WIDE SCATTERED SPACES AND MORASSES

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Abstract. We show that it is relatively consistent with ZFC that 2^ω is arbitrarily large and every sequence $\mathbf{s} = \langle s_\alpha : \alpha < \omega_2 \rangle$ of infinite cardinals with $s_\alpha \le 2^\omega$ is the cardinal sequence of some locally compact scattered space.

1. Introduction

If X is a scattered topological space, and α is an ordinal, denote by $\mathrm{I}_\alpha(X)$ the αth Cantor-Bendixson level of X. The *cardinal sequence* of X, $\mathrm{SEQ}(X)$, is the sequence of the cardinalities of the infinite Cantor-Bendixson levels of X, i.e.

$$\mathrm{SEQ}(X) = \Big\langle\ |I_\alpha(X)| : \alpha < ht^-(X)\ \Big\rangle,$$

where $ht^-(X)$, the *reduced height* of X, is the minimal β such that $\mathrm{I}_\beta(X)$ is finite. If δ is an ordinal, we denote by $\mathcal{C}(\delta)$ the class of all cardinal sequences of length δ of locally compact scattered (LCS, in short) spaces.

Let $\langle\kappa\rangle_\alpha$ denote the constant κ-valued sequence of length α.

Theorem 1.1 (Baumgartner, Shelah, [2])**.** *It is relatively consistent with ZFC that $\langle\omega\rangle_{\omega_2} \in \mathcal{C}(\omega_2)$.*

Refining their argument, first Bagaria, [1], proved that ${}^{\omega_2}\{\omega,\omega_1\} \subset \mathcal{C}(\omega_2)$ in some ZFC model, then Martinez and Soukup, [9], showed that $2^\omega = \omega_2$ and ${}^{\omega_2}\{\omega,\omega_1,\omega_2\} \subset \mathcal{C}(\omega_2)$ is also consistent.

For a long time ω_2 was a mystique barrier in both height and width. In this paper we can construct wider spaces.

Theorem 1.2. *If GCH holds and $\lambda \ge \omega_2$ is a regular cardinal, then in some cardinal preserving generic extension $2^\omega = \lambda$ and every sequence $\mathbf{s} = \langle s_\alpha : \alpha < \omega_2\rangle$ of infinite cardinals with $s_\alpha \le \lambda$ is the cardinal sequence of some locally compact scattered space.*

We will find the suitable generic extension in three steps:

(I) The first extension adds a "*strongly stationary strong* (ω_1,λ)*-semimorass*" to the ground model (see Definition 2.1 and Theorem 2.3).

(II) Using that strong semimorass the second extension adds a $\Delta(\omega_2\times\lambda)$*-function* to the first extension (see Definition 3.1 and Theorem 3.2).

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(III) Using the $\Delta(\omega_2\times\lambda)$-function we add an "*LCS space with stem*" to the second model and we show that those2 space alone guarantees that every sequence $\mathbf{s}=\langle s_\alpha:\alpha<\omega_2\rangle$ of infinite cardinals with $s_\alpha\le\lambda$ is the cardinal sequence of some locally compact scattered space (see Theorem 4.2).

Steps (I) and (II) are based on works of P. Koszmider, see [7] and [8].

2. Strong semimorasses

If ρ is a function and X is set, write $\rho[X]=\{\rho(\xi):\xi\in X\}$.

If X and Y are sets of ordinals with $\operatorname{tp}(X)=\operatorname{tp}(Y)$, denote the unique order preserving bijection between X and Y by $\rho_{X,Y}$.

For $X\in\left[\lambda\right]^\omega$ and $\mathcal{F}\subset\left[\lambda\right]^\omega$ let $\mathcal{F}\restriction X=\{Y\in\mathcal{F}:Y\subsetneq X\}$.

If X, X_1 and X_2 are sets of ordinals, we write

$X=X_1\oplus X_2$ iff $\operatorname{tp}(X_1)=\operatorname{tp}(X_2)$, $X=X_1\cup X_2$ and $\rho_{X_1,X_2}\restriction X_1\cap X_2=\mathrm{id}$;

$X=X_1\odot X_2$ iff $\operatorname{tp}(X_1)=\operatorname{tp}(X_2)$, $X=X_1\cup X_2$ and $X_1\cap X_2<X_1\setminus X_2<X_2\setminus X_1$;

and

$X=X_1\otimes X_2$ iff $X=X_1\oplus X_2$ and $X\cap\omega_2=(X_1\cap\omega_2)\odot(X_2\cap\omega_2)$.

In [7] Koszmider introduced the notion of semimorasses and proved several properties concerning that structures. Unfortunately, in our proof we need structures with a bit stronger properties.

Definition 2.1. Let $\omega_1\le\lambda$ be a cardinal. A family $\mathcal{F}\subset\left[\lambda\right]^\omega$ is a *strong* (ω_1,λ)-*semimorass* iff

(M1) $\langle\mathcal{F},\subseteq\rangle$ is *well-founded*, (and so we have the *rank* function on $\mathcal{F}$),
(M2) $\mathcal{F}$ is *locally small*, i.e. $|\mathcal{F}\restriction X|\le\omega$ for each $X\in\mathcal{F}$.
(M3) $\mathcal{F}$ is *homogeneous*, i.e. $\forall X,Y\in\mathcal{F}$ if $\operatorname{rank}(X)=\operatorname{rank}(Y)$ then $\operatorname{tp}(X)=\operatorname{tp}(Y)$ and $\mathcal{F}\restriction Y=\{\rho_{X,Y}[Z]:Z\in\mathcal{F}\restriction X\}$.
(M4) $\mathcal{F}$ is *directed*, i.e. $\forall X,Y\in\mathcal{F}$ $(\exists Z\in\mathcal{F})$ $X\cup Y\subset Z$.
(M5) $\mathcal{F}$ is *strongly locally semidirected*, i.e. $\forall X\in\mathcal{F}$ either (a) or (b) holds:
  (a) $\mathcal{F}\restriction X$ is directed,
  (b) $\exists X_1,X_2\in\mathcal{F}$ $\operatorname{rank}(X_1)=\operatorname{rank}(X_2)$, $X=X_1\otimes X_2$, and $\mathcal{F}\restriction X=(\mathcal{F}\restriction X_1)\cup(\mathcal{F}\restriction X_2)\cup\{X_1,X_2\}$.
(M6) $\mathcal{F}$ *covers* λ, i.e. $\cup\mathcal{F}=\lambda$.

If in (M5)(b) we weaken the assumption $X=X_1\otimes X_2$ to $X=X_1\oplus X_2$ then we obtain the definition of an (ω_1,λ)-*semimorass* (see [7, Definition 1]). Moreover, a strong (ω_1,ω_2)-semimorass is just Velleman's simplified (ω_1,ω_2)-morass.

Definition 2.2. A family $\mathcal{F}\subset\left[\lambda\right]^\omega$ is *strongly stationary* iff for each function $c:\left[\mathcal{F}\right]^{<\omega}\to\left[\lambda\right]^\omega$ there are stationary many $X\in\mathcal{F}$ such that X is c-*closed*, i.e. $c(X^*)\subset X$ for each $X^*\in\left[\mathcal{F}\restriction X\right]^{<\omega}$.

Theorem 2.3. *If $2^\omega=\omega_1<\lambda=\lambda^{\omega_1}$ then there is a σ-complete ω_2-c.c. forcing notion P such that*

$V^P\models$ "*$\lambda^{\omega_1}=\lambda$ and there is a strongly stationary strong (ω_1,λ)-semimorass $\mathcal{F}$.* "

We say that a family $p\subset\left[\lambda\right]^\omega$ is *neat* iff $\cup p=\cup(p\setminus\{\cup p\})$.

