MORE ON CARDINAL INVARIANTS OF ANALYTIC P-IDEALS

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Abstract. Given an ideal $\mathcal{I}$ on ω let $\mathfrak{a}(\mathcal{I})$ ($\bar{\mathfrak{a}}(\mathcal{I})$) be minimum of the cardinalities of infinite (uncountable) maximal $\mathcal{I}$-almost disjoint subsets of $[\omega]^\omega$. We show that $\mathfrak{a}(\mathcal{I}_h) > \omega$ if $\mathcal{I}_h$ is a summable ideal; but $\mathfrak{a}(\mathcal{Z}_{\vec{\mu}}) = \omega$ for any tall density ideal $\mathcal{Z}_{\vec{\mu}}$ including the density zero ideal $\mathcal{Z}$. On the other hand, you have $\mathfrak{b} \le \bar{\mathfrak{a}}(\mathcal{I})$ for any analytic P-ideal $\mathcal{I}$, and $\bar{\mathfrak{a}}(\mathcal{Z}_{\vec{\mu}}) \le \mathfrak{a}$ for each density ideal $\mathcal{Z}_{\vec{\mu}}$.

For each ideal $\mathcal{I}$ on ω denote $\mathfrak{b}_\mathcal{I}$ and $\mathfrak{d}_\mathcal{I}$ the unbounding and dominating numbers of $\langle \omega^\omega, \le_\mathcal{I}\rangle$ where $f \le_\mathcal{I} g$ iff $\{n \in \omega : f(n) > g(n)\} \in \mathcal{I}$. We show that $\mathfrak{b}_\mathcal{I} = \mathfrak{b}$ and $\mathfrak{d}_\mathcal{I} = \mathfrak{d}$ for each analytic P-ideal $\mathcal{I}$.

Given a Borel ideal $\mathcal{I}$ on ω we say that a poset $\mathbb{P}$ is *$\mathcal{I}$-bounding* iff $\forall x \in \mathcal{I} \cap V^\mathbb{P}\ \exists y \in \mathcal{I} \cap V\ x \subseteq y$. $\mathbb{P}$ is *$\mathcal{I}$-dominating* iff $\exists y \in \mathcal{I} \cap V^\mathbb{P}\ \forall x \in \mathcal{I} \cap V\ x \subseteq^* y$.

For each analytic P-ideal $\mathcal{I}$ if a poset $\mathbb{P}$ has the Sacks property then $\mathbb{P}$ is $\mathcal{I}$-bounding; moreover if $\mathcal{I}$ is tall as well then the property $\mathcal{I}$-bounding/$\mathcal{I}$-dominating implies ω^ω-bounding/adding dominating reals, and the converses of these two implications are false.

For the density zero ideal $\mathcal{Z}$ we can prove more: (i) a poset $\mathbb{P}$ is $\mathcal{Z}$-bounding iff it has the Sacks property, (ii) if $\mathbb{P}$ adds a slalom capturing all ground model reals then $\mathbb{P}$ is $\mathcal{Z}$-dominating.

1. Introduction

In this paper we investigate some properties of some cardinal invariants associated with analytic P-ideals. Moreover we analyze related "bounding" and "dominating" properties of forcing notions.

Let us denote fin the Frechet ideal on ω, i.e. $\mathsf{fin} = [\omega]^{<\omega}$. Further we always assume that if $\mathcal{I}$ is an ideal on ω then the ideal is *proper*, i.e. $\omega \notin \mathcal{I}$, and $\mathsf{fin} \subseteq \mathcal{I}$, so especially $\mathcal{I}$ is *non-principal*. Write $\mathcal{I}^+ = \mathcal{P}(\omega) \setminus \mathcal{I}$ and $\mathcal{I}^* = \{\omega \setminus X : X \in \mathcal{I}\}$.

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An ideal $\mathcal{I}$ on ω is *analytic* if $\mathcal{I} \subseteq \mathcal{P}(\omega) \simeq 2^\omega$ is an analytic set in the usual product topology. $\mathcal{I}$ is a *P-ideal* if for each countable $\mathcal{C} \subseteq \mathcal{I}$ there is an $X \in \mathcal{I}$ such that $Y \subseteq^* X$ for each $Y \in \mathcal{C}$, where $A \subseteq^* B$ iff $A \setminus B$ is finite. $\mathcal{I}$ is *tall* (or *dense*) if each infinite subset of ω contains an infinite element of $\mathcal{I}$.

A function $\varphi : \mathcal{P}(\omega) \to [0, \infty]$ is a *submeasure on* ω iff $\varphi(X) \leq \varphi(Y)$ for $X \subseteq Y \subseteq \omega$, $\varphi(X \cup Y) \leq \varphi(X) + \varphi(Y)$ for $X, Y \subseteq \omega$, and $\varphi(\{n\}) < \infty$ for $n \in \omega$. A submeasure φ is *lower semicontinuous* iff $\varphi(X) = \lim_{n\to\infty} \varphi(X \cap n)$ for each $X \subseteq \omega$. A submeasure φ is *finite* if $\varphi(\omega) < \infty$. Note that if φ is a lower semicontinuous submeasure on ω then $\varphi(\bigcup_{n\in\omega} A_n) \leq \sum_{n\in\omega} \varphi(A_n)$ holds as well for $A_n \subseteq \omega$. We assign the *exhaustive ideal* $\mathrm{Exh}(\varphi)$ to a submeasure φ as follows

$$\mathrm{Exh}(\varphi) = \big\{X \subseteq \omega : \lim_{n\to\infty} \varphi(X \setminus n) = 0\big\}.$$

Solecki, [So, Theorem 3.1], proved that an ideal $\mathcal{I} \subsetneq \mathcal{P}(\omega)$ is an analytic P-ideal or $\mathcal{I} = \mathcal{P}(\omega)$ iff $\mathcal{I} = \mathrm{Exh}(\varphi)$ for some lower semicontinuous finite submeasure. Therefore each analytic P-ideal is $F_{\sigma\delta}$ (i.e. Π^0_3) so a Borel subset of 2^ω. It is straightforward to see that if φ is a lower semicontinuous finite submeasure on ω then the ideal $\mathrm{Exh}(\varphi)$ is tall iff $\lim_{n\to\infty} \varphi(\{n\}) = 0$.

Let $\mathcal{I}$ be an ideal on ω. A family $\mathcal{A} \subseteq \mathcal{I}^+$ is $\mathcal{I}$*-almost-disjoint* ($\mathcal{I}$-AD in short), if $A \cap B \in \mathcal{I}$ for each $\{A, B\} \in [\mathcal{A}]^2$. An $\mathcal{I}$-AD family $\mathcal{A}$ is an $\mathcal{I}$*-MAD family* if for each $X \in \mathcal{I}^+$ there exists an $A \in \mathcal{A}$ such that $X \cap A \in \mathcal{I}^+$, i.e. $\mathcal{A}$ is $\subseteq$-maximal among the $\mathcal{I}$-AD families.

Denote $\mathfrak{a}(\mathcal{I})$ the minimum of the cardinalities of infinite $\mathcal{I}$-MAD families. In Theorem 2.2 we show that $\mathfrak{a}(\mathcal{I}_h) > \omega$ if $\mathcal{I}_h$ is a summable ideal; but $\mathfrak{a}(\mathcal{Z}_{\vec{\mu}}) = \omega$ for any tall density ideal $\mathcal{Z}_{\vec{\mu}}$ including the *density zero ideal*

$$\mathcal{Z} = \Big\{A \subseteq \omega : \lim_{n\to\infty} \frac{|A \cap n|}{n} = 0\Big\}.$$

On the other hand, if you define $\bar{\mathfrak{a}}(\mathcal{I})$ as minimum of the cardinalities of uncountable $\mathcal{I}$-MAD families then you have $\mathfrak{b} \leq \bar{\mathfrak{a}}(\mathcal{I})$ for any analytic P-ideal $\mathcal{I}$, and $\bar{\mathfrak{a}}(\mathcal{Z}_{\vec{\mu}}) \leq \mathfrak{a}$ for each density ideal $\mathcal{Z}_{\vec{\mu}}$ (see Theorems 2.6 and 2.8).

