A symplectic restriction problem

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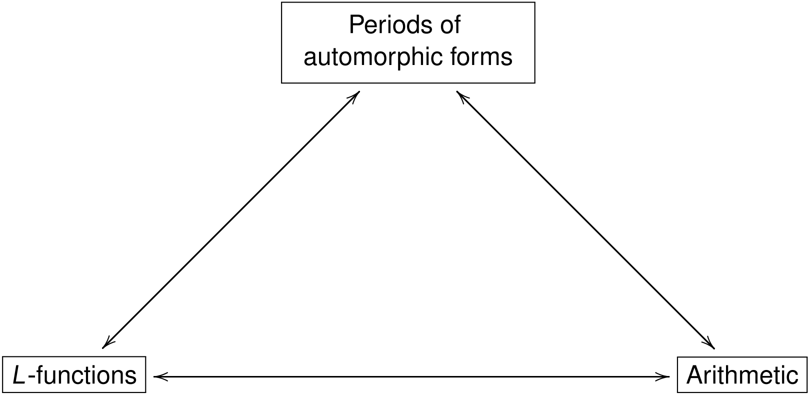
Automorphic Forms in Budapest


Periods of
automorphic forms
L-functions
Arithmetic

Examples

Example 1: (Riemann)

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \frac{(2\pi)^{s/2}}{2\Gamma(s/2)} \int_0^{\infty} \big(\theta(iy) - 1\big) y^{s/2} \frac{dy}{y}$$

$$\theta(z) = \sum_{n \in \mathbb{Z}} e(n^2 z)$$

analytic properties of $\zeta \longleftrightarrow$ analytic properties of θ.

Example 2: (Hecke) Let $f \in S_k$ be a Hecke eigenform. We have

$$L(f, s)\Gamma\Big(s + \frac{k-1}{2}\Big)(2\pi)^{-s} = \int_0^{\infty} f(iy) y^s \frac{dy}{y}$$

... continued

Example 3: Let $X = \Gamma\backslash G/K \supseteq X_0$. Let Φ, ϕ_0 be constituents ("harmonics") in spectral decomposition of $L^2(X)$ resp. $L^2(X_0)$. Is there a connection

$$\Big|\int_{X_0} \Phi(y)\phi_0(y)dy\Big|^2 \quad \longleftrightarrow \quad L-\text{value ?}$$

$\longrightarrow$ Gross-Prasad conjecture

Application: Apply Parseval for the restriction norm:

$$\int_{X_0} |\Phi(y)|^2 dy = \int_{\widehat{X_0}} \Big|\int_{X_0} \Phi(y)\phi_0(y)dy\Big|^2 d\mu(\phi_0)$$

Examples 1 and 2: $G = \mathrm{SL}_2(\mathbb{R})$, $G_0 = \left\{\begin{pmatrix} * & \\ & * \end{pmatrix} \subseteq \mathrm{SL}_2(\mathbb{R})\right\} \cong \mathbb{R}^*$

Siegel modular forms

Let

$$G = \mathrm{Sp}_4(\mathbb{R}), \quad G_0 = \left\{ \begin{pmatrix} \boxed{*} & \\ & \boxed{*} \end{pmatrix} \subseteq \mathrm{Sp}_4(\mathbb{R}) \right\} \cong \mathrm{GL}_2(\mathbb{R}).$$

The Siegel upper half space is

$$\mathbb{H}^{(2)} = \left\{ Z = X + iY \in \mathrm{Sym}_2(\mathbb{C}) \mid Y > 0 \right\}.$$

The group G acts on $\mathbb{H}^{(2)}$ by Möbius transforms

$$MZ = (AZ + B)(CZ + D)^{-1}, \quad M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in G.$$

We define

$$S_k^{(2)} = \{\text{Siegel modular forms } F \text{ for } \mathrm{Sp}_4(\mathbb{Z}) \text{ of weight } k\}.$$

Fourier coefficients of F are indexed by integral positive binary quadratic forms Q.

