Fodor-type Reflection Principle
and
reflection of metrizability and meta-Lindelöfness

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Abstract

We introduce a new reflection principle which we call *"Fodor-type Reflection Principle"* (FRP). This principle follows from but is strictly weaker than Fleissner's Axiom R. For instance, FRP does not impose any restriction on the size of the continuum, while Axiom R implies that the continuum has size $\leq \aleph_2$.

We show that FRP implies that every locally separable countably tight topological space X is meta-Lindelöf if all of its subspaces of cardinality $\leq \aleph_1$ are (Theorem 4.3). It follows that, under FRP, every locally (countably) compact space is metrizable if all of its subspaces of cardinality $\leq \aleph_1$ are (Corollary 4.4). This improves a result of Balogh who proved the same assertion under Axiom R.

We also give several other results in this vein, some in ZFC, others in some further extension of ZFC. For example, we prove in ZFC that if X is a locally (countably) compact space of singular cardinality in

which every subspace of smaller size is metrizable then X itself is also metrizable (Corollary 5.2).

1 Introduction

In this note, we consider the following type of reflection phenomenon in topological spaces. Let $\mathcal{P}$ be a property of a topological space and κ a cardinal.

(1.1) If a topological space X satisfies the property $\mathcal{P}$, then there is a subspace of X of size $< \kappa$ satisfying the property $\mathcal{P}$.

For the negation $\mathcal{Q}$ of $\mathcal{P}$, (1.1) can be reformulated as the following transfer property of $\mathcal{Q}$:

(1.2) If every subspace of X of size $< \kappa$ satisfies the property $\mathcal{Q}$, then X also satisfies $\mathcal{Q}$.

The instance of (1.2), where $\mathcal{Q}$ equals "*metrizable*", is studied extensively in the literature which started with Hajnal and Juhász [15]. The most prominent result in this context is perhaps the theorem of Dow cited below (Theorem 1.2).

Definition 1.1. A topological space X is called $\aleph_1$-*metrizable* if every subspace of X of size $\leq \aleph_1$ is metrizable. More generally, X is said to be κ-*metrizable* ($< \kappa$-*metrizable* resp.) for a cardinal κ if every subspace of X of size $\leq \kappa$ ($< \kappa$ resp.) is metrizable.

A κ-metrizable space satisfies a certain amount of separation axioms:

Lemma 1.1. (1) *A topological space is $< \aleph_0$-metrizable if and only if it is T_1.*

(2) *If a topological space is first countable and $\aleph_0$-metrizable then it is Hausdorff.*

Proof. (1): If X is T_1 then every finite subspace of X is discrete and hence metrizable. If X is not T_1 then there are x, $y \in X$, $x \neq y$ such that every neighborhood of x contains y. Then the subspace topology of $\{x, y\} \subseteq X$ is trivial and hence non-metrizable.

(2): Suppose that X is first countable but not Hausdorff. Let x, $y \in X$, $x \neq y$ be such that any neighborhoods of x and y intersect. Let $\{U_n : n \in \omega\}$ and $\{V_n : n \in \omega\}$ be neighborhood bases of x and y respectively, and let $z_{m,n} \in U_m \cap V_n$ for n, $m \in \omega$. Then $Y = \{x, y\} \cup \{z_{m,n} : m, n \in \omega\}$ as a countable subspace of X is not Hausdorff and hence non-metrizable. This shows that X is not $\aleph_0$-metrizable. ❑ (Lemma 1.1)

On the other hand, a κ-metrizable space for any cardinal κ need not to be first countable in general: for example, the topological space $X = (X, \tau)$ with $X = \kappa^+$ and $\tau = \{\emptyset\} \cup \{O \subseteq \kappa^+ : |\kappa^+ \setminus O| \leq \kappa\}$ for a cardinal κ is κ-metrizable since every subspace of X of cardinality $\leq \kappa$ is discrete but we have $\chi(x, X) > \kappa$ for all $x \in X$.

Theorem 1.2. (A. Dow [6, Theorem 3.1]) *Every countably compact $\aleph_1$-metrizable space is metrizable.* ❑

In particular, every compact $\aleph_1$-metrizable space is metrizable.

There are countably compact $\aleph_0$-metrizable spaces which are not metrizable: $\omega_1 + 1$ with the canonical order topology is such an example. This shows that $\aleph_1$-metrizability in Theorem 1.2 is optimal.

Theorem 1.2 implies that locally countably compact $\aleph_1$-metrizable spaces have many properties common with metrizable spaces. For example:

Lemma 1.3. *A locally countably compact $\aleph_1$-metrizable space is first countable and hence Hausdorff.*

Proof. Suppose that X is a locally countably compact $\aleph_1$-metrizable space. Then X is locally metrizable by Theorem 1.2. Since metrizable spaces are first countable, X is also first countable. By Lemma 1.1, (2), it follows that X is Hausdorff. ❑ (Lemma 1.3)

Arhangelskii [1] asked if every locally compact $\aleph_1$-metrizable space is metrizable. Balogh proved that the answer is affirmative under Fleissner's Axiom R. In fact he proved the following stronger result.

Theorem 1.4. (Z. Balogh [3, Theorem 2.2]) *Assume Axiom R. For a locally compact regular Hausdorff space X, if every subspace of cardinality $\leq \aleph_1$ has a point countable base then X is metrizable. In particular, every locally compact and $\aleph_1$-metrizable space is metrizable.* ❑

Recall that *Axiom R* is the principle asserting that $\mathrm{AR}([\kappa]^{\aleph_0})$ holds for all cardinals $\kappa \geq \aleph_2$, where

$\mathrm{AR}([\kappa]^{\aleph_0})$: For any stationary $S \subseteq [\kappa]^{\aleph_0}$ and ω_1-club $T \subseteq [\kappa]^{\aleph_1}$, there is $I \in T$ such that $S \cap [I]^{\aleph_0}$ is stationary in $[I]^{\aleph_0}$.

Here, $T \subseteq [X]^{\aleph_1}$ for an uncountable set X is said to be ω_1-*club* (or *tight and unbounded* in Fleissner's terminology in [10]) if

(1.3) T is cofinal in $[X]^{\aleph_1}$ with respect to $\subseteq$ and

(1.4)   for any increasing chain $\langle I_\alpha : \alpha < \omega_1 \rangle$ in T of length ω_1, we have
$\bigcup_{\alpha < \omega_1} I_\alpha \in T$.

The assumption of Axiom R cannot be simply dropped from Theorem 1.4 since, as the next proposition shows, one obtains a counterexample to Arhangelskii's question in a very strong sense under the existence of a non-reflecting stationary set of ordinals of countable cofinality. However, we prove that, in Balogh's result, Axiom R can be replaced by Fodor-type Reflection Principle which will be defined in Section 2 (Corollary 4.4) and that this principle is substantially weaker than Axiom R (see Section 3).

Given a topological space X and a family $\mathcal{F}$ of open sets, let $\mathrm{ord}(x, \mathcal{F}) = |\{F \in \mathcal{F} : x \in F\}|$ for $x \in X$ and $\mathrm{ord}(\mathcal{F}) = \sup\{\mathrm{ord}(x, \mathcal{F}) : x \in X\}$. We say that $\mathcal{F}$ is *point countable* if $\mathrm{ord}(\mathcal{F}) \leq \aleph_0$. We shall also use the notation

$$\widehat{\mathrm{ord}}(\mathcal{F}) = \min\{\kappa : \forall\, x \in X \ (\mathrm{ord}(x, \mathcal{F}) < \kappa)\}.$$

Note that $\mathrm{ord}(\mathcal{F}) \leq \widehat{\mathrm{ord}}(\mathcal{F})$ and $\widehat{\mathrm{ord}}(\mathcal{F}) = (\mathrm{ord}(\mathcal{F}))^+$ if $\mathrm{ord}(\mathcal{F}) = \mathrm{ord}(x, \mathcal{F})$ for some $x \in X$.

Recall that a topological space X is said to be *meta-Lindelöf* if every open cover $\mathcal{B}$ of X has a point countable open refinement. It is clear that every paracompact space is meta-Lindelöf; every metrizable space is paracompact by Stone's theorem, and hence meta-Lindelöf.

For a cardinal κ, E_ω^κ denotes the set of all ordinals $< \kappa$ of countable cofinality. A subset S of κ is said to be *non-reflecting stationary* if S is a stationary subset of κ but $S \cap \delta$ is non-stationary in δ for all limit ordinal $\delta < \kappa$.

It was proved in Hajnal and Juhász [15] that if $\kappa > \aleph_1$ is regular and $S \subseteq E_\omega^\kappa$ is a non-reflecting stationary set then the usual order topology on S is $< \kappa$-metrizable but not meta-Lindelöf and hence non-metrizable. This space is not locally compact. However its natural modification as in the following proposition *is*.

Proposition 1.5. *If there is a non-reflecting stationary set $S \subseteq E_\omega^\kappa$ for a regular cardinal $\kappa \geq \aleph_2$ then there is a non-meta-Lindelöf (and hence non-metrizable), locally compact and locally countable $< \kappa$-metrizable space X of size κ.*

Proof. Let $I = \{\xi + 1 : \xi < \kappa\}$. The underlying set of X is $S \cup I$. For each $\alpha \in S$, choose a countable subset $a_\alpha \in [I \cap \alpha]^{\aleph_0}$ of order type ω which is cofinal in α. Now define the topology of X as follows:

(1.5)   the elements of I are isolated;

(1.6)   a neighborhood base of $\alpha \in S$ is $\{\{\alpha\} \cup (a_\alpha \setminus \beta) : \beta < \alpha\}$.

By Fodor's (or even Neumer's) theorem, for every open refinement $\mathcal{B}$ of the open cover $\{\alpha + 1 : \alpha \in S\}$ of X, there is a point $x \in X$ such that $\text{ord}(x, \mathcal{B}) = \kappa$. It follows that X is not meta-Lindelöf (and hence, in particular, not metrizable).

X is clearly locally compact and Hausdorff, so it is regular. It is also clear that X is locally countable.

The rest can be done similarly to the proof of [15, Theorem 2]: we show by transfinite induction that $X \upharpoonright \delta$ is metrizable for all $\delta < \kappa$. If δ is a limit ordinal, there is a club $C \subseteq \delta$ disjoint from S. Let $\langle \gamma_\nu : \nu < \lambda \rangle$ be the increasing enumeration of C. Then $\{X \upharpoonright [\gamma_\nu, \gamma_{\nu+1}) : \nu < \lambda\}$ is a partition of $X \upharpoonright \delta$ into clopen sets and each of these clopen sets is metrizable by the induction hypothesis. So $X \upharpoonright \delta$ is metrizable as well. The successor case $\delta = \alpha + 1$ follows directly from [15, Lemma 2] because $X \upharpoonright \delta = X \upharpoonright (\alpha \cup \{\alpha\})$ is regular and first countable while $X \upharpoonright \alpha$ is metrizable by the induction hypothesis.