In Theorem 3.1 we prove under CH the existence of an uncountable Cohen-indestructible $\mathcal{I}$-MAD families for each analytic P-ideal $\mathcal{I}$.

A sequence $\langle A_\alpha : \alpha < \kappa\rangle \subset [\omega]^\omega$ is a *tower* if it is $\subseteq^*$*-descending,* i.e. $A_\beta \subseteq^* A_\alpha$ if $\alpha \leq \beta < \kappa$, and it has no *pseudointersection*, i.e. a set $X \in [\omega]^\omega$ such that $X \subseteq^* A_\alpha$ for each $\alpha < \kappa$. In Section 4 we show it is consistent that the continuum is arbitrarily large and for each tall analytic P-ideal $\mathcal{I}$ there is towers of height ω_1 whose elements are in $\mathcal{I}^*$.

Given an ideal $\mathcal{I}$ on ω if $f, g \in \omega^\omega$ write $f \leq_{\mathcal{I}} g$ if $\{n \in \omega : f(n) > g(n)\} \in \mathcal{I}$. As usual let $\leq^* = \leq_{\mathsf{fin}}$. The unbounding and dominating numbers of the partially ordered set $\langle \omega^\omega, \leq_{\mathcal{I}} \rangle$, denoted by $\mathfrak{b}_{\mathcal{I}}$ and $\mathfrak{d}_{\mathcal{I}}$ are defined in the natural way, i.e. $\mathfrak{b}_{\mathcal{I}}$ is the minimal size of a $\leq_{\mathcal{I}}$-unbounded family, and $\mathfrak{d}_{\mathcal{I}}$ is the minimal size of a $\leq_{\mathcal{I}}$-dominating family. By these notations $\mathfrak{b} = \mathfrak{b}_{\mathsf{fin}}$ and $\mathfrak{d} = \mathfrak{d}_{\mathsf{fin}}$. In Section 5 we show that $\mathfrak{b}_{\mathcal{I}} = \mathfrak{b}$ and $\mathfrak{d}_{\mathcal{I}} = \mathfrak{d}$ for each analytic P-ideal $\mathcal{I}$. We also prove, in Corollary 6.8, that for any analytic P-ideal $\mathcal{I}$ a poset $\mathbb{P}$ is $\leq_{\mathcal{I}}$-bounding iff it is ω^ω-bounding, and $\mathbb{P}$ adds $\leq_{\mathcal{I}}$-dominating reals iff it adds dominating reals.

In Section 6 we introduce the $\mathcal{I}$-bounding and $\mathcal{I}$-dominating properties of forcing notions for Borel ideals: $\mathbb{P}$ is *$\mathcal{I}$-bounding* iff any element of $\mathcal{I} \cap V^{\mathbb{P}}$ is contained in some element of $\mathcal{I} \cap V$; $\mathbb{P}$ is *$\mathcal{I}$-dominating* iff there is an element in $\mathcal{I} \cap V^{\mathbb{P}}$ which mod-finite contains all elements of $\mathcal{I} \cap V$.

In Theorem 6.2 we show that for each tall analytic P-ideal $\mathcal{I}$ if a forcing notion is $\mathcal{I}$-bounding then it is ω^ω-bounding, and if it is $\mathcal{I}$-dominating then it adds dominating reals. Since the random real forcing is not $\mathcal{I}$-bounding for each tall summable and tall density ideal $\mathcal{I}$ by Proposition 6.3, the converse of the first implication is false. Since a σ-centered forcing can not be $\mathcal{I}$-dominating for a tall analytic P-ideal $\mathcal{I}$ by Theorem 6.4, the standard dominating real forcing $\mathbb{D}$ witnesses that the converse of the second implication is also false.

We prove in Theorem 6.5 that the Sacks property implies the $\mathcal{I}$-bounding property for each analytic P-ideal $\mathcal{I}$.

Finally, based on a theorem of Fremlin we show that the $\mathcal{Z}$-bounding property is equivalent to the Sacks property.

2. Around the almost disjointness number of ideals

For any ideal $\mathcal{I}$ on ω denote $\mathfrak{a}(\mathcal{I})$ the minimum of the cardinalities of infinite $\mathcal{I}$-MAD families.

To start the investigation of this cardinal invariant we recall the definition of two special classes of analytic P-ideals: the density ideals and the summable ideals (see [Fa]).

Definition 2.1. Let $h : \omega \to \mathbb{R}^+$ be a function such that $\sum_{n\in\omega} h(n) = \infty$. The *summable ideal corresponding to h* is

$$\mathcal{I}_h = \Big\{ A \subseteq \omega : \sum_{n \in A} h(n) < \infty \Big\}.$$

Let $\langle P_n : n < \omega\rangle$ be a decomposition of ω into pairwise disjoint nonempty finite sets and let $\vec{\mu} = \langle \mu_n : n \in \omega\rangle$ be a sequences of probability measures, $\mu_n : \mathcal{P}(P_n) \to [0,1]$. The *density ideal generated by* $\vec{\mu}$ is

$$\mathcal{Z}_{\vec{\mu}} = \big\{A \subseteq \omega : \lim_{n\to\infty} \mu_n(A \cap P_n) = 0\big\}.$$

A summable ideal $\mathcal{I}_h$ is tall iff $\lim_{n\to\infty} h(n) = 0$; and a density ideal $\mathcal{Z}_{\vec{\mu}}$ is tall iff

$$(\dagger) \qquad \lim_{n\to\infty} \max_{i\in P_n} \mu_n(\{i\}) = 0.$$

Clearly the density zero ideal $\mathcal{Z}$ is a tall density ideal, and the summable and the density ideals are proper ideals.

Theorem 2.2. *(1)* $\mathfrak{a}(\mathcal{I}_h) > \omega$ *for any summable ideal* $\mathcal{I}_h$.
(2) $\mathfrak{a}(\mathcal{Z}_{\vec{\mu}}) = \omega$ *for any tall density ideal* $\mathcal{Z}_{\vec{\mu}}$.

Proof. (1): We show that if $\{A_n : n < \omega\} \subseteq \mathcal{I}_h^+$ is $\mathcal{I}$-AD then there is $B \in \mathcal{I}_h^+$ such that $B \cap A_n \in \mathcal{I}$ for $n \in \omega$.

For each $n \in \omega$ let $B_n \subseteq A_n \setminus \cup\{A_m : m < n\}$ be finite such that $\sum_{i\in B_n} h(i) > 1$, and put

$$B = \cup\{B_n : n \in \omega\}.$$

(2): Write $\vec{\mu} = \langle \mu_n : n \in \omega\rangle$ and μ_n concentrates on P_n. By (†) we have $\lim_{n\to\infty} |P_n| = \infty$.

Now for each n we can choose $k_n \in \omega$ and a partition $\{P_{n,k} : k < k_n\}$ of P_n such that

(a) $\lim_{n\to\infty} k_n = \infty$,
(b) if $k < k_n$ then $\mu_n(P_{n,k}) \geq \frac{1}{2^{k+1}}$.

Put $A_k = \cup\{P_{n,k} : k < k_n\}$ for each $k \in \omega$. We show that $\{A_k : k \in \omega\}$ is a $\mathcal{Z}_{\vec{\mu}}$-MAD family.

If $k_n > k$ then $\mu_n(A_k \cap P_n) = \mu_n(P_{n,k}) \geq \frac{1}{2^{k+1}}$. Since for an arbitrary k for all but finitely many n we have $k_n > k$ it follows that

$$\limsup_{n\to\infty} \mu_n(A_k \cap P_n) = \limsup_{n\to\infty} \mu_n(P_{n,k}) \geq \limsup_{n\to\infty} \frac{1}{2^{k+1}} = \frac{1}{2^{k+1}} > 0,$$

thus $A_k \in \mathcal{Z}_{\vec{\mu}}^+$.

