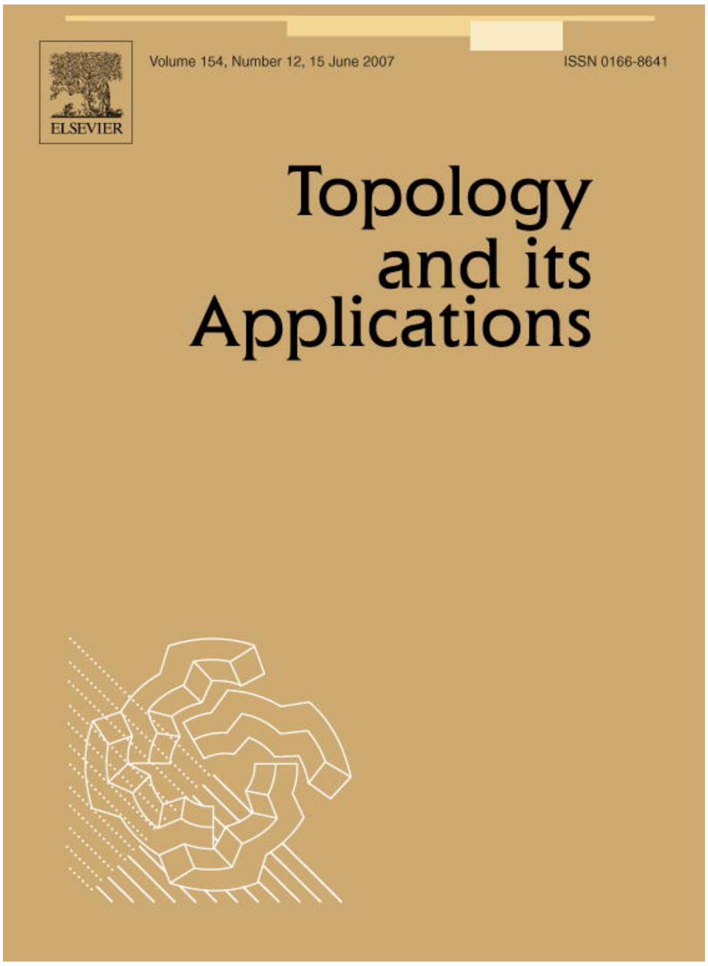
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Countably compact hyperspaces and Frolík sums

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Abstract

Let $H^0(X)$ ($H(X)$) denote the set of all (nonempty) closed subsets of X endowed with the Vietoris topology. A basic problem concerning $H(X)$ is to characterize those X for which $H(X)$ is countably compact. We conjecture that u-compactness of X for some $u \in \omega^*$ (or equivalently: all powers of X are countably compact) may be such a characterization. We give some results that point into this direction.

We define the property $R(\kappa)$: for every family $\{Z_\alpha\colon \alpha < \kappa\}$ of closed subsets of X separated by pairwise disjoint open sets and any family $\{k_\alpha\colon \alpha < \kappa\}$ of natural numbers, the product $\prod_{\alpha<\kappa} Z_\alpha^{k_\alpha}$ is countably compact, and prove that if $H(X)$ is countably compact for a T_2-space X then X satisfies $R(\kappa)$ for all κ. A space has $R(1)$ iff all its finite powers are countably compact, so this generalizes a theorem of J. Ginsburg: if X is T_2 and $H(X)$ is countably compact, then so is X^n for all $n < \omega$. We also prove that, for $\kappa < \mathfrak{t}$, if the T_3 space X satisfies a weak form of $R(\kappa)$, the orbit of every point in X is dense, and X contains κ pairwise disjoint open sets, then X^κ is countably compact. This generalizes the following theorem of J. Cao, T. Nogura, and A. Tomita: if X is T_3, homogeneous, and $H(X)$ is countably compact, then so is X^ω.

Then we study the Frolík sum (also called "one-point countable-compactification") $F(X_\alpha\colon \alpha < \kappa)$ of a family $\{X_\alpha\colon \alpha < \kappa\}$. We use the Frolík sum to produce countably compact spaces with additional properties (like first countability) whose hyperspaces are not countably compact. We also prove that any product $\prod_{\alpha<\kappa} H^0(X_\alpha)$ embeds into $H(F(X_\alpha\colon \alpha < \kappa))$.

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1. Introduction

For a topological space X, let $H(X)$ denote the set of all nonempty closed subsets of X with the Vietoris topology. A subbase for the Vietoris topology consists of sets of the form U^+, U^- where U is open in X and

$$U^+ = \{C \in H(X)\colon C \subset U\} \quad \text{and} \quad U^- = \{C \in H(X)\colon C \cap U \neq \emptyset\}.$$

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Thus a base for the Vietoris topology on $H(X)$ consists of all sets of the form

$$\langle U_0, \ldots, U_n \rangle = \left\{ C \in H(X)\colon\ C \subset \bigcup_{i \leqslant n} U_i \text{ and } C \cap U_i \neq \emptyset \text{ for all } i \leqslant n \right\},$$

where the U_i are open subsets of X. The space $H(X)$ is called the hyperspace of X (for example, see [22,24,6]).

Some authors (e.g., Kuratowski [22]) include the empty set as a member of their hyperspace. We let $H^0(X)$ denote the set of *all* closed subsets of X with the Vietoris topology. Thus $H^0(X) = H(X) \cup \{\emptyset\}$ and the empty set is an isolated point in $H^0(X)$.

Other common notation for the hyperspaces are 2^X, $\exp(X)$, and $\mathrm{CL}(X)$. In this paper we do not assume any separation axioms except where specifically stated.

The results in this paper derived from the question: What relations exist between the countable compactness of X and $H(X)$? This question was studied by J. Ginsburg [11, p. 199] who proved the following two theorems: (1) if X is u-compact (for any $u \in \omega^*$) then $H(X)$ is u-compact, hence countably compact, and (2) if $H(X)$ is countably compact, then every finite power of X is countably compact. Ginsburg also proved the converse of (1) assuming T_2, however, no separation axioms are needed: if (y_n) is a sequence in X then $(\overline{\{y_n\}})$ is a sequence in $H(X)$, and any point $y \in C = u\text{-limit } (\overline{\{y_n\}})$ is easily seen to be a u-limit of (y_n). Thus for all spaces X, X is u-compact if and only if $H(X)$ is u-compact.

One question we considered, but could not fully answer, asks "if $H(X)$ is countably compact must X be u-compact for some u?" By a theorem of Ginsburg and Saks [12, Theorem 2.6], this is equivalent to asking whether X^κ is countably compact for all κ. We obtain some partial information on this question by proving that if $H(X)$ is countably compact, then the product of any finite powers of closed subspaces of X that are separated by disjoint open sets is countably compact (Theorem 2.18). This generalizes the result (2) of Ginsburg.

Cao, Nogura and Tomita considered a specific question of Ginsburg: What relations exist between the countable compactness of X^ω and $H(X)$ [11, 3.2]? They prove that if X is regular, homogeneous and $H(X)$ is countably compact, then X^ω is countably compact [3, 3.1]. Motivated by their paper, we extend this result (see Theorem 2.23) and other results concerning products.

In the first part of Section 2, leading up to Theorem 2.18, we consider products of totally countably compact spaces. We prove that the product of less than $\mathfrak{n}$ (where $\mathfrak{n}$ is the Novak number of ω^*) totally countably compact spaces is countably compact (see Theorem 2.5). This generalizes several known results.

In Section 3 we study a construction first used by Z. Frolík [10], and later by others, which we call the Frolík sum (see 3.1). We show that there is a close relationship between the countable compactness of the product of a family of spaces and that of the hyperspace of their Frolík sum. Among other results, we prove that for any family of T_2 spaces $\{X_\alpha\colon \alpha < \kappa\}$, the product $\prod_{\alpha<\kappa} H(X_\alpha)$ can be embedded as a closed subspace in the hyperspace of the Frolík sum of the family. This generalizes to all "countable compactness type" properties the following result of Cao, Nogura, and Tomita:

Theorem 1.1. *(See [3].) If the hyperspace of the Frolík sum of a family of T_2 spaces $\{X_\alpha\colon \alpha < \kappa\}$ is countably compact, then so is the product $\prod_{\alpha<\kappa} X_\alpha$.*

We recall a few terms used in this paper. A space is called *good* provided it is a countably compact, locally countable, T_3-space (hence first countable) [20]. For a family of spaces $\{X_\alpha\colon \alpha < \kappa\}$, $\bigoplus_{\alpha<\kappa} X_\alpha$ denotes the disjoint topological sum of the family (see [6, 2.2]). The set of natural numbers is denoted by ω, and the cardinality of the continuum is denoted by $\mathfrak{c}$. The cardinality of a set X is denoted by $|X|$. A sequence s in a set X is a function $s : \omega \to X$, which we also denote by (s_n).