❏ (Proposition 1.5)

In Section 2, we introduce a new type of stationary reflection principle which we dubbed "Fodor-type Reflection Principle" and denote FRP. The principle asserts that its local version FRP(κ) holds for all regular cardinals $\kappa \geq \aleph_1$. We show that FRP(κ) follows from RP($[\kappa]^{\aleph_0}$) (Theorem 2.5) where RP($[\kappa]^{\aleph_0}$) is a slight strengthening of what is called "Reflection Principle" and denoted by RP(κ) in Jech's Millennium Book [16].

Since Axiom R implies RP($[\kappa]^{\aleph_0}$) for all cardinals κ of cofinality $\geq \omega_1$, FRP is a consequence of Axiom R.

On the other hand, we show in Section 3 that FRP(κ) is preserved in generic extensions by c.c.c. poset (Theorem 3.4). It is easy to see that this is not the case for RP($[\kappa]^{\aleph_0}$). In particular, it is consistent that FRP(κ) for all cardinal κ of cofinality $\geq \omega_1$ holds while RP($[\kappa]^{\aleph_0}$) does not hold for all $\kappa \geq \aleph_2$ (Theorem 3.5, (1)). From these results we can conclude that FRP is a significantly weaker principle than Axiom R.

In Section 4, we prove that, under FRP, the transfer property (1.2) holds for meta-Lindelöfness of locally separable and countably tight spaces (Theorem 4.3). The assertion of Balogh's theorem (Theorem 1.4) is then deduced from FRP via Theorem 4.3 (Corollary 4.4). Since FRP is much weaker than Axiom R, it is fair to say that Corollary 4.4 is an essential improvement of Balogh's theorem. In particular, Theorem 3.5 (2) implies that the topological transfer properties in these theorems under FRP do not impose any restriction on the size of the continuum.

Since $\text{FRP}(\omega_1)$ is simply equivalent to Fodor's theorem for ω_1, we can easily single out the ZFC part of the proofs of these transfer theorems to obtain the corresponding ZFC results (Corollaries 4.5 and 4.6).

For a property $\mathcal{Q}$, let us say that a topological space be *almost* $\mathcal{Q}$ if every subspace Y of X of cardinality $< |X|$ satisfies the property $\mathcal{Q}$. In particular, X is almost metrizable if and only if X is $< |X|$-metrizable. Note that this terminology conflicts with some established notions of covering properties like "almost compact", "almost Lindelöf" etc. However there will be no ambiguity here as our "almost Q" terminology will be never used in connection with covering properties.

A natural variant of (1.2) would be:

(1.7) If X is almost $\mathcal{Q}$, then X satisfies $\mathcal{Q}$.

Note that, in this terminology, the topological space constructed in Proposition 1.5 is almost metrizable.

For various properties $\mathcal{Q}$, we can ask whether (1.7) holds for all members of a given class $\mathcal{C}$ of topological spaces. We can consider this problem as a question on compactness of $\mathcal{C}$ (in the sense of abstract model theory) with respect to the property $\mathcal{Q}$.

In Section 5, we present miscellaneous results concerning the metrizability (resp. meta-Lindelöfness) of almost metrizable (resp. almost meta-Lindelöf) spaces X in various classes $\mathcal{C}$ of topological spaces.

In Section 6, we show that the same kind of anticompactness of metrizability as Proposition 1.5 can also hold without the existence of non-reflecting stationary sets.

2   Fodor-type Reflection Principle

In this section, we introduce the principle which we call "Fodor-type Reflection Principle"(FRP) and show that this principle follows from Axiom R. We show in the next section that FRP is strictly weaker than Axiom R and even some other weakenings of Axiom R.

The applications of FRP on reflection properties of topological spaces mentioned in the introduction will be given in Section 4. Actually, it appears that most of the known applications of Axiom R are already provable under FRP (see also Fuchino [13] and Fuchino, Sakai, Soukup and Usuba [14]).

Definition 2.1. Let κ be a cardinal of cofinality $\geq \omega_1$. *The Fodor-type Reflection Principle for κ (FRP(κ)) is the following statement:*

FRP(κ): For any stationary $S \subseteq E_\omega^\kappa$ and mapping $g : S \to [\kappa]^{\leq\aleph_0}$ there is $I \in [\kappa]^{\aleph_1}$ such that

(2.1) $\mathrm{cf}(I) = \omega_1$;

(2.2) $g(\alpha) \subseteq I$ for all $\alpha \in I \cap S$;

(2.3) for any regressive $f : S \cap I \to \kappa$ such that $f(\alpha) \in g(\alpha)$ for all $\alpha \in S \cap I$, there is $\xi^* < \kappa$ such that $f^{-1}''\{\xi^*\}$ is stationary in $sup(I)$.

Note that, for S and I as above, $S \cap I$ is stationary in $\sup(I)$.

Fact 2.1. FRP(ω_1) *holds in* ZFC.

Indeed, for $I = \omega_1$, (2.3) follows immediately from Fodor's theorem.

Lemma 2.2. FRP(κ) *fails for a singular cardinal* κ.

Proof. Suppose that $\lambda = \mathrm{cf}(\kappa) < \kappa$. Let $\langle \alpha_\xi : \xi < \lambda \rangle$ be a continuously and strictly increasing sequence of ordinals cofinal in $\kappa \setminus \lambda$. Let $S = \{\alpha_\xi : \xi \in E_\omega^\lambda\}$. Then S is a stationary subset of E_ω^κ. Let $g : S \to \kappa$ be defined by

(2.4) $g(\alpha_\xi) = \{\xi\}$ for $\xi \in E_\omega^\kappa$.

Since the mapping $f : S \to \lambda$; $\alpha_\xi \mapsto \xi$ is regressive but strictly increasing, there is no $I \in [\kappa]^{\aleph_1}$ satisfying (2.3). This shows that FRP(κ) fails.   ❑ (Lemma 2.2)

Definition 2.2. Let FRP be the assertion: FRP(κ) holds for all regular cardinals $\kappa \geq \aleph_1$.

For any regular $\kappa \geq \aleph_2$, FRP(κ) is not provable in ZFC since the existence of a non-reflecting subset of E_ω^κ refutes FRP(κ). In Section 6, we show that the non-existence of non-reflecting subset of E_ω^κ does not even guarantee FRP(κ).

For a cardinal $\kappa \geq \aleph_2$, let RP($[\kappa]^{\aleph_0}$) be the following principle:

RP($[\kappa]^{\aleph_0}$): For any stationary $S \subseteq [\kappa]^{\aleph_0}$, there is an $I \in [\kappa]^{\aleph_1}$ such that

(2.5) $\omega_1 \subseteq I$;

(2.6) $\mathrm{cf}(I) = \omega_1$;

(2.7) $S \cap [I]^{\aleph_0}$ is stationary in $[I]^{\aleph_0}$.

The following is well-known and easy to prove.

Lemma 2.3. RP($[\kappa]^{\aleph_0}$) *is equivalent to the assertion that for any stationary* $S \subseteq [\kappa]^{\aleph_0}$, *there are stationarily many* $I \in [\kappa]^{\aleph_1}$ *satisfying* (2.5), (2.6) *and* (2.7).

❑

AR($[\kappa]^{\aleph_0}$) implies RP($[\kappa]^{\aleph_0}$) for a cardinal κ of cofinality $\geq \omega_1$ since $T = \{I \in [\kappa]^{\aleph_1} : \omega_1 \subseteq I$ and $\mathrm{cf}(I) = \omega_1\}$ is ω_1-club. Jech [16] called the weakening of RP($[\kappa]^{\aleph_0}$) "*Reflection Principle*" which is obtained by dropping the condition (2.6) from the definition of RP($[\kappa]^{\aleph_0}$). Jech's reflection principle is sometimes also called "*Weak Reflection Principle*" in the literature (see, e.g. König, Larson and Yoshinobu [18]) and so we shall denote this principle here by WRP($[\kappa]^{\aleph_0}$).

Axiom R follows from $\mathrm{MA}^+(\sigma\text{-closed})$ (see Beaudoin [4]) which in turn is a consequence of Martin's Maximum (see Foreman, Magidor and Shelah [11]). In the terminology of Foreman and Todorcevic [12], Axiom R is equivalent to the stationary reflection to an internally unbounded structure (this fact is stated essentially in Dow [7] under the definition of Axiom R which is slightly stronger than ours). Since $\mathrm{MA}^+(\sigma\text{-closed})$ is consistent with CH (under a large cardinal hypothesis), all the reflection principles we treat here are compatible with CH.

It is still open if WRP($[\kappa]^{\aleph_0}$), RP($[\kappa]^{\aleph_0}$) and AR($[\kappa]^{\aleph_0}$) can ever be separated. In fact, this seems to be quite a difficult problem: it is known that RP($[\omega_2]^{\aleph_0}$) and AR($[\omega_2]^{\aleph_0}$) are equivalent; under $2^{\aleph_1} = \aleph_2$, WRP($[\omega_2]^{\aleph_0}$) and RP($[\omega_2]^{\aleph_0}$) are equivalent and, e.g. under GCH, WRP($[\omega_n]^{\aleph_0}$) and RP($[\omega_n]^{\aleph_0}$) for all $n \in \omega$ are equivalent (see König, Larson and Yoshinobu [18]). On the other hand, our Fodor-type Reflection Principle can be easily separated from these reflection principles as we will see in the next section.

The following is a useful characterization of FRP(κ).

Lemma 2.4. *For a regular cardinal $\kappa \geq \aleph_2$, FRP(κ) is equivalent to the following* FRP$^\bullet$(κ)*:*

FRP$^\bullet$(κ)*: For any stationary $S \subseteq E^\kappa_\omega$ and mapping $g : S \to [\kappa]^{\leq \aleph_0}$ there is a continuously increasing sequence $\langle I_\xi : \xi < \omega_1 \rangle$ of countable subsets of κ such that*

$\quad$ (2.8) $\quad \langle \sup(I_\xi) : \xi < \omega_1 \rangle$ *is strictly increasing;*

$\quad$ (2.9) $\quad$ *each I_ξ is closed with respect to g and*

$\quad$ (2.10) $\quad \{\xi < \omega_1 : \sup(I_\xi) \in S$ and $g(\sup(I_\xi)) \cap \sup(I_\xi) \subseteq I_\xi\}$ *is stationary in ω_1.*

Proof. First, assume FRP(κ). Let $S \subseteq E^\kappa_\omega$ be stationary and $g : S \to [\kappa]^{\aleph_0}$. Without loss of generality, we may assume that $g(\alpha) \cap \alpha \neq \emptyset$ for all $\alpha \in S$.

Let $I \in [\kappa]^{\aleph_1}$ be as in the definition of FRP(κ) for these S and g. Then, by (2.1) and (2.2), there is a filtration $\langle I_\xi : \xi < \omega_1 \rangle$ of I, that is, a continuously increasing sequence $\langle I_\xi : \xi < \omega_1 \rangle$ of subsets of I of cardinality $< |I|$ with $I = \bigcup_{\xi < \omega_1} I_\xi$, satisfying (2.8) and (2.9).