Assume that $X \in \mathcal{Z}_{\vec{\mu}}^+$. Pick $\varepsilon > 0$ with $\limsup_{n\to\infty} \mu_n(X \cap P_n) > \varepsilon$. For a large enough k we have $\frac{1}{2^{k+1}} < \frac{\varepsilon}{2}$ so if $k < k_n$ then

$$\mu_n(P_n \setminus \cup\{P_{n,i} : i \leq k\}) \leq \frac{1}{2^{k+1}} < \frac{\varepsilon}{2}.$$

So for each large enough n there is $i_n \leq k$ such that $\mu_n(X \cap P_{n,i_n}) > \frac{\varepsilon}{2(k+1)}$. Then $i_n = i$ for infinitely many n, so $\limsup_{n\to\infty} \mu_n(X \cap A_i) \geq \frac{\varepsilon}{2(k+1)}$, and so $X \cap A_i \in \mathcal{Z}_{\vec{\mu}}^+$. □

This Theorem gives new proof of the following well-known fact:

Corollary 2.3. *The density zero ideal $\mathcal{Z}$ is not a summable ideal.*

Given two ideals $\mathcal{I}$ and $\mathcal{J}$ on ω write $\mathcal{I} \leq_{\mathrm{RK}} \mathcal{J}$ (see [Ru]) iff there is a function $f : \omega \to \omega$ such that

$$\mathcal{I} = \{I \subseteq \omega : f^{-1}I \in \mathcal{J}\},$$

and write $\mathcal{I} \leq_{\mathrm{RB}} \mathcal{J}$ (see [LaZh]) iff there is a finite-to-one function $f : \omega \to \omega$ such that

$$\mathcal{I} = \{I \subseteq \omega : f^{-1}I \in \mathcal{J}\}.$$

The following Observations imply that there are $\mathcal{I}$-MAD families of cardinality $\mathfrak{c}$ for each analytic P-ideal $\mathcal{I}$.

Observation 2.4. *Assume that $\mathcal{I}$ and $\mathcal{J}$ are ideals on ω, $\mathcal{I} \leq_{\mathrm{RK}} \mathcal{J}$ witnessed by a function $f : \omega \to \omega$. If $\mathcal{A}$ is an $\mathcal{I}$-AD family then $\{f^{-1}A : A \in \mathcal{A}\}$ is a $\mathcal{J}$-AD family.*

Observation 2.5. $\mathsf{fin} \leq_{\mathrm{RB}} \mathcal{I}$ *for any analytic P-ideal $\mathcal{I}$.*

Proof. Let $\mathcal{I} = \mathrm{Exh}(\varphi)$ for some lower semicontinuous finite submeasure φ on ω. Since $\omega \notin \mathcal{I}$ we have $\lim_{n\to\infty} \varphi(\omega \setminus n) = \varepsilon > 0$. Hence by the lower semicontinuous property of φ for each $n > 0$ there is $m > n$ such that $\varphi([n, m)) > \varepsilon/2$.

So there is a partition $\{I_n : n < \omega\}$ of ω into finite pieces such that $\varphi(I_n) > \varepsilon/2$ for each $n \in \omega$. Define the function $f : \omega \to \omega$ by the stipulation $f''I_n = \{n\}$. Then f witnesses $\mathsf{fin} \leq_{\mathrm{RB}} \mathcal{I}$. □

For any analytic P-ideal $\mathcal{I}$ denote $\bar{\mathfrak{a}}(\mathcal{I})$ the minimum of the cardinalities of uncountable $\mathcal{I}$-MAD families.

Clearly $\mathfrak{a}(\mathcal{I}) > \omega$ implies $\mathfrak{a}(\mathcal{I}) = \bar{\mathfrak{a}}(\mathcal{I})$, especially $\mathfrak{a}(\mathcal{I}_h) = \bar{\mathfrak{a}}(\mathcal{I}_h)$ for summable ideals.

Theorem 2.6. $\bar{\mathfrak{a}}(\mathcal{Z}_{\vec{\mu}}) \leq \mathfrak{a}$ *for each density ideal $\mathcal{Z}_{\vec{\mu}}$.*

Proof. Let $f : \omega \to \omega$ be the finite-to-one function defined by $f^{-1}\{n\} = P_n$ where $\vec{\mu} = \langle \mu_n : n \in \omega \rangle$ and $\mu_n : \mathcal{P}(P_n) \to [0, 1]$. Specially f witnesses $\mathsf{fin} \leq_{\mathrm{RB}} \mathcal{Z}_{\vec{\mu}}$.

Let $\mathcal{A}$ be an uncountable (fin-)MAD family. We show that $f^{-1}[\mathcal{A}] = \{f^{-1}A : A \in \mathcal{A}\}$ is a $\mathcal{Z}_{\vec{\mu}}$-MAD family.

By Observation 2.4, $f^{-1}[\mathcal{A}]$ is a $\mathcal{Z}_{\vec{\mu}}$-AD family.

To show the maximality let $X \in \mathcal{Z}^+_{\vec{\mu}}$ be arbitrary, $\limsup_{n\to\infty} \mu_n(X \cap P_n) = \varepsilon > 0$. Thus

$$J = \{n \in \omega : \mu_n(X \cap P_n) > \varepsilon/2\}$$

is infinite. So there is $A \in \mathcal{A}$ such that $A \cap J$ is infinite.

Then $f^{-1}A \in f^{-1}[\mathcal{A}]$ and $X \cap f^{-1}A \in \mathcal{Z}_{\vec{\mu}}^{+}$ because there are infinitely many n such that we have $P_n \subseteq f^{-1}A$ and $\mu_n(X \cap P_n) > \varepsilon/2$. □

Problem 2.7. Does $\bar{\mathfrak{a}}(\mathcal{I}) \le \mathfrak{a}$ hold for each analytic P-ideal $\mathcal{I}$?

Theorem 2.8. $\mathfrak{b} \le \bar{\mathfrak{a}}(\mathcal{I})$ *provided that* $\mathcal{I}$ *is an analytic P-ideal.*

Remark. If $\mathcal{X} \subset \left[\omega\right]^{\omega}$ is an infinite almost disjoint family then there is a tall ideal $\mathcal{I}$ such that $\mathcal{X}$ is $\mathcal{I}$-MAD. So the Theorem above does not hold for an arbitrary tall ideal on ω.

Proof. $\mathcal{I} = \mathrm{Exh}(\varphi)$ for some lower semicontinuous finite submeasure φ.

Let $\mathcal{A}$ be an uncountable $\mathcal{I}$-AD family of cardinality smaller than $\mathfrak{b}$. We show that $\mathcal{A}$ is not maximal.

There exists an $\varepsilon > 0$ such that the set

$$\mathcal{A}_\varepsilon = \left\{A \in \mathcal{A} : \lim_{n\to\infty} \varphi(A \backslash n) > \varepsilon\right\}$$

is uncountable. Let $\mathcal{A}' = \{A_n : n \in \omega\} \subseteq \mathcal{A}_\varepsilon$ be a set of pairwise distinct elements of $\mathcal{A}_\varepsilon$. We can assume that these sets are pairwise disjoint. For each $A \in \mathcal{A}\backslash\mathcal{A}'$ choose a function $f_A \in \omega^\omega$ such that

$(*_A)$ $\varphi\big((A \cap A_n) \setminus f_A(n)\big) < 2^{-n}$ for each $n \in \omega$.

Using the assumption $|\mathcal{A}| < \mathfrak{b}$ there exists a strictly increasing function $f \in \omega^\omega$ such that $f_A \le^* f$ for each $A \in \mathcal{A}\backslash\mathcal{A}'$. For each n pick $g(n) > f(n)$ such that $\varphi\big(A_n \cap [f(n), g(n))\big) > \varepsilon$, and let

$$X = \bigcup_{n\in\omega} \big(A_n \cap [f(n), g(n))\big).$$

Clearly $X \in \mathcal{Z}_{\vec{\mu}}^{+}$ because for each $n < \omega$ there is m such that $A_m \cap [f(m), g(m)) \subseteq X\backslash n$ and so $\varphi(X \setminus n) \ge \varphi\big(A_m \cap [f(m), g(m))\big) > \varepsilon$, i.e. $\lim_{n\to\infty} \varphi(X\backslash n) \ge \varepsilon$.

