Spectral decomposition formula and moments of symmetric square L-functions

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For $\Re s > 1$, $n \in \mathbf{Z}$

$$\mathscr{L}_n(s) = \frac{\zeta(2s)}{\zeta(s)} \sum_{q=1}^{\infty} \frac{a_q(n)}{q^s},$$

where

$$a_q(n) := \#\{x \pmod{2q} : x^2 \equiv n \pmod{4q}\}.$$

Theorem (Zagier, 1976)

The function $\mathscr{L}_n(s)$ has a meromorphic continuation to the whole complex plane. The completed L-series

$$\mathscr{L}_n^*(s) = (\pi/|n|)^{-s/2}\Gamma(s/2 + 1/4 - \operatorname{sgn} n/4)\mathscr{L}_n(s)$$

satisfies the functional equation

$$\mathscr{L}_n^*(s) = \mathscr{L}_n^*(1-s).$$

Cohen's numbers

For non negative integers N and $r \geq 1$, set

$$H(r,N) := \mathscr{L}_{(-1)^r N}(1-r).$$

The values of the function $H(r,N)$ are always rational;

For fixed r, the denominator of $H(r,N)$ is bounded;

$H(1,N) = H(N)$ is the class number of quadratic forms of discriminant $-N$.

Theorem (Cohen, 1975)

For $r \geq 2$

$$\sum_{N \geq 0} H(r,N) \exp(2\pi i N z)$$

is a modular form of weight $r+1/2$ on $\Gamma_0(4)$.

Properties

- $\mathscr{L}_n(s) = 0$ if n is not a discriminant ($n \equiv 2, 3 \pmod 4$);
- $\mathscr{L}_0(s) = \zeta(2s-1), \quad \mathscr{L}_1(s) = \zeta(s)$;
- If n is a fundamental discriminant

$$\mathscr{L}_n(s) = L(s, \chi_n),$$

where $\chi_n = \left(\frac{n}{*}\right)$ is the Kronecker symbol.

- If $n = Dl^2$ with D fundamental discriminant, then

$$\mathscr{L}_n(s) = l^{1/2-s} T_l^{(D)}(s) L(s, \chi_D),$$

$$T_l^{(D)}(s) = \sum_{l_1 l_2 = l} \chi_D(l_1) \frac{\mu(l_1)}{\sqrt{l_1}} \tau_s(l_2), \quad \tau_s(k) = k^{s-1/2} \sum_{a|k} a^{1-2s}.$$

Averages of Zagier L-series

For a suitable test function ω, consider the following averages:

$$\sum_{n=1}^{\infty} \omega(n) \mathscr{L}_{n^2-4l^2}(s), \tag{1}$$

$$\sum_{l=1}^{\infty} \omega(l) \mathscr{L}_{n^2-4l^2}(s). \tag{2}$$

Relation to Kloosterman sums:

$$\mathscr{L}_{n^2-4l^2}(s) = \zeta(2s) \sum_{q=1}^{\infty} \frac{1}{q^{1+s}} \sum_{c \pmod q} S(l^2, c^2; q) e\left(\frac{nc}{q}\right).$$

Kuznetsov formula for counting prime geodesics

Let $\Gamma = \mathbb{PSL}_2(Z)$. Consider the function

$$\Psi_\Gamma(x) = \sum_{NP \leq x} \Lambda(P),$$

where $\Lambda(P) = \log NP_0$ if $\{P\}$ is a power of a primitive hyperbolic class $\{P_0\}$.

Theorem (Kuznetsov, 1979)

Let $X := \sqrt{x} + 1/\sqrt{x}$. *Then*

$$\Psi_\Gamma(x) = 2 \sum_{n \leq X} \sqrt{n^2 - 4}\,\mathscr{L}_{n^2-4}(1).$$

Prime geodesic theorem

- The Kuznetsov formula was used by Bykovskii (1994) to establish the prime geodesic theorem in short intervals
- and by Soundararajan-Young (2013) to prove that

$$\Psi_\Gamma(x) = x + O(x^{2/3+\theta/6+\epsilon}), \quad \theta = 1/6.$$

- Wu-Zábrádi (2019) generalized the Kuznetsov formula for any number field and for any congruence subgroup.
- The method of Soundararajan-Young was extended to the case of Gaussian integers by Balog-Biró-Cherubini-Laaksonen (2019).

