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UNIVERSAL LOCALLY COMPACT SCATTERED SPACES

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ABSTRACT. If δ is an ordinal, we denote by $\mathcal{C}(\delta)$ the class of all cardinal sequences of length δ of locally compact scattered (in short: LCS) spaces. If λ is an infinite cardinal, we write

$$\mathcal{C}_\lambda(\delta) = \{s \in \mathcal{C}(\delta) : s(0) = \lambda = \min[s(\zeta) : \zeta < \delta]\}.$$

An LCS space X is called $\mathcal{C}_\lambda(\delta)$-*universal* if $\mathrm{SEQ}(X) \in \mathcal{C}_\lambda(\delta)$, and for each sequence $s \in \mathcal{C}_\lambda(\delta)$ there is an open subspace Y of X with $\mathrm{SEQ}(Y) = s$.

We show that

- there is a $\mathcal{C}_\omega(\omega_1)$-universal LCS space,
- under CH there is a $\mathcal{C}_\omega(\delta)$-universal LCS space for every ordinal $\delta < \omega_2$,
- under GCH for every infinite cardinal λ and every ordinal $\delta < \omega_2$, there is a $\mathcal{C}_\lambda(\delta)$-universal LCS space,
- there may exist a $C_\omega(\omega_2)$-universal LCS space.

As a consequence, we obtain that it is consistent that $2^\omega = \omega_2$ and $\mathcal{C}_\omega(\omega_2)$ is large as possible, i.e.

$$\mathcal{C}_\omega(\omega_2) = \{s \in {}^{\omega_2}\{\omega, \omega_1, \omega_2\} : s(0) = \omega\}.$$

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1. Introduction

If X is a locally compact, scattered (in short: LCS) space and α is an ordinal, we denote by $I_\alpha(X)$ the α^{th} Cantor-Bendixson level of X. If $x \in X$, we define the *rank of* x *in* X as

$$\mathrm{rk}(x) = \text{ the ordinal } \alpha \text{ such that } x \in I_\alpha(X).$$

If U is a neighbourhood of x, we say that U is a *cone on* x, if x is the only point of U of rank $\geq \mathrm{rk}(x)$. It is well-known that the collection formed by the compact open cones on a point x is a neighbourhood base of x.

We define the *height of* an LCS space X as

$$\mathrm{ht}(X) = \text{ the least ordinal } \alpha \text{ such that } I_\alpha(X) = \emptyset.$$

Then we define the *cardinal sequence of* X, in symbols $\mathrm{SEQ}(X)$, as the sequence formed by the cardinalities of the infinite levels of X.

If δ is an ordinal, we denote by $\mathcal{C}(\delta)$ the class of all cardinal sequences of length δ of LCS spaces. If δ is an ordinal and λ is an infinite cardinal, we write

$$\mathcal{C}_\lambda(\delta) = \{s \in \mathcal{C}(\delta) : s(0) = \lambda = \min[s(\zeta) : \zeta < \delta]\}.$$

In [4], the authors give a full description under GCH of the classes $\mathcal{C}_\lambda(\delta)$ for every ordinal $\delta < \omega_2$ and every infinite cardinal λ, and they also show that for every ordinal α the class $\mathcal{C}(\alpha)$ is characterized if the classes $\mathcal{C}_\lambda(\beta)$ are characterized for every infinite cardinal λ and every ordinal $\beta \leq \alpha$.

Assume that δ is an ordinal and λ is an infinite cardinal. We say that an LCS space X is $\mathcal{C}_\lambda(\delta)$-*universal*, if $\mathrm{SEQ}(X) \in \mathcal{C}_\lambda(\delta)$ and for each sequence $s \in \mathcal{C}_\lambda(\delta)$ there is an open subspace Y of X with $\mathrm{SEQ}(Y) = s$.

The following definitions will be used in the sequel. Assume that κ is an infinite cardinal and α is an ordinal. Assume that L is a subset of α. We say that L is κ-*closed in* α, if $\sup\langle \alpha_i : i < \kappa\rangle \in L \cup \{\alpha\}$ for each increasing sequence $\langle \alpha_i : i < \kappa\rangle \in {}^{\kappa}L$. And we say that L is *successor closed in* α, if $\beta + 1 \in L \cup \{\alpha\}$ for all $\beta \in L$.

Why are the universal spaces important and interesting objects? In the last twenty years there were many results saying that in certain models certain sequences of cardinals are or are not cardinal sequences of LCS spaces. In the recent years the attention was turned to the characterization of whole classes $\mathcal{C}_\lambda(\beta)$. As we will see, the universal spaces are useful tools in such characterizations. The main result of [7] says that for each uncountable regular cardinal λ and ordinal $\alpha < \lambda^{++}$ it is consistent with GCH that $\mathcal{C}_\lambda(\alpha)$ is as large as possible, i.e.

$$(\odot) \qquad \mathcal{C}_\lambda(\alpha) = \mathcal{D}_\lambda(\alpha),$$

where

$$\mathcal{D}_\omega(\alpha) = \{f \in {}^\alpha\{\omega, \omega_1\} : f(0) = \omega\},$$

and if λ is uncountable,

$$\mathcal{D}_\lambda(\alpha) = \{f \in {}^\alpha\{\lambda, \lambda^+\} : f(0) = \lambda, \\ f^{-1}\{\lambda\} \text{ is } < \lambda\text{-closed and successor-closed in } \alpha\}.$$

The natural idea to prove $(\odot)$ above is to try to carry out some iterated forcing in such a way that in each step we add a space X_f to the intermediate model with cardinal sequence f for some $f \in \mathcal{D}_\lambda(\alpha)$. Since typically $|X_f| = \lambda^+$ and we want to preserve the cardinals, we try to find an iteration of λ-complete, λ^+-c.c. posets. However, in each step we introduce new subsets of λ and the length of the iteration is at least $|\mathcal{D}_\lambda(\alpha)| = \lambda^{++}$! Hence in the final model λ will have at least λ^{++} many new subsets, i.e. $2^\lambda > \lambda^+$.

Here come the universal spaces into the picture. A $\mathcal{C}_\lambda(\delta)$-universal space has cardinality λ^+ so we may hope that there is a λ-complete, λ^+-c.c. poset P of cardinality λ^+ such that V^P contains a $\mathcal{C}_\lambda(\delta)$-universal space. In this case $(2^\lambda)^{V^P} \le ((|P|^\lambda)^\lambda)^V = \lambda^+$. So in the generic extension we might have GCH.

As it turned out, this idea worked in the proof of the above mentioned result from [7].

The main result of [7] does not apply to the classes $\mathcal{C}_\omega(\delta)$, $\delta < \omega_2$. However, we shall prove here that CH implies the existence of a $\mathcal{C}_\omega(\delta)$-universal LCS space for every ordinal $\delta < \omega_2$, and that GCH implies the existence of a $\mathcal{C}_\lambda(\delta)$-universal LCS space for each infinite cardinal λ and each ordinal $\delta < \omega_2$.