We have to show that $X \cap A \in \mathcal{Z}_{\vec{\mu}}$ for each $A \in \mathcal{A}$. If $A = A_n$ for some n then $X \cap A = X \cap A_n = A_n \cap [f(n), g(n))$, i.e. the intersection is finite.

Assume now that $A \in \mathcal{A}\backslash\mathcal{A}'$. Let $\delta > 0$. We show that if k is large enough then $\varphi((A \cap X) \setminus k) < \delta$.

There is $N \in \omega$ such that $2^{-N+1} < \delta$ and $f_A(n) \le f(n)$ for each $n \ge N$.

Let k be so large that k contains the finite set $\bigcup_{n<N}[f(n), g(n))$.

Now $(X \cap A)\backslash k = \bigcup_{n\in\omega}\big(A_n \cap A \cap [f(n), g(n))\big)\backslash k$ and $\big(A_n \cap A \cap [f(n), g(n))\big)\backslash k = \emptyset$ if $n < N$ so

$$(X \cap A)\backslash k = \bigcup_{n\geq N}\big(A_n \cap A \cap [f(n), g(n))\big)\backslash k \subseteq \bigcup_{n\geq N}((A_n \cap A)\backslash f(n)) \subseteq \bigcup_{n\geq N}((A_n \cap A)\backslash f_A(n)).$$

Thus by $(*_A)$ we have

$$\varphi((X \cap A) \setminus k) \leq \sum_{n\geq N}\varphi(A_n \cap A \setminus f_A(n)) \leq \sum_{n\geq N}\frac{1}{2^n} = 2^{-N+1} < \delta.$$

□

3. Cohen-indestructible $\mathcal{I}$-mad families

If φ is a lower semicontinuous finite submeasure on ω then clearly φ is determined by $\varphi \restriction [\omega]^{<\omega}$. Using this observation one can define forcing indestructibility of $\mathcal{I}$-MAD families for an analytic P-ideal $\mathcal{I}$. The following Theorem is a modification of Kunen's proof for existence of Cohen-indestructible MAD family from CH (see [Ku] Ch. VIII Th. 2.3.).

Theorem 3.1. *Assume* CH. *For each analytic P-ideal $\mathcal{I}$ then there is an uncountable Cohen-indestructible $\mathcal{I}$-MAD family.*

Proof. We will define the uncountable Cohen-indestructible $\mathcal{I}$-MAD family $\{A_\xi : \xi < \omega_1\} \subseteq \mathcal{I}^+$ by recursion on $\xi \in \omega_1$. The family $\{A_\xi : \xi < \omega_1\}$ will be fin-AD as well. Our main concern is that we do have $\mathfrak{a}(\mathcal{I}) > \omega$ so it is not automatic that $\{A_\eta : \eta < \xi\}$ is not maximal for $\xi < \omega_1$.

Denote $\mathbb{C}$ the Cohen forcing. Let $\mathcal{I} = \mathrm{Exh}(\varphi)$ be an analytic P-ideal. Let $\{\langle p_\xi, \dot{X}_\xi, \delta_\xi\rangle : \omega \leq \xi < \omega_1\}$ be an enumeration of all triples $\langle p, \dot{X}, \delta\rangle$ such that $p \in \mathbb{C}$, $\dot{X}$ is a nice name for a subset of ω, and δ is a positive rational number.

Write $\varepsilon = \lim_{n\to\infty}\varphi(\omega \setminus n) > 0$. Partition ω into infinite sets $\{A_m : m < \omega\}$ such that $\lim_{n\to\infty}\varphi(A_m \setminus n) = \varepsilon$ for each $m < \omega$.

Assume $\xi \geq \omega$ and we have $A_\eta \in \mathcal{I}^+$ for $\eta < \xi$ such that $\{A_\eta : \eta < \xi\}$ is a fin-AD so especially an $\mathcal{I}$-AD family.

Claim: There is $X \in \mathcal{I}^+$ such that $|X \cap A_\zeta| < \omega$ for $\zeta < \xi$.

Proof of the Claim. Write $\xi = \{\zeta_i : i < \omega\}$. Recursion on $j \in \omega$ we can choose $x_j \in \left[A_{\ell_j}\right]^{<\omega}$ for some $\ell_j \in \omega$ such that

(i) $\varphi(x_j) \geq \varepsilon/2$,

(ii) $x_j \cap (\cup_{i\le j} A_{\zeta_i}) = \emptyset$.

Assume that $\{x_i : i < j\}$ is chosen. Pick $\ell_j \in \omega \setminus \{\zeta_i : i < j\}$. Let $m \in \omega$ such that $A_{\ell_j} \cap \cup\{A_{\zeta_i} : i \le j\} \subseteq m$. Since $\varphi(A_{\ell_j} \setminus m) \ge \varepsilon$ there is $x_j \in \left[A_{\ell_j} \setminus m\right]^{<\omega}$ with $\varphi(x_j) \ge \varepsilon/2$.

Let $X = \cup\{x_j : j < \omega\}$. Then $|A_\zeta \cap X| < \omega$ for $\zeta < \xi$ and $\lim_{n\to\infty}(X \setminus n) \ge \varepsilon/2$. □

If p_ξ does not force (a) and (b) below then let A_ξ be X from the Claim.

(a) $\lim_{n\to\infty} \check{\varphi}(\dot{X}_\xi \setminus n) > \check{\delta}_\xi$,

(b) $\forall\, \eta < \check{\xi}\ \dot{X}_\xi \cap \check{A}_\eta \in \mathcal{I}$.

Assume $p_\xi \Vdash$(a)$\wedge$(b). Let $\{B^\xi_k : k \in \omega\} = \{A_\eta : \eta < \xi\}$ and $\{p^\xi_k : k \in \omega\} = \{p' \in \mathbb{C} : p' \le p_\xi\}$ be enumerations. Clearly for each $k \in \omega$ we have

$$p^\xi_k \Vdash \lim_{n\to\infty} \check{\varphi}\big((\dot{X}_\xi \setminus \cup\{\check{B}^\xi_l : l \le \check{k}\})\setminus n\big) > \check{\delta}_\xi,$$

so we can choose a $q^\xi_k \le p^\xi_k$ and a finite $a^\xi_k \subseteq \omega$ such that $\varphi(a^\xi_k) > \delta_\xi$ and $q^\xi_k \Vdash \check{a}^\xi_k \subseteq (\dot{X}_\xi \setminus \cup\{\check{B}^\xi_l : l \le \check{k}\})\setminus \check{k}$. Let $A_\xi = \cup\{a^\xi_k : k \in \omega\}$. Clearly $A_\xi \in \mathcal{I}^+$ and $\{A_\eta : \eta \le \xi\}$ is a fin-AD family.

Thus $\mathcal{A} = \{A_\xi : \xi < \omega_1\} \subseteq \mathcal{I}^+$ is a fin-AD family.

We show that $\mathcal{A}$ is a Cohen-indestructible $\mathcal{I}$-MAD. Assume otherwise there is a ξ such that $p_\xi \Vdash \lim_{n\to\infty} \check{\varphi}(\dot{X}_\xi \setminus n) > \check{\delta}_\xi \wedge \forall\, \eta < \omega_1\ \dot{X}_\xi \cap \check{A}_\eta \in \mathcal{I}$, specially $p_\xi \Vdash$(a)$\wedge$(b). There is a $p^\xi_k \le p_\xi$ and an N such that $p^\xi_k \Vdash \check{\varphi}((\dot{X}_\xi \cap \check{A}_\xi)\setminus \check{N}) < \check{\delta}_\xi$. We can assume $k \ge N$, so $p^\xi_k \Vdash \check{\varphi}((\dot{X}_\xi \cap \check{A}_\xi)\setminus \check{k}) < \check{\delta}_\xi$. By the choice of q^ξ_k and a^ξ_k we have $q^\xi_k \Vdash \check{a}^\xi_k \subseteq (\dot{X}_\xi \cap \check{A}_\xi)\setminus \check{k}$, so $q^\xi_k \Vdash \check{\varphi}((\dot{X}_\xi \cap \check{A}_\xi)\setminus \check{k}) > \check{\delta}_\xi$, contradiction. □

4. Towers in $\mathcal{I}^*$

Let $\mathcal{I}$ be an ideal on ω. A $\subseteq^*$-decreasing sequence $\langle A_\alpha : \alpha < \kappa\rangle$ is a *tower in* $\mathcal{I}^*$ if (a) it is a tower (i.e. there is no $X \in [\omega]^\omega$ with $X \subseteq^* A_\alpha$ for $\alpha < \kappa$), and (b) $A_\alpha \in \mathcal{I}^*$ for $\alpha < \kappa$. Under CH it is straightforward to construct towers in $\mathcal{I}^*$ for each tall analytic P-ideal $\mathcal{I}$. The existence of such towers is consistent with $2^\omega > \omega_1$ as well by the Theorem 4.2 below. Denote $\mathbb{C}_\alpha$ the standard forcing adding α Cohen reals by finite conditions.

