

The sup norm-problem for automorphic forms over function fields

Will Sawin

Columbia University / CMI

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The sup-norm problem for Maass forms

Let N be a natural number and $X_1(N)$ the modular curve of level N . Let $f : X_1(N) \rightarrow \mathbb{R}$ be an eigenfunction of the Laplacian (with eigenvalue λ) and of all Hecke operators. Assume f cuspidal and $\|f\|_2 = 1$.

- What is $\|f\|_\infty$, i.e. what is the maximum value of $|f|$?

Want answer in terms of N and λ .

Can generalize to an arbitrary arithmetic locally symmetric space.

Adelically, f is a function in $L^2(G(F)\backslash G(\mathbb{A}_F)/\mathbb{K})$ for \mathbb{K} a compact subgroup of $G(\mathbb{A}_F)$.

Prior work

Upper bounds for the sup-norm of Maass forms in the level aspect, with squarefree level:

- $\|f\|_\infty \ll N^{\frac{1}{2} - \frac{25}{914} + \epsilon}$ (Blomer-Holowinsky)
- $\|f\|_\infty \ll N^{\frac{1}{2} - \frac{1}{22} + \epsilon}$ (Templier)
- $\|f\|_\infty \ll N^{\frac{1}{2} - \frac{1}{20} + \epsilon}$ (Helfgott-Ricotta)
- $\|f\|_\infty \ll N^{\frac{1}{3} + \epsilon}$ (Harcos-Templier)

For square level, upper bound is $N^{\frac{1}{4} + \epsilon}$ (Saha, or Marshall over division algebras) or $N^{\frac{5}{24} + \epsilon}$ in the depth aspect over division algebras (Hu-Saha). With square level and highly ramified central character, lower bound is $N^{\frac{1}{4} - \epsilon}$ (Templier).

Much more for higher-rank groups!

Function field setup: Adelic picture

Let \mathbb{F}_q be a finite field and let $F = \mathbb{F}_q(T)$.

The adèles \mathbb{A}_F are the restricted product of the local fields F_v of F . All are fields of formal Laurent series.

- $\mathbb{F}_q^{\deg \pi}((\pi))$ for π an irreducible monic polynomial in T over \mathbb{F}_q .
- $\mathbb{F}_q((T^{-1}))$ - the local field at ∞ .

Let N be a polynomial. Let \mathbb{K}_N be the subgroup of $GL_2(\mathbb{A}_F)$ consisting at each place of integral matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ where $c \equiv 0 \pmod N$, $d \equiv 1 \pmod N$.

A modular form of level N is a function on $GL_2(F) \backslash GL_2(\mathbb{A}_F) / \mathbb{K}_N$.

We can define the concepts “Hecke eigenform”, “cusp form”, “newform”, “unitary central character”, “ L^2 -normalized” exactly as we do in the usual (adelic) setting.

Function field setup: Classical picture

Let \mathbb{H} be $PGL_2(\mathbb{F}_q((T^{-1}))) / PGL_2(\mathbb{F}_q[[T^{-1}]])$ - the vertices of the Bruhat-Tits tree of the local field $\mathbb{F}_q((T^{-1}))$. Let

$$\Gamma_1(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbb{F}_q[[T]]) \mid c \equiv 0 \pmod{N}, d \equiv 1 \pmod{N} \right\}$$

Then $\Gamma_1(N)$ acts on \mathbb{H} by left multiplication.

A modular form of level N is a function on $\Gamma_1(N) \backslash \mathbb{H}$.

It is a Hecke eigenform if it is an eigenfunction of the Hecke operators and the combinatorial Laplacian of the Bruhat-Tits tree of $\mathbb{F}_q((T^{-1}))$.

A Hecke eigenform is a cusp form if it is finitely supported.

How should you think about automorphic forms over $\mathbb{F}_q(T)$?

Function fields like $\mathbb{F}_q(T)$ behave a lot like number fields over \mathbb{Q} .

A key difference is that the infinite place in $\mathbb{F}_q(T)$ is still a non-archimedean local field, while the infinite place of \mathbb{Q} is archimedean.

