A STRENGTHENING OF THE ČECH–POSPIŠIL THEOREM

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ABSTRACT. We prove the following result: If in a compact space X there is a λ-branching family of closed sets then X cannot be covered by fewer than λ many discrete subspaces. (A family of sets $\mathcal{F}$ is λ-branching iff $|\mathcal{F}| < \lambda$ but one can form λ many pairwise disjoint intersections of subfamilies of $\mathcal{F}$.) The proof is based on a recent, still unpublished, lemma of G. Gruenhage.

As a consequence, we obtain the following strengthening of the well-known Čech–Pospišil theorem: If X a is compact T_2 space such that all points $x \in X$ have character $\chi(x, X) \geq \kappa$ then X cannot be covered by fewer than 2^κ many discrete subspaces.

In [4] the following problem was formulated and raised: If X is a crowded (i. e. dense-in-itself) compact T_2 space, is then $\operatorname{dis}(X) \geq \mathfrak{c}$? The cardinal function $\operatorname{dis}(X)$ was defined there as the smallest infinite cardinal κ such that X can be covered by κ many discrete subspaces. This problem was answered affirmatively by G. Gruenhage: He showed in [2] that if $f : X \twoheadrightarrow Y$ is a perfect onto map, with X and Y arbitrary topological spaces, then $\operatorname{dis}(X) \geq \operatorname{dis}(Y)$. Since any crowded compact T_2 space maps continuously (and hence perfectly) onto the closed interval $[0, 1]$, this clearly suffices.

Our aim here is to present another solution to the above problem which also may be considered as a significant strengthening of the, by now classical, Čech–Pospišil theorem from [1], see also [3, 3.16]. Although our solution goes in a completely different direction from that of Gruenhage, it makes use of a lemma of his that was playing a crucial role in his solution as well. Since [2] is still unpublished and because we would like to make our paper self-contained, we shall start by presenting a proof of Gruenhage's lemma. Our proof, we think, is also somewhat simpler than the one given in [2].

First we recall that a point x in a space X is a limit point of a set X iff for every neighbourhood U of x in X we have $U \cap A \setminus \{x\} \neq \emptyset$. (Of

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course, if X is T_1 then this is equivalent to $U \cap A$ being infinite.) Also, for any subset A of X we use A' to denote the derived set of all limit points of A. Note that $D \subset X$ is a discrete subspace iff $D \cap D' = \emptyset$.

Lemma 1. *(G. Gruenhage) Let X be any topological space and $K \subset X$ be its non-empty compact subset with $K \subset \cup \mathcal{D}$, where each $D \in \mathcal{D}$ is a discrete subspace of X. Then there exist $D \in \mathcal{D}$ and $\mathcal{E} \in [\mathcal{D}]^{<\omega}$ such that*

$$\emptyset \neq K \cap \overline{D} \cap \bigcap \{E' : E \in \mathcal{E}\} \subset D.$$

Proof. By Zorn's lemma we may choose a *maximal* subfamily $\mathcal{C} \subset \mathcal{D}$ such that $\{K\} \cup \{C' : C \in \mathcal{C}\}$ is centered (i. e. has the finite intersection property). As K is compact and each C' is closed, then we have

$$K \cap \bigcap \{C' : C \in \mathcal{C}\} \neq \emptyset,$$

hence there is some $D \in \mathcal{D}$ with

$$K \cap D \bigcap \{C' : C \in \mathcal{C}\} \neq \emptyset.$$

But then $D \notin \mathcal{C}$, for $D \cap D' = \emptyset$, so by the maximality of $\mathcal{C}$ there is a finite subfamily $\mathcal{E} \subset \mathcal{C}$ with

$$K \cap D' \cap \bigcap \{E' : E \in \mathcal{E}\} = \emptyset.$$

Since $\overline{D} = D \cup D'$, then D and $\mathcal{E}$ are as required.

□

Note that the sets of the form E' are closed, hence so is $\overline{D} \cap \bigcap \{E' : E \in \mathcal{E}\}$. Consequently, $K \cap \overline{D} \cap \bigcap \{E' : E \in \mathcal{E}\}$ is compact and discrete, and hence finite (and non-empty).

Now, to present the main result of this paper we need a simple definition.

Definition 2. Let λ be an infinite cardinal. A family of sets $\mathcal{F}$ is said to be λ-*branching* if $|\mathcal{F}| < \lambda$ but one can form λ many pairwise disjoint intersections of subfamilies of $\mathcal{F}$.

Theorem 3. *If in a compact space X there is a λ-branching family of* closed *sets then* $\mathrm{dis}(X) \geq \lambda$.

Proof. Let $\mathcal{F}$ be a λ-branching family of closed subsets of X. So $|\mathcal{F}| < \lambda$ and we may fix a family $\mathcal{K}$ of pairwise disjoint non-empty sets with $|\mathcal{K}| = \lambda$ such that each $K \in \mathcal{K}$ is obtainable as the intersection of some subfamily $\mathcal{F}_K \subset \mathcal{F}$. Each $K \in \mathcal{K}$ is closed in X and therefore is also compact.

