General relativistic hypercomputing and foundation of mathematics

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Abstract. Looking at very recent developments in spacetime theory, we can wonder whether these results exhibit features of hypercomputation that traditionally seemed impossible or absurd. Namely, we describe a physical device in relativistic spacetime which can compute a non-Turing computable task, e.g. which can decide the halting problem of Turing machines or decide whether ZF set theory is consistent (more precisely, can decide the theorems of ZF). Starting from this, we will discuss the impact of recent breakthrough results of relativity theory, black hole physics and cosmology to well established foundational issues of computability theory as well as to logic. We find that the unexpected, revolutionary results in the mentioned branches of science force us to reconsider the status of the physical Church Thesis and to consider it as being seriously challenged. We will outline the consequences of all this for the foundation of mathematics (e.g. to Hilbert's programme).
Observational, empirical evidence will be quoted to show that the statements above do not require any assumption of some physical universe outside of our own one: in our specific physical universe there seem to exist regions of spacetime supporting potential non-Turing computations. Additionally, new "engineering" ideas will be outlined for solving the so-called blue-shift problem of GR-computing. Connections with related talks at the Physics and Computation meeting, e.g. those of Jerome Durand-Lose, Mark Hogarth and Martin Ziegler, will be indicated.

1 Introduction

We discuss here the impact of very recent developments in spacetime theory and cosmology on well established foundational issues (and interpretations) of logic and computability theory. The connections between computability theory, logic and spacetime theory (general relativity theory, GR) cut both ways: logic provides a tangible foundation for GR, cf. [1], while GR and its new developments might profoundly influence our interpretation of basic results of computability theory, as we will see in this paper. The new computability paradigms in turn offer feedback to the foundation of mathematics and logic.

Because of the interdisciplinary character of this paper, the first two sections are somewhat introductory, explaining the basic ideas for the nonspecialist. We will start speeding up beginning with section 3.

Two major new paradigms of computing arising from new physics are quantum computing and general relativistic computing. Quantum computing challenges complexity barriers in computability, while general relativistic computing challenges the physical Church-Turing Thesis itself. In this paper we concentrate on relativistic computers and on the physical Church-Turing Thesis (PhCT).

The PhCT is the conjecture that whatever physical computing device (in the broader sense) or physical thought-experiment will be designed by any future civilization, it will always be simulateable by a Turing machine. The PhCT was formulated and generally accepted in the 1930's. At that time a general consensus was reached declaring PhCT valid, and indeed in the succeeding decades the PhCT was an extremely useful and valuable maxim in elaborating the foundations of theoretical computer science, logic, foundation of mathematics and related areas.[1] But since PhCT is partly a physical conjecture, we emphasize that this consensus of the 1930's was based on the physical world-view of the 1930's. Moreover, many thinkers considered PhCT as being based on mathematics + common sense. But "common sense of today" means "physics of 100 years ago". Therefore we claim that the consensus accepting PhCT in the 1930's was based on the world-view deriving from Newtonian mechanics. Einstein's equations became known to a narrow circle of specialists around 1920, but about that time the consequences of these equations were not even guessed at. The world-view of modern black hole physics was very far from being generally known until much later, until after 1980.

Our main point is that in the last few decades there has been a major paradigm shift in our physical world-view. This started in 1970 by Hawking's and Penrose's singularity theorem firmly establishing black hole physics and putting general relativity into a new perspective. After that, discoveries and new results have been accelerating. In the last 10 years astronomers have obtained firmer and firmer evidence for the existence of ever larger, more and more exotic black holes [39],[36] not to mention evidence supporting the assumption that the universe is not finite after all [41]. Nowadays the whole field is in a state of constant revolution. If the background foundation on which PhCT was based has changed so fundamentally, then it is desirable to re-examine the status and scope of applicability of PhCT in view of the change of our general world-picture. A relevant perspective is e.g. in Cooper [9]. Cf. also [19], [15], [31], [23], [37].

Assumption of an absolute time scale is a characteristic feature of the Newtonian world-view. Indeed, this absolute time has its mark on the Turing machine as a model for computer. As a contrast, in general relativity there is no absolute time. Kurt Gödel was particularly interested in the exotic behavior of time in general relativity. Gödel [16] was the first to prove that there are models of GR to which one cannot add a partial order satisfying some natural properties of a "global time". In particular, in GR various observers at various points of

[1] As a contrast, one of the founding fathers of PhCT, László Kalmár, always hoped for a refutation of PhCT and to his students he emphasized that PhCT is meant to be a challenge to future generations, it is aimed at "teasing" researchers to put efforts into attacking PhCT. [21]

spacetime in different states of motion might experience time radically differently. Therefore we might be able to speed up the time of one observer, say C (Cecil, for "computer"), relatively to the other observer, say P (Peter, for "programmer"). Thus P may observe C computing very fast. The difference between general relativity and special relativity is (roughly) that in general relativity this speed-up effect can reach, in some sense, infinity assuming certain conditions are satisfied. Of course, it is not easy to ensure that this speed-up effect happens in such a way that we could utilize it for implementing some non-Turing-computable functions.

In sections 2 and 3 we briefly recall from [31],[30] an intuitive idea of how this infinite speed-up can be achieved and how one can implement a computer based on this idea. More concrete technical details can be found in [15],[31] and to some extent in the remaining parts of this paper. For brevity, we call such thought-experiments *relativistic computers*. We will see that it is consistent with Einstein's equations, i.e. with general relativity, that by certain kinds of relativistic experiments, future generations might find the answers to non-computable questions like the halting problem of Turing machines or the consistency of Zermelo Fraenkel set theory (the foundation of mathematics, abbreviated as ZFC set theory from now on). Moreover, the spacetime structure we assume to exist in these experiments is based in [15],[31] on huge slowly rotating black holes the existence of which is made more and more likely (practically certain) by recent astronomical observations [39],[36].

We are careful to avoid basing the beyond-Turing power of our computer on "side-effects" of the idealizations in our mathematical model of the physical world. For example, we avoid relying on infinitely small objects (e.g. pointlike test particles, or pointlike bodies), infinitely elastic balls, infinitely (or arbitrarily) precise measurements, or anything like these. In other words, we make efforts to avoid taking advantage of the idealizations which were made when GR was set up. Actually, this kind of self-constraint is essential for the present endeavor as can be illustrated by [42, pp.446-447].

In sections 4–6 we discuss some essential questions of principle as well as some technical questions in connection with realizability of a relativistic computer, such as e.g. the so-called blue-shift problem, assuming infinity of time and space. Many of these questions come close to the limits of our present scientific knowledge, provoking new research directions or adding new motivations to already existing ones. We show that, at least, the idea of relativistic computers is not in conflict with presently accepted scientific principles. E.g. we recall that the presently accepted standard cosmological model predicts availability of infinite time and space. We also show that the principles of quantum mechanics are not violated, no continuity of time or space is presupposed by a relativistic computer. Discussing physical realizability and realism of our design for a computer is one of the main issues in [31, §5].