Bagaria faced a similar problem in [1]. He proved that if $MA_{\aleph_2}$ holds and there is a Δ-function (see [2]), then

$$\mathcal{C}_\omega(\omega_2) \supseteq \{s \in {}^{\omega_2}\{\omega, \omega_1\} : s(0) = \omega\}.$$

However, $MA_{\aleph_2}$ implies $2^{\omega_0} \geq \omega_3$, and if $2^{\omega_0} = \omega_\alpha$, then the natural "upper bound" of $\mathcal{C}_\omega(\omega_2)$ is a much larger family of sequences:

$$\mathcal{C}_\omega(\omega_2) \subseteq \{s \in {}^{\omega_2}\{\omega_\nu : \nu \leq \alpha\} : s(0) = \omega\}.$$

Using universal spaces we prove Theorem 3.2 claiming that it is consistent that $2^\omega = \omega_2$ and $\mathcal{C}_\omega(\omega_2)$ is large as possible, i.e.

$$\mathcal{C}_\omega(\omega_2) = \{s \in {}^{\omega_2}\{\omega, \omega_1, \omega_2\} : s(0) = \omega\}.$$

Our set-theoretic and topological notation is standard.

We shall use the notation $\langle\kappa\rangle_\alpha$ to denote the constant κ-valued sequence of length α. Let us denote the concatenation of two sequences f and g of by $f^\frown g$.

2. $\mathcal{C}_\lambda(\delta)$-universal spaces for $\delta < \omega_2$

In this section our aim is to carry out some constructions of $\mathcal{C}_\lambda(\delta)$-universal LCS spaces for $\delta < \omega_2$. First, we need some preparation.

Assume that X is an LCS space, $x \in X \setminus I_0(X)$ and σ_x is a neighbourhood base for x. We say that σ_x is an *admissible base for* x, if there is a pairwise disjoint family $\{U_n^{(x)} : n < \omega\}$ such that for every $n < \omega$, $U_n^{(x)}$ is a compact open cone on some point $x_n \in X$ with $\mathrm{rk}(x_n) < \mathrm{rk}(x)$ in such a way that σ_x is the collection of sets of the form

$$\{x\} \cup \bigcup\{U_n^{(x)} : n \geq m\},$$

where $m < \omega$. Then, we will say that σ_x is the *admissible base for* x *given by* $\{U_n^{(x)} : n < \omega\}$.

In what follows, by an *enumeration* of an infinite set a we mean an enumeration of a without repetitions. By a *decomposition* of an infinite set a we mean a partition of a in infinite subsets.

The proof of the following result is essentially contained in the proofs of [5, Lemma 8 and Theorem 9].

Theorem 2.1. *There is a $\mathcal{C}_\omega(\omega_1)$-universal LCS space.*

Proof. We write $C_n = \omega_1 \times \{n\}$ for every $n < \omega$. Our aim is to construct an LCS space X of height ω_1 such that $I_0(X) = \{0\} \times \omega$, $I_\xi(X) = \{\xi\} \times 2^\omega$ for $0 < \xi < \omega_1$ and $I_{\omega_1}(X) = \emptyset$, in such a way that for every $x \in X$ there is a neighbourhood U of x with $U \setminus \{x\} \subseteq \bigcup\{C_n : n < \omega\}$. To check that such a space X is $\mathcal{C}_\omega(\omega_1)$-universal, consider a sequence $s = \langle \kappa_\xi : \xi < \omega_1 \rangle \in \mathcal{C}_\omega(\omega_1)$. Since $\kappa_0 = \omega$, we have $\kappa_\xi \leq 2^\omega$ for each $\xi < \omega_1$. Take $Z = \bigcup\{\{\xi\} \times \kappa_\xi : \xi < \omega_1\}$ with the relative topology of X. Clearly, Z is an open subspace of X with $\mathrm{SEQ}(Z) = s$.

First, we need to construct an LCS space Y such that the following holds:

($*$) (1) The height of Y is ω_1, $I_\xi(Y) = \{\xi\} \times \omega$ for every $\xi < \omega_1$ and $I_{\omega_1}(Y) = \emptyset$.
(2) C_0 is a closed discrete subspace of Y.
(3) Every point $x \in Y \setminus I_0(Y)$ has an admissible base given by a collection $\{U_n^{(x)} : n < \omega\}$ such that $U_n^{(x)} \cap C_0 = \emptyset$ for every $n < \omega$.

Before constructing the space Y, we show how to construct the desired space X from Y. We put $C_0' = C_0 \setminus \{(0,0)\}$. For every $x \in C_0'$ we consider a set $\{x^{(\xi)} : \xi < 2^\omega\}$ of pairwise different elements with $Y \cap \{x^{(\xi)} : \xi < 2^\omega\} = \emptyset$ and such that if $x, y \in C_0'$ with $x \neq y$ then $x^{(\xi)} \neq y^{(\eta)}$ for every $\xi, \eta < 2^\omega$. Also, we consider an almost disjoint family $\{a_\xi : \xi < 2^\omega\}$ of infinite subsets of ω where $a_\xi \neq a_\eta$ for $\xi < \eta < 2^\omega$. Then, the underlying set of X is the set

$$(Y \setminus C_0') \cup \bigcup\{\{x^{(\xi)} : \xi < 2^\omega\} : x \in C_0'\}.$$

If $y \in Y \setminus C_0'$, then a basic neighbourhood of y in X is a basic neighbourhood U of y in Y with $U \cap C_0' = \emptyset$. Now assume that $y = x^{(\xi)}$ for some $x \in C_0'$ and $\xi < 2^\omega$. Consider the pairwise disjoint family $\{U_n^{(x)} : n < \omega\}$ associated with x given by ($*$)(3). Then we take as a basic neighbourhood of y in X a set of the form

$$\{x^{(\xi)}\} \cup \bigcup\{U_n^{(x)} : n \in a_\xi, n \geq m\}$$

where $m < \omega$.

Since Y is Hausdorff and for every $x \in C_0'$ and every basic neighbourhood U of x in Y we have that $U \setminus \{x\}$ is the disjoint union of $\{U_n^{(x)} : n \geq m\}$ for some $m < \omega$, we infer that X is also Hausdorff. Then, it is easy to check that X is the desired space.