Arguing indirectly, assume that $\mathrm{dis}(X) < \lambda$ and fix a family $\mathcal{D}$ of discrete subspaces of X such that $|\mathcal{D}| < \lambda$ and $X = \cup \mathcal{D}$. Applying

Lemma 1, for each set $K \in \mathcal{K}$ there are a member $D_K \in \mathcal{D}$ and a finite subfamily $\mathcal{E}_K \in [\mathcal{D}]^{<\omega}$ such that

$$\emptyset \neq S_K = K \cap \overline{D_K} \cap \bigcap\{E' : E \in \mathcal{E}_K\} \subset D_K.$$

Since $K = \cap\mathcal{F}_K$ and D'_K is also compact, we may find a *finite* subfamily $\mathcal{G}_K \subset \mathcal{F}_K$ such that $\bigcap \mathcal{G}_K \cap \bigcap\{E' : E \in \mathcal{E}_K\} \cap D'_K = \emptyset$ and hence

$$S_K \subset T_K = \bigcap \mathcal{G}_K \cap \overline{D_K} \cap \bigcap\{E' : E \in \mathcal{E}_K\} \subset D_K.$$

But then T_K is a compact and discrete set as well, and so it is finite. Consequently, we may extend $\mathcal{G}_K$ with finitely many further members of $\mathcal{F}_K$ to obtain a finite family $\mathcal{H}_K$ with $\mathcal{G}_K \subset \mathcal{H}_K \subset \mathcal{F}_K \subset \mathcal{F}$ in such a way that

$$S_K = \bigcap \mathcal{H}_K \cap D_K \cap \bigcap\{E' : E \in \mathcal{E}_K\}.$$

Now, we have both $|\mathcal{F}| < \lambda$ and $|\mathcal{D}| < \lambda$, so there are only fewer than λ many choices for D_K, $\mathcal{E}_K$ and $\mathcal{H}_K$, while $|\mathcal{K}| = \lambda$, hence there must be distinct sets $K, L \in \mathcal{K}$ such that $D_K = D_L$, $\mathcal{E}_K = \mathcal{E}_L$ and $\mathcal{H}_K = \mathcal{H}_L$, consequently $S_K = S_L$. But this is a contradiction because S_K and S_L are disjoint non-empty sets. □

Let us emphasize that in Theorem 3 no separation axiom had to be assumed about our compact space X. In contrast to this, the following result, the promised strengthening of the Čech–Pospišil theorem, seems to require that our compact space be also Hausdorff.

Corollary 4. *If X is a compact T_2 space such that all points $x \in X$ have character $\chi(x, X) \geq \kappa$ then* $\operatorname{dis}(X) \geq 2^\kappa$.

Proof. The proof starts out exactly as in the proof of the original Čech–Pospišil theorem, that is one builds a Cantor-tree $\mathcal{T} = \{F_s : s \in 2^{<\kappa}\}$ of non-empty closed sets as e. g. in [3, 3.16]. For $\kappa = \omega$, each F_s is regular closed. For $\kappa > \omega$ we have $\chi(F_s, X) \leq |s| \cdot \omega < \kappa$ whenever $s \in 2^{<\kappa}$.

Now let $\mu = \log(2^\kappa) \leq \kappa$ (that is, $\mu = \min\{\lambda : 2^\lambda = 2^\kappa\}$) and set $\mathcal{F} = \{F_s : s \in 2^{<\mu}\}$. Then, by $\operatorname{cf}(2^\mu) > \mu$, we have

$$|\mathcal{F}| = 2^{<\mu} = \sum\{2^\lambda : \lambda < \mu\} < 2^\mu = 2^\kappa,$$

moreover the sets $F_t = \bigcap\{F_{t\restriction\alpha} : \alpha < \mu\}$ for $t \in 2^\mu$ are non-empty and pairwise disjoint, hence $\mathcal{F}$ is 2^κ-branching. Thus it follows from Theorem 3 that $\operatorname{dis}(X) \geq 2^\kappa$. □

In [4, Theorem 3] the following related result was proved: If X is a compact T_2 space such that all points $x \in X$ have character $\chi(x, X) \geq$

κ then $\mathrm{rs}(X) > \kappa$, where

$$\mathrm{rs}(X) = \min\{|\mathcal{R}| : X = \cup\mathcal{R} \text{ and each } R \in \mathcal{R} \text{ is right separated}\}.$$

Since every discrete (sub)space is right separated, this result is stronger than corollary 4 provided that $2^\kappa = \kappa^+$. On the other hand, now the following interesting open question can be raised.

Problem 5. *Can one replace in Corollary 4* $\mathrm{dis}(X)$ *with* $\mathrm{rs}(X)$*, even if* $2^\kappa > \kappa^+$*?*

Finally, we would like to formulate one more open problem. Note that it follows immediately even from the original Čech–Pospišil theorem that if X is a compact T_2 space in which all points have character $\geq \kappa$ then

$$\Delta(X) = \min\{|G| : G \text{ is non-empty open in } X\} \geq 2^\kappa.$$

Consequently, an affirmative answer to the following question would yield another strengthening of Corollary 4.

Problem 6. *For X compact T_2, is* $\mathrm{dis}(X) \geq \Delta(X)$ *?*

References

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