Proof of Theorem 2.3. Define $P=\langle P,\le\rangle$ as follows. Let

$$P=\{p\subset\left[\lambda\right]^\omega:|p|\le\omega,\cup p\in p,p\text{ is neat and satisfies (M1)–(M5)}\}.$$

Write $\operatorname{supp}(p) = \cup p$ for $p \in P$. Clearly $\operatorname{supp}(p)$ is the $\subset$-largest element of p. Put

$$p \le q \text{ iff } \operatorname{supp}(q) \in p \wedge q = (p \restriction \operatorname{supp}(q)) \cup \{\operatorname{supp}(q)\}. \tag{1}$$

P is σ-complete. Indeed, if $p_0 \ge p_1 \ge p_2 \dots$ then let

$$p = \cup_{n<\omega} p_n \cup \{\cup_{n<\omega} \operatorname{supp}(p_n)\}.$$

Then $p \in P$ and $p \le p_n$ for each n.

Definition 2.4. We say that two conditions p and p' are twins iff
- (i) $\operatorname{tp}(\operatorname{supp}(p)) = \operatorname{tp}(\operatorname{supp}(p'))$,
- (ii) $\operatorname{supp}(p) \cup \operatorname{supp}(p') = \operatorname{supp}(p) \otimes \operatorname{supp}(p')$,
- (iii) $p' = \rho_{\operatorname{supp}(p),\operatorname{supp}(p')}[p]$.

Lemma 2.5. *If p and p' are twins then they have a common extension in P*

Proof. Write $D = \operatorname{supp}(p)$ and $D' = \operatorname{supp}(p')$. Put $r = p \cup p' \cup \{D \cup D'\}$. We show that r is a common extension of p and p'.
Claim: $p = (r \restriction D) \cup \{D\}$ *and* $p' = (r \restriction D') \cup \{D'\}$.

Indeed, assume that $X \in r \restriction D$. Then $X \in p$ or $X \in p'$. If $X \in p'$ then $X \subset D'$, and so $X \subset D \cap D'$. Since $\rho_{D,D'} \restriction D \cap D' = \mathrm{id}$ it follows that $X = \rho^{-1}_{D,D'}[X] \in p$. So $r \restriction D \subset p$, which proves the Claim.

First we check that that $r \in P$. (M1) and (M2) are clear. Since $\operatorname{supp}(r) = \operatorname{supp}(p) \cup \operatorname{supp}(p')$, r is neat. r has the largest element $\operatorname{supp}(r) = D \cup D' \in r$, and so (M4) also holds. In (M5) we have just one new instance $X = \operatorname{supp}(r)$. But in this case the choice $X_1 = D$ and $X_2 = D'$ works. To check (M3) assume that $X, Y \in r$, $\operatorname{rank}(X) = \operatorname{rank}(Y)$. If $X, Y \in p$ or $X, Y \in p'$ then we can apply that p and p' satisfy (M3). So we can assume that $X \in p \setminus p'$ and $Y \in p' \setminus p$. Let $X' = \rho_{D,D'}[X]$. Then $\operatorname{rank}(X') = \operatorname{rank}(X) = \operatorname{rank}(Y)$ and $X', Y \in p'$. Since p' satisfies (M3), we have $\operatorname{tp}(X') = \operatorname{tp}(Y)$, and so $\operatorname{tp}(X) = \operatorname{tp}(Y)$. Since $\rho_{X',Y} : p' \restriction X' \to p' \restriction Y$ is an isomorphism, and $\rho_{X,Y} = \rho_{D,D'} \circ \rho_{X',Y}$ it follows that $\rho_{X,Y} : p \restriction X \to p' \restriction Y$ is also an isomorphism. However : $p \restriction X = r \restriction X$ and $p' \restriction Y = r \restriction Y$ by the Claim, and so $\rho_{X,Y} : r \restriction X \to r \restriction Y$ is also an isomorphism, which proves (M3).

Finally $r \le p, p'$ follows immediately from the Claim. □

Lemma 2.6. *P satisfies ω_2-c.c.*

Proof. Assume that $\{r_\alpha : \alpha < \omega_2\} \subset P$. Write $D_\alpha = \operatorname{supp}(r_\alpha)$ for $\alpha < \omega_2$. By standard argument we can find $I \in \left[\omega_2\right]^{\omega_2}$ such that
- (a) $\{D_\alpha : \alpha \in I\}$ forms a Δ-system with kernel D, and $\operatorname{tp}(D_\alpha) = \operatorname{tp}(D_\beta)$ for $\alpha, \beta \in I$,
- (b) For $\alpha < \beta \in I$ we have $D \cap \omega_2 < D_\alpha \setminus D < D_\beta \setminus D$,
- (c) $\rho_{D_\alpha,D_\beta}[D_\alpha \cap \omega_2] = D_\beta \cap \omega_2$ and $\rho_{D_\alpha,D_\beta} \restriction D = \mathrm{id}$,
- (d) $r_\beta = \{\rho_{D_\alpha,D_\beta}[X] : X \in r_\alpha\}$.

Then for each $\alpha \ne \beta \in I$ the conditions r_α and r_β are twins, so they are compatible by Lemma 2.5. □

Lemma 2.7. *(a)* $\forall \alpha \in \omega_2$

$$D_\alpha = \{p \in P : \operatorname{supp}(p) \cap (\omega_2 \setminus \alpha) \ne \emptyset\}$$

is dense in P.
(b) $\forall \beta \in \lambda \setminus \omega_2$

$$E_\beta = \{p \in P : \beta \in \operatorname{supp}(p)\}$$

is dense in P.

Proof. (a) For each $q \in P$ and $\alpha < \omega_2$ there is q' such that q and q' are twins and $\mathrm{supp}(q') \cap (\omega_2 \setminus \alpha) \neq \emptyset$. Then q and q' has a common extension $p \in D_\alpha$ by Lemma 2.5.
(b) For all q and $\beta \in \lambda \setminus \omega_2$ there is q' such that q and q' are twins and $\beta \in \mathrm{supp}(q')$. Then the common extension p of q and q' is in E_β. □

Let $\mathcal{G}$ be a P-generic filter over V, and put $\mathcal{F}' = \cup\mathcal{G}$ and $F = \cup\mathcal{F}'$. Then $\mathcal{F}' \subset \left[\lambda\right]^\omega$ and so $F \subset \lambda$. By the previous lemma, $F \supset \lambda \setminus \omega_2$ and $|F \cap \omega_2| = \omega_2$. So $\mathcal{F} = \{\rho_{F,\lambda}[X] : X \in \mathcal{F}'\}$ is a strong (ω_1, λ)-semimorass. To complete the proof of 2.3 it is enough to prove the following lemma.

Lemma 2.8. $V^P \models$ *"$\mathcal{F}$ is strongly stationary".*

Proof. It is enough to prove that if

$$q \Vdash \dot{\mathcal{C}} \subset \left[F\right]^\omega \text{ is club and } \dot{c} : \left[\mathcal{F}'\right]^{<\omega} \to \left[F\right]^\omega$$

then there are $p \le q$ and $C \in \left[\lambda\right]^\omega$ such that $p \Vdash$ "$\check{C} \in \mathcal{F}' \cap \dot{\mathcal{C}}$ is $\dot{c}$-closed".

First we need a claim.

