On the first negative Hecke eigenvalue of automorphic forms on GL(2,R)

June 1, 2020

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Content

0. The least quadratic nonresidue

Let $p \geq 2$ be a prime. Linnik considered in 1942 the least quadratic non-residue problem – to evaluate the smallest size of n_p for which $\left(\frac{n_p}{p}\right) = -1$.

Under GRH, Ankeny showed $n_p \ll (\log p)^2$.

Unconditionally, $n_p \ll p^{\frac{1}{2}} \log p$ by Pólya-Vinogradov's inequality.

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Anonymous on 247B, Notes 3:

The least quadratic nonresidue, and the square root barrier

18 August, 2009 in expository, math.NT | Tags: Burgess inequality, least quadratic nonresidue, Polya-Vinogradov inequality, square root barrier

Probabilistic heuristics (presuming that each non-square integer has a 50-50 chance of being a quadratic residue) suggests that n_p should have size $O(\log p)$, and indeed Vinogradov conjectured that $n_p = O_\varepsilon(p^\varepsilon)$ for any $\varepsilon > 0$. Using the Pólya-Vinogradov inequality, one can get the bound $n_p = O(\sqrt{p} \log p)$ (and can improve it to $n_p = O(\sqrt{p})$ using smoothed sums); combining this with a sieve theory argument (exploiting the multiplicative nature of quadratic residues) one can boost this to $n_p = O(p^{\frac{1}{2\sqrt{e}}} \log^2 p)$. Inserting Burgess's amplification trick one can boost this to $n_p = O_\varepsilon(p^{\frac{1}{4\sqrt{e}}+\varepsilon})$ for any $\varepsilon > 0$. Apart from refinements to the ε factor, this bound has stood for five decades as the "world record" for this problem, which is a testament to the difficulty in breaching the square root barrier.

1.1 Set-up

Let $S_k(N) :=$ the space of holo. cusp forms of weight k, level N.

and $\{T_n\}_{n\geq 1}$ be the family of all Hecke operators.

The space $S_k(N)$ contains a special set of Hecke eigenforms, called the primitive forms, which are the common eigenfunctions of all T_n's and whose Fourier coefficents $a(n) = \lambda_f(n)n^{(k-1)/2}$ where $T_n f = \lambda_f(n) f$.

Write $H_k^*(N) = \{\text{all primitive forms of weight } k \text{ for } \Gamma_0(N)\}$.

Then $$S_k(N) = \bigoplus_{M|N} \mathrm{Span}\{f(dz) : d\big|\tfrac{N}{M},\ f \in H_k^*(M)\}$$

1.2 Problem – The first negative Hecke eigenvalue

For $f \in H_k^*(N)$, let $\lambda_f(n)$ be its nth Hecke eigenvalue.

Then $\lambda_f(n)$ is a real multiplicative function.

<u>Question</u>:

What's the smallest size of n in terms of k and N such that $\lambda_f(n) < 0$? Write n_- for the first n that $\lambda_f(n) < 0$.

Want to get $n_- \ll k^{?} N^{?}$ with ? and ? as small as possible.

1.3 Known results on the first negative Hecke eigenvalue

Let $f \in H_k^*(N)$, $\lambda_f(n_-)$ = 1st negative Hecke eigenvalue.

Kohnen & Sengupta (2006): $n_- \ll kN(\log k)^{27} \exp(c\sqrt{\frac{\log N}{\log\log 3N}})$.

Iwaniec, Kohnen & Sengupta (2007): $n_- \ll (k^2N)^{\frac{1}{2}}$.

Note: k^2N is the (analytic) conductor of $L(s, f)$.

$$n_- \ll (k^2N)^{\frac{1}{2}-\frac{1}{60}}.$$

Kowalski, L., Soundararajan & Wu (2010): $n_- \ll (k^2N)^{\frac{1}{2}-\frac{1}{20}}$.

Matomaki (2012): $n_- \ll (k^2N)^{\frac{1}{2}-\frac{1}{8}}$.

2. Our interest and New result

<u>Question</u>: Study the same problem for Hecke-Maass primitive forms

Qu (2010): $n_- \ll (|\frac{1}{2}+i\nu|^2 N)^{\frac{1}{2}-\delta}$ for some unspecified $\delta > 0$.

This $\delta > 0$ comes from the subconvexity bound of Michel & Venkatesh for $L(s, f)$ uniformly for all aspects. Its value is very small.

<u>Goal</u>: Adapt the method for holomorphic forms to Maass forms.

L., Ng, Tang, Wang (submitted) : $n_- \ll (|\frac{1}{2}+i\nu|^2 N)^{\frac{1}{2}-\frac{1}{10}}$.

