TWIN PARADOX AND THE LOGIC FOUNDATION OF SPACE-TIME

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Abstract. We study the foundation of space-time theory in the framework of first-order logic (FOL). Since the foundation of mathematics has been successfully carried through (via set theory) in FOL, it is not entirely impossible to do the same for space-time theory (or relativity). First we recall a simple and streamlined FOL-axiomatization Specrel of special relativity from the literature. Specrel is complete with respect to questions about inertial motion. Then we ask ourselves whether we can prove usual relativistic properties of accelerated motion (e.g., clocks in acceleration) in Specrel. As it turns out, this is practically equivalent to asking whether Specrel is strong enough to "handle" (or treat) accelerated observers. We show that there is a mathematical principle called induction (IND) coming from real analysis which needs to be added to Specrel in order to handle situations involving relativistic acceleration. We present an extended version AccRel of Specrel which is strong enough to handle accelerated motion, in particular, accelerated observers. Among others, we show that the Twin Paradox becomes provable in AccRel, but it is not provable without IND.

1. Introduction

The idea of elaborating the foundation of space-time (or foundation of relativity) in a spirit analogous with the rather successful foundation of mathematics (FOM) was initiated by several authors including, e.g., David Hilbert or leading contemporary logician Harvey Friedman [9, 10]. Foundation of mathematics has been carried through strictly within the framework of first-order logic (FOL), for certain reasons. The same reasons motivate the effort of keeping the foundation of space-time also inside FOL. One of the reasons is that staying inside FOL helps us to avoid tacit assumptions, another reason is that FOL has a complete inference system while higher order logic cannot have one by Gödel's incompleteness theorem. For more motivation for staying inside FOL (as opposed to higher-order logic), cf. e.g., Ax [3], Pambuccian [15], [2, Appendix 1: "Why exactly FOL"], [1], but the reasons in Väänänen [21], Ferreirós [8], or Woleński [23] also apply.

Following the above motivation, we begin at the beginning, namely first we recall a streamlined FOL axiomatization Specrel of special relativity theory, from the literature. Specrel is complete with respect to (w.r.t.) questions about inertial motion. Then we ask ourselves whether we can prove usual relativistic properties of accelerated motion (e.g., clocks in acceleration) in Specrel. As it turns out, this is practically equivalent to asking whether Specrel is strong enough to "handle" (or treat) accelerated observers. We show that there is a mathematical principle called induction (IND) coming from real analysis

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which needs to be added to Specrel in order to handle situations involving relativistic acceleration. We present an extended version AccRel of Specrel which is strong enough to handle accelerated clocks, in particular, accelerated observers.

We show that the so-called Twin Paradox becomes provable in AccRel. It also becomes possible to introduce Einstein's equivalence principle for treating gravity as acceleration and proving the Tower Paradox, i.e. that gravity "causes time to run slow".

What we are doing here is not unrelated to Field's "Science without numbers" programme and to "reverse mathematics" in the sense of Harvey Friedman and Steven Simpson. Namely, we systematically ask ourselves which mathematical principles or assumptions (like, e.g., IND) are really needed for proving what observational predictions of relativity. (It was this striving for economy in axioms or assumptions which we alluded to when we mentioned, way above, that Specrel was "streamlined".)

The interplay between logic and relativity theory goes back to around 1920 and has been playing a non-negligible role in works of researchers like Reichenbach, Carnap, Suppes, Ax, Szekeres, Malament, Walker, and of many other contemporaries.[1]

In Section 2 we recall the FOL axiomatization Specrel complete w.r.t. questions concerning inertial motion. There we also introduce an extension AccRel of Specrel (still inside FOL) capable for handling accelerated clocks and also accelerated observers. In Section 3 we formalize the Twin Paradox in the language of FOL. We formulate Theorems 3.1, 3.2 stating that the Twin Paradox is provable from AccRel and the same for related questions for accelerated clocks. Theorems 3.5, 3.6 state that Specrel is not sufficient for this, more concretely that the induction axiom IND in AccRel is needed. In Sections 4, 5 we prove these theorems.

Motivation for the research direction reported here is nicely summarized in Ax [3], Suppes [18]; cf. also the introduction of [2]. Harvey Friedman's [9, 10] present a rather convincing general perspective (and motivation) for the kind of work reported here.