Claim 2.8.1. *If $p \Vdash \check{A} \in \left[F\right]^\omega$ then $\exists p' \le p$ such that $A \subset \mathrm{supp}(p')$.*

Proof. If $\alpha \in A$ and $p \Vdash \check{\alpha} \in F$ then there are $p' \le p$ and $p'' \in P$ such that $\alpha \in \mathrm{supp}(p'')$ and $p' \Vdash p'' \in F$. Then p' and p'' have a common extension q, and then $\alpha \in \mathrm{supp}(q)$ and $q \Vdash \check{\alpha} \in F$.

Since P is σ-complete and A is countable, we are done by a straightforward induction. □

We will choose a decreasing sequence $\langle p_n : n < \omega\rangle \subset P$ and an increasing sequence $\langle C_n : n < \omega\rangle \subset \left[\lambda\right]^\omega$ as follows. Let $C_0 = \emptyset$ and $p_0 = q$. If p_n and C_n are given, let $Z_n \supset C_n \cup \mathrm{supp}(p_n)$ and $p'_n \le p_n$ s.t.

$$p'_n \Vdash \cup\{\dot{c}(X) : X \in \left[\check{p}_n\right]^{<\omega}\} \subset Z_n \in \left[F\right]^\omega.$$

Let $p_{n+1} \le p'_n$ and $C_{n+1} \supset Z_n \cup C_n$ such that $C_{n+1} \subset \mathrm{supp}(p_{n+1})$ and $p_{n+1} \Vdash \check{C}_{n+1} \in \dot{\mathcal{C}}$.

Having constructed the sequence finally put $C = \cup\{C_n : n < \omega\}$ and $p = \cup_{n<\omega} p_n \cup \{C\}$. Then $p \in P$, $p \le q$, $C = \mathrm{supp}(p)$. Since $p \Vdash$ "$\check{C}_n \in \dot{\mathcal{C}}$ *and* $\dot{\mathcal{C}}$ *is club*", we have $p \Vdash$ "$\check{C} \in \dot{\mathcal{C}}$. Since $p \Vdash \dot{c}''\left[\left[p_n\right]^{<\omega}\right] \subset \mathrm{supp}(p_{n+1})$, we have $p \Vdash \dot{C}$ *is $\dot{c}$-closed.* Moreover $p \Vdash p \subset \mathcal{F}'$, so $p \Vdash \check{C} \in \mathcal{F}'$.

Putting these together we obtain that p and C have the desired properties, which proves the lemma. □

Since $(\lambda^{\omega_1})^{V[\mathcal{F}]} \le ((|P| + \lambda)^{\omega_1})^V = (\lambda^{\omega_1})^V = \lambda$, the proof of Theorem 2.3 is complete. □

Next we investigate some properties of strong semimorasses.

Lemma 2.9. *Let $\mathcal{F} \subset \left[\lambda\right]^\omega$ be a strongly stationary strong (ω_1, λ)-semimorass.*
(1) If $X, Y \in \mathcal{F}$, $\mathrm{rank}(X) = \mathrm{rank}(Y)$, $\alpha \in X \cap Y \cap \omega_2$, then $X \cap \alpha = Y \cap \alpha$.
(2) If $X, Y \in \mathcal{F}$, $X \subset Y$ and $\mathrm{rank}(X) < \alpha < \mathrm{rank}(Y)$ then there is $Z \in \mathcal{F}$ such that $\mathrm{rank}(Z) = \alpha$ and $X \subset Z \subset Y$.

(3) If $X \in \mathcal{F}$ and $\operatorname{rank}(X) < \alpha < \omega_1$ then there is $Z \in \mathcal{F}$ such that $\operatorname{rank}(Z) = \alpha$ and $X \subset Z$.

(4) If $X, Y \in \mathcal{F}$, $\operatorname{rank}(X) \le \operatorname{rank}(Y)$, and $\alpha \in X \cap Y \cap \omega_2$, then $X \cap \alpha \subset Y \cap \alpha$.

Proof. (1) We prove the statement by induction on the minimal rank of $Z \in \mathcal{F}$ with $Z \supset X \cup Y$.

If rank of Z is minimal, then clearly $Z = Z_1 \otimes Z_2$ where $X \subset Z_1$ and $Y \subset Z_2$. Let $X' = \rho_{Z_1,Z_2}[X] \in \mathcal{F} \restriction Z_2$. Since $\alpha \in Z_1 \cap Z_2 \cap \omega_2$, we have $Z_1 \cap \alpha = Z_2 \cap \alpha$ and so $\rho_{Z_1,Z_2} \restriction (\alpha + 1) = \mathrm{id}$. Thus $X' \cap \alpha = X \cap \alpha$ and $\alpha \in X'$. Since $X', Y \in \mathcal{F} \restriction Z_2$, $\alpha \in X' \cap Y$ and $\operatorname{rank}(Z_2) < \operatorname{rank}(Z)$, by the inductive assumption we have $X' \cap \alpha = Y \cap \alpha$. Thus $X \cap \alpha = Y \cap \alpha$.

(2) Easy by straightforward induction on $\operatorname{rank}(Y)$.

(3) By straightforward induction on α there is $Y \in \mathcal{F}$ such that $X \subset Y$ and $rank(Y) \ge \alpha$. Then apply (2).

(4) By (3) there is $Y' \supset X$ such that $\operatorname{rank}(Y) = \operatorname{rank}(Y')$. Then apply (1) for Y and Y'. □

In [8] Koszmider proved several statements for Velleman's simplified morasses. Here we need similar results for strong semimorasses. The following lemma corresponds to [8, Fact 2.6-Fact 2.7].

Lemma 2.10. *Let $\mathcal{F} \subset \left[\lambda\right]^{\omega}$ be a strongly stationary strong (ω_1, λ)-semimorass. Assume $\lambda^{\omega} = \lambda$, fix an injective function $c : \mathcal{F} \to \lambda$, and consider the stationary set*

$$\mathcal{F}' = \{X \in \mathcal{F} : c(X^*) \in X \text{ for each } X^* \in \mathcal{F} \restriction X\}. \tag{2}$$

Assume that $\mathcal{F}, \mathcal{F}', c \in M \prec H(\theta)$, $|M| = \omega$, and $M \cap \lambda \in \mathcal{F}'$. Then

(1) $\mathcal{F} \restriction M \cap \lambda \subset M$.

(2) $\operatorname{rank}(M \cap \lambda) = M \cap \omega_1$.

(3) If $Y \in \mathcal{F}$ with $\operatorname{rank}(Y) < \delta = M \cap \omega_1$ then there is $Z \in M \cap \mathcal{F}$ such that $(M \cap \lambda) \cap Y \subset Z$, and $\operatorname{rank}(Z) = \operatorname{rank}(Y)$.

(4) If $\mathcal{A} \in \left[\mathcal{F}\right]^{<\omega}$ then there is $Z \in \mathcal{F} \cap M$ such that

$$\cup\{X \cap M : X \in \mathcal{A}, \operatorname{rank}(X) < M \cap \omega_1\} \subset Z. \tag{3}$$

Proof. (1) Let $X \in \mathcal{F} \restriction M \cap \lambda$, i.e. $X \in \mathcal{F}$ and $X \subsetneq M \cap \lambda$. Then $X \subsetneq M \cap \lambda \in \mathcal{F}'$ implies $\alpha = c(X) \in M \cap \lambda$. But $c, \alpha \in M$ and c is injective, so $X = c^{-1}\{\alpha\} \in M$.
(2) If $X \subsetneq M \cap \lambda$, $X \in \mathcal{F}$, then $X \in M$ by (1) and so $\operatorname{rank}(X) \in M \cap \omega_1$. Thus $\operatorname{rank}(M \cap \lambda) \le M \cap \omega_1$.