We show that $\langle I_\xi : \xi < \omega_1 \rangle$ satisfies (2.10) as well. Suppose not. Then $\{\xi < \omega_1 : \sup(I_\xi) \notin S$ or $g(\sup(I_\xi)) \cap \sup(I_\xi) \nsubseteq I_\xi\}$ includes a club set $\subseteq \omega_1$. It follows that $S \cap I \setminus S_0$ is non stationary in $\sup(I)$, where

$$S_0 = \{\alpha \in S \cap I : \alpha = \sup(I_\xi) \text{ for some } \xi < \omega_1 \text{ and } g(\alpha) \cap \alpha \nsubseteq I_\xi\}.$$

Let $f : S \cap I \to I$ be defined by

$$(2.11) \quad f(\alpha) = \begin{cases} \min((g(\alpha) \cap \alpha) \setminus I_\xi) & \text{if } \alpha \in S_0 \text{ and } \alpha = \sup(I_\xi); \\ \min(g(\alpha)) & \text{otherwise.} \end{cases}$$

Then f is regressive and $f(\alpha) \in g(\alpha)$ for all $\alpha \in S \cap I$. By the choice of I, there is an $\alpha^* \in I$ such that $f^{-1}{}''\{\alpha^*\}$ is stationary in $\sup(I)$. In particular, $S_0 \cap f^{-1}{}''\{\alpha^*\}$ is stationary in $\sup(I)$. Let $\xi^* \in \omega_1$ be such that $\alpha^* \in I_{\xi^*}$ and let $\beta \in S_0 \cap f^{-1}{}''\{\alpha^*\}$ be such that $\beta > \sup(I_{\xi^*})$. Let $\eta < \omega_1$ be such that $\beta = \sup(I_\eta)$. Then $\alpha^* \in I_{\xi^*} \subseteq I_\eta$. Since $\beta \in S_0$, we have $f(\beta) \notin I_\eta$ by the definition (2.11) of f. It follows that $f(\beta) \neq \alpha^*$. This is a contradiction to the choice of β.

Now, assume $\mathrm{FRP}^\bullet(\kappa)$. Suppose that $S \subseteq E_\omega^\kappa$ is stationary and $g : S \to [\kappa]^{\aleph_0}$. Let $S_0 = \{\alpha \in S : \alpha \text{ is closed with respect to } g\}$. Since κ is regular, S_0 is stationary. Let $\langle I_\xi : \xi < \omega_1 \rangle$ be as in the definition of $\mathrm{FRP}^\bullet(\kappa)$ for S_0 and $g \upharpoonright S_0$. Let I be the closure of $\bigcup_{\xi < \omega_1} I_\xi \cup \{\sup(I_\xi) : \xi < \omega_1\}$ with respect to g. By the definition of S_0 and since $\sup(I_\xi) \in S_0$ for stationarily many $\xi < \omega_1$, we have $\sup(I) = \sup(\bigcup_{\xi < \omega_1} I_\xi)$. Hence $\{\sup(I_\xi) : \xi < \omega_1\}$ is a club subset of $\sup(I)$.

We claim that this I satisfies the conditions in the definition of $\mathrm{FRP}(\kappa)$. It is clear that I satisfies (2.1) and (2.2). To see that it also satisfies (2.3), suppose that $f : S \cap I \to \kappa$ is regressive and $f(\alpha) \in g(\alpha)$ for all $\alpha \in S \cap I$. Let $S_1 = \{\xi \in \omega_1 : f(\sup(I_\xi)) \in I_\xi\}$. Then we have

$$S_1 \supseteq \{\xi \in \omega_1 : g(\sup(I_\xi)) \cap \sup(I_\xi) \subseteq I_\xi\}$$

and thus S_1 is stationary by the choice of I. For each $\xi \in S_1$, let

$$h(\xi) = \min\{\eta < \omega_1 : f(\sup(I_\xi)) \in I_\eta\}.$$

Then the mapping $h : S_1 \to \omega_1$ is regressive. Thus, by Fodor's theorem, there is a stationary $S_2 \subseteq S_1$ such that $h''S_2 = \{\eta^*\}$ for some $\eta^* \in \omega_1$. Since I_{η^*} is countable, there is a stationary $S_3 \subseteq S_2$ such that, for any $\xi \in S_3$, $f(\sup(I_\xi)) = \alpha^*$ for some fixed $\alpha^* \in I_{\eta^*}$. It follows that $f^{-1}{}''\{\alpha^*\} \supseteq \{\sup(I_\xi) : \xi \in S_3\}$ is stationary in $\sup(I)$. $\qquad \qquad \square$ (Lemma 2.4)

Theorem 2.5. *For any regular cardinal $\kappa > \aleph_1$, $\mathrm{RP}([\kappa]^{\aleph_0})$ implies $\mathrm{FRP}(\kappa)$.*

Proof. By Lemma 2.4, it is enough to show that $\mathrm{RP}([\kappa]^{\aleph_0})$ implies $\mathrm{FRP}^\bullet(\kappa)$. Suppose that $S \subseteq E^\kappa_\omega$ is stationary and $g : S \to [\kappa]^{\leq \aleph_0}$. Let

(2.12) $S_0 = \{a \in [\kappa]^{\aleph_0} : \sup(a) \in S \setminus a \text{ and } g(\sup(a)) \cap \sup(a) \subseteq a\}$.

Claim 2.5.1. *S_0 is a stationary subset of $[\kappa]^{\aleph_0}$.*

⊢ Suppose that $C \subseteq [\kappa]^{\aleph_0}$ is a club. We show that $C \cap S_0 \neq \emptyset$.

By Kueker's theorem, there is a mapping $s : \kappa^{<\omega} \to \kappa$ such that $C \supseteq C(s) = \{a \in [\kappa]^{\aleph_0} : s''a^{<\omega} \subseteq a\}$. Let $D = \{\alpha < \kappa : s''\alpha^{<\omega} \subseteq \alpha\}$. Since κ is regular, D is a club subset of κ. So there is an $\alpha^* \in S \cap D$. Let $\langle \alpha_n : n \in \omega \rangle$ be an increasing sequence of ordinals such that $\alpha^* = \sup_{n \in \omega} \alpha_n$. Let a^* be the closure of $a_0 = \{\alpha_n : n \in \omega\} \cup (g(\alpha^*) \cap \alpha^*)$ with respect to s. Since a_0 is cofinal in α^* and $\alpha^* \in D$, we have $\sup(a^*) = \alpha^*$. Hence $a^* \in S_0$. By the definition of a^*, we also have $a^* \in C(s) \subseteq C$.

⊣ (Claim 2.5.1)

By $\mathrm{RP}([\kappa]^{\aleph_0})$, there is $I \in [\kappa]^{\aleph_1}$ such that

(2.13) $\mathrm{cf}(I) = \omega_1$;

(2.14) $g(\alpha) \subseteq I$ for all $\alpha \in I \cap S$;

(2.15) $S_0 \cap [I]^{\aleph_0}$ is stationary in $[I]^{\aleph_0}$.

Note that the additional condition (2.14) is possible by Lemma 2.3.

Let $\langle I_\xi : \xi < \omega_1 \rangle$ be a filtration of I such that each I_ξ is closed with respect to g (this is possible by (2.14)) and $\langle \sup(I_\xi) : \xi < \omega_1 \rangle$ is strictly increasing (possible by (2.13)).

Let

$$S_1 = \{\xi < \omega_1 : \xi \text{ is a limit and } I_\xi \in S_0\} \text{ and}$$
$$S_2 = \{\xi < \omega_1 : g(\sup(I_\xi)) \cap \sup(I_\xi) \subseteq I_\xi\}.$$

By the definition (2.12) of S_0, we have $S_2 \supseteq S_1$ and S_1 is a stationary subset of ω_1 by (2.15). Thus S_2 is stationary as well.   ☐ (Theorem 2.5)

Corollary 2.6. RP *implies* FRP. *In particular,* Axiom R *implies* FRP.   ☐

3   Separation of FRP from WRP

In this section, we prove the consistency of Fodor-type Reflection Principle with the total negation of the Weak Reflection Principle.

The following lemma is well-known and easy to prove:

Lemma 3.1. *For $\aleph_2 \leq \kappa \leq \kappa'$, if $\mathrm{WRP}([\kappa']^{\aleph_0})$ then $\mathrm{WRP}([\kappa]^{\aleph_0})$.* ❑

For a proof of the following proposition, see e.g. Jech [16, Theorem 37.18].

Proposition 3.2 (S. Todorčević). *$\mathrm{WRP}([\aleph_2]^{\aleph_0})$ implies $2^{\aleph_0} \leq \aleph_2$.* ❑

The first author learned the following lemma in one of Shelah's papers:

Lemma 3.3 (S. Shelah). *Suppose that $\mathbb{P}$ is a c.c.c. poset, S a stationary subset of ω_1 and $p_\alpha \in \mathbb{P}$ for $\alpha \in S$. Then, $S \setminus S'$ is non-stationary where*

$$S' = \{\beta \in S \,:\, p_\beta \Vdash_{\mathbb{P}} \text{``}\{\alpha \in S \,:\, p_\alpha \in \dot{G}\} \text{ is stationary in } \omega_1\text{''}\}.$$

Proof. Suppose otherwise. Then $S'' = S \setminus S'$ is stationary. For $\xi \in S''$, let $q_\xi \leq_{\mathbb{P}} p_\xi$ and $C_\xi \subseteq \omega_1$ be such that C_ξ is a club subset of ω_1 and

(3.1) $\quad q_\xi \Vdash_{\mathbb{P}} \text{``}\{\alpha \in S \,:\, p_\alpha \in \dot{G}\} \cap C_\xi = \emptyset\text{''}$

for all $\xi \in S''$. Choose $\beta_\xi \in \omega_1$, $\xi < \omega_1$ inductively such that

(3.2) $\quad \beta_\xi \in S'' \cap \bigcap\{C_{\beta_\eta} \,:\, \eta < \xi\}.$

Claim 3.3.1. *$\{q_{\beta_\xi} \,:\, \xi < \omega_1\}$ is an antichain.*

⊢ For $\xi < \xi' < \omega_1$, we have $\beta_{\xi'} \in C_{\beta_\xi}$ by (3.2). Thus $q_{\beta_\xi} \Vdash_{\mathbb{P}} \text{``}p_{\beta_{\xi'}} \notin \dot{G}\text{''}$ by (3.1). Since $q_{\beta_{\xi'}} \leq_{\mathbb{P}} p_{\beta_{\xi'}}$, it follows that $q_{\beta_\xi} \Vdash_{\mathbb{P}} \text{``}q_{\beta_{\xi'}} \notin \dot{G}\text{''}$. Hence q_{β_ξ} and $q_{\beta_{\xi'}}$ are incompatible. ⊣ (Claim 3.3.1)

But this is a contradiction to the c.c.c. of $\mathbb{P}$. ❑ (Lemma 3.3)

Theorem 3.4. *Suppose that $\mathrm{FRP}(\kappa)$ holds and $\mathbb{P}$ is a c.c.c. poset. Then $\Vdash_{\mathbb{P}} \text{``}\mathrm{FRP}(\kappa)$ holds''.*

Proof. Suppose that $\dot{S}$ is a $\mathbb{P}$-name of a stationary subset of E_ω^κ and $\dot{g}$ a $\mathbb{P}$-name of a mapping from $\dot{S}$ to $[\kappa]^{\aleph_0}$. Let

(3.3) $\quad S = \{\alpha \in \kappa \,:\, p \Vdash_{\mathbb{P}} \text{``}\alpha \in \dot{S}\text{ for some } p \in \mathbb{P}\text{''}\}.$

Then S is a stationary subset of κ. Let $g : S \to [\kappa]^{\aleph_0}$ be defined by

(3.4) $\quad g(\alpha) = \{\beta \in \kappa \,:\, p \Vdash_{\mathbb{P}} \text{``}\beta \in \dot{g}(\alpha)\text{ for some } p \in \mathbb{P}\text{''}\}$

for $\alpha \in S$. g is well-defined by the c.c.c. of $\mathbb{P}$.