2. Axiomatizing special relativity in FOL

In this paper we deal with the kinematics of relativity only, i.e. we deal with motion of *bodies* (or *test-particles*). The motivation for our choice of vocabulary (for special relativity) is summarized as follows. We will represent motion as changing spatial location in time. To do so, we will have reference-frames for coordinatizing events and, for simplicity, we will associate reference-frames with special bodies which we will call *observers*. We visualize an observer-as-a-body as "sitting" in the origin of the space part of its reference-frame, or equivalently, "living" on the time-axis of the reference-frame. We will distinguish *inertial* observers from non-inertial (i.e. accelerated) ones. There will be another special kind of bodies which we will call *photons*. For coordinatizing events we will use an arbitrary *ordered field* in place of the field of the real numbers. Thus the elements of this field will be the "*quantities*" which we will use for marking time and space. Allowing arbitrary ordered fields in place of the reals increases flexibility of our theory and minimizes the

[1]In passing we mention that Etesi-Németi [7], Hogarth [12] represent further kinds of *connection* between *logic and relativity* not discussed here.

amount of our mathematical presuppositions. Cf. e.g., Ax [3] for further motivation in this direction. Similar remarks apply to our flexibility oriented decisions below, e.g., keeping the number d of space-time dimensions a variable. Using coordinate systems (or reference-frames) instead of a single observer independent space-time structure is only a matter of didactical convenience and visualization, furthermore it also helps us in weeding out unnecessary axioms from our theories. Motivated by the above, we now turn to fixing the FOL language of our axiom systems.

The first occurrences of concepts used in this work are set by boldface letters to make it easier to find them. Throughout this work, if-and-only-if is abbreviated to **iff**.

Let us fix a natural number $d \geq 2$ for the dimension of the space-time that we are going to axiomatize. Our first-order language contains the following non-logical symbols:

- unary relation symbols B (for **Bodies**), Ob (for **Observers**), IOb (for **Inertial Observers**), Ph (for **Photons**) and F (for **quantities** which are going to be elements of a Field),
- binary function symbols $+$, $\cdot$ and a binary relation symbol $\leq$ (for the field operations and the ordering on F), and
- a $2+d$-ary relation symbol W (for **World-view relation**).

The bodies will play the role of the "main characters" of our space-time models and they will be "observed" (coordinatized using the quantities) by the observers. This observation will be coded by the world-view relation W. Our bodies and observers are basically the same as the "test particles" and the "reference-frames", respectively, in some of the literature.

We read $\mathrm{B}(x), \mathrm{Ob}(x), \mathrm{IOb}(x), \mathrm{Ph}(x), \mathrm{F}(x)$ as "x is a body", "x is an observer", "x is an inertial observer", "x is a photon", "x is a field-element". We use the world-view relation W to talk about coordinatization, by reading $\mathrm{W}(x, y, z_1, \ldots, z_d)$ as "observer x observes (or sees) body y at coordinate point $\langle z_1, \ldots, z_d\rangle$". This kind of observation has no connection with seeing via photons, it simply means coordinatization.

$\mathrm{B}(x), \mathrm{Ob}(x), \mathrm{IOb}(x), \mathrm{Ph}(x), \mathrm{F}(x), \mathrm{W}(x, y, z_1, \ldots, z_d), x = y, x \leq y$ are the so-called atomic formulas of our first-order language, where $x, y, z_1, \ldots, z_d$ can be arbitrary variables or terms built up from variables by using the field-operations "$+$" and "$\cdot$". The **formulas** of our first-order language are built up from these atomic formulas by using the logical connectives *not* ($\neg$), *and* ($\wedge$), *or* ($\vee$), *implies* ($\Longrightarrow$), *if-and-only-if* ($\Longleftrightarrow$) and the quantifiers *exists* x ($\exists x$) and *for all* x ($\forall x$) for every variable x.

Usually we use the variables m, k, h to denote observers, b to denote bodies, ph to denote photons and $p_1, \ldots, q_1, \ldots$ to denote quantities (i.e. field-elements). We write p and q in place of $p_1, \ldots, p_d$ and $q_1, \ldots, q_d$, e.g., we write $\mathrm{W}(m, b, p)$ in place of $\mathrm{W}(m, b, p_1, \ldots, p_d)$, and we write $\forall p$ in place of $\forall p_1, \ldots, p_d$ etc.