Spectral decomposition

We write $X = \mathrm{Sp}_4(\mathbb{Z})\backslash\mathbb{H}^{(2)}$ with $\dim X = 6$ and

$$X_0 := \mathrm{SL}_2(\mathbb{Z})\backslash\mathrm{Pos}_2(\mathbb{R}) \quad\cong\quad \mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}\times\mathbb{R}_{>0} =: X_0^*\times\mathbb{R}_{>0}$$

$$Y = \frac{\sqrt{r}}{y}\begin{pmatrix} 1 & -x \\ -x & x^2+y^2\end{pmatrix} \longleftrightarrow (x+iy, r)$$

with $\dim X_0 = 3$. Harmonics on X_0 are Maaß forms and powers.

Is

$$\int_{X_0} F(iY)\, u(z) r^s \frac{dx\,dy}{y^2}\frac{dr}{r}$$

an L-value?

Yes, it is a Koecher-Maaß series.

The Koecher-Maaß series

For a Siegel modular form F with Fourier coefficients $a(Q)$, $Q = (a, b, c)$ an integral positive binary quadratic form, and a Maaß form u define

$$L(F \times u, s) = \sum_{Q \bmod \mathrm{SL}_2(\mathbb{Z})} \frac{a(Q)u(H_Q)}{(\det Q)^s}$$

where

$$H_Q = \frac{-b + i\sqrt{|b^2 - 4ac|}}{2a} \in \mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}$$

is the Heegner point associated with Q.

This L-function has

- a functional equation,
- but no Euler product.

The restriction norm

Let F be a Siegel modular form of weight k. Then

$$\mathcal{N}(F) := \frac{\text{vol}(X)}{\text{vol}(X_0^*)} \frac{1}{\|F\|_2^2} \int_{X_0} |F(iY)|^2 (\det Y)^k \frac{dY}{(\det Y)^{3/2}}$$

$$= \int_{\widehat{X_0^*}} \int_{\mathbb{R}} |L(F \times u, 1/2 + it)|^2 |G(F \times u, 1/2 + it)|^2 dt\, du$$

where $G(s, F \times u)$ are suitable gamma factors.

For large k, the gamma factors decay at $t, t_u \ll k^{1/2}$, so this is an average of size $k^{3/2}$, and the L-function squared has conductor k^8.

Lindelöf hypothesis: $\mathcal{N}(F) \ll k^{\varepsilon}$.

In absence of an Euler product it is not clear if the Lindelöf hypothesis is true.

But we expect more...

The mass equidistribution conjecture

Let $f \in S_k$ be a classical Hecke eigenform of weight k, h a test function.

Theorem (Holowinsky-Soundararajan 2009)

$$\int_{\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}} \frac{|f(z)|^2 y^k}{\|f\|_2^2} h(z) \frac{dx\,dy}{y^2} \overset{k\to\infty}{\longrightarrow} \frac{1}{\mathrm{vol}(\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H})} \int_{\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}} h(z) \frac{dx\,dy}{y^2}.$$

Is this true on thin subsets?

Is this true in higher rank?

Thin symplectic mass equidistribution conjecture

$$\boxed{\mathcal{N}(F) = 4 \log k + O(1)}, \quad k \to \infty$$

Why? Recall

$$X_0 := \mathrm{SL}_2(\mathbb{Z})\backslash \mathrm{Pos}_2(\mathbb{R}) \quad \cong \quad \mathrm{SL}_2(\mathbb{Z})\backslash \mathbb{H} \times \mathbb{R}_{>0} =: X_0^* \times \mathbb{R}_{>0}$$

$$Y = \frac{\sqrt{r}}{y}\begin{pmatrix} 1 & -x \\ -x & x^2 + y^2 \end{pmatrix} \longleftrightarrow (x + iy, r)$$