3.1 Overview

Kohnen
Sengupta
(2006)

$\Rightarrow$

Iwaniec
Kohnen
Sengupta
(2007)

$\Rightarrow$

Kowalski,
L.
Soundararajan
Wu
(2010)

$\Rightarrow$

Matomaki
(2012)

Basic idea:

Give an upper bound for $\sum_{n \le x} \lambda_f(n)$

Derive a lower bound for $\sum_{n \le x} \lambda_f(n)$ subject to $\lambda_f(n) \ge 0$

3.1 Overview (cont'd)

Upper bound:

$$\sum_{n\leq x}\lambda_f(n)\Phi\left(\frac{n}{x}\right)=\frac{1}{2\pi i}\int_{\Re e\, s=\frac{1}{2}}\varphi(s)L(s,f)x^s\,dx$$

Known: $L(s,f)\ll_\varepsilon (|s|^2k^2N)^{\eta+\varepsilon}$ with $0\leq\eta\leq\frac{1}{4}$ on $\Re e\, s=\frac{1}{2}$

$$\begin{aligned}\text{Convexity bound}: &\quad \eta=\tfrac{1}{4}\\ \text{Subconvexity bound}: &\quad \eta=\tfrac{1}{4}-\delta\\ \text{GRH}: &\quad \eta=0\end{aligned}$$

Consequence: $$\sum_{n\leq x}\lambda_f(n)\Phi\left(\frac{n}{x}\right)\ll_\varepsilon x^{1/2}(k^2N)^{\eta+\varepsilon}$$

3.2 Kohnen & Sengupta's method

Let $x \leq n_-$. So $\lambda_f(n) \geq 0$ for $n \leq x$.

$$\sum_{\substack{n \leq x \\ (n,N)=1}} \lambda_f(n)^2 \log^2(x/n)$$

$$\leq \Bigg(\sum_{\substack{n \leq x \\ (n,N)=1}} \lambda_f(n) \log^2(x/n) \Bigg)^{1/2} \Bigg(\sum_{\substack{n \leq x \\ (n,N)=1}} \lambda_f(n)^3 \log^2(x/n) \Bigg)^{1/2}$$

Use $L(s, \mathrm{sym}^2 f) \ll |s|^{\frac{3}{4}+\varepsilon}(kN)^{\frac{1}{2}+\varepsilon}$ on $\Re e\, s = \frac{1}{2}$

$$\sum_{\substack{n \leq x \\ (n,N)=1}} \lambda_f(n)^2 \log^2(x/n) = \underset{\substack{\gg \\ (kN)^{-\varepsilon}}}{C_{k,N}} x + O((kN)^{\frac{1}{2}+\varepsilon} x^{\frac{1}{2}})$$

$$(kN)^{-\varepsilon} x^{1+\varepsilon} + (kN)^{\frac{1}{2}+\varepsilon} x^{\frac{1}{2}} \ll x^{\frac{3}{4}} (k^2 N)^{\frac{1}{8}+\varepsilon}$$

3.3 Iwaniec, Kohnen & Sengupta's method I

Let $x \le n_-$. Use $\lambda_f(p)^2 = \lambda_f(p^2) + 1$.

$$\sum_{\substack{n \le x \\ (n,N)=1}} \lambda_f(n) \ge \frac{1}{2}\Big(\sum_{\substack{p \le \sqrt{x/3} \\ p \nmid N}} \lambda_f(p)\Big)^2 - \frac{1}{2}\sum_{\substack{p \le \sqrt{x/3} \\ p \nmid N}} \lambda_f(p)^2 + \sum_{\substack{p \le \sqrt{x/3} \\ p \nmid N}} \lambda_f(p^2)$$

$$\gg \Big(\sum_{\substack{p \le \sqrt{x/3} \\ p \nmid N}} \lambda_f(p)\Big)^2 - \sum_{\substack{p \le \sqrt{x/3} \\ p \nmid N}} 1.$$

For $p \le \sqrt{x}/3$, $\lambda_f(p) \ge 1$.

Hence $\dfrac{x}{\log^2 x} \ll x^{\frac{1}{2}}(k^2N)^{\eta+\varepsilon} \quad \Rightarrow \quad x \ll (k^2N)^{2\eta+\varepsilon}$.

