $\mathfrak{a}^* = \mathbb{R}/(\pi/\log p)\mathbb{Z}$.

Note that the spectrum here is bounded: cannot let eigenvalue go to ∞.

Anantharaman–Le Masson (2013)

Let Γ_n be a sequence of uniform lattices in $\mathrm{SL}_2(\mathbb{Q}_p)$ such that

1. $X_n = \Gamma_n \backslash \mathbb{T}_{p+1}$ converges BS toward $\mathbb{T}_{p+1}$,
2. $L^2_0(X_n)$ admits a uniform spectral gap.

Let $\{f_j^{(n)}\}$ be an o.n.b. of $L^2(X_n)$ consisting of eigenfunctions of A_p. Let a_n be a sequence of functions on X_n, with $\|a_n\|_\infty = O(1)$. Then for every compact $\Omega \subset \mathfrak{a}^*$

$$\frac{1}{N(\Omega, \Gamma_n)} \sum_{s_j^{(n)} \in \Omega} \left| \langle a_n f_j^{(n)}, f_j^{(n)} \rangle - \int_{X_n} a_n \right|^2 \longrightarrow 0,$$

as $n \to \infty$, where $N(\Omega, \Gamma_n) = |\{j : s_j^{(n)} \in \Omega\}|$.

Rank 1 locally symmetric spaces (under 1, 2, uniformly discrete):
– Le Masson–Sahlsten (2017): QE for quotients of $\mathbb{H}$
– Abert–Bergeron–Le Masson (2019): QE for quotients of $\mathbb{H}^d$

The space $X = \Gamma\backslash \mathrm{Pos}_d^1$

We fix $S = \mathrm{Pos}_d^1 = G/K$, where $G = \mathrm{SL}_d(\mathbb{R})$ and $K = \mathrm{SO}(d)$.

In higher rank ($d \geq 3$), we have

- S is a Hadamard space (complete, simply connected, of non-positive curvature);
- the geodesic flow is no longer ergodic on $S^*(X)$ but the action by $A \simeq \mathbb{R}^{d-1}$ is;
- the regular representation of G on $\bigoplus_n L_0^2(\Gamma_n\backslash G)$ (for any sequence of lattices Γ_n) has an automatic spectral gap;
- the Laplacian $\Delta = \mathrm{div} \circ \mathrm{grad}$ on S is left G-invariant, but it is not the only one!

Definition

Let $\mathcal{D} = \mathcal{D}(S)$ be the algebra of G-invariant diff ops on S.

Let A be the diag torus in G and $\mathfrak{a} = \mathrm{Lie}(A)$; then $\dim \mathfrak{a} = d - 1$.

$\mathcal{D}$ is a **polynomial algebra in $d - 1$ variables** containing Δ.

Harish-Chandra isomorphism

We have a natural isomorphism $\mathcal{D} \simeq \mathrm{Sym}(\mathfrak{a})^W$.

In particular, $\mathrm{Hom}_{\mathbb{C}-\mathrm{alg}}(\mathcal{D}, \mathbb{C}) \simeq \mathfrak{a}^*_{\mathbb{C}}/W$.

Let $X = \Gamma \backslash S$. Then $\mathcal{D} \curvearrowright L^2(X)$, comm family of normal operators.

Definition

An eigenfunction of $\mathcal{D}$ in $L^2(X)$ is called a **Maass form**.

If $Df = \chi(D)f$ for $\chi \in \mathrm{Hom}_{\mathbb{C}-\mathrm{alg}}(\mathcal{D}, \mathbb{C})$, one obtains $\nu \in \mathfrak{a}^*_{\mathbb{C}}/W$.

Main result: QE for compact quotients of $\mathrm{SL}_d(\mathbb{R})$

B.–Matz (in progress)

Let $d \geq 3$. Let Γ_n be a uniformly discrete sequence of uniform lattices in $\mathrm{SL}_d(\mathbb{R})$ such that $\mathrm{vol}(\Gamma_n \backslash \mathrm{Pos}^1_d) \to \infty$.

Let $\{f_j^{(n)}\}$ be an o.n.b. of $L^2(X_n)$ consisting of Maass forms. Let a_n be measurable functions on X_n with $\|a_n\|_\infty = O(1)$. Then for every compact $\Omega \subset \mathfrak{a}^*$

$$\frac{1}{N(\Omega, \Gamma_n)} \sum_{\mathrm{Re}\, \nu_j^{(n)} \in \Omega} \left| \langle a_n f_j^{(n)}, f_j^{(n)} \rangle - \int_{X_n} a_n \right|^2 \longrightarrow 0,$$

as $n \to \infty$, where $N(\Omega, \Gamma_n) = |\{j : \mathrm{Re}\, \nu_j^{(n)} \in \Omega\}|$.

Remark: uniform discreteness in higher rank follows from a conjecture of Margulis.

Sketch of the proof

We can assume that $\int a_n = 0$: show that

$$\frac{1}{N(\Omega, \Gamma_n)} \sum_{\mathrm{Re}\, \nu_j^{(n)} \in \Omega} |\langle a_n f_j^{(n)}, f_j^{(n)} \rangle|^2 \longrightarrow 0.$$

Strategy (Brooks–Le Masson–Lindenstrauss, Sahsten–Le Masson)

1. Convert the *spectral correlations* $\langle af_j, f_j \rangle$ into a dynamical problem on the mixing of large 'balls' in $\Gamma \backslash G$;
2. Treat the geometric correlations by the mean ergodic theorem of Gorodnik–Nevo.

The first step requires a good understanding of spherical functions.