The **models** of this language are of the form

$$\mathfrak{M} = \langle U; \mathrm{B}, \mathrm{Ob}, \mathrm{IOb}, \mathrm{Ph}, \mathrm{F}, +, \cdot, \leq, \mathrm{W}\rangle,$$

where U is a nonempty set, $\mathrm{B}, \mathrm{Ob}, \mathrm{IOb}, \mathrm{Ph}, \mathrm{F}$ are unary relations on U, etc. A unary relation on U is just a subset of U. Thus we use B, Ob etc. as sets as well, e.g., we write $m \in \mathrm{Ob}$ in place of $\mathrm{Ob}(m)$.

Having fixed our language, we now turn to formulating an axiom system for special relativity in this language. We will make special efforts to keep all our axioms inside the above specified first-order logic language of $\mathfrak{M}$.

Throughout this work, i, j and n denote positive integers. $\mathrm{F}^n := \mathrm{F} \times \ldots \times \mathrm{F}$ (n-times) is the set of all n-tuples of elements of F. If $a \in \mathrm{F}^n$, then we assume that $a = \langle a_1, \ldots, a_n \rangle$, i.e. $a_i \in \mathrm{F}$ denotes the i-th component of the n-tuple a.

The following axiom is always assumed and is part of every axiom system we propose.

AxFrame**:** $\mathrm{Ob} \cup \mathrm{Ph} \subseteq \mathrm{B}$, $\mathrm{IOb} \subseteq \mathrm{Ob}$, $U = \mathrm{B} \cup \mathrm{F}$, $\mathrm{W} \subseteq \mathrm{Ob} \times \mathrm{B} \times \mathrm{F}^d$, $+$ and $\cdot$ are binary operations on F, $\leq$ is a binary relation on F and $\langle \mathrm{F}; +, \cdot, \leq \rangle$ is an **Euclidean ordered field**, i.e. a linearly ordered field in which positive elements have square roots.[2]

In pure first-order logic, the above axiom would look like $\forall x \ \left[\big(\mathrm{Ob}(x) \lor \mathrm{Ph}(x)\big) \Longrightarrow \mathrm{B}(x)\right]$ etc. In the present section we will not write out the purely first-order logic translations of our axioms since they will be straightforward to obtain. The first-order logic translations of our next three axioms AxSelf$^-$, AxPh, AxEv can be found in the Appendix.

Let $\mathfrak{M}$ be a model in which AxFrame is true. Let $\mathfrak{F} := \langle \mathrm{F}; +, \cdot, \leq \rangle$ denote the **ordered field reduct** of $\mathfrak{M}$. Here we list the definitions and notation that we are going to use in formulating our axioms. Let $0, 1, -, /, \sqrt{\ }$ be the usual field operations which are definable from "$+$" and "$\cdot$". We use the vector-space structure of F^n, i.e. if $p, q \in \mathrm{F}^n$ and $\lambda \in \mathrm{F}$, then $p + q, -p, \lambda p \in \mathrm{F}^n$; and $o := \langle 0, \ldots, 0 \rangle$ denotes the **origin**. The **Euclidean-length** of $a \in \mathrm{F}^n$ is defined as $|a| := \sqrt{a_1^2 + \ldots + a_n^2}$. The set of positive elements of F is denoted by $\mathrm{F}^+ := \{x \in \mathrm{F} : x > 0\}$. Let $p, q \in \mathrm{F}^d$. We use the notation $p_s := \langle p_2, \ldots, p_d \rangle$ for the **space component** of p and $p_t := p_1$ for the **time component** of p. We define the **line** through p and q as:

$$pq := \{q + \lambda(p - q) : \lambda \in \mathrm{F}\}.$$

The set of **lines** is defined as:

$$Lines := \{pq : p \neq q \ \land \ p, q \in \mathrm{F}^d\}.$$

The **slope** of p is defined as:

$$slope(p) := \frac{|p_s|}{|p_t|}$$

if $p_t \neq 0$ and is undefined otherwise; furthermore

$$slope(pq) := slope(p - q)$$

if $p_t \neq q_t$ and is undefined otherwise. F^d is called the **coordinate system** and its elements are referred to as **coordinate points**. The **event** (the set of bodies) observed by observer m at coordinate point p is:

$$ev_m(p) := \{b \in \mathrm{B} : \mathrm{W}(m, b, p)\}.$$

[2]For example, the ordered fields of the real numbers, the real algebraic numbers, and the hyper-real numbers are Euclidean but the ordered field of the rational numbers is not Euclidean and the field of the complex numbers cannot be ordered. For the definition of (linearly) ordered field, cf. e.g., Rudin [17] or Chang-Keisler [4].