Assume that $\alpha < M \cap \omega_1$. Then

$$M \models \text{“}\exists X \in \mathcal{F} \operatorname{rank}(X) = \alpha.\text{''}$$

Thus there is $X \in M \cap \mathcal{F}$ such that $\operatorname{rank}(X) = \alpha$. Hence $\operatorname{rank}(M \cap \lambda) \ge M \cap \omega_1$.
(3) There is $Y' \supset Y$, $Y' \in \mathcal{F}$ and $\operatorname{rank}(Y') = \operatorname{rank}(M \cap \lambda)$. Let $Z = \rho_{Y',M\cap\lambda}[Y]$. Since $Y \cap (M \cap \lambda) \subset Y' \cap (M \cap \lambda)$ and $\rho_{Y',M\cap\lambda} \restriction Y' \cap (M \cap \lambda) = \mathrm{id}$, we have $Z \supset Y \cap (M \cap \lambda)$.
(4) Just apply (3) and the fact that $\mathcal{F}$ is directed. □

3. A $\Delta(\omega_2\times\lambda)$-function

Let $\lambda\ge\omega_2$ be an infinite cardinal and let $\pi:\omega_2\times\lambda\to\omega_2$ be the natural projection: $\pi(\langle\xi,\alpha\rangle)=\xi$.

Definition 3.1. (1) Assume that f is a function such that $\operatorname{dom}(f)\subset\left[\omega_2\times\lambda\right]^2$ and $f\{x,y\}\in\left[\pi(x)\cap\pi(y)\right]^{<\omega}$ for each $\{x,y\}\in\operatorname{dom}(f)$. We say that two finite subsets d and d' of $\omega_2\times\lambda$ are *good for* f provided $\left[d\cup d'\right]^2\subset\operatorname{dom}(f)$ and $\forall x\in d'\setminus d$ $\forall y\in d\setminus d'$ $\forall z\in d\cap d'\cap(\omega_2\times\omega)$

(S1) if $\pi(z)<\pi(x),\pi(y)$ then $\pi(z)\in f\{x,y\}$,
(S2) if $\pi(z)<\pi(y)$ then $f\{x,z\}\subset f\{x,y\}$,
(S3) if $\pi(z)<\pi(x)$ then $f\{y,z\}\subset f\{x,y\}$.

(2) A function $f:\left[\omega_2\times\lambda\right]^2\to\left[\omega_2\right]^{<\omega}$ is a $\Delta(\omega_2\times\lambda)$*-function* iff $f\{x,y\}\subset\min(\pi(x),\pi(y))$ and for each sequence $\{d_\alpha:\alpha<\omega_1\}\subset\left[\omega_2\times\lambda\right]^{<\omega}$ there are $\alpha\ne\beta$ such that d_α and d_β are good for f.

Remark . The assumption $|f\{x,y\}|<\omega$, instead of the usual $|f\{x,y\}|\le\omega$, is not a misprint.

Theorem 3.2. *If $2^\omega=\omega_1<\lambda=\lambda^{\omega_1}$ and there is a strongly stationary strong (ω_1,λ)-semimorass, then in some cardinal preserving generic extension $\lambda^{\omega_1}=\lambda$ and there is a $\Delta(\omega_2\times\lambda)$-function.*

Proof. To start with fix a strongly stationary strong (ω_1,λ)-semimorass $\mathcal{F}\subset\left[\lambda\right]^\omega$. We can assume that

$$\omega\subset X\text{ for each }X\in\mathcal{F}.\tag{4}$$

Fix an injective function $c:\mathcal{F}\to\lambda$, and consider the stationary set

$$\mathcal{F}'=\{X\in\mathcal{F}:c(X^*)\in X\text{ for each }X^*\in\mathcal{F}\restriction X\}.\tag{5}$$

Definition 3.3. We define a poset $P=\langle P,\le\rangle$ as follows: P consists of triples $p=\langle a,f,\mathcal{A}\rangle$, where $a\in\left[\omega_2\times\lambda\right]^{<\omega}$, $f:\left[a\right]^2\to\mathcal{P}(\pi[a])$ with $f\{s,t\}\subset\min(\pi(s),\pi(t))$, $\mathcal{A}\in\left[\mathcal{F}\right]^{<\omega}$ such that

$$\forall s,t\in a\ \forall X\in\mathcal{A}\text{ if }s,t\in X\times X\text{ then }f\{s,t\}\subset X.\tag{6}$$

Write $p=\langle a_p,f_p,\mathcal{A}_p\rangle$ for $p\in P$. Put $p\le q$ iff $a_p\supset a_q$, $f_p\supset f_q$ and $\mathcal{A}_p\supset\mathcal{A}_q$.

For $p\in P$ let

$$\operatorname{supp}(p_\nu)=\cup a_\nu\cup\cup\mathcal{A}_\nu.$$

Clearly $\operatorname{supp}(p)\in\left[\lambda\right]^{\le\omega}$.

If ρ is a function and $x=\langle a,b\rangle\in\operatorname{dom}(\rho)^2$, let $\bar\rho(x)=\langle\rho(a),\rho(b)\rangle$. We say $p,q\in P$ are *twins* iff

(A) $\operatorname{supp}(p)$ and $\operatorname{supp}(q)$ have the same order type,
(B) the unique $<_{On}$-preserving bijection ρ between $\operatorname{supp}(p)$ and $\operatorname{supp}(q)$ gives an *isomorphism* between p and q, i.e.
  (a) $a_q=\bar\rho[a_p]$,
  (b) $\{\rho[X]:X\in\mathcal{A}_p\}=\mathcal{A}_q$,
  (c) for each $\{s,t\}\in\left[a_p\right]^2$, $\rho[f_p\{s,t\}]=f_q\{\bar\rho(s),\bar\rho(t)\}$.

Definition 3.4 ([7, Definition 22]). Let $\mathcal{K} \subset [\lambda]^{\omega}$. A poset P is *$\mathcal{K}$-proper* iff for some large enough regular cardinal θ if M is a countable elementary submodel of $\mathcal{H}(\theta)$ with $P \in M$ and $M \cap \lambda \in \mathcal{K}$ then for each $p \in M \cap P$ there is an (M, P)-generic $q \leq p$.

Lemma 3.5 ([7, Fact 23]). *If $\mathcal{K} \subset [\lambda]^{\omega}$ is stationary and a poset P is $\mathcal{K}$-proper, then forcing with P preserves ω_1.*

Definition 3.6 ([7, Definition 24]). Assume that P is a poset, $M \prec \mathcal{H}(\theta)$, $|M| = \omega$, $q \in P$, and $P, \pi_1, \ldots, \pi_n \in M$. We say that *the formula $\Phi(x, \pi_1, \ldots, \pi_n)$ well-reflects q in M* iff

(1) $\mathcal{H}(\theta) \models \Phi(q, \pi_1, \ldots, \pi_n)$,
(2) if $s \in M \cap P$ and $M \models \Phi(s, \pi_1, \ldots \pi_n)$ then q and s are compatible in P.