Now, we construct the space Y satisfying ($*$)(1) $-$ (3). For this, we construct by transfinite induction on $\xi < \omega_1$ an LCS space Y_ξ satisfying the following conditions:

(1) The height of Y_ξ is $(\xi+1)$, $I_\mu(Y_\xi) = \{\mu\}\times\omega$ for each $\mu \le \xi$ and $I_{\xi+1}(Y_\xi) = \emptyset$.
(2) $\{(\alpha, 0) : \alpha \le \xi\}$ is a closed discrete subspace of Y_ξ.
(3) Every point $x \in Y_\xi \setminus I_0(Y_\xi)$ has an admissible base given by a collection $\{U_n^{(x)} : n < \omega\}$ such that $U_n^{(x)} \cap C_0 = \emptyset$ for every $n < \omega$.
(4) If $\xi < \eta$ and $x \in Y_\xi$, then a neighbourhood base of x in Y_ξ is also a neighbourhood base of x in Y_η

We define Y_0 as the set $\{0\}\times\omega$ with the discrete topology. Assume $\xi > 0$. We may suppose that ξ is a limit. If ξ is a successor ordinal, the considerations are similar. Let Z be the direct union of $\{Y_\mu : \mu < \xi\}$. Let $\{x_k : k < \omega\}$ be an enumeration of Z. For each $n < \omega$ we take a compact open cone U_n on some u_n in Z as follows. We take U_0 as a compact open cone on x_0. Suppose that $n > 0$. First, assume that $n = 2k$ for some $k \ge 1$. Let u_n be the first element in the enumeration $\{x_k : k < \omega\}$ such that $u_n \notin (U_0 \cup \cdots \cup U_{n-1})$. Then we choose U_n as a compact open cone on u_n such that $U_n \cap (U_0 \cup \cdots \cup U_{n-1}) = \emptyset$. Now assume that $n = 2k+1$ for some $k \ge 0$. Let u_n be the first element in $\{x_k : k < \omega\}$ such that $u_n \notin C_0$ and $\mathrm{rk}(u_n) > \mathrm{rk}(u_m)$ for all $m < n$. Then we take U_n as a compact open cone on u_n such that $U_n \cap (U_0 \cup \cdots \cup U_{n-1}) = \emptyset$ and $U_n \cap C_0 = \emptyset$.

Let $\{y_n : n < \omega\}$ be an enumeration of $\{\xi\}\times\omega$. Let $\langle \xi_n : n < \omega\rangle$ be a sequence of ordinals converging to ξ in a strictly increasing way. For each $n < \omega$ we consider the first element u_{2k+1} for some $k \ge 0$ such that $\mathrm{rk}(u_{2k+1}) \ge \xi_n$ and then we put $V_n = U_{2k+1}$. Clearly, $\{V_n : n < \omega\}$ is a discrete family in Z. Let $\{a_n : n < \omega\}$ be a decomposition of ω. For each $n < \omega$ we define a basic neighbourhood of y_n in Y_ξ as a set of the form

$$\{y_n\} \cup \bigcup\{V_m : m \in a_n \setminus l\}$$

where $l < \omega$. Also, if $x \in Y_\mu$ for some $\mu < \xi$, then a basic neighbourhood of x in Y_ξ is a basic neighbourhood of x in Y_μ.

Let Y be the direct union of $\{Y_\xi : \xi < \omega_1\}$. Then, Y is as required. □

It was proved in [6] that it is consistent with ZFC that there is no LCS space X of height ω_1+1 such that $|I_\xi(X)| = \omega$ for every $\xi < \omega_1$

and $|I_{\omega_1}(X)| = 2^\omega$. So, we can not extend the general construction given in the proof of Theorem 2.1 to the class $\mathcal{C}_\omega(\omega_1+1)$. However, we can prove the following result.

Theorem 2.2. *(CH) There is a $\mathcal{C}_\omega(\delta)$-universal LCS space for every ordinal $\delta < \omega_2$.*

Proof . For every $n < \omega$, we write $C_n = \delta \times \{n\}$. In order to prove the theorem, we will construct an LCS space X of height δ such that $I_0(X) = \{0\} \times \omega$, $I_\xi(X) = \{\xi\} \times \omega_1$ for $0 < \xi < \delta$ and $I_\delta(X) = \emptyset$, in such a way that for every $x \in X$ there is a neighbourhood U of x with $U \setminus \{x\} \subseteq \bigcup\{C_n : n < \omega\}$. Clearly, such a space is a $\mathcal{C}_\omega(\delta)$-universal LCS space under CH.

Without loss of generality, we may assume that $\omega_1 \leq \delta < \omega_2$ and δ is a limit ordinal. Let $\{\alpha_\xi : \xi < \omega_1\}$ be an enumeration of δ with $\alpha_n = n$ for each $n < \omega$. In order to find the desired space X, we construct by transfinite induction on $\xi \in [\omega, \omega_1]$ an space X_ξ such that the following holds:

(1) The underlying set of X_ξ is $\{0\}\times\omega\cup\bigcup\{\{\alpha_\mu\}\times\xi : 0 < \mu < \xi\}$.

(2) X_ξ is an LCS space such that if $\langle\beta_\zeta : \zeta < \xi'\rangle$ is the strictly increasing enumeration of $\{\alpha_\zeta : \zeta < \xi\}$, we have that $\mathrm{ht}(X_\xi) = \xi'$, $I_0(X) = \{0\}\times\omega$, $I_\zeta(X_\xi) = \{\beta_\zeta\}\times\xi$ for $0 < \zeta < \xi'$ and $I_{\xi'}(X_\xi) = \emptyset$.

(3) For every $x \in X_\xi$ there is a neighbourhood U of x such that $U \setminus \{x\} \subseteq \bigcup\{C_n : n < \omega\}$.

If $\mu < \xi \leq \omega_1, x \in X_\mu$ and U is a basic neighbourhood of x in X_μ, we will define a basic neighbourhood $U^{(\xi)}$ of x in X_ξ with $U \subseteq U^{(\xi)}$ in such a way that the following three conditions hold:

(a) If $\mu < \xi < \eta \leq \omega_1$ and $V = U^{(\xi)}$, then $U^{(\eta)} = V^{(\eta)}$.

(b) If $\mu < \xi \leq \omega_1$, $x, y \in X_\mu$ and U, V are basic neighbourhoods of x, y respectively in X_μ with $U \subseteq V$, then $U^{(\xi)} \subseteq V^{(\xi)}$.

(c) If $\mu < \xi \leq \omega_1$, $x, y \in X_\mu$ and U, V are basic neighbourhoods of x, y respectively in X_μ with $U \cap V = \emptyset$, then $U^{(\xi)} \cap V^{(\xi)} = \emptyset$.

We will have that X_{ω_1} is the required space.

The construction of the space X_ω is easy.