2. Clustering families of sequences

Recall that for an ultrafilter $u \in \omega^* = \beta\omega \setminus \omega$ and a sequence $s : \omega \to X$ in a topological space X, a point $x \in X$ is called a *u-limit of s* (denoted $x = u \lim s$) provided $s^{-1}(U) \in u$ for every neighborhood U of x. A space X is called *u-compact* provided every sequence in X has a u-limit in X. If X is T_2 then a sequence can have at most one u-limit point (see [2]). Clearly, any accumulation point of a sequence s is the u-limit of s for some $u \in \omega^*$.

Definition 2.1. Let $\mathcal{S} = \{s_\alpha\colon \alpha < \kappa\}$ be a family of sequences (more formally, for each $\alpha < \kappa$ there is a space X_α such that $s_\alpha : \omega \to X_\alpha$). We say that $\mathcal{S}$ *is clustering* provided there exists $u \in \omega^*$ such that s_α has a u-limit in X_α for all $\alpha < \kappa$.

For what we called clustering, it may be reasonable to use a term like "simultaneously u-clustering", but we prefer the shorter term.

Clustering of a family of sequences arises naturally in the study of limits of sequences in products. Let $s : \omega \to \prod_{\alpha<\kappa} X_\alpha$ be a sequence in a product of spaces, and let $s_\alpha = \pi_\alpha \circ s$ for all $\alpha < \kappa$, where the π_α denote the usual projection maps (thus $s_\alpha : \omega \to X_\alpha$). Then the family $\mathcal{S} = \{s_\alpha\colon \alpha < \kappa\}$ is clustering if and only if the original sequence s has a cluster point, or equivalently a u-limit in the product for some $u \in \omega^*$. Hence a product space $\prod_{\alpha<\kappa} X_\alpha$ is countably compact if and only if every family of sequences $\mathcal{S} = \{s_\alpha\colon \alpha < \kappa\}$ such that $s_\alpha : \omega \to X_\alpha$ is clustering. Moreover, a space X is u-compact for some $u \in \omega^*$ if and only if the family $\mathcal{S} = X^\omega$ of all sequences in X clusters. With this in mind me make the following definition.

Definition 2.2. A space X satisfies property $Q(\kappa)$ provided every family $\mathcal{S} = \{s_\alpha\colon \alpha < \kappa\}$ of sequences in X is clustering.

Since a space X satisfies property $Q(\kappa)$ if and only if X^κ is countably compact, it is not entirely necessary to introduce Definition 2.2. However, $Q(\kappa)$ is defined on X, rather than X^κ, which is useful. Moreover, we shall define and use below a property $wR(\kappa)$, which is a natural weakening of $Q(\kappa)$.

Definition 2.3. A tower (of height κ) on ω^* is a strictly decreasing family of clopen sets $\{W_\alpha\colon \alpha < \kappa\}$ such that $T = \bigcap\{W_\alpha\colon \alpha < \kappa\}$ is nowhere dense in ω^*. The set T is then called a $\mathfrak{t}$*-set on* ω^* (or more precisely a $\mathfrak{t}(\kappa)$-set). The cardinal $\mathfrak{t}$ is the smallest height of a tower in ω^* (see [4] or [27]). The cardinal $\mathfrak{n}$ is defined to be the smallest cardinality of a family of nowhere dense subsets of ω^* which covers ω^* [1].

Definition 2.4. A space is called *sequentially compact* (respectively, *totally countably compact* (cf. [28, Section 7])) provided every sequence in X has a subsequence that is convergent (respectively, is contained in a compact set).

R. Frič and P. Vojtáš [7, p. 103] and independently P. Nyikos, J. Pelant and P. Simon (unpublished) proved that the product of fewer than $\mathfrak{n}$ sequentially compact spaces is countably compact. Using essentially the same proof, we extend this to

Theorem 2.5. *Let $\kappa < \mathfrak{n}$ and let $\mathcal{S} = \{s_\alpha\colon \alpha < \kappa\}$ with $s_\alpha : \omega \to X_\alpha$ be a family of sequences such that X_α is totally countably compact for all $\alpha < \kappa$. Then $\mathcal{S} = \{s_\alpha\colon \alpha < \kappa\}$ is clustering.*

Proof. For each $\alpha < \kappa$ put

$$\mathcal{D}_\alpha = \big\{D \in [\omega]^\omega\colon\ s_\alpha(D) \text{ is contained in a compact subset of } X_\alpha\big\}.$$

Then by totally countably compact, $\mathcal{D}^*_\alpha = \{D^*\colon D \in \mathcal{D}_\alpha\}$ is a π-base for ω^*; hence $\bigcup \mathcal{D}^*_\alpha$ is an open dense subset of ω^*. By definition of $\mathfrak{n}$ there exists $u \in \bigcap\{\bigcup \mathcal{D}^*_\alpha\colon \alpha < \kappa\}$. Let $\alpha < \kappa$, we show that s_α has a u-limit in X_α. Since $u \in \bigcup \mathcal{D}^*_\alpha$ we may pick $D \in \mathcal{D}_\alpha$ such that $D \in u$. By definition of $\mathcal{D}_\alpha$, there exists a compact set $K \subset X_\alpha$ such that $s_\alpha(D) \subset K \subset X_\alpha$. If s_α has no u-limit in K, then by compactness, there exist finitely many open sets $\{U_i\colon i < n\}$ covering K such that $s_\alpha^{-1}(U_i) \notin u$ for each $i < n$. Put $U = \bigcup\{U_i\colon i < n\}$. Then $s_\alpha^{-1}(U) \notin u$ and $s_\alpha(D) \subset K \subset U$. This implies $D \subset s_\alpha^{-1}(U) \notin u$ which contradicts that $D \in u$. □

By what we said above, the following corollary is actually an equivalent reformulation of Theorem 2.5.

Corollary 2.6. *If $\kappa < \mathfrak{n}$, then the product of any family $\{X_\alpha\colon \alpha < \kappa\}$ of totally countably compact spaces is countably compact.*

Since $\mathfrak{t} < \mathfrak{n}$ [16, 4.11], the previous corollary also improves the following result.

Corollary 2.7. *(See [28, Theorem 3.3(C)].) If $\{X_\alpha\colon \alpha < \mathfrak{t}\}$ is a family of totally countably compact spaces then $\prod_{\alpha<\mathfrak{t}} X_\alpha$ is countably compact.*

Corollary 2.8. *If $|X|^\omega < \mathfrak{n}$ and X is totally countably compact, then X is u-compact for some $u \in \omega^*$.*

We also obtain a short proof of the following known result:

Corollary 2.9. *(See [28, Proposition 3.5].)* $[\mathfrak{t} = \mathfrak{c}]$ *If X is totally countably compact with $|X| \leqslant \mathfrak{c}$ then X is u-compact for some $u \in \omega^*$ (hence $H(X)$ is u-compact and therefore countably compact).*

Proof. Since $\mathfrak{t} < \mathfrak{n}$, the result follows from Corollary 2.8 (and Ginsburg's result mentioned in Section 1). □

Example 2.10. The strict inequality "$<$" in Theorem 2.5 and Corollary 2.6 cannot be improved to "$\leqslant$", moreover the assumption $\mathfrak{t} = \mathfrak{c}$ cannot be dropped in Corollary 2.9.

Proof. In [23] it was proved that if T is a $\mathfrak{t}$-set in ω^*, then the associated Franklin–Rajagopalan space $X(T)$ is sequentially compact but not u-compact for any $u \in T$ (more precisely: the identity sequence on ω has no u-limit in $X(T)$ for any $u \in T$). There are models of S. Hechler (see [23] and [15]) in which there exists a family $\mathcal{F} = \mathcal{F}_0 \cup \mathcal{F}_1$ of towers, with $\mathcal{F}_0 = \{T_\alpha\colon \alpha < \omega_2\}$ a family of $\mathfrak{t}(\omega_1)$-sets, and $\mathcal{F}_1 = \{S_\alpha\colon \alpha < \omega_1\}$ a family of $\mathfrak{t}(\omega_2)$-sets such that ω^* is covered by $\mathcal{F}$. In these models, we have $\mathfrak{t} = \omega_1 < \mathfrak{n} = \omega_2 \leqslant \mathfrak{c}$, and $\{X(T_\alpha)\colon \alpha < \omega_2\} \cup \{X(S_\alpha)\colon \alpha < \omega_1\}$ is a family of $\mathfrak{n}$-many sequentially compact spaces whose product is not countably compact. Regarding Corollary 2.9, the Frolík sum $X = F(X(T)\colon T \in \mathcal{F})$ of these spaces (for the definition of Frolík sum see 3.1) is sequentially compact, satisfies $|X| = \mathfrak{n} = \omega_2 \leqslant \mathfrak{c}$, and X is not u-compact for any $u \in \omega^*$. □

We now make use of the notion of clustering families of sequences to obtain a suitable weakening of property $Q(\kappa)$. We start by considering certain special types of families of sequences in a space.

Definition 2.11. A family $\mathcal{S} = \{S_\alpha\colon \alpha < \kappa\}$ of finite sets of sequences in X is said to be *dispersed in X* if there exists a family $\mathcal{U} = \{U_\alpha\colon \alpha < \kappa\}$ of pairwise disjoint open sets such that $\mathrm{cl}_X(s(\omega)) \subset U_\alpha$ whenever $s \in S_\alpha$ and $\alpha < \kappa$. We say that $\mathcal{S}$ is *dispersed by $\mathcal{U}$*. If each S_α consists of a single sequence s_α, we say *the family of sequences $\mathcal{S} = \{s_\alpha\colon \alpha < \kappa\}$ is dispersed*.