Definition 3.7 ([7, Definition 25]). Assume that P is a poset, $\mathcal{K} \subset [\lambda]^{\omega}$. We say that P is *simply $\mathcal{K}$-proper* if the following holds: for some/each large enough regular cardinal θ
IF

(i) $M \prec \mathcal{H}(\theta)$, $|M| = \omega$,
(ii) $p \in P$, $P, p, \mathcal{K} \in M$,
(iii) $M \cap \lambda \in \mathcal{K}$,

THEN there is $p_0 \leq p$ such that for each $q \leq p_0$ some formula $\Phi(x, \pi_1, \ldots, \pi_n)$ well-reflects q in M.

By lemmas [7, Fact 23 and Lemma 26] we have

Lemma 3.8. *If $\mathcal{K} \subset [\lambda]^{\omega}$ is stationary and a poset P is simply $\mathcal{K}$-proper, then forcing with P preserves ω_1.*

To show that ω_1 is preserved we prove the following lemma.

Lemma 3.9. *P is simply $\mathcal{F}'$-proper.*

Actually we will prove some stronger statement. To formulate it we need some preparation.

If $M \prec \mathcal{H}(\theta)$, $p \in P \cap M$, $M \cap \lambda \in \mathcal{F}$ and $q \in P$ let

$$p^M = \langle a_p, f_p, \mathcal{A}_p \cup \{M \cap \lambda\} \rangle$$

and

$$q \restriction M = \langle a_q \cap M, f_q \restriction M, \mathcal{A}_q \cap M \rangle .$$

Lemma 3.10. *(1) If $M \prec \mathcal{H}(\theta)$ and $p \in P \cap M$ then $p^M \in P$. (2) If $q \leq p^M$ then $q \restriction M \in P \cap M$ as well.*

Proof. (1) We should only check (6) for p^M. Assume that $s, t \in a_p$ and $X \in \mathcal{A}_p \cup \{M \cap \lambda\}$. Since $p \in P$, we can assume $X = M \cap \lambda$. However $s, t \in M$, and so $f_p\{s, t\} \in M$ as well by $p \in M$. Since $|f_p\{s, t\}| \leq \omega$, it follows $f_p\{s, t\} \subset M \cap \lambda = X$ which was to be proved.
(2) It is straightforward that $q \restriction M \in P$. To show $q \restriction M \in M$ we should check that $f_q \restriction M \in M$. So assume that $s, t \in a_q \cap M$. Then $s, t \in (M \cap \lambda) \times (M \cap \lambda)$ and $M \cap \lambda \in \mathcal{A}_{p^M} \subset \mathcal{A}_q$. So, by (6), $f_q\{s, t\} \subset M \cap \lambda$. Since $f_q\{s, t\}$ is finite, we have $f_q\{s, t\} \in M$. □

Lemma 3.11. *Assume that $M \prec \mathcal{H}(\theta)$, $|M| = \omega$, $P, \mathcal{F} \in M$, $p \in P \cap M$, $M \cap \lambda \in \mathcal{F}'$. Let $\delta = \operatorname{rank}(M \cap \lambda) = M \cap \omega_1$. Assume that $Z \in M \cap \mathcal{F}$ such that*

$$Z \supset \cup\{X \cap M : X \in \mathcal{A}_q, \operatorname{rank}(X) < \delta\}. \tag{7}$$

Let $\Phi(x, Z, q \restriction M)$ be the following formula:

$$\text{“ } x \in P,\ x \le q \restriction M,\ (a_x \setminus a_{q\restriction M}) \cap (Z \times Z) = \emptyset.\text{”} \tag{8}$$

Then
(1) $\Phi(q, Z, q \restriction M)$ holds.
(2) If $s \in M$, $M \models \Phi(s, Z, q \restriction M)$ and

$$h : [a_s \setminus a_{q\restriction M}, a_q \setminus a_{q\restriction M}] \to \mathcal{P}(\pi[a_s \cup a_q])$$

such that

$$h\{x, y\} \subset \min(\pi(x), \pi(y)) \cap \bigcap\{X \in A_q : x, y \in X \times X, \operatorname{rank}(X) \ge \delta\}, \tag{9}$$

then $r = \langle a_s \cup a_q, f_s \cup f_q \cup h, \mathcal{A}_s \cup \mathcal{A}_q\rangle \in P$ is a common extension of q and s.
(3) $\Phi(x, Z, q \restriction M)$ well reflects q in M.

Proof. (1) Since $q \restriction M \in P$ by Lemma 3.10(2), we have $q \le q \restriction M$ by the definition of the relation $\le$. Since $Z \in M$, we have $Z \times Z \subset M \times M \subset M$ and $a_q \setminus a_{q\restriction M} = a_q \setminus M$.
(2) To show that $r \in P$ we need to check that $r = \langle a_r, f_r, \mathcal{A}_r\rangle$ satisfies (6). So let $x, y \in a_r$, $X \in \mathcal{A}_r$.

Case 1. *$x, y \in a_s$ and $X \in \mathcal{A}_s$ or $x, y \in a_q$ and $X \in \mathcal{A}_q$.*

Then everything is fine, because $s, q \in P$.

Case 2. *$\{x, y\} \in [a_s]^2$, $x \in a_s \setminus a_q$ and $X \in \mathcal{A}_q$.*

If $\operatorname{rank}(X) < \delta$ then $(a_s \setminus a_q) \cap (X \times X) = \emptyset$ by (7), so $x \notin X \times X$. Thus (6) is void.

If $\operatorname{rank}(X) \ge \delta$ then $\nu = \min(\pi(x), \pi(y)) \in M \cap X$, so $M \cap \nu \subset X \cap \nu$ by Lemma 2.9(4). Thus $f_r\{x, y\} = f_s\{x, y\} \cap \nu \subset M \cap \nu \subset X$.

Case 3. *$\{x, y\} \in [a_q]^2$, $x \in a_q \setminus a_s$ and $X \in \mathcal{A}_s$.*

Since $(a_q \setminus a_s) \cap M = \emptyset$, it is not possible that $X \in \mathcal{A}_s$. Then $x \in a_q \setminus a_s = a_q \setminus a_{q\restriction M}$, so $x \notin M$. However $X \subset M$ and so $x \notin X \times X$, so (6) is void.

Case 4. *$x \in a_q \setminus a_s$ and $y \in a_s \setminus a_q$.*

Then the assumption concerning h in (9) is stronger than (6). Indeed, if $y \in a_s \setminus a_q$ then $y \notin Z \times Z$. So if $y \in X \times X$ for some $X \in \mathcal{A}_q$ then $\operatorname{rank}(X) \ge \delta$.

(3) Define the function

$$h : [a_s \setminus a_{q\restriction M}, a_q \setminus a_{q\restriction M}] \to [\omega_2]^{<\omega}$$

by $h\{x, y\} = \emptyset$. Then (9) holds, so s and q are comparable by (2), which was to be proved. ☐

Proof of Lemma 3.9. We can apply Lemma 3.11 because by Lemma 2.10 we can pick $Z \in M \cap \mathcal{F}$ such that $Z \supset \cup\{X \cap M : X \in \mathcal{A}_q, \operatorname{rank}(X) < \delta\}$. ☐

Lemma 3.12. *P satisfies ω_2-c.c.*

Proof. Let $\{p_\nu : \nu < \omega_2\} \subset P$. Put $p_\nu = \langle a_\nu, f_\nu, \mathcal{A}_\nu \rangle$. Recall that $\operatorname{supp}(p_\nu) = \cup a_\nu \cup \cup \mathcal{A}_\nu$.We can assume that

(i) $\{\operatorname{supp}(p_\nu) : \nu < \omega_2)\}$ forms a Δ-system with kernel D
(ii) the conditions are pairwise twins witnessed by functions $\rho_{\nu,\mu} : \operatorname{supp}(p_\nu) \to \operatorname{supp}(p_\mu)$.