The space X_0 has infinite volume, but a Siegel cusp form of weight k decays quickly as soon as $\lambda_{\min}(Y) \ll 1/k$, $\lambda_{\max}(Y) \gg k$. We obtain roughly an effective volume

$$\int_{1/k^2}^{k^2} \frac{dr}{r} = 4 \log k.$$

Saito-Kurokawa lifts

$$M^+_{k-1/2} \cong S_{2k-2} \quad \overset{SK}{\hookrightarrow} \quad S_k^{(2)}$$

$$\dim \asymp k \qquad\qquad \dim \asymp k^3$$

$$a(\det Q) \qquad\qquad a(Q) \text{ depends only on } \det Q$$

Recall

$$L(F \times u, s) = \sum_{Q \bmod \mathrm{SL}_2(\mathbb{Z})} \frac{a(Q)u(H_Q)}{(\det Q)^s} = \sum_{D<0} \frac{a(|D|)P(D,u)}{|4D|^s}$$

where $P(D, u)$ is the Heegner period

$$P(D,u) = \sum_{z \in H_D} u(z),$$

$$H_D = \left\{ \frac{-b + i\sqrt{|D|}}{2a} \mid ax^2 + bxy + cy^2 \text{ of disc } D \right\} / \mathrm{SL}_2(\mathbb{Z})$$

The main result

Theorem 1: (B-Corbett 2019)
Let W be a smooth weight function with support in $[1,2]$, $\omega = \int W(x)x\,dx$. Then

$$\mathcal{N}_{\text{av}}(K) = \frac{1}{\omega}\frac{12}{K^2}\sum_{k\in 2\mathbb{N}} W\Big(\frac{k}{K}\Big)\sum_{f\in B_{2k-2}} \mathcal{N}(SK(f)) = 4\log K + O(1).$$

This is an averaged version of

- the thin symplectic mass equidistribution conjecture and
- (a strong form of) the Lindelöf hypothesis for $L(F\times u, s)$.

Steps of the proof...

- Apply the Parseval period formula and an approximate functional equation:

$$\sum_{k} \sum_{f \in B_{2k-2}} \sum_{j} \int_{\mathbb{R}} |L(SK(f) \times u_j, 1/2 + it)|^2 |G(\cdots)|^2 dt.$$

- separate treatment of the constant function u_0 where $P(D, u_0) = H(D)$ is the Hurwitz class number

Interlude I: Voronoi formula for Hurwitz class numbers

Let $c \in \mathbb{N}$, $4 \mid c$, $(a, c) = 1$. Let ϕ be a smooth function with compact support in $(0, \infty)$. Then

$$\sum_{D<0} \frac{H(D)}{|D|^{1/4}} e\Big(\frac{a|D|}{c}\Big)\phi(|D|)$$

$$= \frac{\sqrt{2}}{c^{1/2}}\Big(\frac{-c}{a}\Big)\bar{\epsilon}_a e\Big(\frac{3}{8}\Big)\Bigg[\sum_{D<0}\frac{H(D)}{|D|^{1/2}} e\Big(-\frac{\bar{a}|D|}{c}\Big)\int_0^\infty \sin\Big(\frac{4\pi\sqrt{|D|t}}{c}\Big)\phi(t)\frac{dt}{t^{1/4}}$$

$$+ \frac{1}{2}\sum_{n=\square} e\Big(\frac{\bar{a}n}{c}\Big)\int_0^\infty \exp\Big(-\frac{4\pi\sqrt{nt}}{c}\Big)\phi(t)\frac{dt}{t^{1/4}}$$

$$+ \int_0^\infty \phi(x)\Big(\frac{1}{4x^{1/4}} - \frac{\pi}{3c}x^{1/4}\Big)dx\Bigg].$$

Steps of the proof... continued

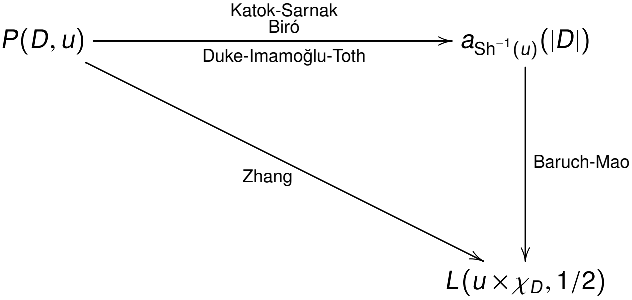
- Apply the Parseval period formula and an approximate functional equation.