Prime Geodesic Theorem

Theorem (B.-Frolenkov-Risager, 2020)

Assume that for some $\alpha > 0$ *the following asymptotic formula holds*

$$\sum_{2<n\leq X} \mathscr{L}_{n^2-4}(1/2+it) = \int_2^X m_t(u)du + O(X^{\alpha+\epsilon})$$

uniformly for $|t| \leq X^{\epsilon}$. *Then*

$$\Psi_\Gamma(x) = x + O(x^{1/2+\alpha/4+\epsilon}).$$

Symmetric square L-functions

Let H_k be an orthonormal basis for the space of holomorphic cusp forms of weight k with respect to the full modular group. For $f \in H_k$, let $\rho_f(n)$ denote the n^{th} Fourier coefficient. For $\Re s > 1$ the associated symmetric square L-function is defined by

$$L(\mathrm{sym}^2 f, s) = \zeta(2s) \sum_{n=1}^{\infty} \frac{\rho_f(n^2)}{n^s}$$

and admits an analytic continuation to the entire complex plane.

Dual moments:

$$\sum_{f \in H_k} \rho_f(l^2) L(\mathrm{sym}^2 f, s) \longleftrightarrow \sum_{2<n<X} \mathscr{L}_{n^2-4l^2}(s).$$

- We multiply by $\zeta(2s)l^{-s}$ the following explicit formula

$$\sum_{f\in H_k}\rho_f(l^2)L(\mathrm{sym}^2 f,s)=\text{Main terms }+$$

$$\sum_{1\le n<2l}\mathscr{L}_{n^2-4l^2}(s)\Phi_k\left(\frac{n^2}{4l^2}\right)+\sum_{n>2l}\mathscr{L}_{n^2-4l^2}(s)\Psi_k\left(\frac{4l^2}{n^2}\right).$$

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- LHS: $\sum_{f\in H_k}L(\mathrm{sym}^2 f,s)^2$.
- RHS: expressions of the form

$$\sum_{l=1}^{\infty}\sum_{n=1}^{\infty}\mathscr{L}_{n^2-4l^2}(s)f(n,l;s).$$

Notation

- Let $H_k(N,\chi_4)$ be an orthonormal basis of holomorphic cusp forms of weight $k \equiv 1 \pmod 2$, level N and nebentypus χ_4.
- Let $H(N,\chi_4)$ be an orthonormal basis of the space of Maass cusp forms. The function $f \in H(N,\chi_4)$ is an eigenfunction of the Laplace-Beltrami operator with eigenvalue $1/4 + t_f^2$.
- For $f \in H_k(N,\chi_4)$ or $f \in H(N,\chi_4)$, we define

$$L(s,\mathrm{sym}^2 f_{\mathfrak{a}}) = \zeta^{(N)}(2s)\sum_{l=1}^{\infty}\frac{\rho_{f_{\mathfrak{a}}}(l^2)}{l^s}, \quad \Re s > 1,$$

where $\rho_{f_{\mathfrak{a}}}(n)$ is the nth coefficient in the Fourier expansion of f around a cusp $\mathfrak{a}$.

Bessel integral transforms

Let $\omega \in C^{\infty}$ be a function of compact support on $[a_1, a_2]$.

Consider the function

$$\psi(x) = \frac{2}{\sqrt{\pi}} \left(\frac{x}{n}\right)^s \int_0^{\infty} \omega(y) \cos\left(\frac{2xy}{n}\right) dy,$$

and the following integral transforms

$$\psi_H(k) := 4 i^k \int_0^{\infty} J_{k-1}(x) \psi(x) \frac{dx}{x},$$

$$\psi_D(t) := \frac{2\pi i t}{\sinh(\pi t)} \int_0^{\infty} (J_{2it}(x) + J_{-2it}(x)) \psi(x) \frac{dx}{x}.$$

Moments

For $\mathfrak{a} = 0, \infty$, let us introduce the following notation

$$\mathfrak{M}_{\mathfrak{a}}(N; n, s) = \mathfrak{M}_{\mathfrak{a}}^{hol}(N; n, s) + \mathfrak{M}_{\mathfrak{a}}^{disc}(N; n, s),$$

where

$$\mathfrak{M}_{\mathfrak{a}}^{hol}(N; n, s) := \sum_{\substack{k>1 \\ k \text{ odd}}} \psi_H(k)\Gamma(k) \sum_{f \in H_k(N, \chi_4)} \rho_{f_{\mathfrak{a}}}(n) \overline{L(s, \mathrm{sym}^2 f_\infty)},$$

$$\mathfrak{M}_{\mathfrak{a}}^{disc}(N; n, s) := \sum_{f \in H(N, \chi_4)} \frac{\psi_D(t_f)}{\cosh(\pi t_f)} \rho_{f_{\mathfrak{a}}}(n) \overline{L(s, \mathrm{sym}^2 f_\infty)}.$$