Now we give our main definitions in this section.

Definition 2.12. Let κ be a (finite or infinite) cardinal number. A space X is said to satisfy property $R(\kappa)$ (respectively, $wR(\kappa)$) provided for every dispersed family $\mathcal{S} = \{S_\alpha\colon \alpha < \kappa\}$ of finite sets of sequences in X (respectively, $\mathcal{S} = \{s_\alpha\colon \alpha < \kappa\}$ of sequences in X), the family $\bigcup \mathcal{S}$ (respectively, $\mathcal{S}$) is clustering.

The assumption in the above definition of $R(\kappa)$ (or $wR(\kappa)$) can be weakened slightly to a mod finite condition:

Lemma 2.13. *A space X satisfies $R(\kappa)$ if and only if for every family $\mathcal{S} = \{S_\alpha\colon \alpha < \kappa\}$ of finite sets of sequences in X, if there exists a family $\{U_\alpha\colon \alpha < \kappa\}$ of pairwise disjoint open sets such that*

(1) *$F_\alpha = \{n \in \omega\colon s(n) \notin U_\alpha$ for some $s \in S_\alpha\}$ is finite for all $\alpha < \kappa$, and*
(2) *$\mathrm{cl}_X(\{s(n)\colon n \in \omega \setminus F_\alpha\}) \subset U_\alpha$ whenever $s \in S_\alpha$ and $\alpha < \kappa$,*

then $\bigcup \mathcal{S}$ clusters in X.

Proof. Let $\mathcal{S} = \{S_\alpha\colon \alpha < \kappa\}$ be a family of finite sets of sequences in X and $\{U_\alpha\colon \alpha < \kappa\}$ a pairwise disjoint family of open sets that satisfies (1) and (2), i.e., $\mathcal{S}$ is dispersed mod finite in $\mathcal{U}$. For each $s \in S_\alpha$ let $m_\alpha \in \omega \setminus F_\alpha$, and define

$$\tilde{s}(n) = \begin{cases} s(m_\alpha) & \text{if } n \in F_\alpha, \\ s(n) & \text{if } n \in \omega \setminus F_\alpha. \end{cases}$$

Then the family $\{\{\tilde{s}\colon s \in S_\alpha\}\colon \alpha < \kappa\}$ is dispersed in $\mathcal{U}$; so by $R(\kappa)$ there is $u \in \omega^*$ such that every sequence $\tilde{s}$ has a u-limit. Since $\tilde{s}$ and s differ only on a finite set, every s has a u-limit. The converse is trivial. □

Recalling again that a family of sequences clusters if and only if the appropriate sequence in the product of the target spaces does, we obtain the following alternative (and perhaps more attractive) characterizations of the properties $wR(\kappa)$ and $R(\kappa)$.

Proposition 2.14. *Property $wR(\kappa)$ (respectively, $R(\kappa)$) is equivalent to the following: For every family $\{Z_\alpha\colon \alpha < \kappa\}$ of closed subsets of X separated by pairwise disjoint open sets (and any family $\{k_\alpha\colon \alpha < \kappa\}$ of natural numbers), the product $\prod_{\alpha<\kappa} Z_\alpha$ (respectively, $\prod_{\alpha<\kappa} Z_\alpha^{k_\alpha}$) is countably compact.*

Clearly $R(\kappa)$ implies $wR(\kappa)$. Although we only need property $wR(\kappa)$ below, property $R(\kappa)$ turns out to be a formally stronger consequence of the hyperspace being countably compact. We also note that $wR(1)$ is equivalent to countable compactness, and $R(1)$ is equivalent to $Q(<\omega)$, that is to all finite powers being countably compact.

If there is no pairwise disjoint family of open sets in X of cardinality κ, or using the notation from [18, 1.22]: $\hat{c}(X) \leqslant \kappa$, we trivially have that X satisfies property $R(\kappa)$. Thus in general, unlike for property $Q(\kappa)$, property $R(\kappa)$ does not imply $R(\lambda)$ or even $wR(\lambda)$ for $\lambda < \kappa$.

However, if our space X satisfies $wR(\kappa)$ nontrivially, that is, it does have κ-many pairwise disjoint open sets (in short: $\hat{c}(X) > \kappa$) we will show that wR demonstrates much monotone behavior. We first present a very simple lemma.

Lemma 2.15. *Let X be T_1, $\lambda < \kappa$ and assume that X has $wR(\kappa)$. If $\{s_\beta\colon \beta < \lambda\}$ is a family of sequences in X, dispersed in a family $\{U_\beta\colon \beta < \lambda\}$ and $\{W_\alpha\colon \alpha < \kappa\}$ are nonempty open sets such that $\{U_\beta\colon \beta < \lambda\} \cup \{W_\alpha\colon \alpha < \kappa\}$ is pairwise disjoint, then $\{s_\beta\colon \beta < \lambda\}$ clusters.*

Proof. For each $\alpha < \kappa$, let t_α be a constant sequence with constant value in W_α. Then $\{s_\beta\colon \beta < \lambda\} \cup \{t_\alpha\colon \alpha < \kappa\}$ is a dispersed family of sequences, hence it clusters (if κ is finite, we select $m = \kappa - \lambda$ many W_α so that we have κ-many dispersed sequences to present to $wR(\kappa)$). Therefore $\{s_\beta\colon \beta < \lambda\}$ clusters. □

We recall that a T_2 space X is called *strongly Hausdorff* provided from every infinite subset $A \subset X$ we can choose a sequence $\{a_n\colon n \in \omega\}$ such that the a_n have pairwise disjoint neighborhoods in X [14], [17, 0.20].

Theorem 2.16.

(i) *If X is strongly Hausdorff and has $wR(\kappa)$ non-trivially for some κ, then X is countably compact (i.e., has $wR(1)$).*
(ii) *If $\lambda < \kappa \leqslant \omega$ and the Urysohn space X has $wR(\kappa)$ then X has $wR(\lambda)$.*
(iii) *If κ is infinite, $\lambda < \mathrm{cf}(\kappa)$ and X is T_3 satisfying both $wR(\kappa)$ and $\kappa < \hat{c}(X)$ then X has $wR(\lambda)$.*

Proof. For (i), we only give the proof for infinite κ and leave the finite case to the reader. Start by fixing a disjoint family $\mathcal{W}$ of open sets with $|\mathcal{W}| = \kappa$.

If X is not countably compact, let $s : \omega \to X$ be a one-to-one sequence whose range is a closed discrete set in X. By strongly Hausdorff, there are two disjoint open sets U and V such that

$$\left|U \cap s(\omega)\right| = \left|V \cap s(\omega)\right| = \omega.$$

Consider $\mathcal{W}_0 = \{W \in \mathcal{W}\colon U \cap W = \emptyset\}$. If $|\mathcal{W}_0| = \kappa$ and we set $A = \{n \in \omega\colon s(n) \in U\}$ then we may apply Lemma 2.15 to $U, s \restriction A$ and $\mathcal{W}_0$ and conclude that $s \restriction A$ is clustering, which is a contradiction. If, however, $|\mathcal{W}_0| < \kappa$ then

$$\mathcal{W}_1 = \{U \cap W\colon W \in \mathcal{W} \setminus \mathcal{W}_0\}$$

has cardinality κ, so we may apply Lemma 2.15 to $V, s \restriction B$ and $\mathcal{W}_1$, where $B = \{n \in \omega\colon s(n) \in V\}$, to get the same kind of contradiction.

(ii) Let $\{s_i\colon i < \lambda\}$ be a family of sequences in X dispersed in $\{U_i\colon i < \lambda\}$. If the range of any s_i has compact closure, then $u \lim s_i$ exists in X for all $u \in \omega^*$; hence we may delete s_i from consideration towards proving that $\{s_i\colon i < \lambda\}$ clusters. Since Urysohn implies strongly Hausdorff, by (i) we may assume that s_0 has at least two cluster points, say p and q. By Urysohn, there are open sets $U, V \subset U_0$ such that $p \in U, q \in V$ and $\overline{U} \cap \overline{V} = \emptyset$. Since V is infinite and Hausdorff, there exists a family $\{W_n\colon n \in \omega\}$ of pairwise disjoint open subsets of V. Let $A = \{n \in \omega\colon s_0(n) \in U\}$. Then $\{s_i \restriction A\colon i < \lambda\}$ is dispersed in $\mathcal{U} = \{U_0 \setminus \overline{V}\} \cup \{U_i\colon 0 < i < \lambda\}$. Also, each W_n is disjoint from the sets in $\mathcal{U}$. By Lemma 2.15 $\{s_i \restriction A\colon i < \lambda\}$ clusters; hence $\{s_i\colon i < \lambda\}$ clusters.