$$\sum_k \sum_{f \in B_{2k-2}} \sum_j \int_{\mathbb{R}} |L(SK(f) \times u_j, 1/2 + it)|^2 |G(\cdots)|^2 dt$$

- separate treatment of the constant function u_0 where $P(D, u_0) = H(D)$ is the Hurwitz class number
- sum over $f \in B_{2k-2}$ by a half-integral weight Kohnen-Petersson formula.
- the diagonal term: by Waldspurger/Zhang we have

$$|P(D, u_j)|^2 = |D|^{1/2} \frac{L(u_j, 1/2) L(u_j \times \chi_D, 1/2)}{4L(\text{sym}^2 u_j, 1)}$$

Interlude II: a commutative diagram



Steps of the proof... continued

- Apply the Parseval period formula and an approximate functional equation.

$$\sum_k \sum_{f \in B_{2k-2}} \sum_j \int_{\mathbb{R}} |L(SK(f) \times u_j, 1/2 + it)|^2 |G(\cdots)|^2 dt$$

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- sum over u_j with the Kuznetsov formula
- the diagonal-diagonal term: explicit evaluation

Interlude III: an Euler product

The various integral and half-integral weight Hecke relations lead to the following Euler product

$$\prod_p \Big(1 - \frac{1}{p^2} - \frac{1}{p^3} + \frac{1}{p^4}\Big)$$

$$\times \sum_{(d\delta\nu,m)=1} \sum_{(f,\delta)=1} \sum_{\substack{d_1 | fd \\ d_2 | f\nu}} \sum_{d_3 | (\frac{fd}{d_1}, \frac{f\nu}{d_2})} \sum_{r_1 r_2 = \frac{f^2 d\nu}{d_1 d_2 d_3^2}} \frac{\mu(d\delta\nu)\mu^2(m)\mu(d_1)\mu(d_2)d_3}{(d\nu)^4 \delta^2 f^3 m^3 \mathrm{rad}(\delta d_1 d_2 r_2)}$$

$$\times \prod_{p | \delta d_1 d_2 r_2} \Big(1 + \frac{1}{p} - \frac{1}{p^3}\Big)^{-1}$$

where $\mathrm{rad}(n)$ is the squarefree kernel. It equals

$$\zeta(4)^{-1}$$

Steps of the proof... continued

- Apply the Parseval period formula and an approximate functional equation.

$$\sum_{k} \sum_{f \in B_{2k-2}} \sum_{j} \int_{\mathbb{R}} |L(SK(f) \times u_j, 1/2 + it)|^2 |G(\cdots)|^2 dt$$

- separate treatment of the constant function u_0 where $P(D, u_0) = H(D)$ is the Hurwitz class number
- sum over $f \in B_{2k-2}$ by a half-integral weight Kohnen-Petersson formula.
- the diagonal term: by Waldspurger/Zhang we have

$$|P(D, u_j)|^2 = |D|^{1/2} \frac{L(u_j, 1/2) L(u_j \times \chi_D, 1/2)}{4L(\text{sym}^2 u_j, 1)}$$

- sum over u_j with the Kuznetsov formula
- the diagonal-diagonal term: explicit evaluation
- the diagonal-offdiagonal term: multiple Poisson summation and Heath-Brown's large sieve for quadratic characters.

The off-diagonal term

- sum over $k \asymp K$
- interpret $P(D, u_j)$ as metaplectic Fourier coefficients (by Katok-Sarnak) and apply half-integral Voronoi summation
- heart of the proof: need cancellation in

$$\sum_u \underbrace{P(D_1, u)P(D_2, u)}_{\text{product of 4 half-integral weight Fourier coefficients}} h(t_u)$$

"*Kuznetsov formula for toric Fourier coefficients*"