Assume that $\xi = \mu + 1$ where $\omega \leq \mu < \omega_1$. Suppose that $\langle\beta_\zeta : \zeta < \mu'\rangle$ is the strictly increasing enumeration of $\{\alpha_\zeta : \zeta < \mu\}$. In order to construct X_ξ, first we define for each ζ with $0 < \zeta < \mu'$ a countable LCS space Y_ζ such that $\mathrm{ht}(Y_\zeta) = \mu'$, $I_\gamma(Y_\zeta) = \{\beta_\gamma\} \times \xi$

for each γ with $0 < \gamma \leq \zeta$ and $I_\gamma(Y_\zeta) = I_\gamma(X_\mu)$ otherwise, and in such a way that for every $x \in Y_\zeta$ there is a neighbourhood U of x with $U \setminus \{x\} \subseteq \bigcup\{C_n : n < \omega\}$. Also, we will have that if $\eta < \zeta < \mu'$ and $y \in Y_\eta$, then a basic neighbourhood of y in Y_ζ is a basic neighbourhood of y in Y_η. We start defining Y_1. Put $Z = X_\mu \setminus (\{0\} \times \omega)$. Let $\{z_k : k < \omega\}$ be an enumeration of Z. For each $n < \omega$, we take a compact open neighbourhood U_n of some y_n in Z as follows. We take U_0 as a compact open cone on z_0. If $n > 0$, let y_n be the first element in the enumeration $\{z_k : k < \omega\}$ such that $y_n \notin U_0 \cup \ldots \cup U_{n-1}$. We choose U_n as a compact open cone on y_n such that $U_n \cap (U_0 \cup \ldots \cup U_{n-1}) = \emptyset$. We take an element $u_k \in I_0(X_\mu) \cap U_k$ for each $k \in \omega$. Then, we define a basic neighbourhood of $(1, \mu)$ as a set of the form $\{(1, \mu)\} \cup \{u_i : i \in \omega \setminus m\}$ where $m < \omega$. Now, assume that $\zeta = \eta + 1$ is a successor ordinal with $\eta \geq 1$. Since Y_η is countable, there is a discrete family $\{V_k : k < \omega\}$ in Y_η such that for every $k < \omega$, V_k is a compact open cone on some $y_k \in \bigcup\{I_\gamma(Y_\eta) : \zeta \leq \gamma < \mu'\}$ in such a way that $V_k \setminus \{y_k\} \subseteq \bigcup\{C_n : n < \omega\}$. Now, for each $k < \omega$, we take an element $u_k \in V_k \cap I_\eta(Y_\eta)$ and a compact open cone U_k on u_k with $U_k \subseteq V_k$. Put $z = (\beta_\zeta, \mu)$. Then, we define a basic neighbourhood of z in Y_ζ as a set of the form $\{z\} \cup \bigcup\{U_k : k > m\}$ where $m < \omega$. Now, assume that ζ is a limit ordinal. Let Z be the direct union of the spaces Y_η for $\eta < \zeta$. By using an argument similar to the one given above, we can define a neighbourhood base for the point (β_ζ, μ). Then, we define Y_ζ as the resulting space.

If μ' is a limit ordinal we define Y as the direct union of the spaces Y_ζ for $\zeta < \mu'$, and if $\mu' = \zeta + 1$ is a successor ordinal we define $Y = Y_\zeta$. We distinguish the following two cases:

Case 1. $\alpha_\mu < \beta_\zeta$ for some $\zeta < \mu'$.

Let γ be the least ordinal ζ such that $\alpha_\mu < \beta_\zeta$. We assume that γ is a successor ordinal $\eta+1$. If γ is a limit ordinal, the considerations are similar. First, we define the space Y' of underlying set $Y \cup (\{\alpha_\mu\} \times (\xi \setminus \omega))$ as follows. If $y \in Y$, a basic neighbourhood of y in Y' is a basic neighbourhood of y in Y. Now, let $\{z_n : n < \omega\}$ be an enumeration of $\{\alpha_\mu\} \times (\xi \setminus \omega)$. Let $\{V'_k : k < \omega\}$ be a discrete family in Y such that, for each $k < \omega$, V'_k is a compact open cone on some point $v'_k \in \bigcup\{I_\zeta(Y) : \gamma \leq \zeta\}$ such that $V'_k \setminus \{v'_k\} \subseteq \bigcup\{C_n : n < \omega\}$. For every $k < \omega$, we take a point $u'_k \in I_\eta(Y) \cap V'_k$ and a compact open cone U'_k on u'_k with $U'_k \subseteq V'_k$. Let $\{a_n : n < \omega\}$ be a decomposition of ω. We define a basic neighbourhood in Y' of a point z_n as a set of the form $\{z_n\} \cup \bigcup\{U'_k : k \in a_n \setminus l\}$ where $l < \omega$.

Now, if $x \in \bigcup\{\{\beta_\zeta\} \times \xi : \zeta \leq \eta\} \cup (\{\alpha_\mu\} \times (\xi \setminus \omega))$, we define a basic neighbourhood of x in X_ξ as a basic neighbourhood of x in Y'. Next, we define a neighbourhood base for each point in $\{\alpha_\mu\} \times \omega$. Let $\{v_k : k < \omega\}$ be an enumeration of $\{\beta_\gamma\} \times \xi$. For each $k < \omega$, we take a compact open cone V_k on v_k in Y' with $V_k \setminus \{v_k\} \subseteq \bigcup\{C_n : n < \omega\}$ such that $\{V_k : k < \omega\}$ is a pairwise disjoint family . Let $\{a_k : k < \omega\}$ be a decomposition of $\{\alpha_\mu\} \times \omega$. For $k < \omega$, put $a_k = \{y_m^{(k)} : m < \omega\}$. Fix $n < \omega$. Put $y_m = y_m^{(n)}$ for $m < \omega$. Let $\{u_k : k < \omega\}$ be an enumeration of $V_n \cap (\{\beta_\eta\} \times \omega)$. For each $k < \omega$, we take a compact open cone U_k on u_k with $U_k \subseteq V_n$ such that $\{U_k : k < \omega\}$ is a discrete family in $V_n \setminus \{v_n\}$. Now, we fix a decomposition $\{b_k : k < \omega\}$ of ω. Then, we define a basic neighbourhood in X_ξ of a point y_m as a set of the form $\{y_m\} \cup \bigcup\{U_l : l \in b_m \setminus k\}$ where $k < \omega$. We put $W_{y_m} = \{y_m\} \cup \bigcup\{U_l : l \in b_m\}$ for each $m < \omega$. Note that since V_n is compact, we have that if U is a neighbourhood of v_n in Y', then there is a $k < \omega$ such that $U_l \subseteq U \cap V_n$ for every $l \in \omega \setminus k$, and so $U_l \subseteq U$ for every $l \in \omega \setminus k$. Then, we define a basic neighbourhood of the point v_n in X_ξ as a set of the form

$$(U \cup \{y_m : m < \omega\}) \setminus (W_{y_1} \cup \cdots \cup W_{y_l})$$

where U is a basic neighbourhood of v_n in Y' and $l < \omega$. Now, assume that $x \in \bigcup\{\{\beta_\zeta\} \times \xi : \zeta > \gamma\}$. Then, we define a basic neighbourhood of x in X_ξ as a set of the form

$$U \cup \bigcup\{\{y_m^{(k)} : m < \omega\} : v_k \in U\}$$

where U is a basic neighbourhood of x in Y'.