Main Theorem

For even n and $0 < \Re s < 1$

$$\sum_{l=1}^{\infty} \omega(l) \mathscr{L}_{n^2-4l^2}(s) = \text{Main terms} + \mathfrak{C}(n,s) + c_1(s)\mathfrak{M}_{\infty}(4; n^2/4, s) + c_2(s)\mathfrak{M}_0(4; n^2/4, s).$$

For odd n and $0 < \Re s < 1$

$$\sum_{l=1}^{\infty} \omega(l) \mathscr{L}_{n^2-4l^2}(s) = \text{Main terms} + \frac{1}{2}\mathfrak{C}(n,s) + c_3(s)\mathfrak{M}_0(16; n^2, s) + c_4(s)\mathfrak{M}_0(64; n^2, s).$$

Continuous spectrum

For $\sigma_s(\chi; n) := \sum_{d|n} \chi(d) d^s$ we have

$$\mathfrak{C}(n,s) = \frac{L(\chi_4, s)}{4\pi^{s-1/2}} \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\psi_D(t)\sinh(\pi t)}{t\cosh(\pi t)} \frac{\zeta(s+2it)\zeta(s-2it)}{L(\chi_4, 1+2it)L(\chi_4, 1-2it)} \times \left(n^{2it}\sigma_{-2it}(\chi_4; n^2) + n^{-2it}\sigma_{2it}(\chi_4; n^2)\right) dt.$$

Remark: we use some results of Kiral-Young (2019).

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$$\mathscr{L}_{n^2-4l^2}(s)=\zeta(2s)\sum_{q=1}^{\infty}\frac{1}{q^{1+s}}\sum_{c \pmod q}S(l^2,c^2;q)e\left(\frac{nc}{q}\right).$$

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- Change the order of summation over q and l and split the range of summation over l into arithmetic progressions:

$$\sum_{l=1}^{\infty}\omega(l)\mathscr{L}_{n^2-4l^2}(s)=\frac{\zeta(2s)}{2\pi i}\int_{(a)}\widehat{\omega}(\alpha)\sum_{q=1}^{\infty}\frac{1}{q^{1+s}}$$
$$\times\sum_{c,d \pmod q}S(d^2,c^2;q)e\left(\frac{nc}{q}\right)\sum_{\substack{l\geq 1\\ l\equiv d \pmod q}}\frac{1}{l^{\alpha}}d\alpha,\quad a>1.$$

The inner sum can be written in terms of the Lerch zeta function $\zeta(a,b;\alpha)$ with the following functional equation

$$\sum_{\substack{l\geq 1\\ l\equiv d \pmod q}} \frac{1}{l^\alpha} = \frac{1}{q^\alpha}\sum_{l=1}^{\infty}\frac{1}{(l+d/q)^\alpha} = \frac{\zeta(d/q,0;\alpha)}{q^\alpha} =$$

$$\frac{\Gamma(1-\alpha)}{q^\alpha(2\pi)^{1-\alpha}}\left[-ie(\alpha/4)\sum_{l=1}^{\infty}\frac{e(ld/q)}{l^{1-\alpha}} + ie(-\alpha/4)\sum_{l=1}^{\infty}\frac{e(-ld/q)}{l^{1-\alpha}}\right].$$

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In order to apply this equation, we move the contour of integration to $\Re\alpha<0$, crossing a simple pole of $\zeta(d/q,0;\alpha)$ at $\alpha=1$.

Contribution of the pole gives rise to the diagonal term:

$$\hat{\omega}(1)\zeta(2s)\sum_{q=1}^{\infty}\frac{1}{q^{2+s}}\sum_{c,d \pmod q} S(d^2,c^2;q)e\left(\frac{nc}{q}\right).$$

Remark: The inner sum was studied by Iwaniec-Michel (2001).

Diagonal term

Recall that $\sigma_s(\chi; n) := \sum_{d|n} \chi(d) d^s$.