Lemma 4.1. *Let $\mathcal{I} = \mathrm{Exh}(\varphi)$ be a tall analytic P-ideal in the ground model V. Then there is a set $X \in V^{\mathbb{C}_1} \cap \mathcal{I}$ such that $|X \cap S| = \omega$ for each $S \in [\omega]^\omega \cap V$.*

Proof. Since $\mathcal{I}$ is tall we have $\lim_{n\to\infty}\varphi(\{n\}) = 0$. Fix a partition $\langle I_n : n \in \omega\rangle$ of ω into finite intervals such that $\varphi(\{x\}) < \frac{1}{2^n}$ for $x \in I_{n+1}$ (we can not say anything about $\varphi(\{x\})$ for $x \in I_0$). Then $X' \in \mathcal{I}$ whenever $|X' \cap I_n| \leq 1$ for each n.

Let $\{i^n_k : k < k_n\}$ be the increasing enumeration of I_n. Our forcing $\mathbb{C}$ adds a Cohen real $c \in \omega^\omega$ over V. Let

$$X_\alpha = \{i^n_k : c(n) \equiv k \bmod k_n\} \in V^{\mathbb{C}} \cap \mathcal{I}.$$

A trivial density argument shows that $|X_\alpha \cap S| = \omega$ for each $S \in V \cap [\omega]^\omega$. □

Theorem 4.2. $\Vdash_{\mathbb{C}_{\omega_1}}$ *"There exists a tower in $\mathcal{I}^*$ for each tall analytic P-ideal $\mathcal{I}$."*

Proof. Let V be a countable transitive model and G be a $\mathbb{C}_{\omega_1}$-generic filter over V. Let $\mathcal{I} = \mathrm{Exh}(\varphi)$ be a tall analytic P-ideal in $V[G]$ with some lower semicontinuous finite submeasure φ on ω. There is a $\delta < \omega_1$ such that $\varphi \restriction [\omega]^{<\omega} \in V[G_\delta]$ where $G_\delta = G \cap \mathbb{C}_\delta$, so we can assume $\varphi \restriction [\omega]^{<\omega} \in V$.

Work in $V[G]$ recursion on ω_1 we construct the tower $\bar{A} = \langle A_\alpha : \alpha < \omega_1\rangle$ in $\mathcal{I}^*$ such that $\bar{A} \restriction \alpha \in V[G_\alpha]$.

Because $\mathcal{I}$ contains infinite elements we can construct in V a sequence $\langle A_n : n \in \omega\rangle$ in $\mathcal{I}^*$ which is strictly $\subseteq^*$-descending, i.e. $|A_n \setminus A_{n+1}| = \omega$ for $n \in \omega$. Assume $\langle A_\xi : \xi < \alpha\rangle$ are done.

Since $\mathcal{I}$ is a P-ideal there is $A'_\alpha \in \mathcal{I}^*$ with $A'_\alpha \subseteq^* A_\beta$ for $\beta < \alpha$.

By lemma 4.1 there is a set $X_\alpha \in V[G_{\alpha+1}] \cap \mathcal{I}$ such that $X_\alpha \cap S \neq \emptyset$ for each $S \in \left[\omega\right]^\omega \cap V[G_\alpha]$.

Let $A_\alpha = A'_\alpha \setminus X_\alpha \in V[G_{\alpha+1}] \cap \mathcal{I}^*$ so $S \not\subseteq^* A_\alpha$ for any $S \in V[G_\alpha] \cap [\omega]^\omega$. Hence $V[G] \models$"$\langle A_\alpha : \alpha < \omega_1\rangle$ is a tower in $\mathcal{I}^*$". □

Problem 4.3. Do there exist towers in $\mathcal{I}^*$ for some tall analytic P-ideal $\mathcal{I}$ in ZFC?

5. Unbounding and dominating numbers of ideals

A *supported relation* (see [Vo]) is a triple $\mathcal{R} = (A, R, B)$ where $R \subseteq A \times B$, $\mathrm{dom}(R) = A$, $\mathrm{ran}(R) = B$, and we always assume that for each $b \in B$ there is an $a \in A$ such that $\langle a, b\rangle \notin R$.

The *unbounding* and *dominating numbers of* $\mathcal{R}$:

$$\mathfrak{b}(\mathcal{R}) = \min\{|A'| : A' \subseteq A \wedge \forall\, b \in B\ A' \not\subseteq R^{-1}\{b\}\},$$

$$\mathfrak{d}(\mathcal{R}) = \min\{|B'| : B' \subseteq B \wedge A = R^{-1}B'\}.$$

For example $\mathfrak{b}_{\mathcal{I}} = \mathfrak{b}(\omega^\omega, \leq_{\mathcal{I}}, \omega^\omega)$ and $\mathfrak{d}_{\mathcal{I}} = \mathfrak{d}(\omega^\omega, \leq_{\mathcal{I}}, \omega^\omega)$. Note that $\mathfrak{b}(\mathcal{R})$ and $\mathfrak{d}(\mathcal{R})$ are defined for each $\mathcal{R}$, but in general $\mathfrak{b}(\mathcal{R}) \leq \mathfrak{d}(\mathcal{R})$ does not hold.

We recall the definition of Galois-Tukey connection of relations.

Definition 5.1. ([Vo]) Let $\mathcal{R}_1 = (A_1, R_1, B_1)$ and $\mathcal{R}_2 = (A_2, R_2, B_2)$ be supported relations. A pair of functions $\phi : A_1 \to A_2$, $\psi : B_2 \to B_1$ is a *Galois-Tukey connection from* $\mathcal{R}_1$ *to* $\mathcal{R}_2$, in notation $(\phi, \psi) : \mathcal{R}_1 \preceq \mathcal{R}_2$ if $a_1 R_1 \psi(b_2)$ whenever $\phi(a_1) R_2 b_2$. In a diagram:

$$\begin{array}{ccc} \psi(b_2) \in B_1 & \xleftarrow{\ \psi\ } & B_2 \ni b_2 \\ R_1 & \Longleftarrow & R_2 \\ a_1 \in A_1 & \xrightarrow{\ \phi\ } & A_2 \ni \phi(a_1) \end{array}$$

We write $\mathcal{R}_1 \preceq \mathcal{R}_2$ if there is a Galois-Tukey connection from $\mathcal{R}_1$ to $\mathcal{R}_2$. If $\mathcal{R}_1 \preceq \mathcal{R}_2$ and $\mathcal{R}_2 \preceq \mathcal{R}_1$ also hold then we say $\mathcal{R}_1$ and $\mathcal{R}_2$ are *Galois-Tukey equivalent*, in notation $\mathcal{R}_1 \equiv \mathcal{R}_2$.

Fact 5.2. *If* $\mathcal{R}_1 \preceq \mathcal{R}_2$ *then* $\mathfrak{b}(\mathcal{R}_1) \geq \mathfrak{b}(\mathcal{R}_2)$ *and* $\mathfrak{d}(\mathcal{R}_1) \leq \mathfrak{d}(\mathcal{R}_2)$.

Theorem 5.3. *If* $\mathcal{I} \leq_{\mathrm{RB}} \mathcal{J}$ *then* $(\omega^\omega, \leq_{\mathcal{I}}, \omega^\omega) \equiv (\omega^\omega, \leq_{\mathcal{J}}, \omega^\omega)$.