(iii) Let $\{s_i\colon i < \lambda\}$ be a family of sequences in X dispersed in $\{U_i\colon i < \lambda\}$ and $\mathcal{W}$ be a family of κ-many pairwise disjoint nonempty open sets in X. If κ-many $W \in \mathcal{W}$ miss $\bigcup\{U_i\colon i < \lambda\}$, then we are done by Lemma 2.15. Thus we assume that $W \cap (\bigcup\{U_i\colon i < \lambda\}) \neq \emptyset$ for all $W \in \mathcal{W}$. By $\lambda < \operatorname{cf}(\kappa)$ then there exists $\alpha < \lambda$ such that $|\{W \in \mathcal{W}\colon W \cap U_\alpha \neq \emptyset\}| = \kappa$.

As above, by (i) we may assume that s_α has two distinct cluster points p and q. Then there exist disjoint open sets $U, V \subset U_\alpha$ such that $p \in U, q \in V$. At least one of U and $U_\alpha \setminus \overline{U}$ intersects κ-many elements of $\mathcal{W}$. Assume first that $U_\alpha \setminus \overline{U}$ does. By T_3, there exists an open set T so that $p \in T \subset \overline{T} \subset U$. Let $A = s^{-1}(T)$ then A is infinite and we have $s_\alpha(A) \subset T \subset \overline{T} \subset U$. By our choice of T and A, $\{s_\beta \restriction A\colon \beta < \lambda\}$ is dispersed by

$$\mathcal{U} = \{U\} \cup \{U_\beta\colon \beta < \lambda, \beta \neq \alpha\}.$$

Moreover, $\mathcal{U} \cup \{W \cap (U_\alpha \setminus \overline{U})\colon W \in \mathcal{W}\}$ is a pairwise disjoint family of size κ. Thus by Lemma 2.15 $\{s_\beta \restriction A\colon \beta < \lambda\}$ is clustering; so $\{s_\beta\colon \beta < \lambda\}$ is clustering.

If $U_\alpha \setminus \overline{U}$ does not intersect κ-many $W \in \mathcal{W}$, then $U \subset U_\alpha \setminus \overline{V}$ does, and we may proceed similarly as above. □

Now we come to the main theorem of this section but before that we need a lemma.

Lemma 2.17. *Let X be a T_2-space, and let C be a cluster point of the sequence (C_n) in the hyperspace $H(X)$. If k is a natural number and U is open in X such that $|C_n \cap U| \leqslant k$ for all $n \in \omega$, then $|C \cap U| \leqslant k$ as well.*

Proof. If the lemma fails, then we may pick $k+1$ distinct points $x_0, \ldots, x_k$ in $C \cap U$. There exists a family of pairwise disjoint open sets $\{V_i\colon i \leqslant k\}$ such that $x_i \in V_i \subset U$ for all $i \leqslant k$. Then $V = \bigcap_{i \leqslant k} V_i^-$, is a neighborhood of C in $H(X)$; so there exists $n < \omega$ such that $C_n \in V$. But then $C_n \cap V_i \neq \emptyset$ for all $i \leqslant k$; so $|C_n \cap U| \geqslant k+1$, which is a contradiction. □

Theorem 2.18. *If X is T_2 and $H(X)$ is countably compact, then X has property $R(\kappa)$ for all cardinals κ.*

Proof. Let $\mathcal{S} = \{S_\alpha\colon \alpha < \kappa\}$ be a family of finite sets of sequences (say $S_\alpha = \{s_\alpha^i\colon i < n_\alpha\}$ for $\alpha < \kappa$) and $\mathcal{U} = \{U_\alpha\colon \alpha < \kappa\}$ a family of pairwise disjoint open sets such that $\mathcal{S}$ is dispersed in $\mathcal{U}$. We must find $u \in \omega^*$ such that every sequence in $\bigcup \mathcal{S}$ has a u-limit in X.

For each $n < \omega$ let $C_n = \operatorname{cl}_X\{s_\alpha^i(n)\colon \alpha < \kappa, i < n_\alpha\}$. Since the U_α are pairwise disjoint we clearly have

$$C_n \cap U_\alpha = \{s_\alpha^i(n)\colon i < n_\alpha\}.$$

Since (C_n) is a sequence in $H(X)$, by hypothesis there exists $u \in \omega^*$ and $C \in H(X)$ such that $C = u \lim C_n$. We show that this u works. Let $\alpha < \kappa$ and let s be a sequence in S_α.

Claim. *s has a u-limit in $C \cap U_\alpha$.*

Suppose otherwise. Since $|S_\alpha| = n_\alpha$, by Lemma 2.17, $|C \cap U_\alpha| \leqslant n_\alpha$ is finite; say $C \cap U_\alpha = \{x_i\colon i < m\}$. By our assumption there exist open sets V_i such that $x_i \in V_i \subset U_\alpha$ and $s^{-1}(V_i) \notin u$ for all $i < m$. Let $W = X \setminus \overline{(\bigcup_{i < n_\alpha} s_\alpha^i(\omega))}$. Then W is open in X and $C \subset W \cup V_0 \cup \cdots \cup V_{m-1}$, i.e., $V = (W \cup V_0 \cup \cdots \cup V_{m-1})^+$ is a neighborhood of C in $H(X)$, so $B = \{n \in \omega\colon C_n \in V\} \in u$. But if $C_n \subset \bigcup\{V_i\colon i < m\} \cup W$ then $C_n \cap U_\alpha \subset \bigcup\{V_i\colon i < m\}$. Thus $n \in B$ implies $s(n) \in \bigcup\{V_i\colon i < m\}$, hence

$$B \subset \bigcup\{s^{-1}(V_i)\colon i < m\}$$

which is a contradiction since $B \in u$ and the larger set is not. This completes the proof. □

In view of Proposition 2.14 this result may be reformulated as follows.

Corollary 2.19. *Assume that X is T_2 and $H(X)$ is countably compact. Then for any family $\{Z_\alpha\colon \alpha < \kappa\}$ of closed subsets of X that can be separated by disjoint open sets and for any family $\{k_\alpha\colon \alpha < \kappa\}$ of natural numbers the product $\prod_{\alpha<\kappa} Z_\alpha^{k_\alpha}$ is countably compact.*

Corollary 2.20. *(See Ginsburg [11].) If X is T_2 and $H(X)$ is countably compact, then X^n is countably compact for all $n < \omega$.*

Note that Theorem 1.1 follows from this corollary as well. We may use Corollary 2.19 to give the following useful condition for $H(X)$ not being countably compact.

Corollary 2.21. *Let X be a T_2 space in which there are a family of pairwise disjoint open sets $\{G_u\colon u \in \omega^*\}$ and a family of closed sets $\{Z_u\colon u \in \omega^*\}$ such that $Z_u \subset G_u$ and Z_u is not u-compact whenever $u \in \omega^*$. Then X fails to have property $wR(2^{\mathfrak{c}})$, hence $H(X)$ is not countably compact.*

Proof. The hypothesis implies that $\prod\{Z_u\colon u \in \omega^*\}$ is not countably compact. □

We will use Corollary 2.21 in proving Corollary 3.7.

Our following theorem generalizes [3, Theorem 3.1] (see Corollary 2.24). For any topological space X we shall use $\mathrm{Aut}(X)$ to denote the set of all autohomeomorphisms of X.

Definition 2.22. We say that a space X is *weakly homogeneous* provided for any nonempty open set $U \subset X$ and any point $x \in X$ there exists $h \in \mathrm{Aut}(X)$ such that $h(x) \in U$.

The compact T_2-space ω^* is weakly homogeneous, but not homogeneous. To see this, recall that the orbit of a point x in a topological space X is defined to be $\{h(x)\colon h \in \mathrm{Aut}(X)\}$. Thus a space is weakly homogeneous if and only if the orbit of every point is dense. It is shown in [30, Corollary 3.20] that the orbit of every point in ω^* is dense; so ω^* is weakly homogeneous. It is a well-known theorem of Z. Frolík that ω^* is not homogeneous (see [30, Corollary 3.46]).

Theorem 2.23. *If $\kappa < \mathfrak{t}$ and X is a weakly homogeneous T_3-space satisfying both property $wR(\kappa)$ and $\hat{c}(X) > \kappa$ then X has $Q(\kappa)$ (i.e., X^κ is countably compact).*

Proof. Let $\mathcal{S} = \{s_\alpha\colon \alpha < \kappa\}$ be a family of sequences in X. We need to show that $\mathcal{S}$ is clustering. Let $\{U_\alpha\colon \alpha < \kappa\}$ be a family of pairwise disjoint nonempty open sets in X. By T_3, for each $\alpha < \kappa$ there exists a nonempty open set V_α such that $V_\alpha \subset \mathrm{cl}_X(V_\alpha) \subset U_\alpha$. By transfinite recursion we construct infinite sets $A_\alpha \subset \omega$ and autohomeomorphisms $h_\alpha \in \mathrm{Aut}(X)$ for $\alpha < \kappa$ such that the following inductive hypotheses hold:

(1) $\alpha < \beta$ implies $A_\beta \subset^* A_\alpha$,
(2) $n \in A_\alpha$ implies $h_\alpha(s_\alpha(n)) \in V_\alpha$.