A trace formula for pairs of Heegner periods

Theorem 2: Let D_1, D_2 be two negative fundamental discriminants. Let

$$F(x,t) = J_{it}(x)\cos(\pi/4 - \pi it/2) - J_{-it}(x)\cos(\pi/4 + \pi it/2),$$

$$W_t(n) = \frac{1}{2\pi i}\int_{(2)} \frac{\Gamma(\frac{1}{2}(\frac{1}{2}+s+2it))\Gamma(\frac{1}{2}(\frac{1}{2}+s-2it))}{\Gamma(\frac{1}{4}+it)\Gamma(\frac{1}{4}-it)\pi^s} e^{s^2} n^{-s}\frac{ds}{s}.$$

Then

$$\boxed{\frac{1}{|D_1D_2|^{1/4}}\int_{\hat{X}_0^*} P(D_1;u)\overline{P(D_2;u)}h(t_u)du}$$

$$= \frac{3}{\pi}\frac{H(D_1)H(D_2)}{|D_1D_2|^{1/4}}h(i/2) \qquad \longleftarrow \text{ constant function}$$

$$+ \int_{-\infty}^{\infty}\left|\frac{D_1D_2}{4}\right|^{it/2}\frac{\Gamma(-\frac{1}{4}+\frac{it}{2})e^{(1/2-it)^2}}{\sqrt{8\pi}\Gamma(\frac{1}{4}+\frac{it}{2})}\frac{L(\chi_{D_1},1/2+it)L(\chi_{D_2},1/2+it)}{\zeta(1+2it)}h(t)\frac{dt}{4\pi} \qquad \leftarrow \text{ polar term}$$

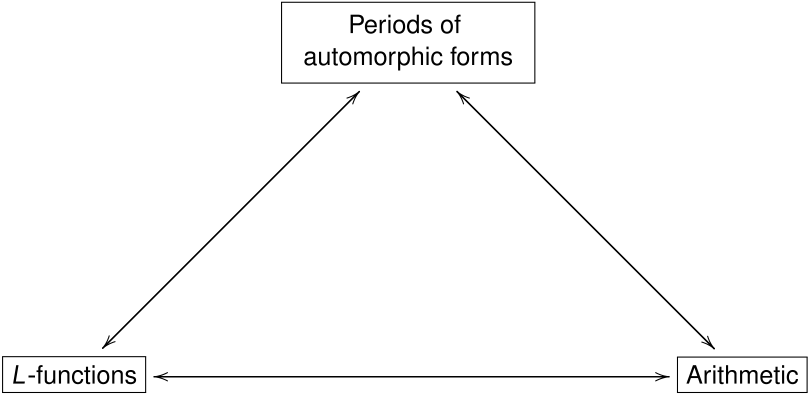
$$+ \delta_{D_1=D_2}\sum_m \frac{\chi_{D_1}(m)}{m}\int_{-\infty}^{\infty} W_t(m)h(t)t\tanh(\pi t)\frac{dt}{4\pi^2} \qquad \longleftarrow \text{ diagonal term}$$

$$+ e(3/8)\sum_{n,c,m}\frac{K^+_{3/2}(|D_1|n^2,|D_2|,c)\chi_{D_1}(m)}{n^{1/2}cm}\int_{-\infty}^{\infty}\frac{F(4\pi n\sqrt{|D_1D_2|}/c,t)}{\cosh(\pi t)}h(t)W_t(nm)t\frac{dt}{\pi}.$$

Idea of proof

We transform $P(D_1, u)P(D_2, u)$ as follows:

- Katok-Sarnak: $a_v(|D_1|)a_v(1)a_v(|D_2|)a_v(1), \quad v = \mathrm{Sh}^{-1}(u)$
- Waldspurger: $a_v(|D_1|)a_v(|D_2|)L(u, 1/2)$
- approximate functional equation $a_v(|D_1|)a_v(|D_2|)\sum_n \lambda(n)n^{-1/2}$
- Shimura: $\sum_n n^{-1/2}a_v(|D_1|n^2)a_v(|D_2|)$
- metaplectic Kuznetsov formula: $\sum_{n,c} K(|D_1|n^2, |D_2|, c)$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