Proof. Fix a finite-to-one function $f : \omega \to \omega$ witnessing $\mathcal{I} \leq_{\mathrm{RB}} \mathcal{J}$.
Define $\phi, \psi : \omega^\omega \to \omega^\omega$ as follows:

$$\phi(x)(i) = \max(x'' f^{-1}\{i\}),$$
$$\psi(y)(j) = y(f(j)).$$

We prove two claims.

Claim 5.3.1. $(\phi, \psi) : (\omega^\omega, \leq_{\mathcal{J}}, \omega^\omega) \preceq (\omega^\omega, \leq_{\mathcal{I}}, \omega^\omega)$.

Proof of the claim. We show that if $\phi(x) \leq_{\mathcal{I}} y$ then $x \leq_{\mathcal{J}} \psi(y)$. Indeed, $I = \{i : \phi(x)(i) > y(i)\} \in \mathcal{I}$. Assume that $f(j) = i \notin I$. Then $\phi(x)(i) = \max(x'' f^{-1}\{i\}) \leq y(i)$. Since $y(i) = \psi(y)(j)$, so

$$x(j) \leq \max(x'' f^{-1}\{f(j)\}) \leq y(f(j)) = \psi(y)(j)$$

Since $f^{-1} I \in \mathcal{J}$ this yields $x \leq_{\mathcal{J}} \psi(y)$. □

Claim 5.3.2. $(\psi, \phi) : (\omega^\omega, \leq_{\mathcal{I}}, \omega^\omega) \preceq (\omega^\omega, \leq_{\mathcal{J}}, \omega^\omega)$.

Proof of the claim. We show that if $\psi(y) \leq_{\mathcal{J}} x$ then $y \leq_{\mathcal{I}} \phi(x)$. Assume on the contrary that $y \not\leq_{\mathcal{I}} \phi(x)$. Then $A = \{i \in \omega : y(i) > \phi(x)(i)\} \in \mathcal{I}^+$. By definition of ϕ, we have $A = \{i : y(i) > \max(x'' f^{-1}\{i\})\}$.
Let $B = f^{-1} A \in \mathcal{J}^+$. For $j \in B$ we have $f(j) \in A$ and so

$$\psi(y)(j) = y(f(j)) > \phi(x)(f(j)) = \max(x'' f^{-1}\{f(j)\}) \geq x(j).$$

Hence $\psi(y) \not\leq_{\mathcal{I}} x$, contradiction. □

These claims prove the statement of the Theorem, so we are done. □

By Fact 5.2 we have:

Corollary 5.4. *If $\mathcal{I} \leq_{\mathrm{RB}} \mathcal{J}$ holds then $\mathfrak{b}_{\mathcal{I}} = \mathfrak{b}_{\mathcal{J}}$ and $\mathfrak{d}_{\mathcal{I}} = \mathfrak{d}_{\mathcal{J}}$.*

By Observation 2.5 this yields:

Corollary 5.5. *If $\mathcal{I}$ is an analytic P-ideal then $(\omega^\omega, \leq^*, \omega^\omega) \equiv (\omega^\omega, \leq_{\mathcal{J}}, \omega^\omega)$, and $\mathfrak{b}_{\mathcal{I}} = \mathfrak{b}$ and $\mathfrak{d}_{\mathcal{I}} = \mathfrak{d}$.*

6. $\mathcal{I}$-bounding and $\mathcal{I}$-dominating forcing notions

Definition 6.1. Let $\mathcal{I}$ be a Borel ideal on ω. A forcing notion $\mathbb{P}$ is *$\mathcal{I}$-bounding* if

$$\Vdash_{\mathbb{P}} \forall\, A \in \mathcal{I}\; \exists\, B \in \mathcal{I} \cap V\; A \subseteq B;$$

$\mathbb{P}$ is *$\mathcal{I}$-dominating* if

$$\Vdash_{\mathbb{P}} \exists\, B \in \mathcal{I}\; \forall\, A \in \mathcal{I} \cap V\; A \subseteq^* B.$$

Theorem 6.2. *Let $\mathcal{I}$ be a tall analytic P-ideal. If $\mathbb{P}$ is $\mathcal{I}$-bounding then $\mathbb{P}$ is ω^ω-bounding as well; if $\mathbb{P}$ is $\mathcal{I}$-dominating then $\mathbb{P}$ adds dominating reals.*

Proof. Assume that $\mathcal{I} = \mathrm{Exh}(\varphi)$ for some lower semicontinuous finite submeasure φ. For $A \in \mathcal{I}$ let

$$d_A(n) = \min\big\{k \in \omega : \varphi(A \setminus k) < 2^{-n}\big\}.$$

Clearly if $A \subseteq B \in \mathcal{I}$ then $d_A \leq d_B$.

It is enough to show that $\{d_A : A \in \mathcal{I}\}$ is cofinal in $\langle \omega^\omega, \leq^* \rangle$. Let $f \in \omega^\omega$. Since $\mathcal{I}$ is a tall ideal we have $\lim_{k\to\infty} \varphi(\{k\}) = 0$ but $\lim_{m\to\infty}(\omega \setminus m) = \varepsilon > 0$. Thus for all but finite $n \in \omega$ we can choose a finite set $A_n \subseteq \omega \setminus f(n)$ such that $2^{-n} \leq \varphi(A_n) < 2^{-n+1}$ so $A = \cup\{A_n : n \in \omega\} \in \mathcal{I}$ and $f \leq^* d_A$.

Why? We can assume if $k \geq f(n)$ then $\varphi(\{k\}) < 2^{-n}$. Let n be so large such that $2^{-n} < \varepsilon$. Now if there is no a suitable A_n then $\varphi(\omega \setminus f(n)) \leq 2^{-n} < \varepsilon$, contradiction. □

The converse of the first implication of Theorem 6.2 is not true by the following Proposition.

Proposition 6.3. *The random forcing is not $\mathcal{I}$-bounding for any tall summable and tall density ideal $\mathcal{I}$.*

Proof. Denote $\mathbb{B}$ the random forcing and λ the Lebesgue-measure.

If $\mathcal{I} = \mathcal{I}_h$ is a tall summable ideal then we can chose pairwise disjoint sets $H(n) \in [\omega]^\omega$ such that $\sum_{l \in H(n)} h(l) = 1$ and $\max\{h(l) : l \in H(n)\} < 2^{-n}$ for each $n \in \omega$. Let $H(n) = \{l^n_k : k \in \omega\}$. For each n fix a partition $\{[B^n_k] : k \in \omega\}$ of $\mathbb{B}$ such that $\lambda(B^n_k) = h(l^n_k)$ for each $k \in \omega$. Let $\dot{X}$ be a $\mathbb{B}$-name such that $\Vdash_{\mathbb{B}} \dot{X} = \{\check{l}^n_k : [\check{B}^n_k] \in \dot{G}\}$. Clearly $\Vdash_{\mathbb{B}} \dot{X} \in \mathcal{I}_h$. $\dot{X}$ shows that $\mathbb{B}$ is not $\mathcal{I}_h$-bounding.

Assume on the contrary that there is a $[B] \in \mathbb{B}$ and an $A \in \mathcal{I}_h$ such that $[B] \Vdash \dot{X} \subseteq \check{A}$. There is an $n \in \omega$ such that

$$\sum_{l^n_k \in A} \lambda(B^n_k) = \sum_{l^n_k \in A} h(l^n_k) < \lambda(B).$$

Choose a k such that $l^n_k \notin A$ and $[B^n_k] \wedge [B] \neq [\emptyset]$. We have $[B^n_k] \wedge [B] \Vdash \check{l}^n_k \in \dot{X} \setminus \check{A}$, contradiction.

If $\mathcal{I} = \mathcal{Z}_{\vec{\mu}}$ is a tall density ideal then for each n fix a partition $\{[B^n_k] : k \in P_n\}$ of $\mathbb{B}$ such that $\lambda(B^n_k) = \mu_n(\{k\})$ for each k. Let $\dot{X}$ be a $\mathbb{B}$-name such that $\Vdash_{\mathbb{B}} \dot{X} = \{\check{k} : [\check{B}^n_k] \in \dot{G}\}$. Clearly $\Vdash_{\mathbb{B}} \dot{X} \in \mathcal{Z}_{\vec{\mu}}$. $\dot{X}$ shows that $\mathbb{B}$ is not $\mathcal{Z}_{\vec{\mu}}$-bounding.