3.3' Iwaniec, Kohnen & Sengupta's method II

Suppose $\lambda_f(n) \geq 0$ for $1 \leq n \leq y$. Write $x = y^{1+\delta}$ where $0 < \delta \leq \frac{1}{4}$.

$$\sum_{\substack{n \leq x \\ (n,N)=1}} \lambda_f(n) \log \frac{x}{n} = \Bigg(\sum_{\substack{n \leq x,\ (n,N)=1 \\ \lambda_f(n)>0}} + \sum_{\substack{n \leq x,\ (n,N)=1 \\ \lambda_f(n)<0}} \Bigg) = S^+(x) + S^-(x)$$

$$S^-(x) = \sum_{\substack{p^a m \leq x,\ \lambda_f(p^a)<0, \\ (p,m)=(mp,N)=1}} \lambda_f(p^a m) \log \frac{x}{p^a m}$$

$$\geq - \sum_{\substack{m \leq y^\delta \\ (m,N)=1}} \lambda_f(m) \sum_{p^a < x/m} (a+1) \log \frac{x/m}{p^a}.$$

Hence $S^-(x) \geq -2 \sum_{\substack{m \leq y^\delta \\ (m,N)=1}} \frac{\lambda_f(m)}{m} \frac{x}{\log y} \Big(1 + O(\frac{1}{\log y}) \Big)$

4.3 Iwaniec, Kohnen & Sengupta's method II (cont'd)

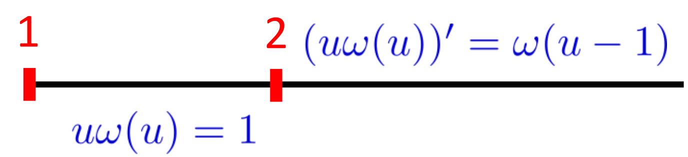
By positivity, for any $Y \geq y^{\delta}$,

$$S^{+}(x) = \sum_{\substack{n \leq x,\, (n,N)=1 \\ \lambda_f(n)>0}} \lambda_f(n) \log \frac{x}{n}$$

$$\geq \sum_{\substack{m<y^{\delta} \\ (m,N)=1}} \lambda_f(m) \sum_{\substack{\ell < x/m \text{ squarefree} \\ p|\ell \Rightarrow y^{\delta}<p\leq Y,\, p\nmid N}} \lambda_f(\ell) \log \frac{x/m}{\ell}$$

Note $\lambda_f(p)^2 = \lambda_f(p^2) + 1$

$\Rightarrow \lambda_f(p) \geq 1$ for $p \leq y^{\frac{1}{2}}$



Set $Y \leq \sqrt{y}$. Use $\displaystyle\sum_{\substack{n \leq x \\ p|n \Rightarrow p>z}} 1 = \omega\left(\frac{\log x}{\log z}\right)\frac{x}{\log z} + O\Big(\frac{z}{\log z} + \frac{x}{\log^2 z}\Big)$

$$\Rightarrow \sum_{\substack{n \leq x \\ p|n \Rightarrow z<p\leq Y}} 1 > \alpha\frac{x}{\log z} - \frac{x}{\log Y} + O\Big(\frac{x}{\log^2 x}\Big) \qquad \Big(\alpha > \frac{1}{7}\Big)$$

3.3' Iwaniec, Kohnen & Sengupta's method II (cont'd)

Consequently,

$$S^+(x) \geq \left(\frac{1}{7\delta} - 2\right) \sum_{\substack{m<y^\delta \\ (m,N)=1}} \frac{\lambda_f(m)}{m} \frac{x}{\log y}\left(1 + O\left(\frac{1}{\log y}\right)\right)$$

When $\frac{1}{7\delta} > 4, \qquad S(x) = S^+(x) + S^-(x) \gg \frac{x}{\log x}.$

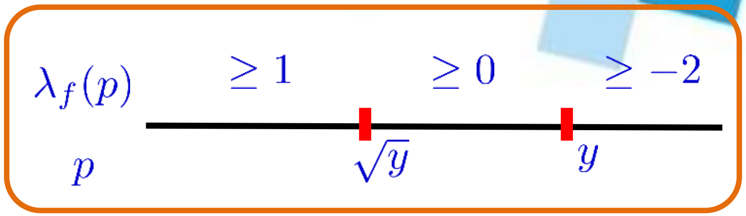
Hence $\frac{x}{\log x} \ll x^{\frac{1}{2}}(k^2N)^{\eta+\varepsilon}$ if $\delta := \frac{1}{29} < \frac{1}{28}.$

Recall $\lambda_f(n) \geq 0$ for $1 \leq n \leq y$ and $x = y^{1+\delta}.$

$$\Rightarrow \quad y \ll (k^2N)^{\frac{2\eta}{1+\delta}+\varepsilon}.$$

4.4 Kowalski, L., Soundararajan & Wu's method

Suppose $\lambda_f(n) \geq 0$ for $1 \leq n \leq y$.



Write $x = y^u$ where $1 < u \leq \frac{3}{2}$. Consider $S^\flat(x) = \sum_{\substack{n \leq x \\ (n,N)=1}}{}^{\flat} \lambda_f(n)$

Upper bound: $S^\flat(x) \ll x^{\frac{1}{2}+\varepsilon}(k^2N)^{\eta+\varepsilon}$.