By Lemma 2.4, there is a continuously increasing sequence $\langle I_\xi \,:\, \xi < \omega_1 \rangle$ with $I_\xi \in [\kappa]^{\aleph_0}$ for $\xi < \omega_1$ such that

(3.5) $\quad \langle \sup(I_\xi) \,:\, \xi < \omega_1 \rangle$ is strictly increasing;

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(3.6) I_ξ is closed with respect to g for all $\xi < \omega_1$, and

(3.7) $S_1 = \{\xi \in \omega_1 : \sup(I_\xi) \in S \text{ and } g(\sup(I_\xi)) \cap \sup(I_\xi) \subseteq I_\xi\}$ is stationary.

For $\xi \in S_1$, since $\sup(I_\xi) \in S$, there is a $p_\xi \in \mathbb{P}$ such that $p_\xi \Vdash_\mathbb{P}$ " $\sup(I_\xi) \in \dot{S}$ ". Hence, by Lemma 3.3, there is a $\xi^* \in S_1$ such that

$$p_{\xi^*} \Vdash_\mathbb{P} \text{ "} \{\xi \in S_1 : p_\xi \in \dot{G}\} \text{ is stationary in } \omega_1 \text{ "}.$$

Let $\dot{S}_2$ be a $\mathbb{P}$-name of " $\{\xi \in S_1 : p_\xi \in \dot{G}\}$ ". Then we have $p_{\xi^*} \Vdash_\mathbb{P}$ " $\dot{S}_2$ is stationary ". By the definition (3.4) of g,

(3.8) $\Vdash_\mathbb{P}$ " $\dot{g}(\alpha) \subseteq g(\alpha)$ for every $\alpha \in \dot{S}$ ".

Since $\Vdash_\mathbb{P}$ " $\dot{S}_2 \subseteq S_1$ ", we have

$$p_{\xi^*} \Vdash_\mathbb{P} \text{ "} \dot{S}_2 \subseteq \{\xi \in \omega_1 : \sup(I_\xi) \in \dot{S} \text{ and } \dot{g}(\sup(I_\xi)) \cap \sup(I_\xi) \subseteq I_\xi\} \text{ "}.$$

Hence

$$p_{\xi^*} \Vdash_\mathbb{P} \text{ "} \{\xi \in \omega_1 : \sup(I_\xi) \in \dot{S} \text{ and } \dot{g}(\sup(I_\xi)) \cap \sup(I_\xi) \subseteq I_\xi\}$$
$$\text{is stationary "}.$$

By (3.8) and (3.6),

$$\Vdash_\mathbb{P} \text{ "} I_\xi \text{ is closed with respect to } \dot{g} \text{ for all } \xi < \omega_1 \text{ "}.$$

Thus p_{ξ^*} forces that $\langle I_\xi : \xi < \omega_1 \rangle$ is as in the definition of $\mathrm{FRP}^\bullet(\kappa)$ for $\dot{S}$ and $\dot{g}$.

Since the argument above can be repeated in $\mathbb{P} \restriction p$ for any $p \in \mathbb{P}$, it follows that $\Vdash_\mathbb{P}$ " $\mathrm{FRP}(\kappa)$ ".   □ (Theorem 3.4)

Theorem 3.5.   (1)   *Suppose that "ZFC + FRP" is consistent. Then so is* "ZFC + FRP + $\neg \mathrm{WRP}([\kappa]^{\aleph_0})$ *for all* $\kappa \geq \aleph_2$".

(2)   *If "ZFC + CH + FRP" is consistent, then "ZFC + FRP" is consistent with any value of the size of continuum possible under* ZFC.

Proof. (1): Suppose that $V \models$ "ZFC + FRP". In V, let $\mathbb{P} = \mathbb{C}_\lambda$ ($=$ the Cohen forcing adding λ many Cohen reals) for some $\lambda \geq \aleph_3$. Then $V^\mathbb{P} \models 2^{\aleph_0} \geq \aleph_3$. Hence, by Proposition 3.2 and Lemma 3.1, $V^\mathbb{P} \models$ "$\neg \mathrm{WRP}([\kappa]^{\aleph_0})$ for all $\kappa \geq \aleph_2$". By Theorem 3.4, $V^\mathbb{P} \models$ "FRP(κ) for all cardinals κ of cofinality $\geq \omega_1$".

(2): Suppose that $V \models$ "ZFC + CH + FRP". In V, let λ be a cardinal such that $\lambda^{\aleph_0} = \lambda$. Then, for $\mathbb{P} = \mathbb{C}_\lambda$, we have $V^\mathbb{P} \models 2^{\aleph_0} = \lambda$ and $V^\mathbb{P} \models$ "FRP".
                    □ (Theorem 3.5)

It seems that we can only establish the consistency of FRP + ¬WRP under $2^{\aleph_0} \geq \aleph_3$ by arguments similar to the one as above. However, it is shown in Fuchino, Sakai, Soukup and Usuba [14] using a completely different method that FRP + ¬WRP is also consistent with $2^{\aleph_0} \leq \aleph_2$ (under a large cardinal hypothesis).

4 Reflection property of meta-Lindelöfness under FRP

Definition 4.1. A topological space X is said to be *small subspaces meta-Lindelöf* (*ssmL* for short) if every subspace of X of size $\aleph_1$ is meta-Lindelöf.

In analogy to "$\aleph_1$-metrizability", the natural wording for this notion might be "$\aleph_1$-meta-Lindelöf". However "$\aleph_1$-meta-Lindelöf" has been already used for a different notion in the literature and hence we decided for the terminology with "*ssmL*". Nevertheless, we shall also say for an uncountable cardinal κ that a topological space X is $<\kappa$-meta-Lindelöf ($\leq\kappa$-meta-Lindelöf resp.) if every subspace Y of X of cardinality $<\kappa$ ($\leq\kappa$ resp.) is meta-Lindelöf.

Before going to the reflection results, let us introduce a notation and a simple but useful lemma which will be applied repeatedly in the following arguments.

For a family $\mathcal{F}$ of sets, let $\sim_{\mathcal{F}}$ be the intersection relation on $\mathcal{F}$, i.e. let $F \sim_{\mathcal{F}} G$ for $F, G \in \mathcal{F}$ if and only if $F \cap G \neq \emptyset$, and let $\approx_{\mathcal{F}}$ be the transitive closure of $\sim_{\mathcal{F}}$. An argument in elementary cardinal arithmetic shows the following:

Lemma 4.1. *Let μ be an uncountable regular cardinal and $\mathcal{F}$ a family of sets such that, for all $F \in \mathcal{F}$, we have $|\{G \in \mathcal{F} : F \sim_{\mathcal{F}} G\}| < \mu$. Then every equivalence class of $\approx_{\mathcal{F}}$ has cardinality $< \mu$.* ❏

The next ZFC result illustrates the use of Lemma 4.1.

Theorem 4.2. *Suppose that X is a locally countably compact and meta-Lindelöf space. If X is $\aleph_1$-metrizable then it is actually metrizable.*

Proof. Let $\mathcal{E}$ be a point countable cover of X consisting of open sets with countably compact closures. By Dow's theorem (Theorem 1.2), $\overline{E}$ is metrizable, and hence compact and second countable, for all $E \in \mathcal{E}$. Note that X is then regular, being locally compact and Hausdorff.

Since every $E \in \mathcal{E}$ is separable and $\mathcal{E}$ is point countable, it is easy to see that $|\{F \in \mathcal{E} : F \sim_{\mathcal{E}} E\}| \leq \aleph_0$ for all $E \in \mathcal{E}$. Thus it follows from Lemma 4.1 (with $\mu = \aleph_1$) that each equivalence class of $\approx_{\mathcal{E}}$ is countable.

Let $\mathbb{E}$ be the set of all equivalence classes of the relation $\approx_{\mathcal{E}}$. Then $\{\bigcup e : e \in \mathbb{E}\}$ is a partition of X into disjoint open sets. For each $e \in \mathbb{E}$, $\bigcup e$ is

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a countable union of second countable open subspaces, so $\bigcup e$ is also second countable (regular) and hence metrizable by Urysohn's metrization theorem. This shows that X can be partitioned into clopen metric subspaces, hence X itself is also metrizable. $\qquad$ ❏ (Theorem 4.2)

In the following, $L(X)$ denotes the *Lindelöf degree* of the topological space X. That is,

$$L(X) = \min\{\kappa \,:\, \text{every open cover of } X, \text{ has a subcover of size } \leq \kappa\}.$$

Theorem 4.3. (1) *Assume that* $\mathrm{FRP}(\kappa)$ *holds for every regular cardinal* κ *with* $\omega_1 < \kappa \leq \lambda$ *and* X *is a locally separable, countably tight space with* $L(X) \leq \lambda$. *If* X *is ssmL then* X *is actually meta-Lindelöf.*

(2) *Under* FRP *every locally separable, countably tight ssmL space is meta-Lindelöf.*

Proof. We shall prove only (1) since (2) trivially follows from (1).

Since we are dealing with spaces with Lindelöf degree $\leq \lambda$, it is enough to show that the following statement $(*)_\kappa$ holds for all $\kappa \leq \lambda$ by induction on κ.

$(*)_\kappa$ For every locally separable, countably tight, ssmL space X, any cover $\mathcal{B}$ of X of cardinality κ that consists of separable open sets has a point countable open refinement.

So assume that $\kappa \leq \lambda$ and $(*)_\mu$ holds for all $\mu < \kappa$. Let X and $\mathcal{B} = \{B_\alpha \,:\, \alpha < \kappa\}$ be as in $(*)_\kappa$.

Case 1. $\kappa = \aleph_0$.

Then $(*)_\kappa$ trivially holds since $\mathcal{B}$ itself is point countable.

Case 2. κ is regular uncountable.

Let $G_\alpha = \bigcup\{B_\beta \,:\, \beta < \alpha\}$ for $\alpha < \kappa$ and $S = \{\alpha < \kappa \,:\, \overline{G_\alpha} \neq G_\alpha\}$.