A virtue of the present research direction is that it establishes connections between central questions of computability theory and logic, foundation of mathematics, foundation of physics, relativity theory, cosmology, philosophy, particle physics, observational astronomy, computer science and Artificial Intelligence [45]. E.g. it gives new kinds of motivation to investigating central questions of these fields like "is the universe finite or infinite (both in space and time) and in what sense", "exactly how do huge Kerr black holes evaporate" (quantum gravity), "how much matter is needed for coding one bit of information (is there such a lower bound at all)", questions concerning the statuses of the various cosmic censor hypotheses, questions concerning the geometry of rotating black holes [5], to mention only a few. The interdisciplinary character of this direction was reflected already in the 1987 course given by the present authors [29] during which the idea of relativistic hypercomputers emerged and which was devoted to connections between the above mentioned areas.

Section 6 is also about the impact of general relativistic computing on the foundation of mathematics.

Section 7 is devoted to the impact of the "new computability paradigm" on spacetime theory. There, we discuss a different kind of motivation for studying relativistic computers. Namely, such a study may have applications to theoretical physics as follows. To GR, there is an infinite hierarchy of hypotheses called causality constraints which can be added to GR as outlined in the monograph [11, §6.3, pp.164-167]. Among these occur the various versions of the cosmic censor hypothesis (CCH) of which the basic reference book of relativity theory [43, p.303] writes "whether the cosmic censor conjecture is correct remains the key unresolved issue in the theory of gravitational collapse". On p.305 [43] writes "... there is virtually no evidence for or against the validity of this second version of CCH". These causality hypotheses play a role in GR analogous with the role formulas like GCH independent of ZF set theory play in set theory (or logic). These causality hypotheses are independent of GR (they are not implied by GR), and their status is the subject of intensive study as op. cit. illustrates this. Now, the study of relativistic computers could, in principle, reveal how the physical Church Thesis PhCT is situated in this hierarchy, in a sense which we will discuss in section 7. If we could find out which one of these constraints imply PhCT (or are implied by PhCT), that could be illuminating in why certain issues are difficult to settle about these constraints, cf. e.g. Etesi [14] and [43, p.303].

Tangible data underlying the above interconnections and also more history, references are available in [31]. The textbook Earman [11, p.119, section 4.9] regards the same interdisciplinary perspective as described above to be one of the main virtues of the present research direction. It is the unifying power of logic which makes it viable to do serious work on such a diverse collection of topics. One of the main aims of the research direction represented by [1]–[3], [24]–[26] is to make relativity theory accessible for anyone familiar with logic.

2 Intuitive idea for non-Turing GR computing

In this section we briefly recall from [31, 30] the ideas of how relativistic computers work, without going into technical details. The technical details are elaborated, among others, in [15], [19], [31]. To make our narrative more tangible, we use the example of huge slowly rotating black holes for our construction of relativistic computers. These are called “slow-Kerr” black holes in the physics literature. There are many more kinds of spacetimes suitable for carrying out essentially the same construction for a relativistic computer. We chose rotating black holes because they provide a tangible example for illustrating the kind of reasoning underlying general relativistic approaches to breaking the “Turing barrier”. Mounting astronomical evidence for their existence makes them an even more attractive choice for our didactic purposes. In passing we note that some intuitively easy to read fine-structure investigations of slowly rotating Kerr-Newman black holes are found in the recent [5].

We start out from the so-called Gravitational Time Dilation effect (GTD). The GTD is a theorem of relativity theory, it says that gravity makes time run slow. Clocks that are deep within gravitational fields run slower than ones that are farther out. Roughly, GTD can be interpreted by the following thought-experiment. Choose a high enough tower on the Earth, put precise enough (say, atomic) clocks at the bottom of the tower and the top of the tower, then wait enough time, and compare the readings of the two clocks. The clock on the top will run faster (show more elapsed time) than the one in the basement. So, gravity causes the clock on the top ticking faster. Therefore computers there also compute faster. If the programmer in the basement would like to use this GTD effect to speed up his computer, he can just send his computer to the top of the tower and he gets some speed-up effect. We want to increase this speed-up effect to the infinity. Therefore, instead of the Earth, we use a huge black hole. A black hole is a region of spacetime with so big “gravitational pull” that even light cannot escape from this region. There are several types of black holes, an excellent source is Taylor and Wheeler [40]. For our demonstration of the main ideas here, we will use a huge, slowly rotating black hole. These black holes have two *event horizons*, these are bubble-like surfaces one inside the other, from which even light cannot escape. See Figures 1–2.

As we approach the outer event horizon from far away outside the black hole, the gravitational “pull” of the black hole approaches infinity as we get closer and closer to the event horizon. This is rather different from the Newtonian case, where the gravitational pull also increases but remains finite everywhere. For a while from now on “event horizon” means “outer event horizon”. Imagine observers suspended over the event horizon. Here, suspended means that the distance between the observer and the event horizon does not change. Equivalently, instead of suspended observers, we could speak about observers whose spaceship is hovering over the event horizon, using their rockets for maintaining altitude. Assume one suspended observer C is higher up and another one, P, is suspended lower down. So, C sees P below her while P sees C above him. Now the gravitational time dilation (GTD) will cause the clocks of C run faster than the clocks of P. They both agree on this if they are watching each other e.g. via photons. Let us keep the height of C fixed. Now, if we gently lower P towards the event horizon, this ratio between the speeds of their clocks increases and, as P approaches the event horizon, this ratio approaches infinity. This means that for any integer n, if we want C's clocks to run n times as fast as P's clocks, then this can be achieved by lowering P to the right position. If we could suspend the lower observer P on the event horizon itself then from the point of view of C, P's clocks would freeze, therefore from the point of view of P, C's clocks (and computers!) would run infinitely fast, hence we would have the desired infinite speed-up upon which we could then start our plan for breaking the Turing barrier. The problem with this plan is that it is impossible to suspend an observer on the event horizon. As a consolation for this, we can suspend observers arbitrarily close to the event horizon. To achieve an "infinite speed-up" we could do the following. We could lower and lower again P towards the event horizon such that P's clocks slow down (more and more, beyond limit) in such a way that there is a certain finite time-bound, say b, such that, roughly, throughout the whole history of the universe P's clocks show a time smaller than b. More precisely, by this we mean that whenever C decides to send a photon to P, then P will receive this photon before time b according to P's clocks. This is possible. See Figure 2.