Lower bound: Construct multiplicative $\mathrm{h}_y(n)$ with $\mathrm{h}_y(p^\nu) = 0$, $\nu \geq 2$

$$\mathrm{h}_y(p) := \begin{cases} -2 & \text{if } p > y, \\ 0 & \text{if } \sqrt{y} < p \leq y, \\ 1 & \text{if } p \leq \sqrt{y}. \end{cases}$$

3.4 Kowalski, L., Soundararajan & Wu's method (cont'd)

For any $y > 0$ and $\varepsilon > 0$,

$$\sum_{\substack{n \le y^u \\ (n,N)=1}} \mathrm{h}_y(n) = C_N \cdot y^u(\rho(2u) - 2\log u)\left\{1 + O(\frac{\log_2^2 y}{\log y})\right\}$$

uniformly for $1 \le u \le \frac{3}{2}$ and $y \ge N^{\frac{1}{3}}$ where $C_N = \prod_{p \nmid N}(1-p^{-2})^{-1}\frac{\phi(N)}{N}$ and $\rho(u)$ is the Dickman function – the unique continuous solution of the difference-differential equation

$$u\rho'(u) + \rho(u-1) = 0 \ (u > 1), \quad \rho(u) = 1 \ \ (0 < u \le 1).$$

Let κ be the solution to $\rho(2\kappa) = 2\log\kappa$. We have $\kappa > \frac{10}{9}$.

Then $\rho(2u) - 2\log u > 0 \ \ \forall\ u < \kappa$.

3.4 Kowalski, L., Soundararajan & Wu's method (cont'd)

Claim: $S^\flat(y^u) \geq \sum_{\substack{n \leq y^u \\ (n,N)=1}} \mathrm{h}_y(n)$ when $u < \kappa$.

Proof: Define $g_y(n)$ so that $\lambda_f = g_y * \mathrm{h}_y$ for squarefree n.

Thus $g_y(p) = \lambda_f(p) - \mathrm{h}_y(p)$, so $g_y(p) \geq 0$ $\forall$ prime p.

Then $S^\flat(x) = \sum_{d \leq y^u}^\flat g_y(d) \sum_{\ell \leq y^u/d}^\flat \mathrm{h}_y(\ell) \geq \sum_{\ell \leq y^u}^\flat \mathrm{h}_y(\ell)$.

3.4 Kowalski, L., Soundararajan & Wu's method (cont'd)

Take $u = \frac{10}{9}^{+} < \kappa$, then

$$\frac{y^u}{\log_2 N} \ll S^\flat(y^u) \ll y^{\frac{u}{2}+\varepsilon}(k^2N)^{\eta+\varepsilon}$$

$$\Rightarrow y \ll (k^2N)^{\frac{2\eta}{u}+\varepsilon}$$

Remark: Use $\lambda_f(p)^2 = \lambda_f(p^2) + 1$ to yield $\lambda_f(p) \geq 1$ for $p \leq \sqrt{y}$.

Deligne's bound $|\lambda_f(p)| \leq 2$ plays a crucial role in the positivity of g_y and hence in $S^\flat(y^u) \geq \sum_{\substack{n \leq y^u \\ (n,N)=1}} \mathrm{h}_y(n)$.

3.5 Matomaki's method

Let $m \in \mathbb{N}$.

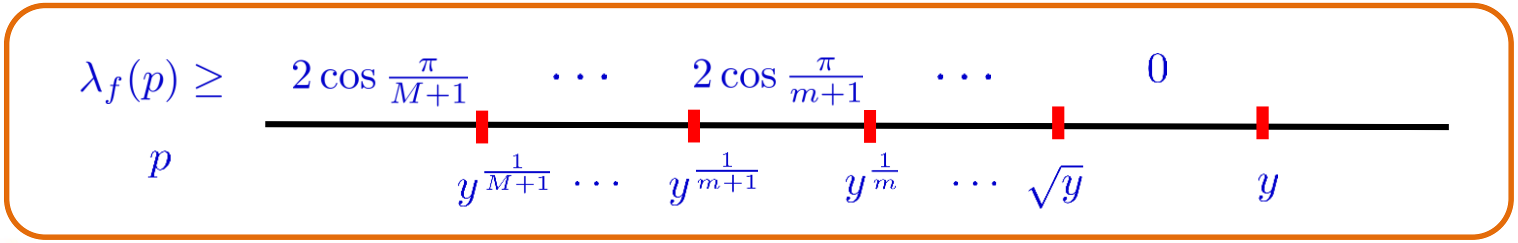
If $\lambda_f(p^j) \geq 0$ where $1 \leq j \leq m$ and $y^{\frac{1}{m+1}} \leq p < y^{\frac{1}{m}}$, then

$$\lambda_f(p) \geq 2\cos\tfrac{\pi}{m+1}.$$

Write $\lambda_f(p) = 2\cos\theta$ with $\theta \in [0, \pi]$. Then,

$$0 \leq \lambda_f(p^j) = \frac{\sin((j+1)\theta)}{\sin\theta} \quad \Rightarrow \quad \theta \leq \frac{\pi}{m+1}$$

Thus $\lambda_f(p) \geq 2\cos\tfrac{\pi}{m+1}$ for $y^{\frac{1}{m+1}} \leq p < y^{\frac{1}{m}}$, $p \nmid N$.