Claim 4.3.1. *S is non-stationary.*

⊢ We prove first the following weaker assertion:

Subclaim 4.3.1.1. *$S \cap E^\kappa_\omega$ is non-stationary.*

⊢ Toward a contradiction, suppose that $S \cap E^\kappa_\omega$ were stationary. For each $\alpha \in S \cap E^\kappa_\omega$, let

(4.1) $\quad p_\alpha \in \overline{G_\alpha} \setminus G_\alpha.$

Fix $h : S \cap E_\omega^\kappa \to \kappa$ such that $p_\alpha \in B_{h(\alpha)}$ for all $\alpha \in S \cap E_\omega^\kappa$. For each $\alpha < \kappa$, let $D_\alpha \in [B_\alpha]^{\aleph_0}$ be dense in B_α. Note that we have $D_\beta \subseteq B_\beta \subseteq G_\alpha$ for all $\beta < \alpha < \kappa$.

Since $p_\alpha \in \overline{G_\alpha} = \overline{\bigcup_{\beta < \alpha} D_\beta}$ and X is countably tight, there is $g_0(\alpha) \in [\alpha]^{\aleph_0}$ such that $p_\alpha \in \overline{\bigcup\{D_\beta : \beta \in g_0(\alpha)\}}$ for all $\alpha \in S \cap E_\omega^\kappa$.

Let $g(\alpha) = g_0(\alpha) \cup \{h(\alpha)\}$. Applying FRP$(\kappa)$ to $S \cap E_\omega^\kappa$ and g, we obtain $I \in [\kappa]^{\aleph_1}$ such that

(4.2) $\operatorname{cf}(I) = \omega_1$;

(4.3) $h(\alpha) \in I$ for all $\alpha \in S \cap E_\omega^\kappa \cap I$;

(4.4) $g_0(\alpha) \subseteq I$ for all $\alpha \in S \cap E_\omega^\kappa \cap I$;

(4.5) if $f : S \cap E_\omega^\kappa \cap I \to \kappa$ is such that $f(\alpha) \in g_0(\alpha)$ for all $\alpha \in S \cap E_\omega^\kappa \cap I$, then there is $\xi^* \in I$ with $\sup(f^{-1}\{\xi^*\}) = \sup(I)$.

Let $Y = \{p_\alpha : \alpha \in S \cap E_\omega^\kappa \cap I\} \cup \bigcup\{D_\beta : \beta \in I\}$. Since $|Y| \leq \aleph_1$, Y is meta-Lindelöf. By (4.3), $\mathcal{G} = \{G_\alpha : \alpha \in I\}$ covers Y. So it follows that $\mathcal{G}$ has an open refinement $\mathcal{E}$ that also covers Y and is point countable on Y. For each $\alpha \in S \cap E_\omega^\kappa \cap I$, let $E_\alpha \in \mathcal{E}$ be such that $p_\alpha \in E_\alpha$. Since $p_\alpha \in \overline{\bigcup\{D_\beta : \beta \in g_0(\alpha)\}}$, we have $E_\alpha \cap \bigcup\{D_\beta : \beta \in g_0(\alpha)\} \neq \emptyset$. Thus, $f(\alpha) \in g_0(\alpha)$ can be chosen such that $E_\alpha \cap D_{f(\alpha)} \neq \emptyset$ for all $\alpha \in S \cap E_\omega^\kappa \cap I$.

By (4.5) there is $\xi^* \in I$ such that $J = f^{-1}\{\xi^*\}$ is unbounded in I. As D_{ξ^*} is countable and $E_\alpha \cap D_{\xi^*} \neq \emptyset$ for all $\alpha \in J$, there is $d \in D_{\xi^*}$ such that $K = \{\alpha \in J : d \in E_\alpha\}$ is also unbounded in I. But since $\operatorname{ord}(d, \mathcal{E})$ is countable, there are $K' \subseteq K$ and $E^* \in \mathcal{E}$ such that K' is still unbounded in I and $E_\alpha = E^*$ for all $\alpha \in K'$.

As $\mathcal{E}$ refines $\mathcal{G}$, there is $\beta \in I$ such that $E^* \subseteq G_\beta$. Then (4.1) implies $E_\gamma \ni p_\gamma \notin G_\beta$ for all $\gamma \in (S \cap E_\omega^\kappa \cap I) \setminus \beta$. In particular, we have $E_\gamma \neq E^*$ for any $\gamma \in K' \setminus \beta$. This is a contradiction to the choice of E^* and K'.

$\dashv$ (Subclaim 4.3.1.1)

Now let C be a club subset of κ consisting of limit ordinals such that $S \cap E_\omega^\kappa \cap C = \emptyset$ and set

$$D = \{\alpha \in C : \alpha \setminus S \text{ is cofinal in } \alpha\}.$$

Then D is also a club subset of κ. So we are done by establishing the following subclaim.

Subclaim 4.3.1.2. $S \cap D = \emptyset$.

⊢ Suppose that $\alpha \in D$. If $\mathrm{cf}(\alpha) = \omega$, then $\alpha \notin S$ since $D \subseteq C$. So assume $\mathrm{cf}(\alpha) > \omega$. For $p \in \overline{G_\alpha}$, there is $Y \in [G_\alpha]^{\aleph_0}$ such that $p \in \overline{Y}$ by the countable tightness of X. By the definition of D there is $\beta \in \alpha \setminus S$ such that $G_\beta \supseteq Y$. It follows that $p \in \overline{G_\beta} = G_\beta \subseteq G_\alpha$. Hence $\overline{G_\alpha} = G_\alpha$, i.e. $\alpha \notin S$.

⊣ (Subclaim 4.3.1.2)

⊣ (Claim 4.3.1)

For every $\gamma \in \kappa \setminus S$ the set G_γ is clopen, so if C is a club in $\kappa \setminus S$ and $\langle \gamma_i : i < \kappa \rangle$ is the increasing enumeration of $C \cup \{0\}$ and $H_i = G_{\gamma_{i+1}} \setminus G_{\gamma_i}$ for all $i < \kappa$ then $\{H_i : i < \kappa\}$ is a partition of X into clopen sets.

Now $\mathcal{U}_i = \{B_\xi \setminus G_{\gamma_i} : \gamma_i \leq \xi < \gamma_{i+1}\}$ is an open cover of H_i with $|\mathcal{U}_i| < \kappa$. So $\mathcal{U}_i$ has a point countable open refinement $\mathcal{F}_i$ by the induction hypothesis. Since H_i's are pairwise disjoint, $\mathcal{F} = \bigcup\{\mathcal{F}_i : i < \kappa\}$ is a point countable open cover of X that refines $\mathcal{B}$.

Case 3. κ is singular.

Let $\langle \kappa_i : i < \mathrm{cf}(\kappa) \rangle$ be a continuously and strictly increasing sequence of cardinals cofinal in κ. Let $G_i = \bigcup\{B_\alpha : \alpha < \kappa_i\}$ for each $i < \mathrm{cf}(\kappa)$.

By the induction hypothesis, there is a point countable open refinement $\mathcal{C}_i$ of $\{B_\alpha : \alpha < \kappa_i\}$ with $\bigcup \mathcal{C}_i = G_i$ for each $i < \mathrm{cf}(\kappa)$. Note that each element C of $\mathcal{C}_i$ is separable since C is an open subset of B_α for some $\alpha < \kappa_i$. Put $\mathcal{C} = \bigcup_{i < \mathrm{cf}(\kappa)} \mathcal{C}_i$, then $\mathcal{C}$ covers X and $\mathrm{ord}(\mathcal{C}) \leq \mathrm{cf}(\kappa)$.

Since each $C \in \mathcal{C}$ is separable, it is easy to see that $|\{C' \in \mathcal{C} : C \sim_{\mathcal{C}} C'\}| \leq \mathrm{cf}(\kappa)$. Hence, by Lemma 4.1, we have $|\{C' \in \mathcal{C} : C \approx_{\mathcal{C}} C'\}| \leq \mathrm{cf}(\kappa)$ as well.

Let $\mathbb{E}$ be the set of all equivalence classes of the relation $\approx_{\mathcal{C}}$, then $\{\bigcup e : e \in \mathbb{E}\}$ is a partition of X into disjoint open sets and every $\bigcup e$ is covered by $e \subset \mathcal{C}$. As $|e| \leq \mathrm{cf}(\kappa) < \kappa$ we can apply the induction hypothesis to get a point countable open refinement $\mathcal{F}_e$ of e which covers $\bigcup e$. Consequently, $\mathcal{F} = \bigcup\{\mathcal{F}_e : e \in \mathbb{E}\}$ is a point countable open refinement of $\mathcal{C}$ and hence of $\mathcal{B}$ as desired.   ❑ (Theorem 4.3)

The promised strengthening of Balogh's theorem (Theorem 1.4) can be obtained now as an easy corollary of Theorems 4.2 and 4.3.

Corollary 4.4. (1) *Let λ be a cardinal such that for each regular cardinal κ with $\omega_1 < \kappa \leq \lambda$ we have $\mathrm{FRP}(\kappa)$. If X is a locally countably compact and $\aleph_1$-metrizable space with $L(X) \leq \lambda$ then X is metrizable.*

(2) *Assume FRP. Then every locally countably compact and $\aleph_1$-metrizable space is metrizable.*

Proof. We prove only (1) since it is clear that (2) follows from (1).

Let X be as in (1). Then every point of X has a countably compact neighborhood, and this neighborhood is compact metrizable by Dow's theorem (Theorem 1.2). It follows that X is both locally separable and countably tight. But X is ssmL since it is $\aleph_1$-metrizable. Hence X is meta-Lindelöf by Theorem 4.3 (1). By Theorem 4.2, it follows then that X is metrizable.

<div align="right">❏ (Corollary 4.4)</div>

As noted in Fact 2.1, $\mathrm{FRP}(\aleph_1)$ is a theorem in ZFC. Thus the proofs of Theorem 4.3 and Corollary 4.4 also establish the following ZFC results:

Corollary 4.5. *Suppose that X is a locally separable and countably tight space with $L(X) \leq \aleph_1$. If X is ssmL, then X is meta-Lindelöf.* ❏

Corollary 4.6. *Suppose that X is a locally countably compact space with $L(X) \leq \aleph_1$. If X is $\aleph_1$-metrizable, then X is metrizable.* ❏

5 Almost metrizability and almost meta-Lindelöfness

The following result may be seen as a singular compactness theorem on the meta-Lindelöfness of locally separable and countably tight spaces, in analogy with Shelah's Singular Compactness Theorem on the notion of freeness (Shelah [21]). It also shows that the regularity of κ in Proposition 1.5 cannot be dropped.

Theorem 5.1. *Every locally separable and almost meta-Lindelöf space of singular cardinality is meta-Lindelöf.*

The proof of Theorem 5.1 will be given after Proposition 5.3.

Corollary 5.2. *Every locally countably compact and almost metrizable space of singular cardinality is metrizable.*

Proof. By Theorem 5.1 and Theorem 4.2 (Repeat the argument of the proof of Corollary 4.4).