There is a remaining problem to solve. As P gets closer and closer to the event horizon, the gravitational pull or gravitational acceleration tends to infinity. If P falls into the black hole without using rockets to slow his fall, then he does not have to withstand the gravitational pull of the black hole. (He would only feel the so-called tidal forces which can be made negligibly small by choosing a large enough black hole.) However, his falling through the event horizon would be so fast that some photons sent after him by C would not reach him outside the event horizon. Thus P has to approach the event horizon relatively slowly in order that he be able to receive all possible photons sent to him by C. In theory he could use rockets for this purpose, i.e. to slow his fall (assuming he has unlimited access to fuel somehow). Because P approaches the event horizon slowly, he has to withstand this enormous gravity (or equivalently acceleration). The problem is that this increasing gravitational force (or acceleration) will kill P before his clock shows time b, i.e. before the planned task is completed. At the outer event horizon of our black hole we cannot compromise between these two requirements by choosing a well-balanced route for P: no matter how he will choose his route, either P will be crashed by the gravitational pull (acceleration), or some photons sent by C would not reach him. (This is the reason why we can not base our relativistic computer on the simplest kind of black holes, called Schwarzschild ones, which have only one event horizon and that behaves as we described above.) To solve this problem, we would like to achieve slowing down the "fall" of P not by brute force (e.g. rockets), but by an effect coming from the structure of spacetime itself. In our slowly rotating black hole, besides the gravitational pull of the black hole (needed to achieve the time dilation effect) there is a counteractive repelling effect coming from the rotation of the black hole. This repelling effect (or cushioning effect) is analogous to "centrifugal force" in Newtonian mechanics and will cause P to slow down in the required rate. So the idea is that instead of the rockets of P, we would like to use for slowing the fall of P this second effect coming from the rotation of the black hole. The *inner event horizon* marks the point where the repelling force overcomes the gravitational force. Inside the inner horizon, it is possible again to "suspend" an observer, say P, i.e. it becomes possible for P to stay at a constant distance from the center of the black hole (or equivalently from the event horizons). It is shown in [15] that the path of the in-falling observer P can be planned in such a way that the event when P reaches the inner event horizon corresponds to the time-bound b (on the wristwatch of P) mentioned above before which P receives all the possible messages sent out by C. In fact, the path of P can be chosen (to be a geodesic, i.e.) so that P does not have to use rockets at all, all the "slowing down" is done by the spacetime itself.

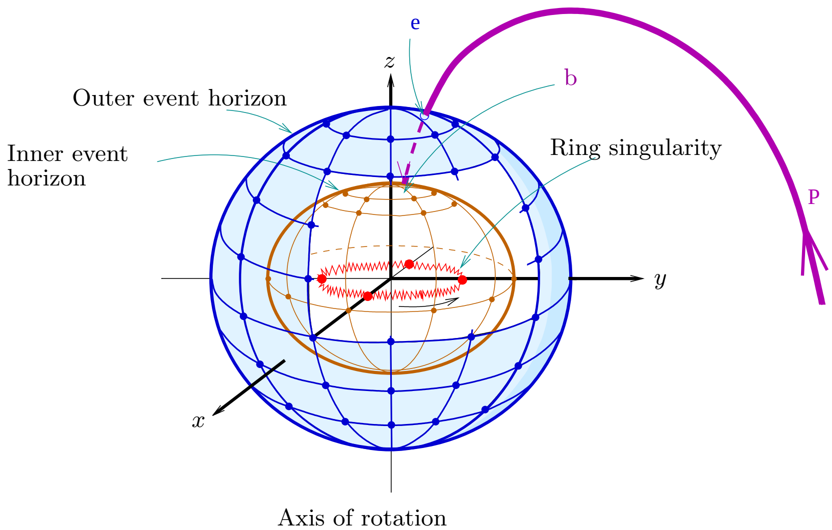

Fig. 1. A slowly rotating (Kerr) black hole has two event horizons and a ring-shape singularity (the latter can be approximated/visualized as a ring of extremely dense and thin "wire"). The ring singularity is inside the inner event horizon in the "equatorial" plane of axes x, y. Time coordinate is suppressed. Figure 2 is a spacetime diagram with x, y suppressed. Rotation of ring is indicated by an arrow. Orbit of in-falling programmer P is indicated, it enters outer event horizon at point e, and meets inner event horizon at point b.

By this we achieved the infinite speed-up we were aiming for. This infinite speed-up is represented in Figure 2 where P measures a finite proper time between its separation from the computer C (this separation point is not represented in the figure) and its touching the inner horizon at proper time b (which point also is not represented in Figure 2). It can be seen in the figure that whenever C decides to send a photon towards P, that photon will reach P before P meets the inner horizon.

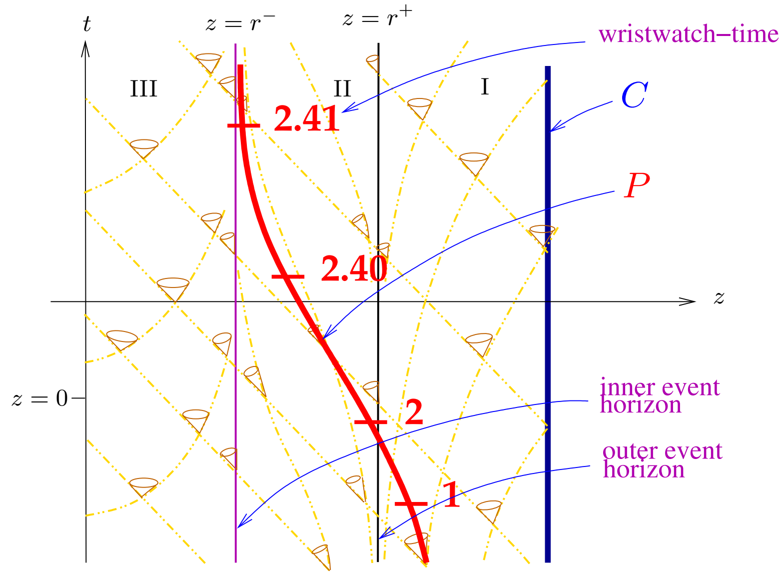

Fig. 2. The "tz-slice" of spacetime of slowly rotating black hole in coordinates where z is the axis of rotation of black hole. The pattern of light cones between the two event horizons r^- and r^+ illustrates that P can decelerate so much in this region that he will receive outside of r^- all messages sent by C. r^+ is the outer event horizon, r^- is the inner event horizon, $z = 0$ is the "center" of the black hole as in Figure 1. The tilting of the light cones indicates that not even light can escape through these horizons. The time measured by P is finite (measured between the beginning of the experiment and the event when P meets the inner event horizon at b) while the time measured by C is infinite.