The mapping $p \mapsto ev_m(p)$ is called the **world-view** (function) of m. The **coordinate domain** of observer m is the set of coordinate points where m observes something:

$$Cd(m) := \{p \in \mathrm{F}^d : ev_m(p) \neq \emptyset\}.$$

The **life-line** (or **trace**) of body b as seen by observer m is defined as the set of coordinate points where b was observed by m:

$$tr_m(b) := \{p \in \mathrm{F}^d : \mathrm{W}(m, b, p)\} = \{p \in \mathrm{F}^d : b \in ev_m(p)\}.$$

The life-line $tr_m(m)$ of observer m as seen by himself is called the **self-line** of m. The **time-axis** is defined as:

$$\bar{t} := \{\langle x, 0, \ldots, 0\rangle : x \in \mathrm{F}\}.$$

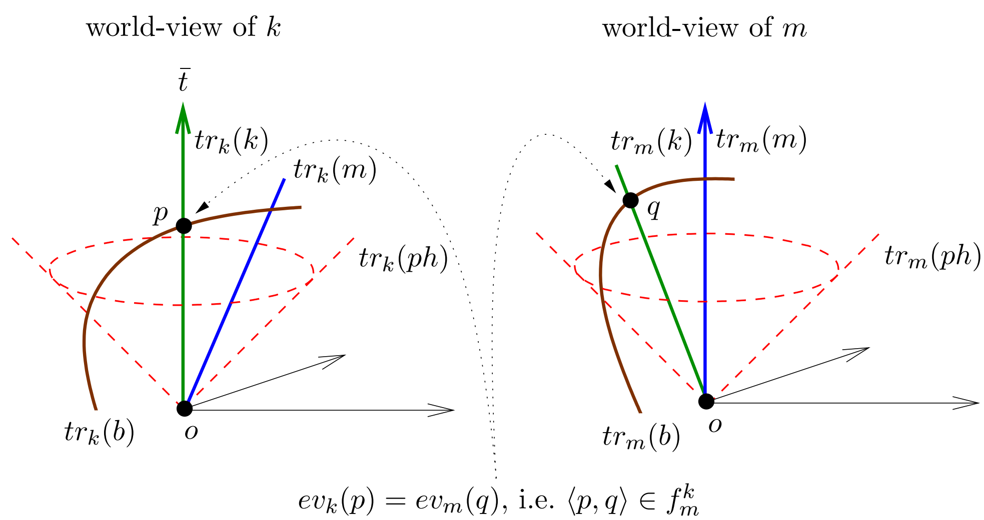

FIGURE 1. for the basic definitions mainly for f^k_m.

Now we are ready to build our space-time theories by formulating our axioms. We formulate each axiom on two levels. First we give an intuitive formulation, then we give a precise formalization using our notation.

The following natural axiom goes back to Galileo Galilei and even to the Norman-French Oresme of around 1350, cf. e.g., [1, p.23, §5]. It simply states that each observer thinks that he rests in the origin of the space part of his coordinate system.

AxSelf$^-$: The self-line of any observer is the time-axis restricted to his coordinate domain:

$$\forall m \in \mathrm{Ob} \quad tr_m(m) = \bar{t} \cap Cd(m).$$

A FOL-formula expressing AxSelf$^-$ can be found in the Appendix.

The next axiom is about the constancy of the speed of the photons, cf. e.g., [5, §2.6]. For convenience, we choose 1 for their speed.

AxPh: For every inertial observer, the lines of slope 1 are exactly the traces of the photons:

$$\forall m \in \mathrm{IOb} \quad \{tr_m(ph) : ph \in \mathrm{Ph}\} = \{l \in Lines : slope(l) = 1\}.$$

A FOL-formula expressing AxPh can be found in the Appendix.