Thus, every phenomenon you know from number fields likely has a very similar analogue over function fields

- except if it depends on local phenomena at the real place, in which case it likely doesn't
 - ▶ except if it already has an analogue at non-archimedean places of \mathbb{Q} , in which case it likely does.

Example: Rudnick and Sarnak found a lower bound for the sup-norm on compact arithmetic 3-manifolds. Generalized by Milićević. Should a function field analogue exist?

Main Theorem

Let N be a polynomial in $\mathbb{F}_q[T]$. Let f be a cuspidal newform of level N .

Assume that f is L^2 -normalized, with unitary central character.

Assume that N is squarefree and either N is prime or f has trivial central character.

Then

$$\|f\|_\infty = O\left(\left(\frac{2\sqrt{q} + 2}{\sqrt{2\sqrt{q} + 1}}\right)^{\deg N} (\log \deg N)^{3/2}\right)$$

Setting $|N| = q^{\deg N}$, this is $O(|N|^{\frac{1}{4} + \epsilon})$ with $\epsilon \rightarrow 0$ as $q \rightarrow \infty$. Beats $N^{1/3}$ for $q > 134$.

Some Ideas of the Proof

Classically, there are two main approaches to studying the sup-norm problem - either via the Fourier expansion and bounding Fourier/Whittaker coefficients or via the trace formula and amplification.

In the function field setting, there is also an algebraic geometry method. Its starting point is the theory of bounds for complete exponential sums using algebraic geometry.

- e.g. Weil bound for Kloosterman sums, Deligne bound for hyper-Kloosterman sums.

Essentially, the method proceeds by viewing the Fourier/Whittaker expansion as a complete exponential sum.

- But an analogous method probably works in the compact case, where there are no Whittaker expansions, via black magic (i.e. geometric Langlands).

The Fourier/Whittaker expansion

Classically

$$f(q) = \sum_{n=1}^{\infty} a_n q^n = \sum_{n=1}^{\infty} a_n e^{2\pi i x n} e^{-2\pi y n}$$

where a_n is a multiplicative function of a natural number n , $e^{2\pi i x n}$ is a character of \mathbb{Z} , and $e^{-2\pi y n}$ is a decaying local term at the infinite place. In the function field world

$$f(n, \alpha) = \sum_{\substack{g \in \mathbb{F}_q[T] \\ \deg g \leq n \\ g \neq 0}} a_g e(g \cdot \alpha) b_{n - \deg g}$$

where

- a_g is a multiplicative function of a nonzero polynomial g ,
- α is a linear function on polynomials of degree n and $e : \mathbb{F}_q \rightarrow \mathbb{C}^\times$ is an additive character, so that $g \mapsto e(g \cdot \alpha)$ is a character,
- and b_d is a function of natural numbers d .

The Fourier/Whittaker expansion

We view

$$f(n, \alpha) = \sum_{\substack{g \in \mathbb{F}_q[T] \\ \deg g \leq n \\ g \neq 0}} a_g e(g \cdot \alpha) b_{n - \deg g}$$

as a complete exponential sum over $n + 1$ variables in \mathbb{F}_q (the $n + 1$ coefficients of f .)

$e(g \cdot \alpha)$ is exactly the kind of function we can handle geometrically - it is an additive character composed with a simple linear map.

Is $a_g b_{n - \deg g}$ the type of function we can handle geometrically?

- We can sum geometrically functions arising from Galois representations.
- Drinfeld proved that f has a Langlands parameter, which is a Galois representation.
- We can compute a_g (and b_d) in terms of the Langlands parameter.
- This gives a nice geometric description of $f(n, \alpha)$ (also Drinfeld).

A bit on geometry

Getting good bounds for higher-dimensional exponential sums is much harder than the one variable case.

To prove our bound, we use mostly classical algebraic geometry tools (Deligne's Weil II in 1980, ℓ -adic Fourier transform from the 70s and 80s, ...) plus one cutting-edge tool - the characteristic cycle in characteristic p constructed by Saito in 2017 (building on earlier work of Beilinson).

The characteristic cycle does something interesting from the automorphic perspective, so I highlight it.