3.5 Matomaki's method (cont'd)

Let $M \geq 2$. Define the multiplicative function $h_y(n)$ by

$$h_y(p) := \begin{cases} -2 & \text{if } p > y, \\ 2\cos\frac{\pi}{m+1} & \text{if } y^{\frac{1}{m+1}} < p \leq y^{\frac{1}{m}},\ 1 \leq m < M, \\ 2\cos\frac{\pi}{M+1} & \text{if } p \leq y^{\frac{1}{M}}, \end{cases}$$

and $h_y(p^\nu) = 0$, $\nu \geq 2$ and $h_y(p) = 0 \ \ \forall\ p|N$

Let $\kappa > 0$, $N \in \mathbb{N}$. $\exists$ constant $C_\kappa > 0$, uniformly for

$x \geq \exp(C_\kappa \log\omega(N) + 3)^{e\kappa+2})$,

$$\sum_{\substack{n \leq x \\ (n,N)=1}}{}^{\flat} \kappa^{\omega(n)} = \frac{\Pi_{N,\kappa}}{\Gamma(\kappa)} x(\log x)^{\kappa-1}(1+o(1))$$

where $\Gamma(\cdot)$ Gamma function and $\Pi_{N,\kappa} = \left(\frac{\varphi(N)}{N}\right)^\kappa \prod_{p\nmid N}\left(1-\frac{1}{p}\right)^\kappa\left(1+\frac{\kappa}{p}\right) \gg (\log\log N)^{-\kappa}$.

3.5 Matomaki's method (cont'd)

Let $\chi_0 = 2\cos\frac{\pi}{M+1}$. Uniformly for $\exp(C_0(\log\omega(N)+3)^{e\chi_0+4}) \le y$ and $u \in [1, \frac{3}{2}]$,

$$\sum_{\substack{n\le y^u\\(n,N)=1}}{}^{\flat} h_y(n) = (\sigma_M(u)+o_M(1))\frac{\Pi_{N,\chi_0}}{\Gamma(\chi_0)}(\log y)^{\chi_0-1}y^u$$

where $C_0 = C_0(M)$ suitably large constant, $\sigma_M(u)$ is a continuous function.

In particular for $M = 100$, we have

u	$=$	$\frac{11}{9}$	$\frac{5}{4}$	$\frac{9}{7}$	$\frac{4}{3}$
$\sigma_{100}(u)$	$\ge$	0.0924	0.0718	0.0445	0.008

Set $\kappa = \frac{4}{3}$. Then $y^\kappa \ll (k^2N)^{\eta+\varepsilon}y^{\kappa/2+\varepsilon}$. Thus $y \ll (k^2N)^{\frac{2\eta}{\kappa}+\varepsilon}$.

4.1 Start-up

Let ϕ be a primitive maass cusp form of eigenvalue $\frac{1}{4}+\nu^2$ and level N.

Write $\lambda_\phi(n)$ for its nth Hecke eigenvalue.

Question: What is the size of n_- in terms of ν and N if $\lambda_\phi(n_-)$ is the first negative Hecke eigenvalue?

Suppose $\lambda_\phi(n) \geq 0$ for $1 \leq n \leq y$. Write $x = y^u$ where $1 \leq u \leq \frac{3}{2}$.

Consider $$S^\flat(x) = \sum_{\substack{n \leq x \\ (n,N)=1}}{}^\flat \lambda_\phi(n)$$

Upper bound: $S^\flat(x) \ll x^{\frac{1}{2}+\varepsilon}(|\nu|^2 N)^{\eta+\varepsilon}$. (Assume $|\nu| \geq 1$)

Iwaniec, Kohnen & Sengupta's method I applies: Take $x = y$, $x \ll (|\nu|^2 N)^{2\eta+\varepsilon}$.