<div align="right">❏ (Corollary 5.2)</div>

Proposition 5.3. *Suppose that X is almost meta-Lindelöf. Then every cover of X of cardinality $< |X|$ consisting of separable open sets has a point countable open refinement.*

Proof. Similarly to Theorem 4.3, it is enough to prove the following assertion $(*)_\kappa$ for all cardinals κ by induction on κ.

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$(*)_\kappa$ For any almost meta-Lindelöf space X with $|X| > \kappa$, if $\mathcal{B}$ is a cover of X of cardinality κ consisting of separable open sets then $\mathcal{B}$ has a point countable open refinement.

Assume that $(*)_{\kappa'}$ holds for all $\kappa' < \kappa$ and $\mathcal{B} = \{B_\alpha : \alpha < \kappa\}$ is a cover of X as in $(*)_\kappa$.

Case 1. $\kappa \leq \aleph_0$.

$\mathcal{B}$ itself is point countable.

Case 2. κ is regular uncountable.

Let $G_\alpha = \bigcup\{B_\beta : \beta < \alpha\}$ for $\alpha < \kappa$ and $S = \{\alpha < \kappa : \overline{G_\alpha} \neq G_\alpha\}$.

Claim 5.3.1. *S is non-stationary.*

⊢ Toward a contradiction, suppose that S were stationary. For each $\alpha \in S$, let $p_\alpha \in \overline{G_\alpha} \setminus G_\alpha$. For $\alpha \in \kappa$, let D_α be a countable dense subset of B_α. Note that we have $\overline{G_\alpha} = \overline{\bigcup_{\beta<\alpha} D_\beta}$ for $\alpha < \kappa$.

Let $A = \{p_\alpha : \alpha \in S\} \cup \bigcup_{\beta<\kappa} D_\beta$. Since X is almost meta-Lindelöf and $|A| = \kappa < |X|$, the subspace A of X is meta-Lindelöf. Thus there is an open refinement $\mathcal{E}$ of $\{G_\alpha : \alpha < \kappa\}$ that covers A and is point countable on A. For each $\alpha \in S$ choose $E_\alpha \in \mathcal{E}$ such that $p_\alpha \in E_\alpha$. Then $p_\alpha \in \overline{\bigcup_{\beta<\alpha} D_\beta}$ implies that there is $f(\alpha) < \alpha$ with $E_\alpha \cap D_{f(\alpha)} \neq \emptyset$.

By Fodor's theorem, there is a $\beta^* < \kappa$ such that $T = \{\alpha \in S : f(\alpha) = \beta^*\}$ is stationary. Since D_{β^*} is countable, there is a $d^* \in D_{\beta^*}$ such that $|\{\alpha \in T : d^* \in D_{\beta^*} \cap E_\alpha\}| = \kappa$ and by the point countability of $\mathcal{E}$ (on A) there is $E^* \in \mathcal{E}$ such that $d^* \in E^*$ and $|\{\alpha \in T : E^* = E_\alpha\}| = \kappa$. Let $\gamma < \kappa$ be such that $E^* \subseteq G_\gamma$ and let $\alpha \in S \setminus \gamma$ be such that $E^* = E_\alpha$. Then $p_\alpha \in E_\alpha = E^* \subseteq G_\gamma \subseteq G_\alpha$. This is a contradiction to the choice of p_α.   ⊣ (Claim 5.3.1)

The rest of the proof for this case can be carried out just as in Case 2 of the proof of Theorem 4.3.

Case 3. κ is singular.

Let $\langle \kappa_i : i < \mathrm{cf}(\kappa) \rangle$ be a continuously and strictly increasing sequence of cardinals cofinal in κ and put $G_i = \bigcup\{B_\alpha : \alpha < \kappa_i\}$ for $i < \mathrm{cf}(\kappa)$.

For each $i < \mathrm{cf}(\kappa)$ there is a point countable open refinement $\mathcal{C}_i$ of the cover $\{B_\alpha : \alpha < \kappa_i\}$ of G_i: if $|G_i| = |X|$ then this follows from the induction hypothesis, since $|G_i| = |X| > \kappa_i \geq |\{B_\alpha : \alpha < \kappa_i\}|$; if $|G_i| < |X|$ then from the almost meta-Lindelöfness of X.

$\mathcal{C} = \bigcup_{i < \mathrm{cf}(\kappa)} \mathcal{C}_i$ is then an open refinement of $\mathcal{B}$ with $\mathrm{ord}(\mathcal{C}) \leq \mathrm{cf}(\kappa)$. Since each member of $\mathcal{C}$ is separable, as we have seen several times already, the intersection relation $\sim_\mathcal{C}$ on $\mathcal{C}$ satisfies $|\{C' \in \mathcal{C} : C' \sim_\mathcal{C} C\}| \leq \mathrm{cf}(\kappa)$ for each $C \in \mathcal{C}$. Consequently, every equivalence class e of the transitive closure $\approx_\mathcal{C}$ of $\sim_\mathcal{C}$ is of size $\leq \mathrm{cf}(\kappa)$ by Lemma 4.1. But then the open cover e of $\bigcup e$ has a point countable open refinement $\mathcal{D}_e$: if $|\bigcup e| = |X|$ then by the induction hypothesis, and if $|\bigcup e| < |X|$ then by the almost meta-Lindelöfness of X. It is now obvious that if $\mathbb{E}$ is the set of all equivalence classes of $\approx_\mathcal{C}$ then $\bigcup\{\mathcal{D}_e : e \in \mathbb{E}\}$ is a point countable open refinement of $\mathcal{C}$ and hence of $\mathcal{B}$.   ❏ (Proposition 5.3)

Proof of Theorem 5.1: Let X be a locally separable and almost meta-Lindelöf space with $|X| = \lambda > \mathrm{cf}(\lambda)$.

First we show that every open cover $\mathcal{B}$ of X has an open refinement $\mathcal{C}$ with $\mathrm{ord}(\mathcal{C}) \leq \mathrm{cf}(\lambda)$. Since X is locally separable, we may assume that $\mathcal{B}$ consists of separable open sets. If $|\mathcal{B}| < \lambda$ then $\mathcal{B}$ has a point countable open refinement by Proposition 5.3. Thus we may assume $|\mathcal{B}| = \lambda$. Let $\mathcal{B} = \{O_\alpha : \alpha \in \lambda\}$.

Fix an increasing sequence of cardinals $\langle \lambda_i : i < \mathrm{cf}(\lambda) \rangle$ cofinal in λ and for each $i < \mathrm{cf}(\lambda)$ let $\mathcal{B}_i = \{O_\alpha : \alpha < \lambda_i\}$. $\mathcal{B}_i$ has a point countable open refinement $\mathcal{C}_i$ for each $i < \lambda$. Indeed, if $|\bigcup \mathcal{B}_i| < \lambda$ this is because X is almost meta-Lindelöf, and if $|\bigcup \mathcal{B}_i| = \lambda$ then this follows from Proposition 5.3. Clearly $\mathcal{C} = \bigcup\{\mathcal{C}_i : i < \mathrm{cf}(\lambda)\}$ is the required refinement.

Now we show that X is meta-Lindelöf. For this, it is enough to show that if $\mathcal{B}$ is a cover of X consisting of separable open sets with $\mathrm{ord}(\mathcal{B}) \leq \mathrm{cf}(\lambda)$ then $\mathcal{B}$ has a point countable open refinement.

This is done simply by repeating the proof of Case 3 of Proposition 5.3 with the only difference that if we have $|\bigcup e| = \lambda$ for an equivalence class e of the transitive closure $\approx_\mathcal{B}$ of the intersection relation $\sim_\mathcal{B}$ on $\mathcal{B}$. Then we apply Proposition 5.3 instead of the induction hypothesis.   ❏ (Theorem 5.1)

Proposition 5.3 also has the following obvious application:

Corollary 5.4. *Suppose that X is locally separable with $L(X) < |X|$. If X is almost meta-Lindelöf then X is meta-Lindelöf.*   ❏

Corollary 5.5. *Suppose that X is a locally countably compact space such that $\max\{L(X), \aleph_1\} < |X|$. If X is almost metrizable then X is metrizable.*

Proof. By Corollary 5.4 and the argument of the proof of Corollary 4.4.
     ❏ (Corollary 5.5)

We note that local (countable) compactness cannot be simply dropped from any of the results above on almost metrizability implying metrizability. In fact, it was observed in Hajnal and Juhász [15] that for every uncountable cardinal κ there is an almost metrizable but non-metrizable space of cardinality κ. Moreover, the space constructed in [15] is "nice" in the sense that it has a single non-isolated point and hence is totally paracompact.

Extending the terminology of local countability, let us say that a topological space X is locally $< \kappa$ if each $x \in X$ has a neighborhood U of cardinality $< \kappa$. Locally $\leq \kappa$ is the same as locally $< \kappa^+$.

Proposition 5.6. *Assume that $\mu > \aleph_1$ is a regular cardinal, $\mu \leq \lambda$ and $\mathrm{FRP}(\delta)$ holds for all regular δ with $\mu \leq \delta \leq \lambda$. If X is a countably tight and $< \mu$-meta-Lindelöf space of size $\leq \lambda$ then every cover $\mathcal{B}$ of X consisting of open sets of size $< \mu$ has an open refinement $\mathcal{B}'$ with $\widehat{\mathrm{ord}}(\mathcal{B}') \leq \mu$.*

Proof. Let $(*)_\kappa$ be the following assertion:

$(*)_\kappa$ Any cover $\mathcal{B}$ of size κ consisting of open sets of size $< \mu$ of a countably tight and $< \mu$-meta-Lindelöf space X has an open refinement $\mathcal{B}'$ with $\widehat{\mathrm{ord}}(\mathcal{B}') \leq \mu$.

Clearly it is enough to prove $(*)_\kappa$ for all cardinals κ by induction on $\kappa \leq \lambda$. So assume that $(*)_{\kappa'}$ holds for all $\kappa' < \kappa \leq \lambda$ and suppose that X and $\mathcal{B}$ are as in $(*)_\kappa$.

Case 1. $\kappa < \mu$.

Then $\mathcal{B}$ is an open refinement of itself with $\widehat{\mathrm{ord}}(\mathcal{B}) \leq \mu$.

Case 2. $\kappa \geq \mu$ is regular.

Note that we have $|X| \leq |\mathcal{B}| \cdot \mu = \kappa$. If $|X| < \kappa$ we are done by the induction hypothesis, so we may assume that $|X| = \kappa$ and in fact that $X = \kappa$. Let $\mathcal{B} = \{B_\alpha : \alpha < \kappa\}$ and $G_\alpha = \bigcup_{\beta < \alpha} B_\beta$ for $\alpha < \kappa$. Since $|B_\alpha| < \mu$ for all $\alpha < \kappa$, we have $|G_\alpha| < \kappa$ for all $\alpha < \kappa$. Thus $C = \{\alpha < \kappa : \alpha$ is a limit or 0 and $G_\alpha = \alpha\}$ is a club in κ.