Assume on the contrary that there is a $[B] \in \mathbb{B}$ and an $A \in \mathcal{Z}_{\vec{\mu}}$ such that $[B] \Vdash \dot{X} \subseteq \check{A}$. There is an $n \in \omega$ such that

$$\sum_{k \in A \cap P_n} \lambda(B^n_k) = \mu_n(A \cap P_n) < \lambda(B).$$

Choose a $k \in P_n \setminus A$ such that $[B^n_k] \wedge [B] \neq [\emptyset]$. We have $[B^n_k] \wedge [B] \Vdash \check{k} \in \dot{X} \setminus \check{A}$, contradiction. □

The converse of the second implication of Theorem 6.2 is not true as well: the Hechler forcing is a counterexample according to the following Theorem.

Theorem 6.4. *If $\mathbb{P}$ is σ-centered then $\mathbb{P}$ is not $\mathcal{I}$-dominating for any tall analytic P-ideal $\mathcal{I}$.*

Proof. Assume that $\mathcal{I} = \mathrm{Exh}(\varphi)$ for some lower semicontinuous finite submeasure φ. Let $\varepsilon = \lim_{n \to \infty} \varphi(\omega \setminus n) > 0$.

Let $\mathbb{P} = \cup\{C_n : n \in \omega\}$ where C_n is centered for each n. Assume on the contrary that $\Vdash_{\mathbb{P}} \dot{X} \in \mathcal{I} \wedge \forall\ A \in \mathcal{I} \cap V\ A \subseteq^* \dot{X}$ for some $\mathbb{P}$-name $\dot{X}$.

For each $A \in \mathcal{I}$ choose a $p_A \in \mathbb{P}$ and a $k_A \in \omega$ such that

$$(\circ) \qquad p_A \Vdash \check{A} \setminus \check{k}_A \subseteq \dot{X} \wedge \varphi(\dot{X} \setminus \check{k}_A) < \varepsilon/2.$$

For each $n, k \in \omega$ let $\mathcal{C}_{n,k} = \{A \in \mathcal{I} : p_A \in C_n \wedge k_A = k\}$, and let $B_{n,k} = \bigcup \mathcal{C}_{n,k}$. We show that for each n and k

$$\varphi(B_{n,k} \setminus k) \leq \varepsilon/2.$$

Assume indirectly $\varphi(B_{n,k} \setminus k) > \varepsilon/2$ for some n and k. There is a k' such that $\varphi(B_{n,k} \cap [k, k')) > \varepsilon/2$ and there is a finite $\mathcal{D} \subseteq \mathcal{C}_{n,k}$ such that $B_{n,k} \cap [k, k') = (\cup \mathcal{D}) \cap [k, k')$. Choose a common extension q of $\{p_A : A \in \mathcal{D}\}$. Now we have $q \Vdash \cup\{A \setminus \check{k} : A \in \check{\mathcal{D}}\} \subseteq \dot{X}$ and so

$$q \Vdash \varepsilon/2 < \varphi(\check{B}_{n,k} \cap [\check{k}, \check{k}')) = \varphi((\cup \check{\mathcal{D}}) \cap [\check{k}, \check{k}')) \leq \varphi(\dot{X} \cap [\check{k}, \check{k}')) \leq \varphi(\dot{X} \setminus \check{k}),$$

which contradicts ($\circ$).

So for each n and k the set $\omega \setminus B_{n,k}$ is infinite, so $\omega \setminus B_{n,k}$ contains an infinite $D_{n,k} \in \mathcal{I}$. Let $D \in \mathcal{I}$ such that $D_{n,k} \subseteq^* D$ for each $n, k \in \omega$.

Then there is no n, k such that $D \subseteq^* B_{n,k}$. Contradiction. □

By this Theorem an by Lemma 4.1 the Cohen forcing is neither $\mathcal{I}$-dominating nor $\mathcal{I}$-bounding for any tall analytic P-ideal $\mathcal{I}$.

Finally in the rest of the paper we compare the Sacks property and the $\mathcal{I}$-bounding property.

Theorem 6.5. *If $\mathbb{P}$ has the Sacks property then $\mathbb{P}$ is $\mathcal{I}$-bounding for each analytic P-ideal $\mathcal{I}$.*

Proof. Let $\mathcal{I} = \text{Exh}(\varphi)$. Assume $\Vdash_{\mathbb{P}} \dot{X} \in \mathcal{I}$. Let $d_{\dot{X}}$ be a $\mathbb{P}$-name for an element of ω^ω such that $\Vdash_{\mathbb{P}} d_{\dot{X}}(\check{n}) = \min\{k \in \omega : \varphi(\dot{X} \setminus k) < 2^{-\check{n}}\}$. We know that $\mathbb{P}$ is ω^ω-bounding. If $p \Vdash d_{\dot{X}} \leq \check{f}$ for some strictly increasing $f \in \omega^\omega$ then by the Sacks property there is a $q \leq p$ and a slalom $S : \omega \to \left[[\omega]^{<\omega}\right]^{<\omega}$, $|S(n)| \leq n$ such that

$$q \Vdash \forall^\infty n \ \dot{X} \cap [f(n), f(n+1)) \in S(n).$$

Now let

$$A = \bigcup_{n \in \omega} \{D \in S(n) : \varphi(D) < 2^{-n}\}.$$

$A \in \mathcal{I}$ because $\varphi(A \setminus f(n)) \leq \sum_{k \geq n} \varphi(A \cap [f(k), f(k+1)) \leq \sum_{k \geq n} \frac{k}{2^k}$. Clearly $q \Vdash \dot{X} \subseteq^* \check{A}$. □

A supported relation $\mathcal{R} = (A, R, B)$ is called *Borel-relation* iff there is a Polish space X such that $A, B \subseteq X$ and $R \subseteq X^2$ are Borel sets. Similarly a Galois-Tukey connection $(\phi, \psi) : \mathcal{R}_1 \preceq \mathcal{R}_2$ between Borel-relations is called *Borel GT-connection* iff ϕ and ψ are Borel functions. To be Borel-relation and Borel GT-connection is absolute for transitive models containing all relevant codes.

Some important Borel-relation:

(A): $(\mathcal{I}, \subseteq, \mathcal{I})$ and $(\mathcal{I}, \subseteq^*, \mathcal{I})$ for a Borel ideal $\mathcal{I}$.

(B): Denote Slm the set of *slaloms on* ω, i.e. $S \in \mathrm{Slm}$ iff $S : \omega \to [\omega]^{<\omega}$ and $|S(n)| = 2^n$ for each n. Let $\sqsubseteq$ and $\sqsubseteq^*$ be the following relations on $\omega^\omega \times \mathrm{Slm}$:

$$f \sqsubseteq^{(*)} S \iff \forall^{(\infty)} n \in \omega\, f(n) \in S(n).$$

The supported relations $(\omega^\omega, \sqsubseteq, \mathrm{Slm})$ and $(\omega^\omega, \sqsubseteq^*, \mathrm{Slm})$ are Borel-relations.

(C): Denote ℓ_1^+ the set of positive summable series. Let $\le$ be the coordinate-wise and $\le^*$ the almost everywhere coordinate-wise ordering on ℓ_1^+. $(\ell_1^+, \le, \ell_1^+)$ and $(\ell_1^+, \le^*, \ell_1^+)$ are Borel-relations.

Definition 6.6. Let $\mathcal{R} = (A, R, B)$ be a Borel-relation. A forcing notion $\mathbb{P}$ is $\mathcal{R}$-*bounding* if

$$\Vdash_{\mathbb{P}} \forall\, a \in A\, \exists\, b \in B \cap V\; aRb;$$

$\mathcal{R}$-*dominating* if

$$\Vdash_{\mathbb{P}} \exists\, b \in B\, \forall\, a \in A \cap V\; aRb.$$

For example the property $\mathcal{I}$-bounding/dominating is the same as $(\mathcal{I}, \subseteq^*, \mathcal{I})$-bounding/dominating.