4.2 Lower bound – Adapting Matomaki's method

(i) Auxiliary function for lower bound estimation:

Let $M \geq 2$. Define the multiplicative function $w_y(n)$ by

$$w_y(p) := \begin{cases} -|\lambda_\phi(p)| & \text{if } p > y, \\ 2\cos\frac{\pi}{m+1} & \text{if } y^{\frac{1}{m+1}} < p \leq y^{\frac{1}{m}},\ 1 \leq m < M, \\ 2\cos\frac{\pi}{M+1} & \text{if } p \leq y^{\frac{1}{M}}, \end{cases}$$

and $w_y(p^\nu) = 0$, $\nu \geq 2$ and $w_y(p) = 0 \ \ \forall\ p|N$

Then $g_y(p) = \lambda_\phi(p) - w_y(p) \geq 0$, and $\lambda_\phi = w_y * g_y$.

u_0 is the first zero of $\sum_{\substack{n \leq y^u \\ (n,N)=1}}^{\flat} w_y(n)$.

Consequently $S^\flat(y^u) \geq \sum_{\substack{n \leq y^u \\ (n,N)=1}}^{\flat} w_y(n)$ when $u < u_0$.

4.2 Lower bound – Adapting Matomaki's method (cont'd)

(ii) Connect $\sum_{\substack{n\le y^u\\(n,N)=1}}^{\flat} w_y(n)$ to $\sum_{\substack{n\le y^u\\(n,N)=1}}^{\flat} h_y(n)$:

$$\sum_{\substack{n\le y^u\\(n,N)=1}}^{\flat} w_y(n) = \sum_{\substack{n\le y^u\\P(n)\le y}}^{\flat} w_y(n) - \sum_{\substack{y<p\le y^u\\(p,N)=1}} |\lambda_\pi(p)| \sum_{\substack{n\le \frac{y^u}{p}\\(n,N)=1}}^{\flat} w_y(n)$$

$$= \sum_{\substack{n\le y^u\\(n,N)=1}}^{\flat} h_y(n) + 2\sum_{\substack{y<p\le y^u\\(p,N)=1}} \sum_{\substack{n\le \frac{y^u}{p}\\(n,N)=1}}^{\flat} h_y(n) - \sum_{\substack{y<p\le y^u\\(p,N)=1}} |\lambda_\phi(p)| \sum_{\substack{n\le \frac{y^u}{p}\\(n,N)=1}}^{\flat} h_y(n)$$

$$= \sum_{\substack{n\le y^u\\(n,N)=1}}^{\flat} h_y(n) - \underbrace{\sum_{\substack{y<p\le y^u\\(p,N)=1}} \left(|\lambda_\phi(p)| - 2\right) \sum_{\substack{n\le \frac{y^u}{p}\\(n,N)=1}}^{\flat} h_y(n)}_{S_2}$$

4.2 Lower bound – Adapting Matomaki's method (cont'd)

(iii) $S_2 = \sum_{\substack{y<p\le y^u \\ (p,N)=1}} (|\lambda_\phi(p)|-2) \sum_{\substack{n\le y^u/p \\ (n,N)=1}}^{\flat} h_y(n) \le ?$:

Write $S_2 = S_{2,1} + S_{2,2}$. Set $\log Z = o(1)(\log y)^{(\chi_0-1/8)/(\chi_0+7/8)}$.

(iii)' $S_{2,2} \le \sum_{\substack{y^u/Z<p\le y^u \\ (p,N)=1}} |\lambda_\phi(p)| \sum_{\substack{n\le y^u/p \\ (n,N)=1}}^{\flat} \chi_0^{\omega(n)}$

$$\le y^u \sum_{\substack{y^u/Z<p\le y^u \\ (p,N)=1}} \frac{|\lambda_\phi(p)|}{p} \sum_{n\le Z}^{\flat} \frac{\chi_0^{\omega(n)}}{n}$$

$$\le y^u \frac{(\log Z)^{\chi_0+7/8}}{(\log y^u)^{7/8}} \Bigg(\sum_{\substack{y^u/Z<p\le y^u \\ (p,N)=1}} \frac{\lambda_\phi(p)^8}{p} \Bigg)^{1/8} = o(1) y^u (\log y)^{\chi_0-1}$$

4.2 Lower bound – Adapting Matomaki's method (cont'd)

(iii)'' $S_{2,1} \leq \sum_{\substack{y<p\leq y^u/Z \\ (p,N)=1}} (|\lambda_\phi(p)|-2)_+ \sideset{}{^\flat}\sum_{\substack{n\leq y^u/p \\ (n,N)=1}} h_y(n)$

$$= (1+o(1))\frac{\Pi_{N,\chi_0}}{\Gamma(\chi_0)} y^u (\log y)^{\chi_0-1} \underbrace{\sum_{\substack{y<p\leq y^u/Z \\ (p,N)=1}} \frac{(|\lambda_\phi(p)|-2)_+}{p}\sigma_M\Big(u-\frac{\log p}{\log y}\Big)}_{\Sigma_2}$$