Claim 5.6.1. $\{\alpha \in C : \overline{G_\alpha} = G_\alpha\}$ *contains a club.*

$\vdash$ Since X is countably tight, the same argument as in the proof of Subclaim 4.3.1.2 can be repeated here to conclude that it suffices to show that $S = \{\alpha \in C \cap E_\omega^\kappa : \overline{G_\alpha} \neq G_\alpha\}$ is non-stationary.

Suppose, toward a contradiction, that S were stationary. Choose

$(5.1) \quad p_\alpha \in \overline{G_\alpha} \setminus G_\alpha$

for each $\alpha \in S$. Since $G_\alpha = \alpha$ and X is countably tight, there is $g_0(\alpha) \in [\alpha]^{\aleph_0}$ such that $p_\alpha \in \overline{g_0(\alpha)}$. Let

$$(5.2) \quad g(\alpha) = g_0(\alpha) \cup \{p_\alpha\}.$$

Now, there is $I \in [\kappa]^{\aleph_1}$ as in the definition of FRP(κ) for these S and g. For every $\alpha \in S \cap I$, we have $p_\alpha \in I$ and $g_0(\alpha) \subseteq I$ by (2.2) and (5.2). Since $|I| = \aleph_1 < \mu$, I as a subspace of X is meta-Lindelöf. Thus the open cover $\mathcal{E} = \{G_\alpha \cap I : \alpha \in I\}$ of I has a point countable open refinement $\mathcal{E}^*$. For each $\alpha \in S \cap I$ let $E_\alpha \in \mathcal{E}^*$ be such that $p_\alpha \in E_\alpha$. Then $p_\alpha \in \overline{g_0(\alpha)}$ implies $g_0(\alpha) \cap E_\alpha \neq \emptyset$, so we may pick $f(\alpha) \in g_0(\alpha) \cap E_\alpha$. By the choice of I then there is $\beta^* \in I$ such that $\{\alpha \in S \cap I : f(\alpha) = \beta^*\}$ is stationary in sup(I).

Now, ord$(\beta^*, \mathcal{E}^*) \leq \aleph_0$ implies that there is $E^* \in \mathcal{E}^*$ such that $J = \{\alpha \in S \cap I : E_\alpha = E^*\}$ is unbounded in sup(I). Let $\gamma \in I$ be such that $E^* \subset G_\gamma$ and $\alpha \in J \setminus \gamma$. Then $p_\alpha \in E_\alpha = E^* \subseteq G_\gamma \subseteq G_\alpha$. This is a contradiction to (5.1). $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\dashv$ (Claim 5.6.1)

The rest of the proof can be carried out as in Case 2 of the proof of Theorem 4.3. Let $D \subseteq \{\alpha \in C : \overline{G_\alpha} = G_\alpha\}$ be a club and $\langle \gamma_i : i < \kappa \rangle$ be the increasing enumeration of $D \cup \{0\}$. Let $H_i = G_{\gamma_{i+1}} \setminus G_{\gamma_i}$ for $i < \kappa$. Then $\{H_i : i < \kappa\}$ forms a partition of X into disjoint clopen sets of size $< \kappa$ and by the induction hypothesis there is an open refinement $\mathcal{B}'_i$ of $\mathcal{B} \restriction H_i$ with $\widehat{\mathrm{ord}}(\mathcal{B}'_i) \leq \mu$. $\mathcal{B}' = \bigcup_{i < \kappa} \mathcal{B}'_i$ is then a refinement of $\mathcal{B}$ as required.

Case 3. $\kappa > \mu$ is singular.

The proof of this case is quite similar to Case 3 in the proof of Theorem 4.3. Let $\langle \kappa_i : i < \mathrm{cf}(\kappa) \rangle$ be a continuously and strictly increasing sequence of cardinals cofinal in κ and put $G_i = \bigcup \{B_\alpha : \alpha < \kappa_i\}$ for each $i < \mathrm{cf}(\kappa)$. By the induction hypothesis, the cover $\{B_\alpha : \alpha < \kappa_i\}$ of G_i has an open refinement $\mathcal{C}_i$ such that $\widehat{\mathrm{ord}}(\mathcal{C}_i) \leq \mu$. Note that each element C of $\mathcal{C}_i$ is of cardinality $< \mu$. Now, if $\mathcal{C} = \bigcup_{i < \mathrm{cf}(\kappa)} \mathcal{C}_i$ then $\mathcal{C}$ covers X and ord$(\mathcal{C}) \leq \max\{\mathrm{cf}(\kappa), \mu\} < \kappa$.

Let $\approx_{\mathcal{C}}$ be the transitive closure of the intersection relation $\sim_{\mathcal{C}}$ on $\mathcal{C}$. Using Lemma 4.1 it is easy to check that each equivalence class of $\approx_{\mathcal{C}}$ has cardinality $\leq \max\{\mathrm{cf}(\kappa), \mu\}$. So if $\mathbb{E}$ is the set of all equivalence classes of $\approx_{\mathcal{C}}$ then $\{\bigcup e : e \in \mathbb{E}\}$ is a partition of X into disjoint clopen sets of size $\leq \max\{\mathrm{cf}(\kappa), \mu\} < \kappa$ and so the inductive hypothesis can be applied as in Case 2 to obtain a desired refinement. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad\square$ (Proposition 5.6)

The following theorem shows that Question 4.3 in Dow [6] can be (consistently) irrelevant:

Theorem 5.7. (1) *Assume that $\aleph_0 < \mu \le \lambda$ with μ regular and $\mathrm{FRP}(\delta)$ holds for all regular δ with $\mu \le \delta \le \lambda$. Then every locally $< \mu$ and $< \mu$-metrizable space of cardinality $\le \lambda$ is metrizable.*

(2) *Assume that FRP holds and μ is an uncountable regular cardinal. Then every locally $< \mu$ and $< \mu$-metrizable space is metrizable.*

Proof. Again, it suffices to prove only (1). Let X be locally $< \mu$ and $< \mu$-metrizable with $|X| \le \lambda$. By Proposition 5.6 and Lemma 4.1, X can be partitioned into open subsets of cardinality $< \mu$. As X is $< \mu$-metrizable, each open set in the partition is metrizable, hence so is X.   ❑ (Theorem 5.7)

Dow proved in [6] the statement of the following corollary under Axiom R.

Corollary 5.8. (1) *Assume that $\mathrm{FRP}(\delta)$ holds for all regular δ for which $\aleph_1 \le \delta \le \lambda$. Then every locally $\le \aleph_1$ and $\aleph_1$-metrizable space of cardinality $\le \lambda$ is metrizable.*

(2) *Assume FRP. Then every locally $\le \aleph_1$ and $\aleph_1$-metrizable space is metrizable.*   ❑

Corollary 5.8 is a variant of 4.4 in which "countable compactness" is replaced by "locally $\le \aleph_1$". Since any compact metric space has cardinality $\le 2^{\aleph_0}$, this corollary extends 4.4 if, in addition to FRP, we also have CH.

The following natural problem remains open.

Problem 1. *Is it consistent that the statement of Theorem 5.7 holds for all uncountable cardinals μ?*

For singular μ, Dow [6] showed in ZFC that if X is of cardinality μ and locally $\le \delta$ for some $\mathrm{cf}(\mu) \le \delta < \mu$ then the $< \mu$-metrizability of X implies the metrizability of X.

The following result was established in Hajnal and Juhász [15]: *If κ is a weakly compact cardinal then every countably tight and almost metrizable (i.e. $< \kappa$-metrizable) space of cardinality κ is metrizable.* Actually, this was stated as [15, Theorem 1] for first countable spaces but the same proof as there also works for countably tight spaces. The next theorem shows that if we assume "more" compactness of κ then we can strengthen this result for countably tight $< \kappa$-metrizable spaces of cardinality $\ge \kappa$.

Theorem 5.9. (1) *Assume that $\kappa \le \lambda$ and κ is λ-compact. Then every countably tight and $< \kappa$-metrizable space of cardinality λ is metrizable.*

(2) *Assume that κ is (strongly) compact. Then every countably tight and $< \kappa$-metrizable space is metrizable.*

Proof. (1): Without loss of generality we may assume that $X = \lambda$ with the topology τ.

Recall that "κ is λ-compact" means there is a κ-complete fine ultrafilter U on $\mathcal{P}_\kappa \lambda$ (for more about λ-compact cardinals see e.g. [16] or [17]). For each $x \in \mathcal{P}_\kappa \lambda$, let d_x be a metric on x compatible with the subspace topology of x induced by τ. Now, we define $d : \lambda \times \lambda \to \mathbb{R}^+$ by

(5.3) $d(\alpha, \beta) = r$ if and only if $\{x \in \mathcal{P}_\kappa \lambda : d_x(\alpha, \beta) = r\} \in U$.

It is easy to check that d is a metric on X. Let τ_d be the topology on X induced from d. We claim that $\tau = \tau_d$.

Since both τ and τ_d are countably tight, it suffices for this to show that $\alpha \in \overline{A}$ if and only if $\alpha \in \overline{A}^d$ whenever $\alpha \in X$ and $A \subseteq X$ is countable where $\overline{A}$ denotes the closure of A with respect to τ while $\overline{A}^d$ the closure of A with respect to d. But this easily follows from the fact that $H_{\alpha,A} = \{x \in \mathcal{P}_\kappa \lambda : \{\alpha\} \cup A \subseteq x\} \in U$ and, for every $x \in H_{\alpha,A}$, the metric d_x is compatible with the τ-subspace topology on x. This completes the proof of (1).

(2) follows immediately from (1) since κ is compact if and only if it is λ-compact for all $\lambda \geq \kappa$. $\square$ (Theorem 5.9)

The assumption of countable tightness in Theorem 5.9 may seem to be restrictive but it actually is not. To explain this let us recall the following piece of notation from cardinal function theory. For a topological space X, let $\widehat{t}(X)$ denote the smallest cardinal μ such that whenever $p \in \overline{A}$ for some $p \in X$ and $A \subset X$ then there is a subset $B \subset A$ with $p \in \overline{B}$ and $|B| < \mu$. Thus X is countably tight if and only if $\widehat{t}(X) \leq \aleph_1$. The following simple proposition implies that in Theorem 5.9 the assumption of countable tightness could have been replaced by the seemingly much weaker condition $\widehat{t}(X) \leq \kappa$.

Proposition 5.10. *For any cardinal κ, if X is $< \kappa$-metrizable and $\widehat{t}(X) \leq \kappa$ then X is actually countably tight.*

Proof.

Assume that $p \in \overline{A}$ in X, then $\widehat{t}(X) \leq \kappa$ implies $p \in \overline{B}$ for some $B \subseteq A$ with $|B| < \kappa$. But the subspace $B \cup \{p\}$ of X is metrizable because of its cardinality being less than κ. Hence there is a countable set $C \subseteq B \subseteq A$ such that $p \in \overline{C}$.
                      $\square$ (Proposition 5.10)

The following is mentioned in [22] as Hamburger's problem:

Problem 2. *Is it consistent that every regular, first countable and $\aleph_1$-metrizable space is metrizable?*

The assertion in Problem 2 with "$\aleph_1$-metrizable" replaced by "$< 2^{\aleph_0}$-metrizable" is known to be consistent. In Dow, Tall and Weiss [8, Theorem 5.2] it is shown that the model of ZFC obtained by adding supercompact many Cohen reals satisfies that every first countable $< 2^{\aleph_0}$-metrizable space is metrizable. Note however that the continuum in this model is fairly large. Theorem 5.9 above can be also seen as a special case of this theorem since it is proved in [8, Lemma 5.4] that non-metrizability of a topological space in the ground model is preserved in Cohen extensions.