Now, for every $x \in X_\mu$ and every compact open cone U on x in X_μ we define $U^{(\xi)}$ as follows. If $x \in I_\zeta(X_\mu)$ for some $\zeta \geq \gamma$ we put

$$U^{(\xi)} = U \cup \bigcup\{\{y_m^{(k)} : m < \omega\} : v_k \in U\}$$

and we put $U^{(\xi)} = U$ otherwise. Also, if $x \in X_\zeta$ with $\zeta < \mu$ and U is a compact open cone on x in X_ζ, we consider $V = U^{(\mu)}$ and then we define $U^{(\xi)} = V^{(\xi)}$.

Case 2. $\beta_\zeta < \alpha_\mu$ for every $\zeta < \mu'$.

Let $\{z_n : n < \omega\}$ be an enumeration of $\{\alpha_\mu\} \times \xi$. Without loss of generality, we may assume that μ' is a limit ordinal. Let $\langle \mu_k : k < \omega\rangle$ be a strictly increasing sequence of ordinals converging to μ'. For each $k < \omega$ we take a compact open cone V_k on some point v_k in Y such that $\operatorname{rk}(v_k) > \mu_k$, $V_k \setminus \{v_k\} \subseteq \bigcup\{C_n : n \geq 0\}$ and $\{V_k : k < \omega\}$ is a discrete family in Y. Now, for every $k \in \omega$ we choose a point $u_k \in V_k \setminus \{v_k\}$ with $\operatorname{rk}(u_k) \geq \mu_k$ and we take a compact open cone U_k on u_k with $U_k \subseteq V_k$. Consider a decomposition $\{a_n : n < \omega\}$ of ω. Fix $n < \omega$. We define a basic neighbourhood of z_n in X_ξ as a set of the form $\{z_n\} \cup \bigcup\{U_k : k \in a_n \setminus l\}$ where $l < \omega$.

Also, if $x \in Y$ we define a basic neighbourhood of x in X_ξ as a basic neighbourhood of x in Y.

If $x \in X_\mu$ and U is a compact open cone on x in X_μ, we define $U^{(\xi)} = U$. And if $x \in X_\zeta$ for some $\zeta < \mu$ and U is a compact open cone on x in X_ζ, we put $U^{(\xi)} = U^{(\mu)}$.

Next, assume that ξ is a limit ordinal. We want to define the space X_ξ. The underlying set of X_ξ is the union of the underlying sets of the spaces X_μ for $\mu < \xi$. If U is a compact open cone on a point in X_μ for $\mu < \xi$, we define

$$U^{(\xi)} = U \cup \bigcup\{U^{(\eta)} : \mu < \eta < \xi\}.$$

Assume that $x \in X_\xi$. We define a basic neighbourhood of x in X_ξ as a set of the form

$$U^{(\xi)} \setminus (V_1^{(\xi)} \cup \ldots \cup V_n^{(\xi)})$$

where U is a compact open cone on x in some space X_ζ with $\zeta < \xi$, $n < \omega$ and there are $\mu_1, \ldots, \mu_n < \xi$ such that for every $i = 1, \ldots, n$, V_i is a compact open cone on some $y_i \in U^{(\xi)} \setminus \{x\}$ in the space X_{μ_i}. It can be verified that if U is a compact open cone on x in some space X_ζ with $\zeta < \xi$, then $U^{(\xi)}$ is a compact open cone on x in X_ξ. It is clear that $U^{(\xi)}$ is an open cone on x. To show compactness, we consider the strictly increasing enumeration $\langle \beta_\mu : \mu < \xi'\rangle$ of $\{\alpha_\mu : \mu < \xi\}$ and then, proceeding by transfinite induction on $\mu < \xi'$, we can prove that if $x \in \{\beta_\mu\} \times \xi$ and U is a compact open cone on x in an space X_ζ with $\zeta < \xi$, then $U^{(\xi)}$ is compact in X_ξ.

It can be checked that X_{ω_1} is the required space. $\square$

Theorem 2.3. *(GCH) For every infinite cardinal λ and every ordinal $\delta < \omega_2$, there is a $\mathcal{C}_\lambda(\delta)$-universal LCS space.*

Proof . Assume that λ is an infinite cardinal and δ is an ordinal $< \omega_2$. Since GCH holds, for every $s \in \mathcal{C}_\lambda(\delta)$ we have $s(\alpha) \in \{\lambda, \lambda^+\}$ for each $\alpha < \delta$. If $\lambda = \omega$, we are done by Theorem 2.2. Assume $\lambda = \omega_1$. It follows from [4, Theorem 3.9] that if $\alpha < \beta \leq \delta$ and $\operatorname{cf}(\alpha) = \omega_1$ then there is an LCS space $X_{\alpha,\beta}$ of height β such that $I_\mu(X_{\alpha,\beta}) = \omega_1$ for each $\mu < \alpha$, $I_\mu(X_{\alpha,\beta}) = \omega_2$ for $\alpha \leq \mu < \beta$ and $I_\beta(X_{\alpha,\beta}) = \emptyset$. Let X be the disjoint union of $\{X_{\alpha,\beta} : \alpha < \beta \leq \delta,\ \operatorname{cf}(\alpha) = \omega_1\}$. Clearly, $X \in \mathcal{C}_{\omega_1}(\delta)$. Now consider a sequence $s \in \mathcal{C}_{\omega_1}(\delta)$. It follows from GCH that $s^{-1}(\omega_1)$ is successor closed and ω-closed in δ. Then, it is not difficult to see that there is a $\mathcal{Y} \subseteq \{X_{\alpha,\beta} : \alpha < \beta \leq \delta,\ \operatorname{cf}(\alpha) = \omega_1\}$ such that $\operatorname{SEQ}(\bigcup \mathcal{Y}) = s$. Hence, X is $\mathcal{C}_{\omega_1}(\delta)$-universal.

Finally, assume that $\lambda \geq \omega_2$. Note that $|\mathcal{C}_\lambda(\delta)| \leq 2^{|\delta|} \leq 2^{\omega_1} = \omega_2$. Then for each $s \in \mathcal{C}_\lambda(\delta)$ pick an LCS space X_s with $\operatorname{SEQ}(X_s) = s$, and take X as the disjoint union of the spaces X_s. Clearly, X is $\mathcal{C}_\lambda(\delta)$-universal. □

3. A $\mathcal{C}_\omega(\omega_2)$-universal space

Baumgartner and Shelah introduced the notion of Δ-functions in [2, Section 8]. In that paper they also proved that (a) *the existence of a Δ-function is consistent with ZFC + GCH*, (b) *if there is a Δ-function then $\langle\omega\rangle_{\omega_2} \in \mathcal{C}_\omega(\omega_2)$ holds in a "natural" c.c.c forcing extension of the ground model.* "Natural" means that the elements of the posets are just finite approximations of the locally compact right-separating neighbourhoods of the points of the desired space. Building on their method, Bagaria, [1], proved that

$$(\dagger)\qquad \mathcal{C}_\omega(\omega_2) \supseteq \{s \in {}^{\omega_2}\{\omega, \omega_1\} : s(0) = \omega\}.$$

is also consistent. More precisely, he showed that if there is a Δ-function and $MA_{\aleph_2}$ holds (which is a consistent assumption), then (†) above holds.