Note:

- $\sideset{}{^\flat}\sum_{\substack{n\leq y^u/p \\ (n,N)=1}} h_y(n) = \Big(\sigma_M(u-\frac{\log p}{\log y})+o_M(1)\Big)\frac{\Pi_{N,\chi_0}}{\Gamma(\chi_0)}(\log y)^{\chi_0-1}\frac{y^u}{p}$
- $0 \leq \sigma_M(u) \leq u^{\chi_0-1}$

4.2 Lower bound – Adapting Matomaki's method (cont'd)

(iv) Bounding Σ_2

$$\sum_{\substack{y<p\le y^u/Z\\(p,N)=1}} \frac{(|\lambda_\phi(p)|-2)_+}{p}\sigma_M\Big(u-\frac{\log p}{\log y}\Big)$$

$$\le \Bigg(\sum_{\substack{y<p\le y^u\\(p,N)=1}} \frac{(|\lambda_\phi(p)|-2)_+^8}{p}\Bigg)^{\frac18}\Bigg(\sum_{y<p\le y^u}\sigma_M\Big(u-\frac{\log p}{\log y}\Big)^{\frac87}\frac1p\Bigg)^{\frac78}$$

$L(s,\mathrm{sym}^j\phi)$, $1\le j\le 4$, is automorphic.

$L(s,\mathrm{sym}^j\phi\times\mathrm{sym}^j\phi)$ has the standard properties.

4.2 Lower bound – Adapting Matomaki's method (cont'd)

(v) Connect $\displaystyle\sum_{\substack{y<p\le y^u \\ (p,N)=1}} \frac{(|\lambda_\phi(p)|-2)_+^8}{p}$ to $\displaystyle\sum_{\substack{y<p\le y^u \\ p\nmid N}} \lambda_{\mathrm{sym}^4\phi\times\mathrm{sym}^4\phi}(p)\frac{1}{p}$

$$\lambda_\phi(p)^{2R} = \sum_{j=0}^{R} a_{R,j}\lambda_{\mathrm{sym}^{2(R-j)}\phi}(p)$$

$$\lambda_{\mathrm{sym}^{2r}\phi}(p) = \lambda_{\mathrm{sym}^r\phi\times\mathrm{sym}^r\phi}(p) - \lambda_{\mathrm{sym}^{r-1}\phi\times\mathrm{sym}^{r-1}\phi}(p)$$

$$\mathrm{sym}^r\pi_p \otimes \mathrm{sym}^r\pi_p = \bigoplus_{j=0}^{r} \mathrm{sym}^{2j}\pi_p$$

We have no good control on the conductors of $L(s,\mathrm{sym}^r\phi)$ and $L(s,\mathrm{sym}^r\phi\times\mathrm{sym}^r\phi)$ for large r.

4.2 Lower bound – Adapting Matomaki's method (cont'd)

Note $(x-2)_+^8 \le x^8 - 6x^6 + \frac{3^7}{16} + \frac{1}{2^8}$

$$\lambda_\phi(p)^8 - 6\lambda_\phi(p)^6 = \sum_{j=0}^{4} b_{4,j}\lambda_{\mathrm{sym}^{4-j}\phi\times\mathrm{sym}^{4-j}\phi}(p)$$

where $b_{4,0}=1$, $b_{4,1}=0$, $b_{4,2}=-11$, $b_{4,3}=-16$, $b_{4,4}=10$.

$$\sum_{\substack{y<p\le y^u\\(p,N)=1}} \frac{(|\lambda_\phi(p)|-2)_+^8}{p} \le \sum_{\substack{y<p\le y^u\\p\nmid N}} \lambda_{\mathrm{sym}^4\phi\times\mathrm{sym}^4\phi}(p)\frac{1}{p} + B_8 \sum_{y<p\le y^u} p^{-1}$$

$$B_8 = \frac{3^7}{16} + \frac{1}{2^8} + 10$$

4.2 Lower bound – Adapting Matomaki's method (cont'd)

(vi) Relate $\displaystyle\sum_{\substack{y<p\le y^u\\ p\nmid N}} \frac{\lambda_{\mathrm{sym}^4\phi\times\mathrm{sym}^4\phi}(p)}{p}$ to $L(s, \mathrm{sym}^4\phi \times \mathrm{sym}^4\phi)$

$$\text{Rankin's trick:} \quad \le \frac{y^{\delta u}}{\log y} \underbrace{\sum_p \lambda_{\mathrm{sym}^4\phi\times\mathrm{sym}^4\phi}(p) \frac{\log p}{p^{1+\delta}}}_{\Vert}$$

$$-\frac{L'}{L}(1+\delta, \mathrm{sym}^4\phi \times \mathrm{sym}^4\phi) + O(1)$$

$$\le \frac{y^{\delta u}}{\log y} \left\{ \delta^{-1} + \tfrac{1}{2} \log \mathcal{Q}_{\mathrm{sym}^4\phi\times\mathrm{sym}^4\phi} + O(1) \right\}$$

Remark we shall take $y = (|\nu|^2 N)^?$. Want $\log \mathcal{Q} \le \boxed{?} \log(|\nu|^2 N)$.