At step γ, pick an infinite set $B \subset \omega$ such that $B \subset^* A_\alpha$ for all $\alpha < \gamma$. Let x be a cluster point of $s_\gamma \restriction B$ (this exists because X is countably compact by Theorem 2.16(i)), and let $h_\gamma \in \mathrm{Aut}(X)$ be chosen so that $h_\gamma(x) \in V_\gamma$. Finally, let $A_\gamma = \{n \in B\colon h_\gamma(s(n)) \in V_\gamma\}$. Then A_γ is infinite because $h_\gamma(x)$ is a cluster point of $h_\gamma \circ s_\gamma \restriction B$, and V_γ is open. This completes the induction.

Since $\kappa < \mathfrak{t}$, there exists an infinite set $A \subset \omega$ such that $A \subset^* A_\alpha$ for all $\alpha < \kappa$. Let $\phi : \omega \to A$ be any one–one and onto function. Define

$$r_\alpha = h_\alpha \circ s_\alpha \circ \phi.$$

Then $\{r_\alpha\colon \alpha < \kappa\}$ is dispersed mod finite in $\{U_\alpha\colon \alpha < \kappa\}$. By Lemma 2.13, there exists $u \in \omega^*$ such that r_α has a u-limit in X for all $\alpha < \kappa$. To complete the proof we show that if $x = u \lim r_\alpha$ then $h_\alpha^{-1}(x) = \phi(u) \lim s_\alpha$. By continuity,

$h_\alpha^{-1}(x) = u \lim h^{-1} \circ r_\alpha = u \lim s_\alpha \circ \phi$. Since ϕ is one–one and onto, it follows that $h_\alpha^{-1}(x) = \phi(u) \lim s_\alpha$. This shows that $\mathcal{S} = \{s_\alpha\colon \alpha < \kappa\}$ is clustering, and that completes the proof. □

Corollary 2.24. *(See Cao et al. [3].) If X is a homogeneous T_3 space such that the hyperspace $H(X)$ is countably compact, then X^ω is countably compact.*

Proof. It is known that every infinite T_2-space has an infinite family of pairwise disjoint open sets. Hence if X is infinite then it satisfies both $wR(\omega)$ and $\hat{c}(X) > \omega$, moreover $\omega < \mathfrak{t}$. □

We conclude this section by considering the following question: If X is countably compact and T_4, is then its hyperspace $H(X)$ countably compact? Consistently, the answer to this question is known to be negative, as is shown by the following example of van Douwen.

Example 2.25. (See Van Douwen [5].) Under Martin's Axiom (MA) there exists a countably compact T_4-space X such that X^2 is not countably compact (hence neither is $H(X)$).

In his (unpublished) dissertation [25, Chapter 5] Oleg Pavlov constructed another countably compact T_4-space X such that X^2 is not countably compact using the continuum hypothesis (CH). In fact, as we shall show, Pavlov's construction works under the (much weaker) assumption that there exists an HFD. (Note that in the model obtained by adding ω_1 Cohen reals to an arbitrary ground model there are HFD's, see [19, 4.7], [21, 2.3].) Also, Pavlov's example is unpublished and rather involved because his space was constructed to have other properties which go beyond those stated in Example 2.25 (his space X satisfies the properties that X^ω is normal and the free topological group $F(X)$ is not). Our construction follows that of Pavlov. First we recall the definition of an HFD (see [13] or [19]).

Definition 2.26. A set $X \subset 2^\lambda$ $(\lambda > \omega)$ is called an HFD provided for every countably infinite $A \subset X$ there is a countable $b \subset \lambda$ such that A (i.e., $A \restriction (\lambda \setminus b)$) is dense in $2^{\lambda \setminus b}$.

Theorem 2.27. *If there exists an HFD of size κ, then there exists a countably compact T_4-space X such that X^2 has a closed discrete subset of cardinality κ; hence X^2 (and so $H(X)$) is very much not countably compact.*

Proof. Let $Z \subset 2^{\omega_1}$ be an HFD (it is known that if there is an HFD of any size $\kappa \geqslant \omega$ then there is one of the same size κ in 2^{ω_1}, see [19, 2.8(ii)]). We need to choose Z in a special way: First, note that $\{f \in Z\colon f \text{ is eventually constant}\}$ is clearly finite. Any subset of an HFD is an HFD, so we may assume that no $f \in Z$ is eventually constant. We may also assume that $z(0) = 0$ for all $z \in Z$.

We then consider the "mirror" of Z, namely the set $Z + 1 = \{z + 1\colon z \in Z\}$, where addition is co-ordinatewise modulo 2. Then $Z + 1$ is easily seen to be an HFD as well [19, 2.8], contains no function that is eventually constant, and $z(0) = 1$ for all $z \in Z + 1$. Clearly, then $Z \cap (Z + 1) = \emptyset$.

Let

$$\Sigma = \left\{f \in 2^{\omega_1}\colon f \text{ is eventually constant with constant value } 0\right\}$$

be the usual Σ-product in 2^{ω_1}. It is well known that Σ is countably compact, T_4 and dense in 2^{ω_1}. Also, we have

$$\Sigma \cap Z = \emptyset = \Sigma \cap (Z + 1).$$

For the underlying set of our example, put $X = \Sigma \cup Z \cup (Z + 1)$. The topology on X is obtained from the subspace topology in 2^{ω_1} by declaring all points in $Z \cup (Z + 1)$ isolated. Since Σ retains its subspace topology from 2^{ω_1}, it remains countably compact and T_4, but is closed in X with this finer topology.

X is countably compact: Let $A \subset X$ be countably infinite. It suffices to consider three cases: $A \subset \Sigma$, $A \subset Z$ and $A \subset (Z + 1)$. If $A \subset \Sigma$ then A has a limit point in Σ since Σ is countably compact. Suppose $A \subset Z$. By [19, 2.12] we may pick $\alpha < \omega_1$ so that α has the property that $A \restriction \alpha$ is infinite and $\alpha \in J(A)$. By [19, 2.13] $\alpha \in J(A)$ implies that if h is a limit point of $A \restriction \alpha$ in 2^α then $h^\frown 0 \in \Sigma$ is a limit point of A in 2^{ω_1}. The case $A \subset (Z + 1)$ is similar.

X is T_4: In general (as is probably known) if D is a dense T_4 subset of a space Y, and X is obtained from Y by isolating all points of $Y \setminus D$ then X is T_4. To see this, let H and K be disjoint closed sets in X. Then there exist disjoint open subsets U', $V'$of D such that $H \cap D \subset U'$ and $K \cap D \subset V'$. Let U, V be open in Y such that $U \cap D = U'$ and $V \cap D = V'$. Since D is dense in Y, $U \cap V = \emptyset$. Thus $(U \cup H) \setminus K$ and $(V \cup K) \setminus H$ are disjoint open sets in X that separate H and K.

The set $D = \{(z, z+1)\colon z \in Z\}$ (clearly of cardinality κ) is closed discrete in X^2: Since D consists of isolated points of X^2, it remains to show that D is closed in X^2. Let $(x, y) \in X^2 \setminus D$. Since $X = \Sigma \cup Z \cup (Z+1)$ is the union of three pairwise disjoint sets, X^2 is the union of nine corresponding pairwise disjoint sets. We proceed with these nine cases. *Case* 1: $(x, y) \in \Sigma \times \Sigma$: There exists $\alpha < \kappa$ such that $x(\alpha) = y(\alpha) = 0$ (in fact a final set of such α). Put $\varepsilon = \{(\alpha, 0)\}$, then $[\varepsilon] \times [\varepsilon]$ is a neighborhood of (x, y) missing D. This completes Case 1. Since $Z \cap (Z+1) = \emptyset$, $(Z+1) \times X$ and $X \times Z$ are open sets in X^2 that miss D, and this takes care of 5 more cases. If $(x, y) \in Z \times (Z+1)$, then (x, y) is isolated; so this leaves us with two cases: $(x, y) \in \Sigma \times (Z+1)$ or $(x, y) \in Z \times \Sigma$. If $(x, y) \in Z \times \Sigma$, then $\{x\} \times (X \setminus \{x+1\})$ is an open neighborhood of (x, y) missing D. The last case is proved similarly, and this completes the proof of the theorem. □

The spaces discussed in both Example 2.25 and Theorem 2.27 are both consistent examples. We do not know of a ZFC example of a countably compact T_4-space X such that X^2 (or just $H(X)$) is not countably compact.

3. Frolík sums

Z. Frolík [10], and others have used the following construction, and refer to it as a "one-point countable-compactification" of a family of spaces $\{X_\alpha\colon \alpha < \kappa\}$. The definition involves the disjoint topological sum $\bigoplus\{X_\alpha\colon \alpha < \kappa\}$ (see [6, 2.2]).