We will also assume the following axiom:

AxEv: All inertial observers observe the same events:

$$\forall m, k \in \mathrm{IOb} \ \forall p \in \mathrm{F}^d \ \exists q \in \mathrm{F}^d \quad ev_m(p) = ev_k(q).$$

A FOL-formula expressing AxEv can be found in the Appendix.

$$\mathsf{Specrel_0} := \{\mathsf{AxSelf^-}, \mathsf{AxPh}, \mathsf{AxEv}, \mathsf{AxFrame}\}.$$

Since, in some sense, AxFrame is only an "auxiliary" (or book-keeping) axiom about the "mathematical frame" of our reasoning, the heart of $\mathsf{Specrel_0}$ consists of three very natural axioms, $\mathsf{AxSelf^-}$, AxPh, AxEv. These are really intuitively convincing, natural and simple assumptions. From these three axioms one can already prove the most characteristic predictions of special relativity theory. What the average layperson usually knows about relativity is that "moving clocks slow down", "moving spaceships shrink", and "moving pairs of clocks get out of synchronism". We call these the **paradigmatic effects** of special relativity. All these can be proven from the above three axioms, in some form, cf. Theorem 2.2. E.g., one can prove that "if m, k are any two observers not at rest relative to each other, then one of m, k will "see" or "think" that the clock of the other runs slow". However, $\mathsf{Specrel_0}$ does not imply yet the inertial approximation of the so-called Twin Paradox.[3] In order to prove the inertial approximation of the Twin Paradox also, and to prove all the paradigmatic effects in their strongest form, it is enough to add one more axiom AxSym to $\mathsf{Specrel_0}$. This is what we are going to do now.

We will find that studying the relationships between the world-views is more illuminating than studying the world-views in themselves. Therefore the following definition is fundamental. The **world-view transformation** between the world-views of observers k and m is the set of pairs of coordinate points $\langle p, q\rangle$ such that m and k observe the same nonempty event in p and q, respectively:

$$f_m^k := \{\langle p, q\rangle \in \mathrm{F}^d \times \mathrm{F}^d : ev_k(p) = ev_m(q) \neq \emptyset\},$$

cf. Figure 1. We note that although the world-view transformations are only binary relations, axiom AxPh turns them into functions, cf. (iii) of Proposition 5.1 way below.

CONVENTION 2.1. Whenever we write "$f_m^k(p)$", we mean that there is a unique $q \in \mathrm{F}^d$ such that $\langle p, q\rangle \in f_m^k$, and $f_m^k(p)$ denotes this unique q. I.e. if we talk about the value $f_m^k(p)$ of f_m^k at p, we tacitly postulate that it exists and is unique (by the present convention).

[3] This inertial approximation of the twin paradox is formulated as $\mathsf{AxTp^{in}}$ at the end of Section 3 below Theorem 3.6.

The following axiom is an instance (or special case) of the Principle of Special Relativity, according to which the "laws of nature" are the same for all inertial observers, in particular, there is no experiment which would decide whether you are in absolute motion, cf. e.g., Einstein [6] or [5, §2.5] or [13, §2.8]. To explain the following formula, let $p, q \in \mathrm{F}^d$. Then $p_t - q_t$ is the time passed between the events $ev_m(p)$ and $ev_m(q)$ as seen by m and $f_k^m(p)_t - f_k^m(q)_t$ is the time passed between the same two events as seen by k. Hence $|(f_k^m(p)_t - f_k^m(q)_t)/(p_t - q_t)|$ is the rate with which k's clock runs slow as seen by m. The same explanation applies when m and k are interchanged.

AxSym**:** Any two inertial observers see each other's clocks go wrong in the same way:

$$\forall m, k \in \mathrm{IOb}\ \forall p, q \in \bar{t} \quad \left|f_m^k(p)_t - f_m^k(q)_t\right| = \left|f_k^m(p)_t - f_k^m(q)_t\right|.$$

All the axioms so far talked about inertial observers, and they in fact form an axiom system complete w.r.t. the inertial observers, cf. Theorem 2.2 below.

$$\mathsf{Specrel} := \{\mathsf{AxSelf}^-, \mathsf{AxPh}, \mathsf{AxEv}, \mathsf{AxSym}, \mathsf{AxFrame}\}.$$