However, $MA_{\aleph_2}$ implies $2^{\omega_0} \geq \omega_3$, and if $2^{\omega_0} = \omega_\alpha$, then the natural "upper bound" of $\mathcal{C}_\omega(\omega_2)$ is a much larger family of sequences:

$$(\ddagger)\qquad \mathcal{C}_\omega(\omega_2) \subseteq \{s \in {}^{\omega_2}\{\omega_\nu : \nu \leq \alpha\} : s(0) = \omega\}.$$

These results naturally raised the following questions.

Problem 3.1. *Does* $\langle\omega\rangle_{\omega_2} \in \mathcal{C}_\omega(\omega_2)$ *imply* (†)*, or even*

$$(*) \qquad \mathcal{C}_\omega(\omega_2) \supseteq \{s \in {}^{\omega_2}\{\omega, \omega_1, \omega_2\} : s(0) = \omega\}.$$

Although these questions remain still open we prove Theorem 3.10 claiming that if there is a "natural" poset P such that $\langle\omega\rangle_{\omega_2} \in \mathcal{C}_\omega(\omega_2)$ holds in V^P then there is a natural poset Q such that $(*)$ holds in V^Q. Especially, the posets used by Bagaria can be constructed directly from the poset applied by Baumgartner and Shelah without even mentioning a Δ-function.

Moreover,

Theorem 3.2. Con*(ZFC)* $\longrightarrow$ Con*(ZFC +* $2^\omega = \omega_2$ *+ there is a* $\mathcal{C}_\omega(\omega_2)$*-universal LCS space witnessing that* $\mathcal{C}_\omega(\omega_2)$ *is as large as possible, i.e.*

$$\mathcal{C}_\omega(\omega_2) = \{s \in {}^{\omega_2}\{\omega, \omega_1, \omega_2\} : s(0) = \omega\}.$$

Before proving these results we need some preparation.

Let $T_0 = \{0\} \times \omega$, $T_\alpha = \{\alpha\} \times \omega_2$ for $1 \le \alpha < \omega_2$, and

$$T = \bigcup\{T_\alpha : \alpha < \omega_2\}.$$

Let $\pi : T \to \omega_2$ be the natural projection: $\pi(\langle\alpha, \xi\rangle) = \alpha$.

Definition 3.3. Define the poset $\mathcal{P}^* = \langle P^*, \preceq\rangle$ as follows. The underlying set P^* consists of triples $p = \langle a_p, \le_p, i_p\rangle$ satisfying the following requirements:

(1) $a_p \in \left[T\right]^{<\omega}$,
(2) $\le_p$ is a partial ordering on a_p with the property that if $x <_p y$ then $x \in \omega_2 \times \omega$ and $\pi(x) < \pi(y)$,
(3) $i_p : \left[a_p\right]^2 \to \left[a_p\right]^{<\omega}$ is such that
    (3.1) if $\{x, y\} \in \left[a_p\right]^2$ then
        (3.1.1) if $x, y \in \omega_2 \times \omega$ and $\pi(x) = \pi(y)$ then $i_p\{x, y\} = \emptyset$,
        (3.1.2) if $x <_p y$ then $i_p\{x, y\} = \{x\}$.
    (3.2) if $\{x, y\} \in \left[a_p\right]^2$ and $z \in a_p$ then
        $\big((z \le_p x \wedge z \le_p y)$ iff $\exists t \in i\{x, y\}\ z \le_p t\big)$.

Set $p \preceq q$ iff $a_p \supseteq a_q$, $\le_p\restriction a_q = \le_q$ and $i_p \restriction \left[a_q\right]^2 = i_q$.

Let $P^*_\omega = \{p \in P^* : a_p \subseteq \omega_2 \times \omega\}$ and $\mathcal{P}^*_\omega = \langle P^*_\omega, \preceq\rangle$.

Consider a function $d : \left[\omega_2\right]^2 \to \left[\omega_2\right]^{\le\omega}$. An element $p \in P^*_\omega$ is *d-good* iff

($\star_d$) if $\{x,y\} \in \big[a_p\big]^2$, $\pi(x) < \pi(y)$ and $x \not\leq_p y$ then
$$\pi'' i_p\{x,y\} \subseteq d(\pi(x), \pi(y)).$$
Let P^*_d be the family of d-good elements of P^*_ω and put $\mathcal{P}^*_d = \langle P^*_d, \preceq\rangle$.

Observation 3.4. Our poset P^*_d is just "the poset P defined from d" in [2, Section 7]. (Stipulation ($\star_d$) corresponds to (3.1.3), the other requirements have the same numbering here as in [2].)

A condition $r \in P$ is an *amalgamation of conditions p and q* iff $r \prec p$ and $r \prec q$, $a_r = a_p \cup a_q$, and $\leq_r$ is the partial ordering on a_r generated by $\leq_p \cup \leq_q$. Let $Q \subseteq P^*$. The poset $\mathcal{Q} = \langle Q, \prec\rangle$ has the *amalgamation property* iff every uncountable subset of Q contains two elements which have an amalgamation in Q.

Clearly the amalgamation property implies the countable chain condition.

Baumgartner and Shelah proved, [2, Theorem 8.1], that if $d : \big[\omega_2\big]^2 \to \big[\omega_2\big]^{\leq\omega}$ is a Δ-function then $\mathcal{P}^*_d$ has the countable chain condition. Actually, they proved the following:

Proposition 3.5. *If d is a Δ-function then $\mathcal{P}^*_d$ has the amalgamation property.*

For a condition $p \in P^*$ and $x \in (T) \setminus a_p$ define the condition $q = p \uplus \{x\}$, as follows. Let $a_q = a_p \cup \{x\}$. Put $u \leq_q t$ iff $u \leq_p t$ or $u = t = x$. Let $i_q\{u,t\} = i_p\{u,t\}$ unless u or t is x. Let $i_q\{u,x\} = \emptyset$. Clearly $q = p \uplus \{x\} \in P^*$.

Let $P' \subseteq P^*$. The poset $\mathcal{P}' = \langle P', \preceq\rangle$ has the *density property D* (or the *density property D_ω*) iff $p \uplus \{x\} \in P$ for each $p \in P$ and for each $x \in (T) \setminus a_p$ (or for each $x \in (\omega_2 \times \omega) \setminus a_p$, respectively).

For $p \in P^*$, $y \in a_p$ and $x \in (\omega_2 \times \omega) \setminus a_p$ with $\pi(x) < \pi(y)$ define the condition $q = p \uplus_y \{x\}$ as follows. Let $a_q = a_p \cup \{x\}$. Put $u \leq_q t$ iff $u \leq_p t$, or $u = x$ and $y \leq_p t$. Let $i_q\{u,t\} = i_p\{u,t\}$ unless u or t, say t, is x. Let $i_q\{x,u\} = x$ if $x \leq_q u$ and $i_q\{x,u\} = \emptyset$ otherwise.