Definition 3.1. Let $\mathcal{F} = \{X_\alpha\colon \alpha < \kappa\}$ be a family of topological spaces. The *Frolík sum* of $\mathcal{F}$, denoted $F(X_\alpha\colon \alpha < \kappa)$, is defined in two cases: If κ is finite, it is defined to be equal to the topological sum: $F(X_\alpha\colon \alpha < \kappa) = \bigoplus\{X_\alpha\colon \alpha < \kappa\}$. If κ is infinite, the Frolík sum is defined to be $\bigoplus\{X_\alpha\colon \alpha < \kappa\} \cup \{*\}$ where $*$ is a point not in $\bigoplus\{X_\alpha\colon \alpha < \kappa\}$, and a neighborhood base of $(*)$ is formed by the unions of all but finitely many X_α. We let $F(\kappa \cdot X)$ denote the Frolík sum of κ copies of the same space X.

Definition 3.2. A topological property $\mathcal{P}$ is said to be (countably) *F-additive* provided the Frolík sum of (countably many) spaces with property $\mathcal{P}$ has property $\mathcal{P}$.

Theorem 3.3. *The following properties are all F-additive: countable compactness, sequential compactness, total countable compactness, u-compactness, ω-boundedness, pseudocompactness, in Frolík's class C, and in Frolík's class P.*

Proof. These are all easy to prove. We only check this for Frolík's class C and leave all other cases to the reader. Recall that a space X is said to be in Frolík's class C provided $X \times Y$ is countably compact for every countably compact space Y (see [8].) (Frolík's class P is defined similarly for pseudocompact spaces, see [9]). Let $\{X_\alpha\colon \alpha < \kappa\}$ be a family of spaces, each in Frolík's class C. We show that $X = F(X_\alpha\colon \alpha < \kappa)$ is also in Frolík's class C. Let Y be countably compact, and let $((x_n, y_n))$ be a sequence in $X \times Y$. We may assume that $x_n \neq *$ for all $n < \omega$. If $*$ is a cluster point of the sequence (x_n) then there is a subsequence of $(x_n)_{n \in A}$ that actually converges to $*$. It follows that $((x_n, y_n))_{n \in A}$ has a cluster point in $\{*\} \times Y$. If $*$ is not a cluster point of the sequence (x_n), then there exists a basic neighborhood U of $*$ such that $\{n \in \omega\colon x_n \in U\} = \emptyset$. Thus there exists a finite $F \subset \kappa$ such that $((x_n, y_n))$ is contained in $(\bigcup_{\alpha \in F} X_\alpha) \times Y$. Hence there exists $\alpha \in F$ such that $\{n \in \omega\colon (x_n, y_n) \in X_\alpha \times Y\}$ is infinite, and since X_α is in class C, $((x_n, y_n))$ has a cluster point in the countably compact subspace $X_\alpha \times Y \subset X \times Y$. □

We note that first countability is countably F-additive, but the Frolík sum of uncountably many spaces is never first countable (at $*$). Also the Frolík sum of even countably many locally countable spaces is not locally countable if infinitely many of the spaces are uncountable.

Let us recall at this point our initial problem: Is the countable compactness of the hyperspace $H(X)$ equivalent to X being u-compact for some u in ω^*? Our next result may be considered as a partial answer to this problem and is also an analogue to a theorem of Ginsburg and Saks [12]:

Theorem 3.4. *For any T_2-space X the following are equivalent*:

(1) $H(F(\kappa \cdot X))$ *is countably compact for every cardinal* κ.
(2) $H(F(2^{\mathfrak{c}} \cdot X))$ *is countably compact.*
(3) $H(F(|X|^{\omega} \cdot X))$ *is countably compact.*
(4) X *is* u*-compact for some* $u \in \omega^*$.

Proof. This follows from 2.19 and 3.3 and the analogous result of Ginsburg–Saks for powers of X, see [12]. For example, both (2) and (3) imply (4) because they imply that either $X^{2^{\mathfrak{c}}}$ or $X^{|X|^{\omega}}$ is countably compact (by 2.19); so by the result of Ginsburg–Saks, X is u-compact for some ultrafilter $u \in \omega^*$. □

The following result will turn out to be instrumental in constructing examples of countably compact spaces with nice additional properties (like first countability or even local countability) and having hyperspaces that are not countably compact.

Theorem 3.5. *Let $\mathcal{P}$ be a property that is countably F-additive, hereditary to clopen sets, and preserved by well-ordered increasing unions of clopen subsets of length of uncountable cofinality, where the union carries the direct limit topology. Then for every κ and every pairwise disjoint family of spaces $\{X_\alpha\colon \alpha < \kappa\} \subset \mathcal{P}$, there exists a space $Z \in \mathcal{P}$ such that each X_α is a clopen subspace of Z.*

Proof. Let $Z_0 = \emptyset$, and assume we have constructed spaces $Z_\alpha \in \mathcal{P}$ for $\alpha < \gamma$ (where $\gamma \leqslant \kappa$) such that $\beta < \alpha < \gamma$ implies

(1) $Z_\beta \subset Z_\alpha$,
(2) Z_β and X_β are clopen in Z_α.

We then define Z_γ in three cases. If $\gamma = \alpha + 1$, then put $Z_\gamma = Z_\alpha \oplus X_\alpha$. If $\mathrm{cf}(\gamma) > \omega$, put $Z_\gamma = \bigcup\{Z_\alpha\colon \alpha < \gamma\}$ with the direct limit topology. If $\mathrm{cf}(\gamma) = \omega$, then fix a strictly increasing sequence $(\beta_n\colon n < \omega)$ such that $\sup\{\beta_n\colon n < \omega\} = \gamma$, and define $Y_n = Z_{\beta_{n+1}} \setminus Z_{\beta_n}$. Each Y_n is clopen in $Z_{\beta_{n+1}}$ and so has property $\mathcal{P}$. Put $Z_\gamma = F(Y_n\colon n \in \omega)$. It is easy to check that $Z_\gamma \in \mathcal{P}$ in all three cases, using our hypotheses, and that the inductive assumptions (1) and (2) remain valid. Finally, $Z = Z_\kappa$ is the desired space. □

Corollary 3.6. *Let $\mathcal{P}$ be a property of Hausdorff spaces as in Theorem* 3.5. *Then the following are equivalent*: (i) *there is a* (*transfinite*) *sequence of spaces in $\mathcal{P}$ whose product is not countably compact*; (ii) *there is a space Z in $\mathcal{P}$ such that $H(Z)$ is not countably compact.*

Proof. (i) ⇒ (ii) follows immediately from Theorem 3.5 and Corollary 2.19. (ii) ⇒ (i) holds because if $H(Z)$ is not countably compact then Z is not u-compact for any $u \in \omega^*$, hence some power of Z is not countably compact. □

Corollary 3.7. *If $\mathfrak{b} = \mathfrak{c}$ then there exists a first countable, countably compact* 0*-dimensional T_2-space X such that $H(X)$ is not countably compact. If, in addition to $\mathfrak{b} = \mathfrak{c}$, we also have $2^{\mathfrak{c}} < \mathfrak{c}^{+\omega}$ then there is even a good space X such that $H(X)$ is not countably compact.*

Proof. Let $\{X_u\colon u \in \omega^*\}$ be the family of good spaces constructed by van Douwen under $\mathfrak{b} = \mathfrak{c}$ such that X_u is not u-compact for $u \in \omega^*$ [4, Section 13]. Without loss of generality, the family $\{X_u\colon u \in \omega^*\}$ is pairwise disjoint.

The property $\mathcal{P} \equiv$ "first countable, countably compact 0-dimensional T_2" is countably F-additive, hereditary to closed sets, and preserved by increasing unions of uncountable cofinality of clopen sets with the direct limit topology. By Theorem 3.5 there exists a space Z having property $\mathcal{P}$ in which $\{X_u\colon u \in \omega^*\}$ is a family of pairwise disjoint clopen sets. But then $H(Z)$ is not countably compact by Corollary 2.21.

To see the second part, we use theorem [20, Theorem 16] that, under the given conditions, states the existence of a good space Z with a pairwise disjoint family $\{X_u\colon u \in \omega^*\}$ of clopen sets such that X_u is not u-compact for all $u \in \omega^*$, and then we apply Corollary 2.21 to that family. □

Our next result fits here because it is closely related to van Douwen's construction of non-u-compact good spaces from [4] that was used in the previous theorem.

Theorem 3.8. *If $\mathfrak{b} = \mathfrak{c}$, then for any tower $\{W_\alpha^*\colon \alpha < \mathfrak{c}\}$ in ω^*, there exists a good space X which is not u-compact for any $u \in T = \bigcap\{W_\alpha^*\colon \alpha < \mathfrak{c}\}$.*

Proof. The construction of our good space X is an Ostaszewski type construction of a not u-compact space (see [26]). We follow the version of this construction given by van Douwen [4, Section 13] which uses the assumption $\mathfrak{b} = \mathfrak{c}$. We only need to change his Case 2, subcase B [4, p. 160], but for clarity we sketch the entire construction.