Problem 3. *Is every regular, first countable, and almost metrizable space of singular cardinality metrizable?*

Koszmider [19, Theorem 35] showed that the negative answer to the problem above is consistent for spaces of singular cardinality of uncountable cofinality.

6 Non-reflection for locally compact spaces under the reflection of all stationary sets

For a regular cardinal κ, let $\mathrm{ADS}^-(\kappa)$ denote the following principle:

$\mathrm{ADS}^-(\kappa)$: there are a stationary set $S \subseteq \kappa$ and a sequence $\langle a_\alpha : \alpha \in S \rangle$ such that

(6.1) $a_\alpha \subseteq \alpha$ and $\mathrm{otp}(a_\alpha) = \omega$ for all $\alpha \in S$;

(6.2)   for any $\beta < \kappa$, there is a mapping $f : S \cap \beta \to \beta$ such that $f(\alpha) < \sup(a_\alpha)$ for all $\alpha \in S \cap \beta$ and $a_\alpha \setminus f(\alpha)$, $\alpha \in S \cap \beta$ are pairwise disjoint.

Let $\mathrm{ADS}^{-*}(\kappa)$ be the assertion that there are a stationary $S \subseteq E^\kappa_\omega$ and a sequence $\langle a_\alpha : \alpha \in S \rangle$ such that (6.1) and (6.2) hold.

Lemma 6.1. *For any regular κ, $\mathrm{ADS}^-(\kappa)$ is equivalent to $\mathrm{ADS}^{-*}(\kappa)$.*

Proof. It is clear that $\mathrm{ADS}^-(\kappa)$ follows from $\mathrm{ADS}^{-*}(\kappa)$. So assume $\mathrm{ADS}^-(\kappa)$ with $S \subseteq \kappa$ and $\langle a_\alpha : \alpha \in S \rangle$ witnessing this. We have to show that there are a stationary $S^* \subseteq E^\kappa_\omega$ and a sequence $\langle a^*_\alpha : \alpha \in S^* \rangle$ such that they satisfy (6.1) and (6.2).

Case 1. $\{\sup(a_\alpha) : \alpha \in S'\}$ is bounded in κ for some stationary $S' \subseteq S$.

Let $\alpha^* = \sup\{\sup(a_\alpha) : \alpha \in S'\} < \kappa$ and $S^* = E^\kappa_\omega \setminus \alpha^*$. Let $\langle a^*_\alpha : \alpha \in S^* \rangle$ be any one-to-one re-enumeration of $\langle a_\alpha : \alpha \in S' \rangle$. Then this S^* and $\langle a^*_\alpha : \alpha \in S^* \rangle$ are as desired.

Case 2. $\{\sup(a_\alpha) : \alpha \in S'\}$ is unbounded in κ for all stationary $S' \subseteq S$.

Claim 6.1.1. $S^* = \{\alpha \in S : \sup(a_\alpha) = \alpha\}$ *is stationary.*

⊢ Otherwise there is a club C such that $C \cap S^* = \emptyset$. Then for every $\alpha \in S \cap C$ we have $\sup(a_\alpha) < \alpha$. By Fodor's theorem there are a stationary set $S' \subseteq S \cap C$ and a $\delta < \kappa$ such that $\sup(a_\alpha) = \delta$ for all $\alpha \in S'$ which is a contradiction to the assumption of the case. ⊣ (Claim 6.1.1)

Clearly $S^* \subseteq E^\kappa_\omega$. Thus these S^* and $\langle a_\alpha : \alpha \in S^* \rangle$ are as desired. ❏ (Lemma 6.1)

Proposition 6.2. *For a regular cardinal $\kappa > \aleph_1$, $\mathrm{ADS}^-(\kappa)$ implies $\neg\mathrm{FRP}(\kappa)$.*

Proof. Suppose that a stationary $S \subseteq \kappa$ and a sequence $\langle a_\alpha : \alpha \in S \rangle$ witness $\mathrm{ADS}^-(\kappa)$. By Lemma 6.1, we may assume that $S \subseteq E^\kappa_\omega$. Let $g : S \to [\kappa]^{\aleph_0}$ be defined by $g(\alpha) = a_\alpha$ for $\alpha \in S$. For any $I \in [\kappa]^{\aleph_1}$, since $\sup(I) < \kappa$, there is an $f : S \cap I \to \kappa$ such that $f(\alpha) \in g(\alpha) \subset \alpha$ for all $\alpha \in S \cap I$ and the sets $g(\alpha) \setminus f(\alpha)$ are pairwise disjoint for $\alpha \in S \cap I$. But since $f(\alpha) \in g(\alpha) \setminus f(\alpha)$, f is one-to-one. This shows that $\mathrm{FRP}(\kappa)$ fails. ❏ (Proposition 6.2)

$\mathrm{ADS}^-(\kappa)$ for a regular $\kappa > \aleph_1$ not only negates $\mathrm{FRP}(\kappa)$ but actually it also implies the existence of a space as in Proposition 1.5.

Proposition 6.3. *Suppose that $\mathrm{ADS}^-(\kappa)$ holds for a regular uncountable κ. Then there is a locally countable, locally compact, and almost metrizable space of cardinality κ that is not meta-Lindelöf.*

Proof. Let $\langle a_\alpha : \alpha \in S \rangle$ be a sequence as in the definition of $\mathrm{ADS}^-(\kappa)$. Without loss of generality we may assume that all members of S are limit ordinals while the elements of the a_α's are successors. For the latter condition note that we may simply replace each a_α by $a'_\alpha = \{\xi + 1 : \xi \in a_\alpha\}$.

Let $X = S \cup \{\xi + 1 : \xi \in \kappa\}$ with the topology defined as follows: All successors are isolated and a basic neighborhood of $\alpha \in S$ is of the form $\{\alpha\} \cup (a_\alpha \setminus \beta)$ where $\beta < \sup(a_\alpha)$. Just as in the proof of Proposition 1.5, it is easy to show that X is not meta-Lindelöf, locally countable, and locally compact. Thus the following claim completes the proof.

Claim 6.3.1. X *is almost metrizable.*

⊢ It is enough to show that $X \cap \beta$ is metrizable for every limit ordinal $\beta < \kappa$. To see this, take an $f : S \cap \beta \to \beta$ such that $f(\alpha) < \sup(a_\alpha)$ for all $\alpha \in S \cap \beta$

and the sets $a_\alpha \backslash f(\alpha)$ for $\alpha \in S \cap \beta$ are pairwise disjoint. Let $I = \{\xi + 1 : \xi \in \beta\} \backslash \bigcup\{a_\alpha \backslash f(\alpha) : \alpha \in S \cap \beta\}$. Then $\mathcal{U} = \{\{\alpha\} \cup (a_\alpha \backslash f(\alpha)) : \alpha \in S \cap \beta\} \cup \{\{\alpha\} : \alpha \in I\}$ is a partition of $X \cap \beta$ into countable open sets. Each element of $\mathcal{U}$ is second countable and regular and hence metrizable. It follows that $X \cap \beta$ is metrizable as well. $\qquad\qquad \dashv$ (Claim 6.3.1)

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ ❏ (Proposition 6.3)

The following principle ADS(λ) was studied by S. Shelah in [20].

ADS(λ): there is a sequence $\langle a_\alpha : \alpha < \lambda^+ \rangle$ such that

(6.3) $a_\alpha \subseteq \lambda$, $\sup(a_\alpha) = \lambda$ and $\operatorname{otp}(a_\alpha) = \operatorname{cf}(\lambda)$ for all $\alpha < \lambda^+$;

(6.4) for any $\beta < \lambda^+$, there is a mapping $f : \beta \to \lambda$ such that $a_\alpha \backslash f(\alpha)$, $\alpha < \beta$ are pairwise disjoint.

The following is immediate from the definitions of the principles ADS(λ) and ADS$^-(\lambda^+)$:

Proposition 6.4. *Suppose that* $\operatorname{cf}(\lambda) = \omega$. *Then* ADS($\lambda$) *implies* ADS$^-(\lambda^+)$.

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ ❏

Note that, even if $\operatorname{cf}(\lambda) = \omega$, ADS($\lambda$) and ADS$^-(\lambda^+)$ are not quite the same: while, in ADS(λ), the "almost" pairwise essentially disjoint family $\langle a_\alpha : \alpha < \lambda^+ \rangle$ consists of subsets of λ, there is no such restriction on the corresponding family in ADS$^-(\lambda^+)$. Actually, it is proved in [14] that, for a cardinal κ, ADS$^-(\lambda)$ for all regular cardinal $\lambda < \kappa$ is equivalent to $\neg$FRP(λ) for all regular cardinal $\lambda < \kappa$. From Theorem 4.3, Corollary 4.4 and Proposition 6.3 together with this result from [14], it follows that FRP is equivalent with both of the assertions of Theorem 4.3, (2) and Corollary 4.4 over ZFC.

Let us denote with ORP(κ) the assertion that (the stationarity of) every stationary subset of E^κ_ω reflects down to an ordinal of cofinality ω_1 (in the notation of [5], this is Refl$(1, E^\kappa_\omega, \omega_1)$). Clearly FRP($\kappa$) implies ORP($\kappa$) but the converse is false, as we shall see.

The following two results from [5] now provide what we need to show the consistency (modulo consistency strength of some large cardinal) of ADS$^-(\kappa)$ + ORP(κ) for a regular cardinal $\kappa > \aleph_1$.

Theorem 6.5. (J. Cummings, M. Foreman and M. Magidor [5, Theorem 7 and Theorem 21])

(1) $\square^*_\lambda$ *implies* ADS(λ).

(2) *If* ZFC + *"there are infinitely many supercompact cardinals" is consistent then so* ZFC + $\square^*_{\omega_\omega}$ + ORP($\aleph_{\omega+1}$). $\qquad\qquad$ ❏

Actually, [5] proves the consistency of $\square^*_{\omega_\omega}$ with a reflection property that looks much stronger than ORP($\aleph_{\omega+1}$).

Corollary 6.6. *It is consistent (modulo the large cardinal assumption of Theorem 6.5,(2)) that there is a locally countable, locally compact Hausdorff space of cardinality $\aleph_{\omega+1}$ which is almost metrizable but not meta-Lindelöf (in particular, then FRP($\aleph_{\omega+1}$) fails), while ORP($\aleph_{\omega+1}$) holds.* ❏

In Fuchino, Sakai, Soukup and Usuba [14], the consistency of ORP($\aleph_2$) with ¬FRP($\aleph_2$) is proved relative to a single supercompact cardinal.

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