We can reformulate some classical properties of forcing notions:

ω^ω-bounding	$\equiv$	$(\omega^\omega, \le^{(*)}, \omega^\omega)$-bounding
adding dominating reals	$\equiv$	$(\omega^\omega, \le^*, \omega^\omega)$-dominating
Sacks property	$\equiv$	$(\omega^\omega, \sqsubseteq^{(*)}, \mathrm{Slm})$-bounding
adding a slalom capturing all ground model reals	$\equiv$	$(\omega^\omega, \sqsubseteq^*, \mathrm{Slm})$-dominating

If $\mathcal{R} = (A, R, B)$ is a supported relation then let $\mathcal{R}^\perp = (B, \neg R^{-1}, A)$ where $b(\neg R^{-1})a$ iff not aRb. Clearly $(\mathcal{R}^\perp)^\perp = \mathcal{R}$ and $\mathfrak{b}(\mathcal{R}) = \mathfrak{d}(\mathcal{R}^\perp)$. Now if $\mathcal{R}$ is a Borel-relation then $\mathcal{R}^\perp$ is a Borel-relation too, and a forcing notion is $\mathcal{R}$-bounding iff it is not $\mathcal{R}^\perp$-dominating.

Fact 6.7. *Assume $\mathcal{R}_1 \preceq \mathcal{R}_2$ are Borel-relations with Borel GT-connection and $\mathbb{P}$ is a forcing notion. If $\mathbb{P}$ is $\mathcal{R}_2$-bounding/dominating then $\mathbb{P}$ is $\mathcal{R}_1$-bounding/dominating.*

By Corollary 5.5 this yields

Corollary 6.8. *For each analytic P-ideal $\mathcal{I}$ (1) a poset $\mathbb{P}$ is $\le_{\mathcal{I}}$-bounding iff it is ω^ω-bounding, (2) forcing with a poset $\mathbb{P}$ adds $\le_{\mathcal{I}}$-dominating reals iff this forcing adds dominating reals.*

We will use the following Theorem.

Theorem 6.9. ([Fr] 526B, 524I) *There are Borel GT-connections $(\mathcal{Z}, \subseteq, \mathcal{Z}) \preceq (\ell_1^+, \le, \ell_1^+)$ and $(\ell_1^+, \le^*, \ell_1^+) \equiv (\omega^\omega, \sqsubseteq^*, \mathrm{Slm})$.*

Note that there is no any Galois-Tukey connection from $(\ell_1^+, \leq, \ell_1^+)$ to $(\mathcal{Z}, \subseteq, \mathcal{Z})$ so they are not GT-equivalent (see [LoVe]) Th. 7.).

Corollary 6.10. *If $\mathbb{P}$ adds a slalom capturing all ground model reals then $\mathbb{P}$ is $\mathcal{Z}$-dominating.*

Proof. By Fact 6.7 and Theorem 6.9 adding slalom is the same as $(\ell_1^+, \leq^*, \ell_1^+)$-dominating. Let $\dot{x}$ be a $\mathbb{P}$-name such that $\Vdash_{\mathbb{P}} \dot{x} \in \ell_1^+ \wedge \forall\ y \in \ell_1^+ \cap V\ y \leq^* \dot{x}$. Moreover let $\dot{X}$ be a $\mathbb{P}$-name such that $\Vdash_{\mathbb{P}} \dot{X} = \{z \in \ell_1^+ : |z \backslash \dot{x}| < \omega,\ \forall\ n\ (z(n) \neq \dot{x}(n) \Rightarrow z(n) \in \omega)\}$. Let $(\phi, \psi) : (\mathcal{Z}, \subseteq, \mathcal{Z}) \preceq (\ell_1^+, \leq, \ell_1^+)$ be a Borel GT-connection. Now if $\dot{A}$ is a $\mathbb{P}$-name such that $\Vdash_{\mathbb{P}} \forall\ z \in \dot{X}\ \psi(z) \subseteq^* \dot{A}$ then $\dot{A}$ shows that $\mathbb{P}$ is $\mathcal{Z}$-dominating. □

Denote $\mathbb{D}$ the dominating forcing and $\mathbb{LOC}$ the Localization forcing.

Observation 6.11. *If $\mathcal{I}$ is an arbitrary analytic P-ideal then two step iteration $\mathbb{D} * \mathbb{LOC}$ is $\mathcal{I}$-dominating.*

Indeed, let $\mathcal{I} \in V \subseteq M \subseteq N$ be transitive models, $d \in M \cap \omega^\omega$ be strictly increasing and dominating over V, and $S \in N$, $S : \omega \to \big[[\omega]^{<\omega}\big]^{<\omega}$, $|S(n)| \leq n$ a slalom which captures all reals from M. Now if

$$X_n = \cup\{A \in S(n) \cap \mathcal{P}([d(n), d(n+1)) : \varphi(A) < 2^{-n}\}$$

then it is easy to see that $Y \subseteq^* \cup\{X_n : n \in \omega\} \in \mathcal{I} \cap N$ for each $Y \in V \cap \mathcal{I}$.

Problem 6.12. For which analytic P-ideal $\mathcal{I}$ does $(\mathcal{I}, \subseteq^{(*)}, \mathcal{I}) \preceq (\ell_1^+, \leq^{(*)}, \ell_1^+)$ hold, or "adding slaloms" imply $\mathcal{I}$-dominating, or at least $\mathbb{LOC}$ is $\mathcal{I}$-dominating?

Problem 6.13. Does $\mathcal{Z}$-dominating (or $\mathcal{I}$-dominating) imply adding slaloms?

We will use the following deep result of Fremlin to prove Theorem 6.15.

Theorem 6.14. ([Fr] 526G) *There is a family $\{P_f : f \in \omega^\omega\}$ of Borel subsets of ℓ_1^+ such that the following hold:*

(i) $\ell_1^+ = \cup\{P_f : f \in \omega^\omega\}$,
(ii) *if $f \leq g$ then $P_f \subseteq P_g$,*
(iii) $(P_f, \leq, \ell_1^+) \preceq (\mathcal{Z}, \subseteq, \mathcal{Z})$ *with a Borel GT-connection for each f.*

Theorem 6.15. *$\mathbb{P}$ is $\mathcal{Z}$-bounding iff $\mathbb{P}$ has the Sacks property.*

Proof. Let $\{P_f : f \in \omega^\omega\}$ be a family satisfying (i), (ii), and (iii) in Theorem 6.14, and fix Borel GT-connections $(\phi_f, \psi_f) : (P_f, \leq, \ell_1^+) \preceq (\mathcal{Z}, \subseteq, \mathcal{Z})$ for each $f \in \omega^\omega$. Assume $\mathbb{P}$ is $\mathcal{Z}$-bounding and $\Vdash_{\mathbb{P}} \dot{x} \in \ell_1^+$. $\mathbb{P}$ is ω^ω-bounding by Theorem 6.2 so using (ii) we have $\Vdash_{\mathbb{P}} \ell_1^+ = \cup\{P_f : f \in \omega^\omega \cap V\}$. We can choose a $\mathbb{P}$-name $\dot{f}$ for an element of $\omega^\omega \cap V$ such that $\Vdash_{\mathbb{P}} \dot{x} \in P_{\dot{f}}$. By $\mathcal{Z}$-bounding property of $\mathbb{P}$ there is a $\mathbb{P}$-name $\dot{A}$ for an element of $\mathcal{Z} \cap V$ such that $\Vdash_{\mathbb{P}} \phi_{\dot{f}}(\dot{x}) \subseteq \dot{A}$, so $\Vdash_{\mathbb{P}} \dot{x} \leq \psi_{\dot{f}}(\dot{A}) \in \ell_1^+ \cap V$. So we have $\mathbb{P}$ is $(\ell_1^+, \leq^{(*)}, \ell_1^+)$-bounding. By Theorem 6.9 and Fact 6.7 $\mathbb{P}$ has the Sacks property.

The converse implication was proved in Theorem 6.5.

□

Problem 6.16. Does the $\mathcal{I}$-bounding property imply the Sacks property for each tall analytic P-ideal $\mathcal{I}$?

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