Fix $\nu < \mu < \omega_2$. Define the function e as follows:

$$\operatorname{dom}(e) = [a_\nu \setminus a_\mu, a_\mu \setminus a_\nu] \text{ and } e\{s,t\} = \emptyset.$$

We claim that

$$r = \langle a_\nu \cup a_\mu, f_\nu \cup f_\mu \cup e, \mathcal{A}_\nu \cup \mathcal{A}_\mu \rangle \tag{10}$$

is a common extension of p_ν and p_μ. We need to show that $r \in P$. Since p_ν and p_μ are twins, we should check only (6). So let $t, s \in a_\nu \cup a_\mu$ and $X \in \mathcal{A}_\nu \cup \mathcal{A}_\mu$ with $s, t \in X \times X$. We can assume e.g. $X \in \mathcal{A}_\nu$. Since $X \subset \operatorname{supp}(p_\nu)$, we have $s, t \in \operatorname{supp}(p_\nu) \times \operatorname{supp}(p_\nu)$. Then $\operatorname{supp}(p_\nu) \times \operatorname{supp}(p_\nu) \cap a_\mu \subset a_\nu$ because $\operatorname{supp}(p_\nu) \times \operatorname{supp}(p_\nu) \cap a_\mu \subset D$ and a_ν, a_μ are twins. Thus we have $s, t \in a_\nu$, and so we are done because p_ν satisfies (6). □

To complete the proof of Theorem 3.2 we claim that if $\mathcal{G}$ is a P-generic filter then the function

$$f = \cup\{f_p : p \in \mathcal{G}\} \tag{11}$$

is a $\Delta(\omega_2 \times \lambda)$-function.

Assume that

$$p \Vdash \{\dot{d}_\xi : \xi < \omega_1\} \subset \left[\omega_2 \times \lambda\right]^{<\omega}.$$

We can assume

$$p \Vdash \{\dot{d}_\xi : \xi < \omega_1\} \text{ is a } \Delta\text{-system with kernel } \check{d}.$$

Assume $M \prec H(\theta)$, $|M| = \omega$, $p, \mathcal{F}' \left\langle \dot{d}_\xi : \xi < \omega_1 \right\rangle \in M$ and $X_0 = M \cap \lambda \in \mathcal{F}'$. Let

$$p^M = \langle a_p, f_p, \mathcal{A}_p \cup \{X_0\} \rangle.$$

Let $q \le p^M$, $\xi_1 < \omega_1$ and $e_1 \in \left[\omega_2 \times \lambda\right]^{<\omega}$ that

$$q \Vdash \dot{d}_{\xi_1} = \check{e}_1 \wedge (e_1 \setminus d) \cap M = \emptyset \wedge e_1 \subset a_q.$$

Put $\delta = \operatorname{rank}(M \cap \lambda)$. By Lemma 2.10 we can pick $Z \in M \cap \mathcal{F}$ such that

$$Z \supset \cup\{X \cap M : X \in \mathcal{A}_q, \operatorname{rank}(X) < \delta\}.$$

Consider the following formula $\Psi(x, \xi, e)$ with free variables x, ξ and e and parameters $Z, q \restriction M, \left\langle \dot{d}_\xi : \xi < \omega_1 \right\rangle, d \in M$:

$$\Phi(x, Z, q \restriction M) \wedge \xi \in \omega_1 \wedge (x \Vdash \dot{d}_\xi = e) \wedge e \subset a_x \wedge (e \setminus d) \subset a_x \setminus a_{q \restriction M}$$

where the formula $\Phi(x, Z, q \restriction M)$ was defined in (8) in Lemma 3.11. Then $\Psi(q, \xi_1, e_1)$ holds. Thus $\exists x \exists \xi \exists e \Psi(x, \xi, e)$ also holds. Since the parameters are all in M, we have

$$M \models \exists x\ \exists \xi\ \exists e\ \Psi(x, \xi, e). \tag{12}$$

Thus there are $s, \xi_2, e_2 \in M$ such that

$$\Phi(s, Z, q \restriction M) \wedge (s \Vdash \dot{d}_{\xi_2} = e_2) \wedge e_2 \subset a_s \wedge (e_2 \setminus d) \subset a_s \setminus a_{q \restriction M}.$$

Since $e_2 \setminus d \subset a_s \setminus a_{q \restriction M}$ and $(a_s \setminus a_{q \restriction M}) \cap Z \times Z = \emptyset$ by $\Phi(s, Z, q \restriction M)$, we have

$$(e_2 \setminus d) \cap Z \times Z = \emptyset.$$

Define the function

$$h : [a_s \setminus a_{q \restriction M}, a_q \setminus a_{q \restriction M}] \to \mathcal{P}(\pi[a_s \cup a_q])$$

by the formula

$$\text{(13)} \quad h\{x, y\} = \pi[a_s \cup a_q] \cap \min(\pi(x), \pi(y)) \cap \bigcap \{X \in \mathcal{A}_q : x, y \in X \times X, \operatorname{rank}(X) \geq \delta\}.$$

So $h\{x, y\}$ is as large as it is allowed by (9).

Then, by Lemma 3.11, the condition $r = \langle a_r, f_r, \mathcal{A}_r \rangle$, where $a_r = a_s \cup a_q$, $f_r = f_s \cup f_q \cup h$ and $\mathcal{A}_r = \mathcal{A}_s \cup \mathcal{A}_q$, is a common extension of q and s.

Lemma 3.13. *e_1 and e_2 are good for f_r.*

Proof. We should check conditions (S1)-(S3).

Assume that $z \in e_1 \cap e_2 \cap (\omega_2 \times \omega)$, $x \in e_1 \setminus e_2 \subset a_q \setminus a_{q \restriction M}$, and $y \in e_2 \setminus e_1 \subset a_s \setminus a_{q \restriction M}$. Observe that $z, y \in M$, and so $\pi(z), \pi(y) \in M$ as well.

(S1): Assume that $\pi(z) < \pi(x), \pi(y)$.

We should show that $\pi(z) \in f_r\{x, y\}$. However, $f_r\{x, y\}$ was defined by (13). So we should show that

$$\text{if } X \in \mathcal{A}_q,\ \operatorname{rank}(X) \geq \delta,\ x, y \in X \times X \text{ then } \pi(z) \in X.$$

Since $\pi(y) \in M \cap X \cap \omega_2$ and $\operatorname{rank}(M \cap \lambda) \leq \operatorname{rank}(X)$ we have $M \cap \pi(y) \subset X \cap \pi(y)$ by Lemma 2.9(4). Since $\pi(z) < \pi(y)$ and $\pi(z) \in M$ it follows that $\pi(z) \in X$.

(S2): Assume that $\pi(z) < \pi(y)$.

We need to show that $f_r\{x, z\} \subset f_r\{x, y\}$. Since $f_r\{x, z\} = f_q\{x, z\}$ and $f_r\{x, y\}$ was defined by (13) we should show that

$$\text{if } X \in \mathcal{A}_q,\ \operatorname{rank}(X) \geq \delta,\ x, y \in X \times X \text{ then } f_q\{x, z\} \subset X.$$

Since $\pi(y) \in M \cap X$ we have $\pi(z) \in M \cap \pi(y) \subset X \cap \pi(y)$ by Lemma 2.9(4).