The second require good estimates on intersections volumes:

$$(\text{large 'ball' } B) \cap (\text{large translate of } B).$$

In rank 1 the intersection of 2 metric balls is comparable to a ball.

Let $\rho_{\Gamma \backslash G}$ be the reg rep on $L^2(\Gamma \backslash G)$. For a measurable $E \subset G$, let

$$\rho_{\Gamma \backslash G}(E) f(x) = \frac{1}{m_G(E)} \int_E f(xg) dg.$$

Let $\{E_t\}_{t \geq 0}$ be exhaustive with $E_t^{-1} = E_t$. Let $U_t = \rho_{\Gamma \backslash G}(E_t)$:

$$U_t f(x) = \frac{1}{\sqrt{\mathrm{vol}(E_t)}} \int_{E_t} f(xg) dg,$$

a self-adjoint propagation operator. Let

$$\mathbf{A}(T) = \int_0^T U_t a U_t dt.$$

We find E_t such that the following holds:

Spectral proposition

Let $\Omega \subset \mathfrak{a}^*$ cmpt. There are $c, T_0 > 0$ such that for $T \geq T_0$:

$$\sum_{\mathrm{Re}\, \nu_j \in \Omega} |\langle a f_j, f_j \rangle|^2 \leq c \sum_{\mathrm{Re}\, \nu_j \in \Omega} |\langle \mathbf{A}(T) f_j, f_j \rangle|^2.$$

Definition of E_t

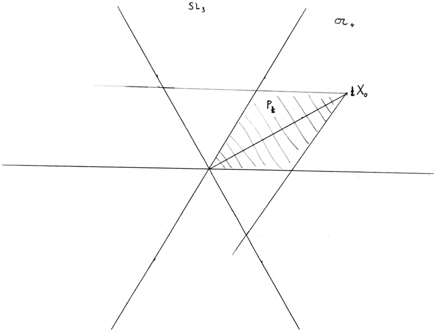
For $t > 1$ we let $E_t = K \exp \mathcal{P}_t^+ K$, where

$$\mathcal{P}_t = \{X \in \mathfrak{a}^+ : |X|_\infty \leq t\}.$$

Let $X_0 \in \mathfrak{a}^+$ realize $\max_{X \in \mathcal{P}_t^+} \langle \rho, X \rangle$. Then

$$\mathcal{P}_t = \mathfrak{a}_+ \cap (tX_0 - C),$$

where C is a cone in $\mathfrak{a}$.



Proof of the spectral proposition

Action on eigenfunctions: $U_t f_\nu = h_t(\nu) f_\nu$, where

$$h_t(\nu) = \frac{1}{\sqrt{m_G(E_t)}} \int_{E_t} \varphi_\nu(g) dg,$$

and φ_ν the Harish-Chandra spherical function. Thus

$$|\langle \mathbf{A}(T) f_j, f_j \rangle|^2 = \left(\frac{1}{T} \int_0^T |h_t(\nu)|^2 dt \right)^2 |\langle a f_j, f_j \rangle|^2.$$

Lemma

There are $C, T_0 > 0$ such that for $T \geq T_0$ and $\nu \in \Omega$:

$$\frac{1}{T} \int_0^T |h_t(\nu)|^2 dt \geq C.$$

Proof of the lemma

We have to control the oscillatory behavior of $\varphi_\nu(e^X)$ for $X \in \mathcal{P}_t^+$.

Recall: $\mathcal{P}_t^+ = \mathfrak{a}_+ \cap (tX_0 - C)$ is *directed* by X_0.

Let M be the centralizer of X_0 in G.

1. we replace φ_ν by the main term Φ_ν in the Harish-Chandra expansion relative to M;
2. when ν is rationnal, we show that Φ_ν behaves like a sum of characters along the line directed by X_0;
3. we use a linear independence of characters argument to conclude.

Second step (geometric)

We can write $\mathbf{A}(T)$ as an operator with kernel $A(T)$, where

$$A(T)(g,h) = \frac{1}{T}\int_0^T \frac{1}{m_G(E_t)} \int_{gE_t \cap hE_t} a(x)dxdt.$$

Now,

$$\sum_{\mathrm{Re}\,\nu_j \in \Omega} |\langle \mathbf{A}(T)f_j, f_j\rangle|^2 \leq \|\mathbf{A}(T)\|_{HS} = \int_{X\times X} |\sum_{\gamma\in\Gamma} A(T)(g,\gamma h)|^2 dhdg.$$

A thick-thin decomposition of the integral gives

$$\|\mathbf{A}(T)\|_{HS} \leq \int_{X\times S} |A(T)(g,h)|^2 dgdh + e(T),$$

where the contribution from the thin part $e(T)$ is controlled by the hypotheses and a lower bound on $N(\nu,\Gamma)$.

The main term

We bound the main term by applying a mean ergodic theorem.

Theorem (Gorodnik–Nevo)

Assume $\rho_{\Gamma\backslash G}$ has a spectral gap. There are $C, \theta > 0$ such that

$$\left\| \rho_{\Gamma\backslash S}(E)(f) - \int_{\Gamma\backslash G} f \right\|_2 \leq C m_G(E)^{-\theta} \|f\|_2, \qquad f \in L^2(\Gamma\backslash G)$$

We must control the intersection volume

$$m_G(E_t \cap e^X E_t)$$

uniformly in $X \in \mathfrak{a}_+$ and $t > 0$.

Geometric proposition

We have $m_G(E_t \cap e^X E_t) \ll e^{-\langle \rho, X \rangle} m_G(E_t)$.

The proof relies on the Iwasawa decomposition.