3 Implementation for a relativistic computer

We now use the above to describe a computer that can compute tasks which are beyond the Turing limit. To break the Turing limit, let us choose the task, for an example, to decide whether ZFC set theory is consistent. I.e. we want to learn whether from the axioms of set theory one can derive the formula FALSE. (This formula FALSE can be taken to be $x \neq x$.) The programmer P and his computer C are together (on Earth), not moving relative to each other, and P uses a finite time-period for transferring input data to the computer C as well as for programming C. After this, P boards a huge spaceship, taking all his mathematical friends with him (like a Noah's Ark), and chooses an appropriate route towards a huge slowly rotating black hole, entering the inner event horizon when his wrist-watch shows time b. While he is on his journey towards the black hole, the computer that remained on the Earth checks one by one the theorems of set theory, and as soon as the computer finds a contradiction in set theory, i.e. a proof of the formula FALSE from the axioms of set theory, the computer sends a signal to the programmer indicating that set theory is inconsistent. If it does not find a proof for FALSE, the computer sends no signal.

The programmer falls into the inner event horizon of the black hole and after he has crossed the inner event horizon, he can evaluate the situation. If a light signal has arrived from the direction of the computer, of an agreed color and agreed pattern, this means that the computer found an inconsistency in ZFC set theory, therefore the programmer will know that set theory is inconsistent. If the light signal has not arrived, and the programmer is already inside the inner event horizon, then he will know that the computer did not find an inconsistency in set theory, did not send the signal, therefore the programmer can conclude that set theory is consistent. So he can build the rest of his mathematics on the secure knowledge of the consistency of set theory. We will return to the issue of whether the programmer has enough space and time and resources for using the just gained information at the end of section 5.

The above outlined train of thought can be used to show that any recursively enumerable set can be decided by a relativistic computer [15]. Actually, more than that can be done by relativistic computers. Welch [44] shows that the arrangement described in section 3 using Kerr black holes can solve exactly the Δ_2-hard problems in the technical sense of the arithmetical hierarchy (degrees of unsolvability) used in logic and computability theory (under some mild extra assumptions). Computability limits connected with such relativistic computers are also addressed in [19], [20], [37], [45].

Relativistic computers are not tied to rotating black holes, there are other general relativistic phenomena on which they can be based. An example is anti-de-Sitter spacetime which attracts more and more attention in explaining recent discoveries in cosmology (the present acceleration of the expansion of the universe, cf. [31]). Roughly, in anti-de-Sitter spacetime, time ticks faster and faster at farther away places in such a way that P can achieve infinite speed-up by sending away the computer C and waiting for a signal from her. This scenario is described and is utilized for computing non-Turing computable functions in [19]. This example shows that using black holes (or even singularities) is not inherent in relativistic computers.

Spacetimes suitable for an implementation of relativistic computation like described in this section are called Malament-Hogarth spacetimes in the physics literature. A relativistic spacetime is called Malament-Hogarth (MH) if there is an event (called MH-event) in it which contains in its causal past a worldline of infinite proper length. The spacetime of ordinary Schwarzschild black hole is not MH, the spacetime of rotating Kerr black hole is MH and any event within the inner event horizon is MH, in anti-de-Sitter spacetime every event is an MH-event, the spacetime of an electrically charged BH (called Reissner-Nordström spacetime) is MH and there are many other examples for MH.

We note that using MH spacetimes does not entail faith in some exotically "benevolent" global property of the whole of our universe. Instead, most of the MH spacetimes, like rotating BH's, can be built by a future, advanced civilization inside our usual "standard" universe of high precision cosmology. Namely, such MH spacetimes do not necessarily refer to the whole universe, but instead, to some "local" structure like a rotating ring of gravitationally collapsed matter in a "spatially finite part" of a more or less usual universe involving no particular global "witchcraft", so-to-speak. We are writing this because the word "spacetime" in the expression "MH spacetime" might be misleading in that it might suggest to the reader that it is an exotic unlikely property of the whole of God's creation, namely, the whole universe. However, in most MH spacetimes this is not the case, they are (in some sense) finite structures that can be built, in theory, by suitably advanced civilizations in a standard kind of universe like the one which is predicted by the present-day standard version of cosmology. Indeed, one of them, namely large rotating black holes, have been actually observed by astronomers many times and with great confidence. A typical example is the rotating 100 million solar mass black hole at the center of galaxy WCG-6-30-15. In other words, nothing fancy is required from the whole universe, the "fancy part" is a structure which can, in theory, be manufactured in an ordinary infinite universe. Therefore in the present context it would be more fortunate to talk about MH regions of spacetime than about MH spacetimes.

4 Two sides of the coin and the blue-shift problem

A relativistic computer as we described in section 3 is a team consisting of a Computer (C, for Cecil) and a Programmer (P, for Peter).

How does the computer C experience the task of this computing? C will see (via photons) that the programmer P approaches the black hole (BH), and as he approaches it, his wristwatch ticks slower and slower, never reaching wristwatch time b. C will see the Programmer approaching the BH in all her infinite time. For C, the Programmer shines on the sky for eternity. The only effect of C's time passing is that this image gets dimmer and dimmer, but it will never disappear. Under this sky, C computes away her task consisting of potentially infinitely many steps, i.e. checking the theorems of ZFC one by one, in an infinite amount of time.

How does the Programmer experience the task of this computing? He is traveling towards the black hole, and he only has to check whether he received a special signal from the Computer or not. For this task, which consists of finitely many steps, he has a finite amount of time.

What would he see would he watch his team-member, the Computer? He would see the Computer computing faster and faster, speeding up so that when his (P's) wristwatch time reaches b, C would just flare up and disappear. Well, this flare-up would burn P, because it carries the energy of photons emitted during the whole infinite life of C, thus the total amount of this energy is infinite. In fact, we have to design a shield (or mirror) so that only intended signals from C can reach P. This means that we have to ensure that P does not see C! P's task is to watch whether there is one special kind of signal coming through this shield. All in all, P's task is to do finitely many steps in a finite amount of time.

A task in the literature is called supertask if it involves one to carry out infinitely many steps in a finite amount of time [12]. Therefore, by the above, we think that the relativistic computer need not implement a supertask.

The above led us to the so-called blue-shift problem [11]. This is the following. The frequency of light-signals (photons) sent by C to P gets increased (i.e. blue-shifted) by the time they reach P because of the infinite speed-up we worked so hard to achieve! Thus, if we do nothing about this, the one signal that C sends, can kill P. Further, P may not be able to recognize the blue-shifted signal. There are many solutions for this problem, two such solutions can be found in sections 5.3.1 and 5.4.1 of [31]. For example, C can arrange sending the signal to P such that C asks her sister C' to embark on a spaceship S which speeds up in the direction opposite to the direction of the Kerr hole, and send the signal from this spaceship. If S moves fast enough, then any signal sent from S to P will be red-shifted because of the speed of S. Then C chooses the speed of S to be such that the red-shift caused by this speed exactly cancels out the blue-shift caused by the gravitational effects at the event when P receives the signal.