Since x is a minimal element in $\leq_q$ we have $q = p \uplus_y \{x\} \in P^*$.

Let $P \subseteq P^*$. The poset $\mathcal{P} = \langle P, \prec\rangle$ has the *density property E* iff $p \uplus_x \{y\} \in P$ for each $p \in P$, $x \in a_p$ and $y \in (\omega_2 \times \omega) \setminus a_p$ with $\pi(y) < \pi(x)$.

The following claim is straightforward from the definition of $\mathcal{P}^*_d$.

Proposition 3.6. *For each function* $d : [\omega_2]^2 \to [\omega_2]^{\leq\omega}$ *the poset* $\mathcal{P}^*_d$ *has the density properties* D_ω *and* E.

Definition 3.7. (a) Let $Q \subseteq P^*_\omega$. We say that the poset $\mathcal{Q} = \langle Q, \preceq\rangle$ is a *BS-poset* iff $\mathcal{Q}$ has the amalgamation property, and the density properties D_ω and E.
(b) Let $P \subseteq P^*$. We say that the poset $\mathcal{P} = \langle P, \preceq\rangle$ is a *U-poset* iff $\mathcal{P}$ has the amalgamation property, and the density properties D and E.

In [2, Section 9] Baumgartner and Shelah also proved that

Proposition 3.8. *It is consistent that* $2^\omega \leq \omega_2$ *and there is a* Δ-*function* d.

Putting together Propositions 3.5, 3.6 and 3.8 we obtain

Proposition 3.9. *It is consistent that* $2^\omega \leq \omega_2$ *and there is a BS-poset* $\mathcal{Q} = \langle Q, \preceq\rangle$.

As we will see, Theorem 3.2 follows almost immediately from the Proposition above and from the next two theorems.

Theorem 3.10. *If there is a BS-poset then there is a U-poset as well.*

Theorem 3.11. *If* $\mathcal{P}$ *is a U-poset then in* $V^{\mathcal{P}}$ *there is an LCS space* X *such that* $SEQ(X) = \langle\omega\rangle ^\frown \langle\omega_2\rangle_{\omega_2} \in \mathcal{C}_\omega(\omega_2)$, *and for every* $s \in {}^{\omega_2}\{\omega, \omega_1, \omega_2\}$ *with* $s(0) = \omega$ *there is an open subspace* $Y \subseteq X$ *with* $SEQ(Y) = s$.

Proof of Theorem 3.2. By Proposition 3.9 and Theorem 3.10 we can assume that in the ground model we have $2^\omega \leq \omega_2$ and there is a U-poset $\mathcal{P} = \langle P, \preceq\rangle$. We show that the model $V^{\mathcal{P}}$ satisfies the requirements.

Since $|P| = \omega_2$, $\mathcal{P}$ satisfies c.c.c and $2^\omega \leq \omega_2$, we have $(2^\omega)^{V^{\mathcal{P}}} \leq ((|\mathcal{P}|^\omega)^\omega)^V = \omega_2$.

By Theorem 3.11, in $V^{\mathcal{P}}$ there is an LCS space X such that $SEQ(X) \in \mathcal{C}_\omega(\omega_2)$ and

($\bullet$) $\{\mathrm{SEQ}(Y) : Y \subseteq X \text{ is open }\} \supseteq \{s \in {}^{\omega_2}\{\omega, \omega_1, \omega_2\} : s(0) = \omega\}$.

Since $|X| \geq \omega_2$ and $|\,\mathrm{I}_0(X)| = \omega$, we have $2^\omega \geq \omega_2$ in $V^{\mathcal{P}}$. So $2^\omega = \omega_2$ in $V^{\mathcal{P}}$.

Thus

$$(\circ) \qquad \mathcal{C}_\omega(\omega_2) \subseteq \{s \in {}^{\omega_2}\{\omega, \omega_1, \omega_2\} : s(0) = \omega\}.$$

But $(\bullet)$ and $(\circ)$ together yield that

$$\mathcal{C}_\omega(\omega_2) = \{s \in {}^{\omega_2}\{\omega, \omega_1, \omega_2\} : s(0) = \omega\},$$

and that X is a $\mathcal{C}_\omega(\omega_2)$-universal space. $\square_{3.2}$

Proof of Theorem 3.11. Let $\mathcal{G}$ be a $\mathcal{P}$-generic filter. Recall that $T_0 = \{0\} \times \omega$, $T_\alpha = \{\alpha\} \times \omega_2$ for $1 \le \alpha < \omega_2$, and $T = \bigcup\{T_\alpha : \alpha < \omega_2\}$ Let

$$\le_{\mathcal{G}} = \bigcup\{\le_p : p \in \mathcal{G}\}$$

and for each $x \in T$ put

$$U(x) = \{z \in T : z \le_{\mathcal{G}} x\}.$$

Let $X = \langle T, \tau\rangle$ be the LCS space generated by the family $\{U(x) : x \in T\}$. Density properties D and E imply that $\mathrm{I}_\alpha(X) = T_\alpha$ for $\alpha < \omega_2$.

Let $s \in {}^{\omega_2}\{\omega, \omega_1, \omega_2\}$ with $s(0) = \omega$. Put

$$Y = \{\langle \alpha, \xi\rangle : \alpha < \omega_2, \xi < s(\alpha)\}.$$

If $x \in \mathrm{I}_\alpha(X)$ then $U(x) \setminus \{x\} \subseteq \alpha \times \omega$. Hence

$$Y = \bigcup\{U(y) : y \in Y\},$$

therefore Y is open. Thus $\mathrm{I}_\alpha(Y) = \mathrm{I}_\alpha(X) \cap Y$, that is, $\mathrm{I}_\alpha(Y) = \{\alpha\} \times s(\alpha)$. Thus the cardinal sequence of Y is exactly s. $\square_{3.11}$

Proof of Theorem 3.10. Fix an injective function $\varphi : \omega_2 \times \omega_2 \xrightarrow{1\text{–}1} \omega_2 \times \omega$ such that

(A) if $\pi(x) < \pi(y)$ and $x \in \omega_2 \times \omega$ then $\pi(\varphi(x)) < \pi(\varphi(y))$,

(B) if $x \ne y$ then $\pi(\varphi(x)) = \pi(\varphi(y))$ iff $\pi(x) = \pi(y)$ and $x, y \in \omega_2 \times \omega$.

"Lift" this φ to a function $\varphi : P^* \to P^*_\omega$ in the natural way: for $p = \langle a_p, \le_p, i_p\rangle \in P^*$ let $\varphi(p) = \left\langle a_{\varphi(p)}, \le_{\varphi(p)}, i_{\varphi(p)}\right\rangle$, where $a_{\varphi(p)} = \varphi'' a_p$, $\varphi(x) \le_{\varphi(p)} \varphi(y)$ iff $x \le_p y$ and $i_{\varphi(p)}\{\varphi(x), \varphi(y)\} = \varphi'' i_p\{x, y\}$.