The general definition is abstract but in this case it corresponds to a concrete object called the global nilpotent cone.

The global nilpotent cone

The modular curve $X_1(N)$ parameterizes pairs of lattices $\Lambda_1 \subseteq \Lambda_2 \subset \mathbb{R}^2$ with $\Lambda_2/\Lambda_1 \cong \mathbb{Z}/N$, up to the action of \mathbb{C}^\times .

The fiber of the global nilpotent cone over a point $(\Lambda_1, \Lambda_2) \in X_1(N)$ is the set of nilpotent linear maps $\phi : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ such that $\phi(\Lambda_1) \subseteq \Lambda_2$ and $|\phi(x)| \leq |x|$ for all x . This is a finite set.

The fiber of the global nilpotent cone is large for points near one or more cusps and small otherwise.

More generally, for G a semisimple group over F with Lie algebra \mathfrak{g} , the fiber over $g \in G(F) \backslash G(\mathbb{A}_F) / \mathbf{K}$ of the global nilpotent cone consists of nilpotent elements of $\mathfrak{g}(\mathbb{A}_F)$ which lie in the orthogonal spaces to $g^{-1}\mathfrak{g}(F)g$, whose non-Archimedean part is orthogonal to the Lie algebra of \mathbf{K} , and whose Archimedean part lies in some fixed ball.

How the global nilpotent cone appears

In function field, the global nilpotent cone is a geometric object cut out by polynomial equations (via the Hitchin fibration).

I first prove a bound for $|f(x)|$ that depends on the fiber of the global nilpotent cone over x - the bigger the fiber, the worse the bound.

I then explicitly calculate the global nilpotent cone. Key point is that we can reduce from arbitrary N to $N = 1$, and then use the fundamental domain.

The global nilpotent cone has one contribution for each cusp, which grows larger as the point is closer to the cusp.

What about points near a cusp?

The global nilpotent cone has one contribution for each cusp, which grows larger as the point is closer to the cusp.

For points near a cusp, we would get a very bad bound. However, working with the characteristic cycle restricted to a smaller space allows us to ignore the component coming from the standard cusp.

My assumptions on the level and character are sufficient to ensure every cusp is conjugate under Atkin-Lehner operators to the standard cusp. Hence we can ignore the largest component. This leads to the stated bound.

Worst case is points equidistant between two cusps (e.g. midpoint of a geodesic).

Alternate approach: Use purely analytic Whittaker expansion method to bound f near the cusps.

Can cusps really explain large values?

In the compact case, there are no cusps, and no nilpotent elements in the Lie algebra, so one might expect a very strong bound.

However, Rudnick-Sarnak showed that a very strong bound is not true in some compact examples.

What explains this?

- The method is geometric, so it “sees” cusps defined not just over $\mathbb{F}_q(T)$, $\mathbb{F}_{q^2}(T)$, \dots
- Similarly the nilpotent cone may not contain elements over $\mathbb{F}_q(T)$ but its base change to $\mathbb{F}_{q^2}(T)$ might have a lot of elements.
- I checked that this does happen in the function field analogue of Rudnick-Sarnak examples (and Milićević’s generalization).
- I didn’t check that this happens in the further generalization by Brumley-Marshall - it would be interesting to do so.

Varying f versus varying x

This geometric method proves bounds for $|f(x)|$ that vary with x , not with f . Instead of bounds for $\|f\|_\infty = \sup_{x \in X} |f(x)|$, these results are strongest when expressed as bounds for $\sup_f |f(x)|$. (The sup is taken over all automorphic forms satisfying certain local conditions.)

Good: Maybe interesting structure exists in $\sup_f |f(x)|$, that is hidden in $\sup_{x \in X} |f(x)|$, over number fields, as well.

Bad: This means if we have f_1, f_2 with the same local properties, applying this method directly will never give an upper bound for $\|f_1\|_\infty$ less than the true value of $\|f_2\|_\infty$.

We could get around this problem by dividing our space X into two parts, a large one U where we prove a bound for an arbitrary form by geometric methods, and a small one Z where we prove an upper bound for f_2 , and maybe even a lower bound for f_1 , using analytic methods (e.g. period formulas.)