Some new ideas on the blue-shift problem using mirrors. Programmer P sends out a second spaceship P' inhabited by robot P' running ahead of P in the same direction, i.e. towards the inner horizon of the black hole. So, P' travels faster than P and P' is between P and the inner horizon (in the relevant time period, of course). When computer C sends out the message (that e.g. an inconsistency was found in ZFC), the message is "beamed" to P' and not to P. Careful engineering is needed to ensure that the photons coming from C (or from anywhere the outside area) avoid P. See the Penrose diagram in Figure 3.

Bold solution: P' uses a huge mirror, by which P' reflects the message "back" to P. Since P' is moving extremely fast, we can say that P' is "running away" from the incoming photons. Therefore by the Doppler Effect, the frequency of the reflected photons (message) will be arbitrarily smaller (depending on the relative velocity[2] of P' and P) and so the energy of the message received by P from P' will be suitably small to ensure (i) recognizability and (ii) not burning P to death. For this, the pilots of P and P' should adjust their relative velocities appropriately. This seems to be possible as indicated in Figure 3. For the case something would go wrong with the above "bold plan", we include below a cautious plan.

Cautious plan: Instead of carrying a huge mirror, P' carries a large banner that can be spared. Now, the light signal (message) from C is directed to the banner carried by P'. So by the blue-shift effect, the message burns (destroys) the banner of P'. Even simpler solution is obtained by burning the whole of P'. Now, P is watching P' for the message. If P' disappears, then P concludes that P' was burned by the message, hence there was a message, hence P concludes that ZFC is inconsistent. If ZFC *is* consistent then C does not send a message to P', hence P' does not get burned, hence P "sees" that P' is still there, hence after both P and P' crossed the inner event horizon, P concludes that ZFC is consistent. Although this will happen only some time after P crosses the inner

[2] measurable e.g. by radar

horizon, but that is consistent with our plan. The above plan of burning P' by the message might contain possibilities for error, e.g. P' might disappear because of some completely irrelevant accident (without C's sending a message). While this is so, the likelihood of such an accident can be minimized by the usual techniques of careful engineering, e.g. using redundancy (several copies of P' moving in different directions all of which are targeted by C if C finds the inconsistency etc).

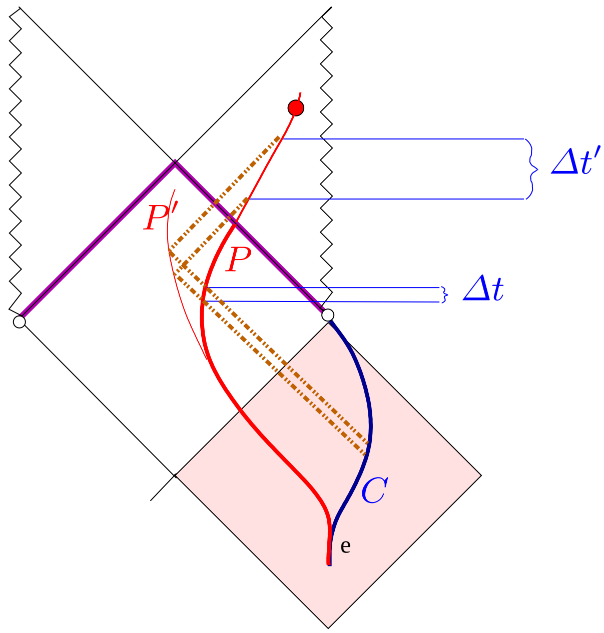

Fig. 3. Attacking the blue-shift problem by "mirrors" and by an "escaping" second spaceship P'. The "decoded" message is received by P only after he crossed the inner event horizon but not later than P''s crossing the inner event horizon is observed by P.

5 Some questions naturally arising

The following questions come up in connection with realizability of the plan described in section 3.

- can the programmer check whether the distant object he chose for a slowly rotating black hole is indeed one (whether it has the spacetime structure needed for his purposes)?
- can he check when he passed the event horizon?

– can he survive passing the event horizon?
– can he receive and recognize the signal sent by the computer?
– how long can he live inside the black hole?
– is there a way for the programmer to know that absence of signal from the computer is not caused by some catastrophe in the life of the computer?
– is it possible for a civilization to exist for an infinite amount of time?
– can the programmer repeat the computation or is this a once-for-a-lifetime computation for him?

Here we just assert that the answers to all these questions are in the affirmative, or at least do not contradict present scientific knowledge. These questions are discussed in detail in [31]. Below we address three of these questions.

On the question of traverseability of the event horizon: We chose the black hole to be large. If the black hole is huge[3], the programmer will feel nothing when he passes either event horizon of the black hole—one can check that in case of a huge black hole the so-called tidal forces on the event horizons of the black hole are negligibly small [33], [15].

On the question of how long the programmer can live after crossing the event horizon: The question is whether the programmer can use this new information, namely that set theory is consistent, or whatever he wanted to compute, for his purposes. A pessimist could say that OK they are inside a black hole, so—now we are using common sense, we are not using relativity theory—common sense says that the black hole is a small unfriendly area and the programmer will sooner or later fall into the middle of the black hole where there is a singularity and the singularity will kill the programmer and his friends. The reason why we chose our black hole to be a huge slowly rotating one, say of mass $10^{10} m_{\odot}$, is the following. If the programmer falls into a black hole which is as big as this and it rotates slowly, then the programmer will have quite a lot of time inside the black hole because the center of the black hole is relatively far from the event horizon. But this is not the key point. If it rotates, the "matter content", the so-called singularity, which is the source of the gravitational field of the black hole so-to-speak, is not a point but a ring (see Fig.1). So if the programmer chooses his route in falling into the black hole in a clever way, say, relatively close to the north pole instead of the equatorial plane, then the programmer can comfortably pass through the middle of the ring, never get close to the singularity and happily live on forever (see Fig.s 1,2). We mean, the rules of relativity will not prevent him from happily living forever. He may have descendants, he can found society, he can use and pass on the so obtained mathematical knowledge.

On the question of whether the computation can be repeated: Let us look at the extension of slow Kerr spacetime in [32, §3.3, pp.116-140]. Especially, consider the maximal slow Kerr spacetime (MSK) on Fig. 3.16, p.139. By considering this MSK, we can convince ourselves that our GR-computation is repeatable and with appropriate care it can be made deterministic. Some further meditation on this repeatability of GR-computing can lead to new perspectives

[3] this is a technical expression in observational astronomy

on the Platonism - formalism debates and views in the philosophical schools of the foundations of mathematics. E.g. one of the ages old arguments fades away, namely the argument saying that we cannot have access to any instances of actual infinity.