Since $\pi(z) \in X$, $z \in \omega_2 \times \omega$ and $\omega \subset X$ by (4), it follows that $z \in X \times X$. Since $x, z \in X \times X$ and $X \in \mathcal{A}_q$, we have $f_q\{x, z\} \subset X$ by (6), which was to be proved.

(S3): Assume that $\pi(z) < \pi(x)$.

We need to show that $f_r\{y, z\} \subset f_r\{x, y\}$. Since $f_r\{y, z\} = f_s\{y, z\}$ and $f_r\{x, y\}$ was defined by (13) we should show that

$$\text{if } X \in \mathcal{A}_q,\ \operatorname{rank}(X) \geq \delta,\ x, y \in X \times X \text{ then } f_s\{y, z\} \subset X.$$

Since $z, y \in M$ we have $f_s\{y, z\} \subset M$.

Moreover $y \in X \times X$, and so $\pi(y) \in M \cap X \cap \omega_2$, which implies $M \cap \pi(y) \subset X \cap \pi(y)$ by Lemma 2.9(4) .

Thus $f_s\{y, z\} = f_s\{y, z\} \cap \pi(y) \subset M \cap \pi(y) \subset X \cap \pi(y) \subset X$, which was to be proved. □

Since $r \Vdash \dot{d}_{\xi_1} = \check{e}_1 \wedge \dot{d}_{\xi_2} = \check{e}_2 \wedge f \supset \check{f}_r$, by Lemma 3.13 $r \Vdash$ "$\dot{d}_{\xi_1}$ and $\dot{d}_{\xi_2}$ are good for f". So f is a $\Delta(\omega_2 \times \lambda)$-function in $V[\mathcal{G}]$.

Since $|P| \leq \lambda$ and so $(\lambda^{\omega_1})^{V[\mathcal{G}]} \leq ((|P| + \lambda)^{\omega_1})^V = (\lambda^{\omega_1})^V = \lambda$, the proof of Theorem 3.2 is complete. □

4. Space construction

Assume that X is a scattered space. We say that a subspace $Y \subset X$ is a *stem* of X provided

(i) $ht(Y) = ht(X)$,
(ii) $X \setminus Y$ is closed discrete in X.

Clearly (ii) holds iff every $x \in X$ has a neighborhood U_x such that $U_x \setminus \{x\} \subset Y$.

Proposition 4.1. *Assume that X is an LCS space, $Y \subset X$ is a stem, $SEQ(X) = \langle \kappa_\nu : \nu < \mu \rangle$ and $SEQ(Y) = \langle \lambda_\nu : \nu < \mu \rangle$. Then*

$$\{SEQ(Z) : Y \subset Z \subset X\} = \{s \in {}^{\mu}Card : \lambda_\nu \le s(\nu) \le \kappa_\nu \text{ for each } \nu < \mu\}. \tag{14}$$

Proof. Assume that $s \in {}^{\mu}\mathrm{Card}$ such that $\lambda_\nu \le s(\nu) \le \kappa_\nu$ for each $\nu < \mu$. For $\nu < \mu$ pick $Z_\nu \in [I_\nu(X)]^{s(\nu)}$ with $Z_\nu \supset I_\nu(Y)$. Put $Z = \bigcup\{Z_\nu : \nu < \mu\}$. Since $Y \subset Z$ and Y is a stem, we have $I_\nu(Z) = Z_\nu$ for $\nu < \mu$, and so $SEQ(Z) = s$. □

Theorem 4.2. *If there is a $\Delta(\omega_2 \times \lambda)$-function, then there is a c.c.c poset P such that in V^P there is an LCS space X with stem Y such that $SEQ(X) = \langle \lambda \rangle_{\omega_2}$ and $SEQ(Y) = \langle \omega \rangle_{\omega_2}$.*

Corollary 4.3. *If there is a $\Delta(\omega_2 \times \lambda)$-function, then there is a c.c.c poset P such that in V^P every sequence $\mathbf{s} = \langle s_\alpha : \alpha < \omega_2 \rangle$ of infinite cardinals with $s_\alpha \le \lambda$ is the cardinal sequence of some locally compact scattered space.*

Proof of Theorem 4.2. Instead of constructing the topological space directly, we actually produce a certain "graded poset" which guarantees the existence of the desired locally compact scattered space. We use the ideas from [1] to formulate the properties of our required poset.

Definition 4.4. Given two sequences $\mathfrak{t} = \langle \kappa_\alpha : \alpha < \delta \rangle$ and $\mathfrak{s} = \langle \lambda_\alpha : \alpha < \delta \rangle$ of infinite cardinals with $\lambda_\alpha \le \kappa_\alpha$, we say that a poset $\langle T, \prec \rangle$ is a $\mathfrak{t}$*-poset with an* $\mathfrak{s}$*-stem* iff the following conditions are satisfied:

(T1) $T = \bigcup\{T_\alpha : \alpha < \delta\}$ where $T_\alpha = \{\alpha\} \times \kappa_\alpha$ for each $\alpha < \delta$. Let $S_\alpha = \{\alpha\} \times \lambda_\alpha$, and $S = \bigcup\{S_\alpha : \alpha < \delta\}$.
(T2) For each $s \in T_\alpha$ and $t \in T_\beta$, if $s \prec t$ then $\alpha < \beta$ and $s \in S_\alpha$.
(T3) For every $\{s, t\} \in [T]^2$ there is a finite subset $i\{s, t\}$ of S such that for each $u \in T$:

$$(u \preceq s \wedge u \preceq t) \text{ iff } u \preceq v \text{ for some } v \in i\{s, t\}.$$

(T4) For $\alpha < \beta < \delta$, if $t \in T_\beta$ then the set $\{s \in S_\alpha : s \prec t\}$ is infinite.

Lemma 4.5. *If there is a $\mathfrak{t}$-poset with an $\mathfrak{s}$-stem then there is an LCS space X with stem Y such that $SEQ(X) = \mathfrak{t}$ and $SEQ(Y) = \mathfrak{s}$.*

Indeed, if $\mathcal{T} = \langle T, \prec \rangle$ is an $\mathfrak{s}$-poset, we write $U_{\mathcal{T}}(x) = \{y \in T : y \preceq x\}$ for $x \in T$, and we denote by $X_{\mathcal{T}}$ the topological space on T whose subbase is the family

$$\{U_{\mathcal{T}}(x), T \setminus U_{\mathcal{T}}(x) : x \in T\}, \tag{15}$$

then $X_{\mathcal{T}}$ is our desired LCS-space with stem.

So, to prove Theorem 4.2 it will be enough to show that a $\langle \lambda \rangle_{\omega_2}$-poset with an $\langle \omega \rangle_{\omega_2}$-stem may exist.

We follow the ideas of [2] to construct P. Fix a $\Delta(\omega_2\times\lambda)$-function $f : \big[\omega_2 \times \lambda\big]^2 \to \big[\omega_2\big]^{<\omega}$.