Let $\mathcal{Q} = \langle Q, \preceq\rangle$ be a BS-poset. Take

$$P = \{p \in P^* : \varphi(p) \in Q\}$$

and $\mathcal{P} = \langle P, \preceq\rangle$.

We claim that $\mathcal{P}$ is a U-poset. Before proving it we need some preparation.

Claim 3.12. $\varphi(p) \in P^*_\omega$ for each $p \in P^*$.

Proof. To check that $\varphi(p)$ satisfies (2), assume that $\varphi(x) <_{\varphi(p)} \varphi(y)$. Then $\{x, y\} \in \left[a_p\right]^2$ with $x <_p y$ and hence $\pi(x) < \pi(y)$ and $x \in \omega_2 \times \omega$ by applying (2) for p. Thus, by (A), we have $\pi(\varphi(x)) < \pi(\varphi(y))$, and so (2) holds for $\varphi(p)$.

To check that $\varphi(p)$ satisfies (3.1.1), assume that $\pi(\varphi(x)) = \pi(\varphi(y))$. Then, by (B), $\pi(x) = \pi(y)$ and $x, y \in \omega_2 \times \omega$. Hence, applying (3.1.1) for p, we have $i_p\{x, y\} = \emptyset$. Thus $i_{\varphi(p)}\{\varphi(x), \varphi(y)\} = \varphi'' i_p\{x, y\} = \emptyset$.

$\varphi(p)$ clearly satisfies the other requirements. □

Claim 3.13. $p \prec q$ iff $\varphi(p) \prec \varphi(q)$.

Straightforward. □

Claim 3.14. If $p \uplus_y \{x\}$ is defined then $\varphi(p) \uplus_{\varphi(y)} \{\varphi(x)\}$ is also defined and $\varphi(p \uplus_y \{x\}) = \varphi(p) \uplus_{\varphi(y)} \{\varphi(x)\}$.

Proof. If $p \uplus_y \{x\}$ is defined then $\pi(x) < \pi(y)$ and $x \in \omega_2 \times \omega$. Hence, by (A), $\pi(\varphi(x)) < \pi(\varphi(y))$. Since $\varphi(x) \in \omega_2 \times \omega$, $\varphi(p) \uplus_{\varphi(y)} \{\varphi(x)\}$ is defined. The equality is clear. □

Claim 3.15. If $\varphi(p)$ and $\varphi(q)$ have an amalgamation s, then p and q have an amalgamation r with $\varphi(r) = s$.

Proof. Since we want $s = \varphi(r)$, we should define $r = \langle a_r, \leq_r, i_r \rangle$ as follows: $a_r = a_p \cup a_q$, $x \leq_r y$ iff $\varphi(x) \leq_s \varphi(y)$, $i_r\{x, y\} = \varphi^{-1} i_s\{\varphi(x), \varphi(y)\}$.

To check that r satisfies (2), assume that $x <_r y$. If $x <_p y$ or $x <_q y$ then $\pi(x) < \pi(y)$ and $x \in \omega_2 \times \omega$. Thus we can assume that e.g. $x \in a_p \setminus a_q$ and $y \in a_q \setminus a_p$. Since $\varphi(x) <_s \varphi(y)$ and $\leq_s$ is generated by $\leq_{\varphi(p)} \cup \leq_{\varphi(q)}$ there is $t \in a_{\varphi(p)} \cap a_{\varphi(q)}$ such that $\varphi(x) \leq_{\varphi(p)} t \leq_{\varphi(q)} \varphi(y)$. Let $u = \varphi^{-1}(t)$. Then as $u \in a_p \cap a_q$, we have $x <_p u <_q y$. Hence, applying (2) for p and q we have $\pi(x) < \pi(u) < \pi(y)$ and $x \in \omega_2 \times \omega$.

As for (3.1.1), assume that $\{x, y\} \in \left[a_r \cap (\omega_2 \times \omega)\right]^2$ with $\pi(x) = \pi(y)$. Then, by (B), $\pi(\varphi(x)) = \pi(\varphi(y))$. Since $\varphi(x), \varphi(y) \in a_s \subseteq \omega_2 \times \omega$, we can apply (3.1.1) for s to get $i_s\{\varphi(x), \varphi(y)\} = \emptyset$. Hence, we have $i_r\{x, y\} = \varphi^{-1} i_s\{\varphi(x), \varphi(y)\} = \emptyset$.

The other requirements are clear, so $r \in P^*$.

By the construction, it is also clear that r is an amalgamation of p and q. $\square$

Claim 3.16. $\mathcal{P}$ has the amalgamation property.

Indeed, let S be an uncountable subset of P. Then $\{\varphi(p) : p \in S\}$ is an uncountable subset of Q and $\mathcal{Q}$ has the amalgamation property, so there are $p \neq q \in S$ such that $\varphi(p)$ and $\varphi(q)$ have an amalgamation s in Q. But then, by (3.15), p and q have an amalgamation r in P^* with $\varphi(r) = s \in Q$. Thus $r \in P$, i.e. p and q have an amalgamation in P. $\square$

Claim 3.17. $\mathcal{P}$ has the density property D.

Indeed, let $p \in P$ and $x \in (T) \setminus a_p$. Then $\varphi(p) \in Q$ and $\varphi(x) \in (\omega_2 \times \omega) \setminus a_{\varphi(p)}$. Since $\mathcal{Q}$ has the density property D_ω, we have $\varphi(p) \uplus \{\varphi(x)\} \in Q$. Since $\varphi(p \uplus \{x\}) = \varphi(p) \uplus \{\varphi(x)\}$, we have $\varphi(p \uplus \{x\}) \in Q$ and so $x \uplus \{p\} \in P$. $\square$

Claim 3.18. $\mathcal{P}$ has the density property E.

Indeed, let $p \in P$, $y \in a_p$ and $x \in (\omega_2 \times \omega) \setminus a_p$ with $\pi(x) < \pi(y)$. Then $\varphi(p) \in Q$, $\varphi(x) \in (\omega_2 \times \omega) \setminus a_{\varphi(p)}$ and $\pi(\varphi(x)) < \pi(\varphi(y))$ by (A). Then, by (3.14), $\varphi(p) \uplus_{\varphi(y)} \{\varphi(x)\}$ is defined and $\varphi(p \uplus_y \{x\}) = \varphi(p) \uplus_{\varphi(y)} \{\varphi(x)\}$. Since $\mathcal{Q}$ has the density property E, we have $\varphi(p \uplus_y \{x\}) = \varphi(p) \uplus_{\varphi(y)} \{\varphi(x)\} \in Q$. Thus $p \uplus_y \{x\} \in P$. $\square$

Claims 3.16-3.18 above give that $\mathcal{P} = \langle P, \prec \rangle$ is a U-poset. $\square_{3.10}$

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