6 Can we learn something about infinity? Impact on the foundation of mathematics.

The relativistic computer as we implemented it in section 3 assumes that an infinite amount of time is available for C for computing. This seems to be essential for breaking the Turing barrier (by our construction). We are in a good position here, because of the following. As a result of very recent revolution in cosmology, there is a so-called standard model of cosmology. This standard model is based on matching members of a family of GR spacetimes against a huge number of observational data obtained by three different astronomical projects. This huge number of measurements (made and processed by using computers) all point amazingly to one specific GR spacetime. This spacetime is called the standard cosmological model, and in accordance with the so far highly successful scientific practice of the last 2500 years, we regard this standard model of the latest form of high-precision cosmology as the model best suited to explain observations and experience collected so far. According to this standard model, our universe is infinite both in regard of time and space, moreover there is an infinite amount of matter-energy available in it. We will see soon that the latter infinity is not needed for our construction. For more on this see Dávid [10], [31] and the references therein. Our point here is not about believing that our universe indeed has infinite time or not. The point is that assuming availability of an infinite amount of time for computing is not in contradiction with our present-day scientific knowledge.

We would like to say some words on the question of how much matter/energy is needed for storing, say, 10 bits of information. Although this question is not essential for the realizability of the relativistic computer (because of availability of infinite energy in the standard model of cosmology), we still find this question interesting for purely intellectual/philosophical reasons.

Is information content strongly tied to matter/energy content? Is there a lower bound to mass which is needed to store 10 bits of information? This is a question which has nagged one of the authors ever since he wrote his MsC thesis [28] where a separate section was devoted to this issue. The question is: "If I want to write more, do I need more paper to write on?" Right now it seems to us that the answer is in the negative. Matter and information might be two independent (orthogonal) "dimensions" of reality. The reason for this is the following. One might decide to code data by photons. Then the amount of matter/energy used is the energy total of these photons. But the energy of a photon is inversely proportional with its wavelength. So, one might double the wavelength of all photons and then one halved the energy needed to carry the same information one coded originally. If this is still too much energy expense, then one can double the wavelength again. Since there is no upper bound to the wavelengths of photons, there is no lower bound for the energy needed for storing 10 bits of data.

So, it seems to us that energy and information are not as strongly linked entities as energy and mass are (via $E = mc^2$). In the above argument when we said that there was no upper bound to the wavelength of possible photons, we used that according to the standard cosmological model the Universe is infinite in space. We note that Einstein when inventing photons did not say that there is a smallest nonzero value for energy. He said this only for light of a fixed color, i.e. fixed wavelength.

We would like to emphasize that we did not use that space is continuous. We seem to have used that time is continuous, but we can avoid that assumption by refining the implementation of relativistic computer. Constructions for this are in [31]. Thus, no contradictions with the principles of quantum mechanics seems to be involved in the idea of relativistic computer.

In the above, we argued that in principle, one can even build a relativistic computer sometime in the future. However, a fascinating aspect of relativistic computers for us is that they bring up mind-boggling questions about the nature of infinity. These questions would be worth thinking over even if our present-day science would predict a finite universe. We seem to understand and be familiar with the use of potential infinity in science. However, the above thought-experiment seems to use the notion of actual infinity. Is infinity a mental construction only or does it exist in a more tangible way, too? Can we learn something about actual infinity by making physical experiments? This leads to questions inherent in foundational issues in mathematics and physics. For more about this and about connection with Hilbert's Programme for mathematics we refer to [4].

7 Relativistic Computers and Causality Hypotheses in Physics

Let us consider the hierarchy of causality hypotheses $C0, \ldots, C6$ summarized in the monograph Earman [11, §6.3, pp.164-166]. None of these follow from GR (cf. e.g. [43, p.303]), they function as extra possible hypotheses for narrowing the scope of the theory. The strongest of these is the strong cosmic censor hypothesis $C6$ saying that spacetime is globally hyperbolic. A spacetime is called *globally hyperbolic* if it contains a spacelike hypersurface that intersects every causal curve without endpoint exactly once. This implies that the "temporal" structure of spacetime is basically the same as that of the Newtonian world in that it admits a "global time" associating a real number $t(p)$ to every point p of spacetime. In other words, $C6$ implies that spacetime admits a "global foliation", i.e. it is a disjoint union of global time-slices or "global now"-s. This is a quite extreme assumption and its role is more of a logical status, i.e. one investigates questions of what follows if $C6$ is assumed rather than assuming that it holds for the actual universe. Recall that Wald [43, p.305 lines 7-8] wrote about the cosmic censor hypothesis that "there is virtually no evidence for or against the validity of this", as we quoted around the end of section 1. Cf. also [11, pp.97,99] for further doubts on $C6$.

In section 1 we said that the assumption of absolute time has its marks on the Turing Machine as a model for computer, and now we see that $C6$ provides a kind of global time. Indeed, the construction of a GR-computer in this paper relies heavily on failure of $C6$, because Hogarth proved that no MH-spacetime is globally hyperbolic [11, Lemma 4.1, p.107]. This motivates the following question.

Question 1. Does (a carefully formulated variant of) PhCT follow from GR+$C6$?

On this question: PhCT has not been formalized precisely yet, this is part of why this question is asking for a formulation of PhCT which would follow from GR+$C6$. In this question we are asking if there are some natural and convincing extra conditions on physically realistic computability which would yield PhCT from GR+$C6$. The need for such extra realisticity assumptions is demonstrated by e.g. Tipler [42, pp.446-447]. Actually, we started collecting such conditions for physical realisticity in the middle of sec. 1 (in the paragraph beginning with "We are careful to avoid ..."). One might conjecture that, under suitable assumptions on physical realisticity and with a suitable formulation of PhCT, a kind of positive answer to Question 1 might be plausible[4].

Question 1 is about the connection between PhCT and the cosmic censor hypothesis $C6$. Next we concern ourselves with the connection between PhCT and the Malament-Hogarth property of a spacetime. We note that $C6$ implies that the spacetime is not Malament-Hogarth (NoMH for short), [11, Lemma 4.1, p.107] but NoMH does not imply $C6$ [11, p.110, first 2 sentences of sec.4.5]. Hence NoMH is a strictly weaker causality hypothesis than $C6$. We note that [14] explores the connection between $C6$ and MH. Again, we have no reason for believing in NoMH.

Question 2. Under what natural (extra) conditions is NoMH equivalent with what version of PhCT?