The construction starts by listing all countably infinite subsets of $\mathfrak{c}$ as $\{K_\eta\colon \omega \leqslant \eta < \mathfrak{c}\}$ such that $K_\eta \subset \eta$ for $\omega \leqslant \eta < \mathfrak{c}$ [4, 13.2]. The underlying set for X is the cardinal $\mathfrak{c}$. By recursion we construct topologies $\mathcal{T}_\eta$ on the ordinals $\eta \leqslant \mathfrak{c}$ satisfying the following conditions:

(1) $(\eta, \mathcal{T}_\eta)$ is a locally compact T_2-space in which the collection of compact open subsets is a base,
(2) $\forall \xi \in \eta [(\xi, \mathcal{T}_\xi)$ is an open subspace of $(\eta, \mathcal{T}_\eta)]$,
(3) $\forall \xi \in \eta \setminus \omega$ $[K_\xi$ has a cluster point in $(\eta, \mathcal{T}_\eta)]$,
(4) $\forall \xi \in \eta \setminus \omega$ there exists $V \in \mathcal{T}_\xi$ with $\xi \in V$ such that $(V \cap \omega)^* \cap T = \emptyset$.

The first step of the construction yields the discrete topology $\mathcal{T}_\omega$ on ω; so we consider the step for $\omega < \eta \leqslant \mathfrak{c}$, assuming we have $\mathcal{T}_\zeta$ for $\zeta \in \eta$.

Case 1. η is a limit ordinal. This case is the same as in [4].

Case 2. This case concerns the successor ordinal step of the construction, which van Douwen calls $\eta = \xi + 1$.

Subcase A. K_ξ, has a cluster point in $(\xi, \mathcal{T}_\xi)$. This subcase is the same as in [4].

Subcase B. This is the case in the construction that we change. By $\mathfrak{b} = \mathfrak{c}$, property D holds for the space $(\xi, \mathcal{T}_\xi)$, so we choose an indexed discrete family of open sets $\{U_x\colon x \in K_\xi\}$ with $x \in U_x$ for all $x \in K_\xi$ the same way van Douwen did except we add the additional requirement that $(U_x \cap \omega)^* \cap T = \emptyset$ which is possible by (4). Since T is a $\mathfrak{t}$-set, there exists W_γ such that $W_\gamma^* \cap (U_x \cap \omega)^* = \emptyset$ for all $x \in K_\xi$. Hence $F_x = W_\gamma \cap (U_x \cap \omega)$ is finite for all $x \in K_x$. We define the topology on the space $(\xi + 1, \mathcal{T}_{\xi+1})$ to be the smallest topology containing $\mathcal{T}_\xi$ and

$$\Bigl\{B_G = \{\xi\} \cup \bigcup \bigl\{(U_x \setminus F_x)\colon x \in K_\xi \setminus G\bigr\}\colon G \subset K_\xi \text{ finite}\Bigr\}.$$

Then $\mathcal{T}_\xi \cup \{B_G\colon G \in [K_\xi]^{<\omega}\}$ is a base for $\mathcal{T}_\eta$. Since, $(U_x \setminus F_x) \cap W_\gamma = \emptyset$ for all $x \in K_\xi$, we have $(\omega \cap B_G)^* \cap W_\gamma^* = \emptyset$; so $(\omega \cap B_G)^* \cap T = \emptyset$ for any $G \in [K_\xi]^{<\omega}$. This completes the construction.

Now let $u \in T$. Then $X = (\mathfrak{c}, \mathcal{T}_\mathfrak{c})$ is not u-compact because the identity sequence on ω has no u-limit in X by (4). $\square$

Let us now turn to some general results concerning the hyperspaces of Frolík sums of families of spaces. We start with the case of finite families which, of course, just means looking at hyperspaces of finite topological sums. The simplest of these is when we take just two summands.

Lemma 3.9. *For any two spaces X and Y we have*

$$H(X \oplus Y) \simeq H(X) \oplus H(Y) \oplus \bigl(H(X) \times H(Y)\bigr).$$

Proof. This lemma is easily seen to be equivalent to

$$H^0(X \oplus Y) \simeq H^0(X) \times H^0(Y),$$

see [22, Corollary 5a of Section 17]. The apparent difference in form is due to the fact that $H^0(X)$, as opposed to $H(X)$, includes the empty set. (So in this case the choice of H^0 over H seems to be more fortunate.) $\square$

Corollary 3.10. *If $\mathcal{P}$ is a finitely additive and finitely productive property of topological spaces and $H(X_1), \ldots, H(X_n)$ all have property $\mathcal{P}$, then $H(X_1 \oplus \cdots \oplus X_n)$ has $\mathcal{P}$ as well.*

We now come to considering Frolík sums of infinite families of spaces. Our main result is the following embedding theorem.

Theorem 3.11. *Let $\{X_\alpha\colon \alpha < \kappa\}$ be an infinite pairwise disjoint family of topological spaces and $X = F(X_\alpha\colon \alpha < \kappa)$ be their Frolík sum. Then*

$$P = \prod_{\alpha<\kappa} H^0(X_\alpha) \simeq Q = \bigl\{C \in H(X)\colon * \in C\bigr\},$$

moreover Q is closed in $H(X)$.

Proof. Let $\vec{C} = (C_\alpha)$ denote a point (κ-tuple) in the product $P = \prod_{\alpha<\kappa} H^0(X_\alpha)$, i.e., each C_α is a (possibly empty) closed subset of X_α. Define the map $h : \prod_{\alpha<\kappa} H^0(X_\alpha) \to H(X)$ by

$$h(\vec{C}) = \bigcup\{C_\alpha\colon \alpha < \kappa\} \cup \{*\},$$

then clearly $Q = h[P]$. We claim that h is a homeomorphism between P and Q.

h is one–one: if $h(\vec{C}) = h(\vec{D})$ then (since the X_α are pairwise disjoint) $h(\vec{C}) \cap X_\alpha = C_\alpha = h(\vec{D}) \cap X_\alpha = D_\alpha$ for all $\alpha < \kappa$; so $\vec{C} = \vec{D}$.

h is continuous: Let U be a nonempty open set in X, then U^+ and U^- are typical subbasic open sets in $H(X)$, so it suffices to show that $h^{-1}(U^+)$ and $h^{-1}(U^-)$ are both open in the product P. First we consider the case that $* \in U$, hence there exists a finite $E \subset \kappa$ such that $\bigcup\{X_\alpha\colon \alpha \in \kappa \setminus E\} \subset U$. Then for every $\vec{C} \in P$ we have $h(\vec{C}) \in U^+$ if and only if $C_\alpha \subset U \cap X_\alpha$ for every $\alpha \in E$, and therefore

$$h^{-1}(U^+) = \bigcap\bigl\{\pi_\alpha^{-1}\bigl((U \cap X_\alpha)^+\bigr)\colon \alpha \in E\bigr\}$$

is open in P. Concerning U^-, we have $h^{-1}(U^-) = P$ because $Q \subset U^-$ by definition.

If $* \notin U$ then $h^{-1}(U^+) = \emptyset$ because $Q \cap U^+ = \emptyset$. Moreover,

$$h^{-1}(U^-) = \bigl\{\vec{C}\colon (\exists \alpha < \kappa)(C_\alpha \cap U \neq \emptyset)\bigr\} = \bigcup_{\alpha<\kappa} \pi_\alpha^{-1}\bigl((U \cap X_\alpha)^-\bigr)$$

is open in P.

h is open: Since h is one–one, it suffices to show that the h-images from a subbase of P are open in Q. Hence it suffices to show that if $\alpha < \kappa$ and V is open in X_α then both $h[\pi_\alpha^{-1}(V^+)]$ and $h[\pi_\alpha^{-1}(V^-)]$ are open in Q. Indeed, this holds because

$$h\bigl[\pi_\alpha^{-1}(V^+)\bigr] = \bigl\{h(\vec{C})\colon C_\alpha \subset V\bigr\} = (\widetilde{V})^+ \cap Q$$

where $\widetilde{V} = \bigcup\{X_\beta\colon \beta \neq \alpha\} \cup V \cup \{*\}$ is clearly open in X. Also

$$h\bigl[\pi_\alpha^{-1}(V^-)\bigr] = \bigl\{h(\vec{C})\colon C_\alpha \cap V \neq \emptyset\bigr\} = V^- \cap Q,$$

where the last "$^-$" refers to the hyperspace $H(X)$.