If n is even, then

$$M^D(n,s) = \frac{\widehat{\omega}(1)\zeta(2s)}{L(\chi_4, 1+s)} \left(n^{-2s}\sigma_s(\chi_4; n^2) + \sigma_{-s}(\chi_4; n^2)\right).$$

If n is odd, then

$$M^D(n,s) = \frac{\widehat{\omega}(1)\zeta(2s)}{L(\chi_4, 1+s)} \sigma_{-s}(\chi_4; n^2).$$

Remark: Diagonal terms have a pole at $s = 1/2$.

Non-diagonal term

The non-diagonal term can be expressed in terms of the Gauss sums

$$G(a,n;q) = \sum_{x \pmod q} e\left(\frac{ax^2+nx}{q}\right), \quad (a,q)=1.$$

More precisely,

$$M^{ND} = \frac{\zeta(2s)}{\pi^{s-1/2}} \sum_{l=1}^{\infty} \frac{1}{l^s} \sum_{q=1}^{\infty} \frac{f(\omega,s;4\pi nl/q)}{q^2} K(n,l;q),$$

where

$$K(n,l;q) = \sum_{c,d \pmod q} S(c^2,d^2,q) e\left(\frac{nc+ld}{q}\right)$$
$$= \sum_{\substack{a,b \pmod q \\ ab \equiv 1 \pmod q}} G(a,n;q)G(b,l;q).$$

Gauss sums and symmetric square L-functions

The exponential sum $K(n, l; q)$ and Gauss sums appear explicitly or implicitly in various computations related to symmetric square L-functions. For example, in the level aspect:

- Iwaniec-Michel (2001);
- Blomer (2008);
- Munshi (2017).

In order to evaluate

$$K(n,l;q)=\sum_{\substack{a,b \pmod q\\ ab\equiv 1 \pmod q}} G(a,n;q)G(b,l;q)$$

in the non-diagonal term we use the book of Malyshev (1962).

Examples: Suppose q is odd, then

$$K(n,l;q)=q\chi_4(q)S(\overline{4}_q n^2,\overline{4}_q l^2;q).$$

Suppose that $q\equiv 0 \pmod 4$, n and l are even. Then

$$K(n,l;q)=2iq\sum_{\substack{a,b \pmod q\\ ab\equiv 1 \pmod q}} \chi_4(a)e\left(-\frac{a(l/2)^2+b(n/2)^2}{q}\right).$$

Kloosterman sums

Let $\Gamma = \Gamma_0(N)$ and let $\mathfrak{a}$, $\mathfrak{b}$ be two singular cusps for χ with corresponding scaling matrices $\sigma_{\mathfrak{a}}$, $\sigma_{\mathfrak{b}}$. Kloosterman sums associated to $\mathfrak{a}$, $\mathfrak{b}$ are defined by

$$S_{\mathfrak{a}\mathfrak{b}}(m,n;c;\chi) = \sum_{\gamma=\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\in\Gamma_\infty\backslash\sigma_{\mathfrak{a}}^{-1}\Gamma\sigma_{\mathfrak{b}}/\Gamma_\infty} \chi(\mathrm{sgn}(c))\overline{\chi(\sigma_{\mathfrak{a}}\gamma\sigma_{\mathfrak{b}}^{-1})}e\left(\frac{am+dn}{c}\right).$$

The set of allowed moduli is given by

$$C_{\mathfrak{a},\mathfrak{b}}(N) = \{c > 0 \text{ such that } \left(\begin{smallmatrix} * & * \\ c & * \end{smallmatrix}\right) \in \sigma_{\mathfrak{a}}^{-1}\Gamma\sigma_{\mathfrak{b}}\}.$$

Examples:

$$S_{\infty 0}(m,n;c\sqrt{N};\chi) = \overline{\chi}(c)S(\overline{N}m,n;c), \quad (c,N)=1,$$

$$S_{\infty\infty}(m,n;c;\chi) = \sum_{ab\equiv 1 \pmod{c}} e\left(\frac{am+bn}{c}\right)\overline{\chi}(b).$$

Non-diagonal terms: even case

If n is even (let $n_1 := n/2$), then

$$M^{ND} = -\frac{2i\zeta(2s)}{2^s\pi^{s-1/2}}\sum_{l=1}^{\infty}\frac{1}{l^s}\sum_{\gamma\in C_{\infty,\infty}(4)}\frac{1}{\gamma}\psi\left(\frac{4\pi l n_1}{\gamma}\right)S_{\infty\infty}(l^2,n_1^2;\gamma;\chi_4)$$
$$+\frac{2\zeta(2s)}{\pi^{s-1/2}}\sum_{l=1}^{\infty}\frac{1}{l^s}\sum_{\gamma\in C_{\infty,0}(4)}\frac{1}{\gamma}\psi\left(\frac{4\pi l n_1}{\gamma}\right)S_{\infty 0}(l^2,n_1^2;\gamma;\chi_4),$$

where

$$C_{\infty,\infty}(4)=\{\gamma=q>0,\quad q\equiv 0 \pmod 4\},$$
$$C_{\infty,0}(4)=\{\gamma=2q>0,(q,4)=1\}.$$