On this question: In theory, the MH property implies failure of PhCT (i.e. PhCT $\Rightarrow$ NoMH), because in any MH-spacetime one can, at least in theory, construct a GR-computer like in this paper, cf. [11, §4], [19]. However, there is a reason why in the works [15], [31], [30] we chose to implement our relativistic computer via a huge rotating black hole. Namely, huge-ness of the rotating BH was used to ensure that the tidal forces at the event horizons do not kill the programmer. It is possible to construct a toy-example of a MH spacetime

[4] Very tentatively: Recently, emerging new kinds of computing devices like the Internet seem to pose a challenge against the conjecture in Question 1, cf. [45]. However, even the Internet (or even the Human Mind) will probably not prove ZFC consistent (assuming C6). So, perhaps we should separate PhCT into two theses, one about "hard" problems like proving the consistency of ZFC, and the other about problems in general which are not Turing computable.

in which our kind of relativistic computer is not realistic physically. By physical realisticity we mean requirements that we do not use infinitely small computers (objects), infinitely precise measurements, or the like in designing our non-Turing computer, cf. [31] for more detail. We note that if we do not insist on physical realisticity, then already in Newtonian Mechanics PhCT would fail as demonstrated e.g. in Tipler [42, pp.446-447]. This motivates the question of what natural assumptions would ensure PhCT $\Rightarrow$ NoMH or equivalently MH $\Rightarrow$ NotPhCT in a physically realistic way. This is one direction of Question 2 above.

The other direction of Question 2 seems to be the harder one: Under what natural conditions (if any) does NotPhCT imply MH. I.e. under what conditions is

$$(\star) \qquad \text{NotPhCT} \Rightarrow \text{MH}$$

true? One way of rephrasing ($\star$) is to conjecture that if there is a physically realistic non-Turing computer then there must be one which is built up in the style of the present paper utilizing MH property of spacetime. (By non-Turing computer we mean a physical computer that can compute beyond the Turing barrier.) This seems to be a daring conjecture. But let us remember that the question was: under what conditions is statement ($\star$) true. In particular, if the physical non-Turing computer "designed" in the book Pour-El and Richards [35] turns out to be physically realizable, then our conjecture (that under some reasonable conditions ($\star$) might become true) might get refuted.

We note that the tentative conjecture implicit in Question 2 was arrived at jointly with Gábor Etesi.

8 Logic based conceptual analysis of GR and "reverse mathematics" for GR

So far we have been applying general relativity (GR) to logic and to the foundation of mathematics (FOM). In the other direction, logic and FOM are being applied to a conceptual analysis and logical/mathematical foundation of relativity (including GR). In more detail: FOM has an important branch called reverse mathematics. In the latter we ask the question(s) of which axioms of set theory are responsible (needed) for which important theorems of mathematics. In a series of papers, e-books, and book chapters, a team containing the present authors has been working on a programme which could be called "exporting the success story of FOM to a foundation of relativity" (set theorist Harvey Friedman coined this slogan), cf. e.g. [1]–[3], [24]–[26]. Roughly speaking, this group builds up relativity in first-order logic (like FOM is built up as a theory in the sense of mathematical logic), then analyzes the so obtained logical theory from various perspectives and (like in reverse mathematics) attempts to answer the so-called why-type questions asking why a certain prediction of general relativity is being predicted, i.e. which axioms of the theory are responsible for that particular prediction.

Section 7 above can be regarded as a small sample from the above quoted logical/conceptual analysis of GR and its extensions. A particular example of this feedback from logic and FOM to GR is the paper [5].

By the above sketched feedback from logic and FOM to GR, the "circle" GR $\mapsto$ Logic $\mapsto$ FOM $\mapsto$ GR promised in the introduction is completed.

9 History of relativistic computation

The idea of general relativistic computing as described in section 2 was found at different parts of the globe, independently. It was discovered by Németi in 1987 [29], Pitowsky in 1990 [34], Malament in 1989 [27], and Hogarth in 1992 [18] independently. Németi's idea used large slowly rotating black holes (slow Kerr spacetimes) but the careful study of feasibility and transversability of these was done later in Etesi-Németi [15]. All this led to a fruitful cooperation between the parties mentioned above, e.g. between Cambridge (Hogarth et al), Budapest (Németi et al), Pittsburgh (Earman et al). The first thorough and systematic study of relativistic computation was probably Hogarth [19]. Related work on relativistic computing include [44], [37], [11, §4], [12], [13], [17], [30].

Acknowledgements

Special thanks are due to Gyula Dávid for intensive cooperation and ideas on this subject. We are grateful for enjoyable discussions to Attila Andai, John Earman, Gábor Etesi, Stefan Gruner, Balázs Gyenis, Mark Hogarth, Judit Madarász, Victor Pambuccian, Gábor Sági, Endre Szabó, László E. Szabó, Renáta Tordai, Sándor Vályi, Jiří Wiedermann, Chris Wüthrich. We gratefully acknowledge support by the Hungarian National Foundation for scientific research grant No. T73601.

References

1. Andréka, H., Madarász, J. X. and Németi, I., Logic of Spacetime and Relativity Theory. In: Handbook of Spatial Logics, eds: M. Aiello, J. van Benthem, and I. Pratt-Hartman, Springer-Verlag, 2007. pp.607-711.
2. Andréka, H., Madarász, J. X. and Németi, I. with contributions from Andai, A., Sain, I., Sági, G., Tőke, Cs. and Vályi, S., On the logical structure of relativity theories. Internet book, Budapest, 2000. http://www.math-inst.hu/pub/algebraic-logic/olsort.html
3. Andréka, H., Madarász, J. X., Németi, I. and Székely, G., Axiomatizing relativistic dynamics without conservation postulates. Studia Logica 89,2 (2008), 163-186.
4. Andréka, H. and Németi, I., Relativistic computers and Hilbert's Program for mathematics. Manuscript.
5. Andréka, H., Németi, I. and Wüthrich, A twist in the geometry of rotating black holes: seeking the cause of acausality. General Relativity and Gravitation 40,9 (2008), 65-89.