That Q is closed in $H(X)$ is obvious from $Q = H(X) \setminus (X \setminus \{*\})^+$. Note that $h[\prod_{\alpha<\kappa} H(X_\alpha)]$ is also closed in $H(X)$ because

$$H(X) \setminus h\Biggl[\prod_{\alpha<\kappa} H(X_\alpha)\Biggr] = \bigl(X \setminus \{*\}\bigr)^+ \cup \bigcup\bigl\{(X \setminus X_\alpha)^+\colon \alpha < \kappa\bigr\}. \qquad \square$$

Corollary 3.12. *Let $\{X_\alpha\colon \alpha < \kappa\}$ be a family of T_1-spaces. Then $\prod_{\alpha<\kappa} X_\alpha$ is homeomorphic to a subspace of $H(F(X_\alpha\colon \alpha < \kappa))$. If the spaces X_α are T_2, then $\prod_{\alpha<\kappa} X_\alpha$ is homeomorphic to a closed subspace of $H(F(X_\alpha\colon \alpha < \kappa))$.*

Proof. It is well known that if Y is T_1 then the map $j(y) = \{y\}$ is an embedding of Y into $H(Y)$ that maps Y onto a closed set if Y is T_2. Moreover, $H(X)$ is closed in $H^0(X)$. The result then follows from that and Theorem 3.11. $\square$

Corollary 3.13. *Let $\{X_\alpha\colon \alpha<\kappa\}$ be a family of T_2-spaces. If $\mathcal{P}$ is a closed hereditary property and $H(F(X_\alpha\colon \alpha<\kappa))$ has property $\mathcal{P}$, then $\prod_{\alpha<\kappa} X_\alpha$ has property $\mathcal{P}$. In particular, if $H(F(X_\alpha\colon \alpha<\kappa))$ is countably compact, sequentially compact, totally countably compact, etc. then so is $\prod_{\alpha<\kappa} X_\alpha$.*

The statement concerning countably compact spaces in Corollary 3.13 was proved by Cao, Noguro, and Tamano using a different and rather involved proof [3, Theorem 2.1]. As we pointed out above, this special case also follows from Corollary 2.19.

As we have seen in Lemma 3.9 and Theorem 3.11, the product $\prod_{\alpha<\kappa} H^0(X_\alpha)$ embeds as a closed subspace into the hyperspace of the Frolík sum $F(X_\alpha\colon \alpha<\kappa)$, hence any closed hereditary property of the latter is inherited by the former. It may come as a surprise that for several countable compactness type properties (that are in the center of our attention in this paper) the converse of this also holds. Of course, if κ is finite then the two spaces under consideration are actually homeomorphic, hence in what follows we restrict ourselves to the case $\kappa \geqslant \omega$.

Theorem 3.14. *If $\{X_\alpha\colon \alpha<\kappa\}$ is a family of T_2-spaces, then $\prod_{\alpha<\kappa} H^0(X_\alpha)$ is countably compact (respectively, sequentially compact, respectively, totally countably compact) if and only if $H(F(X_\alpha\colon \alpha<\kappa))$ is countably compact (respectively, sequentially compact, respectively, totally countably compact).*

Proof. We prove the theorem for the totally countably compact case, the other cases being easier. Of course, by Theorem 3.11, only the "only if" part needs proof, so assume that $\prod_{\alpha<\kappa} H^0(X_\alpha)$ is totally countably compact.

Let $X = F(X_\alpha\colon \alpha<\kappa)$ and (C_n) be a sequence of points in $H(X)$. If there exists a finite $E\subset\kappa$ such that $\{n\in\omega\colon C_n\subset\bigcup_{\alpha\in E} X_\alpha\}$ is infinite then, as $H(\bigoplus_{\alpha\in E} X_\alpha)$ is totally countably compact by our assumptions, (C_n) has a subsequence contained in a compact subset of $H(\bigoplus_{\alpha\in E} X_\alpha)\subset H(X)$; so we are done.

Therefore we may assume that

(†) for every finite $E\subset\kappa$, $\{n\in\omega\colon C_n\subset\bigcup_{\alpha\in E} X_\alpha\}$ is finite.

For each n let d_n be the point in the product $\prod_{\alpha<\kappa} H^0(X_\alpha)$ whose αth coordinate is $C_n\cap X_\alpha$. By hypothesis the sequence (d_n) has a subsequence contained in a compact sets K. By passing to a subsequence and re-indexing, we may assume the entire sequence is contained in K. Let h be the homeomorphism given by Theorem 3.11. By continuity, $h(K)$ is compact in $H(X)$ and contains $h(d_n)$ for all n. Note that $h(d_n)=C_n$ iff $*\in C_n$, thus if the point $*$ is in C_n for infinitely many n then we are done. Otherwise, we may assume that none of the C_n contain $*$; hence none of the C_n are in the range of h. We show that in this case the sequence of sets (C_n) converges to the compact set $h(K)$; consequently $\{C_n\colon n<\omega\}\cup h(K)$ is a compact set containing all the C_n.

Consider any finite cover of $h(K)$ by basic open sets in $H(X)$, say

$$h(K)\subset\bigcup\{O_i\colon i<t\}$$

where $O_i=\langle U^i_j\colon j<n_i\rangle$ for $i<t$. We may assume that $O_i\cap h(K)\neq\emptyset$ for $i<t$. We also may assume that for each $i<t$ the U^i_j have been arranged so that for some $0<m_i\leqslant n_i$ we have $*\in U^i_j$ iff $j<m_i$. Then there is a finite set E such that $\bigcup\{X_\alpha\colon \alpha\notin E\}\subset\bigcap\{U^i_j\colon j<m_i, i<t\}$. By (†), there is $N<\omega$ such that for all $n>N$ we have

$$C_n\cap\bigcap\bigl\{U^i_j\colon j<m_i, i<t\bigr\}\neq\emptyset.$$

For every n we have $h(d_n)=C_n\cup\{*\}\in h(K)$, hence there is $i<t$ such that $C_n\cup\{*\}\in O_i$. Then we have $C_n\subset C_n\cup\{*\}\subset\bigcup\{U^i_j\colon j<n_i\}$ and $C_n\cap U^i_j\neq\emptyset$ for all $j<m_i$ whenever $n>N$. If $m_i\leqslant j<n_i$ then $*\notin U^i_j$; so $C_n\cup\{*\}\in O_i$ implies $(C_n\cup\{*\})\cap U^i_j\neq\emptyset$, which implies $C_n\cap U^i_j\neq\emptyset$. Thus we conclude that $C_n\in\bigcup\{O_i\colon i<t\}$ for all $n>N$. This completes the proof. □

Corollary 3.15. *Let $\kappa<\mathfrak{t}$, and let $\{X_\alpha\colon \alpha<\kappa\}$ be a family of T_2-spaces such that $H(X_\alpha)$ is sequentially compact (respectively totally countably compact) for all $\alpha<\kappa$. Then $H(F(X_\alpha\colon \alpha<\kappa))$ is sequentially compact (respectively, totally countably compact).*

Proof. First we consider the case that $H(X)$ is totally countably compact. Clearly, so is $H^0(X)$. By [28, 3.3(B)], the product $\prod_{\alpha<\kappa} H^0(X_\alpha)$ is totally countably compact; so the result follows from Theorem 3.14. The same idea works for the case that $H(X)$ is sequentially compact: The product $\prod_{\alpha<\kappa} H^0(X_\alpha)$ is sequentially compact (as remarked after the proof of [28, 3.4(B)]); so again the result follows from Theorem 3.14. □

Our next example shows that the restriction $\kappa < \mathfrak{t}$ in Corollary 3.15 cannot be improved substantially. Recall that *the splitting number* $\mathfrak{s}$ is the minimum cardinal λ such that 2^λ is not sequentially compact. It is well known that $\mathfrak{t} \leqslant \mathfrak{s} \leqslant \mathfrak{c}$ (e.g., see [4, 6.1], [27, 5.12]).

Example 3.16. The family $\{X_\alpha\colon \alpha < \mathfrak{s}\}$, where $X_\alpha = 2$ is a copy of the two-point discrete space for all $\alpha < \mathfrak{s}$, satisfies the hypothesis of Corollary 3.15 (i.e., $H(X_\alpha)$ is sequentially compact for $\alpha < \mathfrak{s}$), but not the conclusion (i.e., $H(F(X_\alpha\colon \alpha < \kappa))$ is not sequentially compact).

Proof. The hyperspace of a finite space is finite, hence sequentially compact. On the other hand, $H(F(X_\alpha\colon \alpha < \mathfrak{s}))$ is not sequentially compact because it contains a closed copy of the product $2^{\mathfrak{s}}$ which is not sequentially compact (see [28, 5.12]). □

We can however step a little further if we are satisfied with the hyperspace of the Frolík sum being only countably compact.

Corollary 3.17. *Let $\{X_\alpha\colon \alpha < \mathfrak{t}\}$ be a family of T_2-spaces such that each $H(X_\alpha)$ is totally countably compact. Then $H(F(X_\alpha\colon \alpha < \mathfrak{t}))$ is countably compact.*

Proof. It is known, see, e.g., [28, 3.3(C)], that the product of $\mathfrak{t}$-many totally countably compact spaces is countably compact. Thus by our assumption so is $\prod_{\alpha<\mathfrak{t}} H^0(X_\alpha)$ and hence, by Theorem 3.14, so is $H(F(X_\alpha\colon \alpha < \mathfrak{t}))$ as well. □

We conclude with listing a few open questions.

(1) If $H(X)$ is countably compact, is X u-compact for some $u \in \omega^*$?
(2) Does there exist in ZFC a countably compact and first countable (or just sequentially compact) space whose hyperspace is not countably compact?
(3) What could be the smallest cardinality of a countably compact, first countable space X such that $H(X)$ is not countably compact?
(4) Is there a ZFC example of a countably compact T_4 space whose hyperspace is not countably compact?

Let us point out that by Corollary 3.6 the sequentially compact case of question (2) is equivalent to the well-known Scarborough–Stone problem (see [29]): Is every product of sequentially compact spaces countably compact?

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