Non-diagonal terms: odd case

If n is odd, then

$$M^{ND} = \frac{8\zeta(2s)}{\pi^{s-1/2}}\sum_{l=1}^{\infty}\frac{1}{l^s}\sum_{\gamma\in C_{\infty,0}(64)}\frac{1}{\gamma}\psi\left(\frac{4\pi ln}{\gamma}\right)S_{\infty 0}(l^2,n^2;\gamma;\chi_4)$$

$$+(1-2^{-s})\frac{4\zeta(2s)}{\pi^{s-1/2}}\sum_{l=1}^{\infty}\frac{1}{l^s}\sum_{\gamma\in C_{\infty,0}(16)}\frac{1}{\gamma}\psi\left(\frac{4\pi ln}{\gamma}\right)S_{\infty 0}(l^2,n^2;\gamma;\chi_4),$$

where

$$C_{\infty,0}(16)=\{\gamma=4q>0,\quad (q,2)=1\},$$

$$C_{\infty,0}(64)=\{\gamma=8q>0,\quad (q,2)=1\}.$$

Kuznetsov trace formula

For $m, n \geq 1$

$$\sum_{c \in C_{\mathfrak{a},\mathfrak{b}}(N)} \frac{S_{\mathfrak{a}\mathfrak{b}}(m,n;c;\chi_4)}{c} \psi\left(\frac{4\pi\sqrt{mn}}{c}\right) = H + D + C,$$

where

$$H := \sum_{\substack{k>1 \\ k \equiv 1 \pmod 2}} \sum_{f \in H_k(N,\chi_4)} \psi_H(k)\Gamma(k)\overline{\rho_{f_\mathfrak{a}}(m)}\rho_{f_\mathfrak{b}}(n),$$

$$D := \sum_{f \in H(N,\chi_4)} \frac{\psi_D(t_f)}{\cosh(\pi t_f)} \overline{\rho_{f_\mathfrak{a}}(m)}\rho_{f_\mathfrak{b}}(n),$$

$$C := \sum_{\mathfrak{c} \text{ sing.}} \frac{\sqrt{mn}}{4\pi} \int_{-\infty}^{\infty} \frac{\psi_D(t)}{\cosh(\pi t)} \overline{\rho_{\mathfrak{a},\mathfrak{c}}(m, 1/2+it)}\rho_{\mathfrak{b},\mathfrak{c}}(n, 1/2+it)dt.$$

Continuous spectrum

It is required to compute explicitly the Fourier coefficients

$$\rho_{\mathfrak{a},\mathfrak{c}}(m,s) = -\frac{\pi^s i|m|^{s-1}}{\Gamma(s+1/2)} \sum_{\gamma \in C_{\mathfrak{c},\mathfrak{a}}(N)} \frac{S_{\mathfrak{c}\mathfrak{a}}(0,m;\gamma;\chi_4)}{\gamma^{2s}},$$

where $\mathfrak{a} = \infty, 0$ and

- For $\Gamma_0(4)$: $\mathfrak{c} = 0, \infty$.
- For $\Gamma_0(16)$: $\mathfrak{c} = 0, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{12}, \infty$.
- For $\Gamma_0(64)$: $\mathfrak{c} = 0, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{12}, \frac{1}{16}, \frac{1}{24}, \frac{1}{32}, \frac{1}{40}, \frac{1}{48}, \frac{1}{56}, \infty$.