4.2 Lower bound – Adapting Matomaki's method (cont'd)

(vii) Evaluate $\boxed{?}$ in $\log \mathcal{Q} \leq \boxed{?} \log(|\nu|^2 N)$: $\boxed{?} = 108$

Bushnell & Henniart: Let q_π, $q_{\pi'}$ be the conductors of $L(s,\pi)$, $L(s,\pi')$.

The conductor $q_{\pi\times\pi'}$ of $L(s,\pi\times\pi')$ is $\leq \dfrac{q_\pi^{d'} q_{\pi'}^{d}}{(q_\pi, q_{\pi'})}$.

$d = \deg L(s,\pi)$
$d' = \deg L(s,\pi')$

Hence $\mathcal{Q}_{\mathrm{sym}^4\phi\times\mathrm{sym}^4\phi} \leq q_{\mathrm{sym}^4\phi}^9 \leq (|\nu|^2 N)^{108}$.

$$L(s,\phi\times\phi) = \zeta(s)L(s,\mathrm{sym}^2\phi)$$
$$L(s,\mathrm{sym}^2\phi\times\mathrm{sym}^2\phi) = \zeta(s)L(s,\mathrm{sym}^2\phi)L(s,\mathrm{sym}^4\phi)$$

$q_{\mathrm{sym}^2\phi} \leq (|\nu|^2 N)^3$ $\qquad$ $q_{\mathrm{sym}^4\phi} \leq q_{\mathrm{sym}^2\phi}^4 \leq (|\nu|^2 N)^{12}$

4.2 Lower bound – Adapting Matomaki's method (cont'd)

(viii) Round up :

By (v)-(vii) :

$$\sum_{\substack{y<p\leq y^u \\ (p,N)=1}} \frac{(|\lambda_\phi(p)|-2)_+^8}{p} \leq 131u + 138\log u$$

With (iv) :

$$\sum_{\substack{y<p\leq y^u/Z \\ (p,N)=1}} \frac{(|\lambda_\phi(p)|-2)_+}{p}\sigma_M\Big(u-\frac{\log p}{\log y}\Big) \leq (131u+138\log u)^{\frac{1}{8}} \times\Big(\int_1^u (u-t)^{\frac{8}{7}(\chi_0-1)}\,\frac{dt}{t}\Big)^{\frac{7}{8}}$$

Recall (iii) :

$$S_2 \leq (1+o(1))(131u+138\log u)^{\frac{1}{8}}\Big(\int_1^u (u-t)^{\frac{8}{7}(\chi_0-1)}\,\frac{dt}{t}\Big)^{\frac{7}{8}} \times\frac{\Pi_{N,\chi_0}}{\Gamma(\chi_0)}y^u(\log y)^{\chi_0-1}.$$

4.2 Lower bound – Adapting Matomaki's method (cont'd)

With (i)-(ii) : $$S^\flat(y^u) \ge \sum_{\substack{n\le y^u\\(n,N)=1}}{}^{\flat} h_y(n) - S_2$$

$$\ge \left(\sigma_M(u) - \ *\,*\,*\right)\frac{\Pi_{N,\chi_0}}{\Gamma(\chi_0)}\, y^u(\log y)^{\chi_0-1}$$

u	$=$	$\frac{11}{9}$	$\frac{5}{4}$	$\frac{9}{7}$	$\frac{4}{3}$
$\sigma_{100}(u)$	$\ge$	0.0924	0.0718	0.0445	0.008

$$*** \;=\; (131u + 138\log u)^{\frac{1}{8}} \times \Big(\int_1^u (u-t)^{\frac{8}{7}(\chi_0-1)}\,\frac{dt}{t}\Big)^{\frac{7}{8}}$$

Take $u = \frac{5}{4}^+$, then $\sigma_{100}(u) - \ *** > 2.5\times 10^{-3}$.

$$y^u(\log y)^{\chi_0-1} \ll S^\flat(y^u) \ll y^{\frac{u}{2}+\varepsilon}(|\nu|^2N)^{\eta+\varepsilon}$$

$$\Rightarrow y \ll (|\nu|^2N)^{\frac{8\eta}{5}-\varepsilon}$$

Hence $n_- \ll (|\nu|^2N)^{\frac{2}{5}}$

Thank you