Definition 4.6. Define the poset $\mathcal{P} = \langle P, \preceq\rangle$ as follows. The underlying set P consists of triples $p = \langle a_p, \le_p, i_p\rangle$ satisfying the following requirements:

(1) $a_p \in \big[\omega_2 \times \lambda\big]^{<\omega}$,
(2) $\le_p$ is a partial ordering on a_p with the property that if $x <_p y$ then $x \in \omega_2 \times \omega$ and $\pi(x) < \pi(y)$,
(3) $i_p : \big[a_p\big]^2 \to \big[a_p\big]^{<\omega}$ is such that
  (3.1) if $\{x, y\} \in \big[a_p\big]^2$ then
    (3.1.1) if $x, y \in \omega_2 \times \omega$ and $\pi(x) = \pi(y)$ then $i_p\{x, y\} = \emptyset$,
    (3.1.2) if $x <_p y$ then $i_p\{x, y\} = \{x\}$,
    (3.1.3) if x and y are $<_p$-incomparable, then $i_p\{x, y\} \subset f\{x, y\} \times \omega$.
  (3.2) if $\{x, y\} \in \big[a_p\big]^2$ and $z \in a_p$ then $\big((z \le_p x \wedge z \le_p y) \text{ iff } \exists t \in i_p\{x, y\}\ z \le_p t\big)$.

Set $p \preceq q$ iff $a_p \supseteq a_q$, $\le_p\restriction a_q = \le_q$ and $i_p \restriction \big[a_q\big]^2 = i_q$.

Lemma 4.7. *P satisfies ω_1-c.c..*

Proof. Let $\{p_\nu : \nu < \omega_1\} \subset P$, $p_v = \langle a_\nu, \le_\nu, i_\nu\rangle$. By thinning out our sequence we can assume that

(i) $\{a_\nu : \nu < \omega_1\}$ forms a Δ-system with kernel a'.
(ii) $i_\nu \restriction \big[a'\big]^2 = i$.
(iii) $\le_\nu\restriction a' \times a' = \le$.
(iv) for each $\nu < \mu < \omega_1$ there is a bijection $\rho_{\nu,\mu} : a_\nu \to a_\mu$ such that
  (a) $\rho_{\nu,\mu} \restriction a' = \mathrm{id}$
  (b) $\pi(x) \le \pi(y)$ iff $\pi(\rho_{\nu,\mu}(x)) \le \pi(\rho_{\nu,\mu}(y))$,
  (c) $x \le_\nu y$ iff $\rho_{\nu,\mu}(x) \le_\mu \rho_{\nu,\mu}(y)$,
  (d) $x \in \omega_2 \times \omega$ iff $\rho_{\nu,\mu}(x) \in \omega_2 \times \omega$,
  (e) $\rho_{\nu,\mu}[i_\nu\{x, y\}] = i_\mu\{\rho_{\nu,\mu}(x), \rho_{\nu,\mu}(y)\}$.

Now it follows from condition (3.1) and condition (iv) that if $\nu < \mu < \omega_2$ and $\{x, y\} \in \big[a'\big]^2$ then $i_\nu\{x, y\} = i_\mu\{x, y\}$.

Since f is a $\Delta(\omega_2 \times \lambda)$-function there is $\nu < \mu < \omega_1$ such that a_ν and a_μ are good for f, i.e. (S1)–(S3) hold. Define $r = \langle a, \le, i\rangle$ as follows:

(a) $a = a_\nu \cup a_\mu$,
(b) $x \le y$ iff $x \le_\nu y$ or $x \le_\mu y$ or there is $s \in a_\nu \cap a_\mu$ such that $x \le_\nu s \le_\mu y$ or $x \le_\mu s \le_\nu y$,
(c) $i \supset i_\nu \cup i_\mu$,
(d) for $x \in a_\nu \setminus a_\mu$ and $y \in a_\mu \setminus a_\nu$, if x and y are $\le$-incomparable then

$$i\{x, y\} = \big(f\{x, y\} \times \omega\big) \cap \{t \in a : t \le x \wedge t \le y\}. \tag{16}$$

(e) for $\{x, y\} \in \big[a\big]^2$ with $x < y$, $i\{x, y\} = \{x\}$.

We claim that $r \in P$.

By the construction, we have $\le\restriction a_\nu \times a_\nu = \le_\nu$ and $\le\restriction a_\mu \times a_\mu = \le_\mu$.

Claim: $\le$ is a partial order.

We should check only the transitivity. Assume $x \le y \le z$. If $x \le_\nu y \le_\nu z$ or $x \le_\mu y \le_\mu z$ then we are done. Assume that $x \le_\nu u \le_\mu y \le_\mu z$ for some $u \in a_\nu \cap a_\mu$. Then $x \le_\nu u \le_\mu z$ so $x \le z$.

If $x \le_\nu u \le_\mu y \le_\mu t \le_\nu z$ for some $u, t \in a_\nu \cap a_\mu$, then $u \le_\mu t$, which implies $u \le_\nu t$. Thus $x \le_\nu u \le_\nu t \le_\nu z$ and so $x \le_\nu z$, and hence $x \le z$.

The other cases are similar to these ones.

(3.1.3) holds by the construction of i.

To show that p is a condition we should finally check (3.2). Let $x, y \in a$ be $\le$-incomparable elements. It is clear that if $u \le t$ for some $t \in i\{x, y\}$ then $u \le x$ and $\le y$. So we should check that

$(*)$ if $z \le x$ and $z \le y$ then there is $t \in i\{x, y\}$ such that $z \le t$.

If $x, y, z \in a_\nu$ or $x, y, z \in a_\mu$ then it is clear because $p_\nu, p_\mu \in P$.

Case 1. $x, y \in a_\nu$ *and* $z \in a_\mu \setminus a_\nu$.

Subcase 1.1 $x, y \in a_\nu \setminus a_\mu$.

There are $x', y' \in a_\nu \cap a_\mu$ such that $z \le_\mu x' \le_\nu x$ and $z \le_\mu y' \le_\nu y$. Then there is $t' \in i_\mu\{x', y'\}$ such that $z \le_\mu t'$. Then $t' \in a_\nu \cap a_\mu$, so $t' \le_\nu x, y$. Thus there is $t \in i_\nu\{x, y\}$ such that $t' \le_\nu t$, and so $z \le t$. Since $i\{x, y\} = i_\nu\{x, y\}$, we are done.

Subcase 1.2 $x \in a_\nu \setminus a_\mu$ *and* $y \in a_\nu \cap a_\mu$

Put $y' = y$, then proceed as in Subcase 1.1.

Case 2. $x, z \in a_\nu \setminus a_\mu$ *and* $y \in a_\mu \setminus a_\nu$.

Then $z \le_\nu y' \le_\mu y$ for some $y' \in a_\nu \cap a_\mu$. Then there is $t \in i_\nu\{x, y'\}$ such that $z \le_\nu t$. Clearly $t \le x, y$. We show that $t \in i\{x, y\}$.

If $t = y'$ then $t \le x, y$ and $\pi(t) \in f\{x, y\}$ by (S1). Thus $t \in i\{x, y\}$.

Assume that $t <_\nu y'$. Then $\pi(t) \in f\{x, y'\} \subset f\{x, y\}$ by (S2), because $y' \in a_\nu \cap a_\mu$ and $\pi(y') < \pi(y)$. Thus $t \in i\{x, y\}$ by (16). □

Assume that $\mathcal{G}$ is a $\mathcal{P}$-generic filter. We claim that if we take

$$\ll = \cup\{\le_p : p \in \mathcal{G}\}.$$

then $\langle \omega_2 \times \lambda, \ll \rangle$ is a $\langle \lambda \rangle_{\omega_2}$-poset with an $\langle \omega \rangle_{\omega_2}$-stem. By standard density arguments, $\ll$ is a partial order on $\omega_2 \times \lambda$ which satisfies (T4). Moreover, every $p \in P$ satisfies (2), so (T2) also holds. Finally the function

$$i = \bigcup \{i_p : p \in \mathcal{G}\}$$

witnesses (T3) because every $p \in P$ satisfies (3.2). □

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