6. Barrow, J. D., How to do an infinite number of things before breakfast. New Scientist, 29 January 2005.
7. Barrow, J. D. and Tipler, F. J., The anthropic cosmological principle. Oxford University Press, 1986.
8. Chown, M., Smash and grab. New Scientist, 2002. April 6, pp.24-28.
9. Cooper, S. B., How can Nature Help Us Compute? In SOFSEM 2006: Theory and Practice of Computer Science - 32nd Conference on Current Trends in Theory and Practice of Computer Science, Merin, Czech Republic, January 2006 (Jiri Wiedermann, Julius Stuller, Gerard Tel, Jaroslav Pokorny, Maria Bielikova, editors), Springer Lecture Notes in Computer Science No. 3831, 2006, pp. 1-13.
10. Dávid, Gy., Modern cosmology - astronomical, phsical and logical approaches. Abstracts for the conference "Logic in Hungary 2005", http://atlas-conferences.com/cgi-bin/abstract/caqb-64
11. Earman, J., Bangs, crunches, whimpers, and shrieks. Singularities and acausalities in relativistic spacetimes. Oxford university Press, Oxford, 1995.
12. Earman, J. and Norton, J., Forever is a day: supertasks in Pitowsky and Malament-Hogarth spacetimes. Philosophy of Science 60 (1993), 22-42.
13. Earman, J. and Norton, J., Infinite pains: the trouble with supertasks. In: S. Stich (ed.), Paul Benacerraf: the Philosopher and his critics. Blackwell, New York, 1994.
14. Etesi, G., Note on a reformulation of the strong cosmic censor conjecture based on computability. Physics Letters B 550 (2002), 1-7. Revised version (2008).
15. Etesi, G. and Németi, I., Turing computability and Malament-Hogarth spacetimes. International Journal of Theoretical Physics 41,2 (2002), 342-370.
16. Gödel, K., Lecture on rotating universes. In: Kurt Gödel Collected Works, Vol. III. Eds.: Feferman, S., Dawson, J. S., Goldfarb, W., Parson, C. and Solovay, R. N., Oxford University Press, New York Oxford 1995. pp. 261-289.
17. Gyenis, B. and Roberts, B., Supertasks: Gödel strikes back. Preprint, Department of History and Philosophy of Science, University of Pittsburgh, April 2007.
18. Hogarth, M. L., Non-Turing computers and non-Turing computability. In: D. Hull, M. Forbes, and R. M. Burian (eds), PSA 1994, Vol. 1. East Lansing: Philosophy of Science Association. pp. 126-138.
19. Hogarth, M. L., Predictability, computability, and spacetime. PhD Dissertation, University of Cambridge, UK, 2000. http://ftp.math-inst.hu/pub/algebraic-logic/Hogarththesis.ps.gz
20. Hogarth. M. L., Deciding arithmetic using SAD computers. Brit. J. Phil. Sci. 55 (2004), 681-691.
21. Kalmár, L., An Argument Against the Plausibility of Church's Thesis. In Heyting, A. (ed.) Constructivity in Mathematics. North-Holland, Amsterdam, 1959.
22. van Leeuwen, J. and Wiedermann, J., The Turing Machine paradigm in comtemporary computing. In B. Enquist & W. Schmid (Eds.), Mathematics Unlimited - 2001 and Beyond. Springer-Verlag, Berlin, 2001. pp. 1139-1155.
23. Loff, B. and Costa, J. F., Five views of hypercomputation. International Journal of Unconventional Computing, Special Issue on Hypercomputation 4,4 (2008), in print.
24. Madarász, J. X., Németi, I. and Székely, G., Twin Paradox and the logical foundation of space-time. Foundation of Physics 36,5 (2006), 681-714. arXiv:gr-qc/0504118.
25. Madarász, J. X., Németi, I. and Székely, G., First-order logic foundation of relativity theories. In: Mathematical problems from applied logic II, International Mathematical Series Vol 5, Springer-Verlag, 2007. pp.217-252.

26. Madarász, J. X. and Székely, G., The effects of gravitation on clocks, proved in axiomatic relativity. Abstracts for the conference "Logic in Hungary 2005", http://atlas-conferences.com/cgi.bin/abstract.caqb-41.
27. Malament, D., Private communication to Earman, J., 1988. Cf. Earman's book in the present list of references, p.238.
28. Németi, I., Computing the disturbance of phone lines caused by high-voltage electrical power lines. Master Thesis, Dept. Electrodynamics, Technical University Budapest, 1965.
29. Németi, I., On logic, relativity, and the limitations of human knowledge. Iowa State University, Department of Mathematics, Ph. D. course during the academic year 1987/88.
30. Németi, I. and Andréka, H., Can general relativistic computers break the Turing barrier? In: Arnold Beckmann, Ulrich Berger, Benedikt Löwe, and John V. Tucker (eds.): Logical Approaches to Computational Barriers, Second Conference on Computability in Europe, CiE 2006, Swansea, UK, July 2006, Proceedings, Lecture Notes in Computer Science 3988, Springer-Verlag Berlin Heidelberg 2006. pp.398-412.
31. Németi, I. and Dávid, Gy., Relativistic computers and the Turing barrier. Journal of Applied Mathematics and Computation 178 (2006), 118-142.
32. O'Neill, B., The geometry of Kerr black holes. A K Peters, Wellesley, 1995.
33. Ori, A., On the traversability of the Cauchy horizon: Herman and Hiscock's argument revisited, in: Internal Structures of Black Holes and Spacetime Singularitites (ed. A. Ori, L.M. Ori), Ann. Isra. Phys. Soc. 13, IOP (1997).
34. Pitowsky, I., The physical Church thesis and physical computational complexity. Iyyun A Jerusalem Philosophical Quarterly 39 (1990), 81-99.
35. Pour-El, M. B. and Richards, J. I., Computability in analysis and physics, Springer-Verlag, Berlin (1989).
36. Reynolds, C. C., Brenneman, L. W. and Garofalo, D., Black hole spin in AGN and GBHCs. Oct. 5, 2004. arXiv:astro-ph/0410116. (Evidence for rotating black holes.)
37. Sági, G. and Etesi, G., Transfinite computations and diameters of spacetimes. Preprint, Hungarian Academy of Sciences, Budapest.
38. Shagrir, O. and Pitowsky, I., Physical hypercomputation and the Church-Turing Thesis. Minds and machines 13 (2003), 87-101.
39. Strohmayer, T. E., Discovery of a 450 HZ quasi-periodic oscillation from the microquasar GRO J1655-40 with the Rossi X-Ray timing explorer. The Astrophysical Journal, 553,1 (2001), pp.L49-L53. (Evidence for rotating black holes.) arXiv:astro-ph/0104487.
40. Taylor, E. F. and Wheeler, J. A., Black Holes. Addison, Wesley, Longman, San Francisco, 2000.
41. Tegmark, M., Parallel Universes. Scientific American May 2003, pp.41-51.
42. Tipler, F. J., The physics of immortality. Anchor Books, New York, 1994.
43. Wald, R. M., General relativity. University of Chicago Press, Chicago 1984.
44. Welch, P. D., Turing unbound: on the extent of computation in Malament-Hogarth spacetimes. The Britisch Journal for the Philosophy of Science, to appear. School of Mathematics, University of Bristol, September 2006. arXiv:gr-qc/0609035v1
45. Wiedermann, J. and van Leeuwen, J., Relativistic computers and non-uniform complexity theory. In: Calude et al (eds.) UMC 2002. Lecture Notes in Computer Science Vol. 2509, Springer-Verlag, Berlin, 2002. pp.287-299.
46. Wischik, L., A formalization of non-finite computation. Dissertation for the degree of Master of Philosophy, University of Cambridge, 1997. www.wischik.com/lu/philosophy/non-finite-computation.html