Example

Let $N = 4$. Then

$$\rho_{0,0}(m,s) = \rho_{\infty,\infty}(m,s) = \frac{2\pi^s \sigma_{2s-1}(\chi_4; m) m^{-s}}{4^{2s}\Gamma(s+1/2)L(\chi_4, 2s)},$$

$$\rho_{0,\infty}(m,s) = -\rho_{\infty,0}(m,s) = -\frac{\pi^s i m^{s-1} \sigma_{1-2s}(\chi_4; m)}{2^{2s}\Gamma(s+1/2)L(\chi_4, 2s)}.$$

Continuous spectrum: even case

For $\Re s > 1$ we have

$$C_{\text{even}} = \frac{L(\chi_4, s)}{4\pi^{s-1/2}} \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\psi_D(t)\sinh(\pi t)}{t\cosh(\pi t)} \frac{\zeta(s+2it)\zeta(s-2it)}{L(\chi_4, 1+2it)L(\chi_4, 1-2it)} \times \left(n^{2it}\sigma_{-2it}(\chi_4; n^2) + n^{-2it}\sigma_{2it}(\chi_4; n^2) \right) dt.$$

Lemma

Suppose that the function $F(s)$ is defined for $\Re s > 1$ by

$$F(s) = \frac{1}{2\pi i} \int_{(0)} f(s,z) dz,$$

where $f(s,z)$ has simple poles at $z_1 = 1 - s$ and $z_2 = s - 1$.

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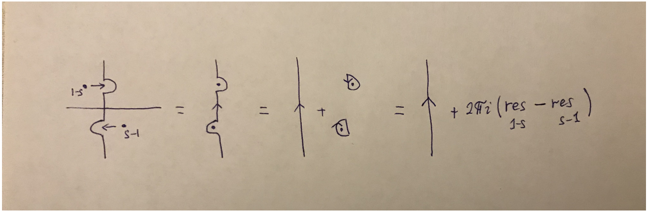

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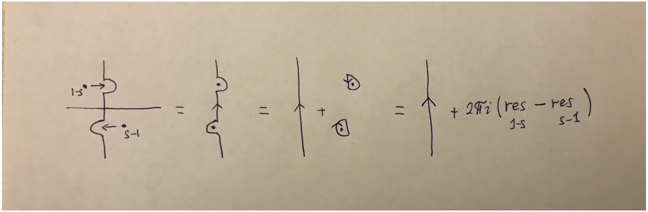

Then for $\Re s < 1$ we have

$$F(s) = \frac{1}{2\pi i}\int_{(0)} f(s,z)dz + \mathrm{Res}_{z_1} f(s,z) - \mathrm{Res}_{z_2} f(s,z).$$

Continuous spectrum on the critical line

For $0<\Re s<1,\ s\neq 1/2$ we have

$$C_{\text{even}}=M^C(n,s)+\frac{L(\chi_4,s)}{4\pi^{s-1/2}}\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{\psi_D(t)\sinh(\pi t)}{t\cosh(\pi t)}$$
$$\times\frac{\zeta(s+2it)\zeta(s-2it)}{L(\chi_4,1+2it)L(\chi_4,1-2it)}\left(n^{2it}\sigma_{-2it}(\chi_4;n^2)+n^{-2it}\sigma_{2it}(\chi_4;n^2)\right)dt,$$

where

$$M^C(n,s)=\frac{\Gamma(s-1/2)}{2^{s-1}\pi^{s-1/2}}\left(\sigma_{s-1}(\chi_4;n^2)+\frac{\sigma_{1-s}(\chi_4;n^2)}{n^{2-2s}}\right)$$
$$\times\frac{\zeta(2s-1)}{L(\chi_4,2-s)}\left(\sin(\pi s/2)\int_0^{n/2}\omega(y)\left(\frac{n^2}{4}-y^2\right)^{1/2-s}dy\right.$$
$$\left.+\cos(\pi s/2)\int_{n/2}^{\infty}\omega(y)\left(y^2-\frac{n^2}{4}\right)^{1/2-s}dy\right).$$

Dual moments

The second moment of symmetric square L-functions associated to cusp forms of even weight k_1 for the full modular group

$$\sum_{f \in H_{k_1}} L(\text{sym}^2 f_\infty, s)^2$$

can be expressed in terms of the following moments

$$\sum_{\substack{k_2 > 1 \\ k_2 \text{ odd}}} \phi_1(k_1, k_2) \sum_{f \in H_{k_2}(N, \chi_4)} L(s, \text{sym}^2 f_{\mathfrak{a}}) \overline{L(s, \text{sym}^2 f_\infty)},$$

$$\sum_{f \in H(N, \chi_4)} L(s, \text{sym}^2 f_{\mathfrak{a}}) \overline{L(s, \text{sym}^2 f_\infty)} \phi_2(k_1, t_f),$$

where

$$(N, \mathfrak{a}) = \{(4, \infty), \quad (4, 0), \quad (16, 0), \quad (64, 0)\}.$$

Thank You!