Let $p, q \in \mathrm{F}^d$. Then

$$\mu(p) := \begin{cases} \sqrt{\left|p_t^2 - |p_s|^2\right|} & \text{if } p_t^2 - |p_s|^2 \geq 0 \\ -\sqrt{\left|p_t^2 - |p_s|^2\right|} & \text{otherwise} \end{cases}$$

is the **Minkowski-length** of p and the **Minkowski-distance** between p and q is defined as follows:

$$\mu(p, q) := \mu(p - q).$$

Let $f : \mathrm{F}^d \to \mathrm{F}^d$ be a function. f is said to be a **Poincaré-transformation** if f is a bijection and it preserves the Minkowski-distance, i.e. $\mu\big(f(p), f(q)\big) = \mu(p, q)$ for all $p, q \in \mathrm{F}^d$. f is called a **dilation** if there is a positive $\delta \in \mathrm{F}$ such that $f(p) = \delta p$ for all $p \in \mathrm{F}^d$ and f is called a **field-automorphism-induced** mapping if there is an automorphism π of the field $\langle \mathrm{F}, +, \cdot \rangle$ such that $f(p) = \langle \pi p_1, \ldots, \pi p_d \rangle$ for all $p \in \mathrm{F}^d$. The following is proved in [2, 2.9.4, 2.9.5] and in [13, 2.9.4–2.9.7].

Let Σ be a set of formulas and $\mathfrak{M}$ be a model. $\mathfrak{M} \models \Sigma$ denotes that all formulas in Σ are true in model $\mathfrak{M}$. In this case we say that $\mathfrak{M}$ is a **model of** Σ.

Theorem 2.2. Let $d > 2$, let $\mathfrak{M}$ be a model of our language and let m, k be inertial observers in $\mathfrak{M}$. Then f_k^m is a Poincaré-transformation whenever $\mathfrak{M} \models \mathsf{Specrel}$. More generally, f_k^m is a Poincaré-transformation composed with a dilation and a field-automorphism-induced mapping, whenever $\mathfrak{M} \models \mathsf{Specrel}_0$.

Remark 2.3. Assume $d > 2$. Theorem 2.2 is best possible in the sense that, e.g., for every Poincaré-transformation f over an arbitrary Euclidean ordered field there are a model $\mathfrak{M} \models \mathsf{Specrel}$ and inertial observers m, k in $\mathfrak{M}$ such that the world-view transformation f_k^m between m's and k's world-views in $\mathfrak{M}$ is f, see [2, 2.9.4(iii), 2.9.5(iii)]. Similarly for the second statement in Theorem 2.2. Hence, Theorem 2.2 can be refined to a pair of completeness theorems, cf. [2, 3.6.13, p.271]. Roughly, for every Euclidean ordered field, its Poincaré-transformations (can be expanded to) form a model of Specrel. Similarly for the other case. $\lhd$

It follows from Theorem 2.2 that the paradigmatic effects of relativity hold in Specrel in their strongest form, e.g., if m and k are observers not at rest w.r.t. each other, then both will "think" that the clock of the other runs slow. Specrel also implies the "inertial approximation" of the Twin Paradox, see e.g., [2, 2.8.18], and [19]. It is necessary to add AxSym to $\mathsf{Specrel}_0$ in order to be able to prove the inertial approximation of the Twin Paradox by Theorem 2.2, cf. e.g., [19].

We now begin to formulate axioms about non-inertial observers. The non-inertial observers are called **accelerated** observers. To connect the world-views of the accelerated and the inertial observers, we are going to formulate the statement that at each moment of his life, each accelerated observer sees the nearby world for a short while as an inertial observer does. To formalize this, first we introduce the relation of being a co-moving observer. The (open) **ball** with center $c \in \mathrm{F}^n$ and radius $\varepsilon \in \mathrm{F}^+$ is:

$$B_\varepsilon(c) := \{a \in \mathrm{F}^n : |a - c| < \varepsilon\}.$$

m is a **co-moving observer** of k at $q \in \mathrm{F}^d$, in symbols $m \succ_q k$, if $q \in Cd(k)$ and the following holds:

$$\forall \varepsilon \in \mathrm{F}^+ \ \exists \delta \in \mathrm{F}^+ \ \ \forall p \in B_\delta(q) \cap Cd(k) \ \ \left|p - f^k_m(p)\right| \le \varepsilon |p - q|.$$

Behind the definition of the co-moving observers is the following intuitive image: as we zoom into smaller and smaller neighborhoods of the given coordinate point, the world-views of the two observers are more and more similar. Notice that $f^k_m(q) = q$ if $m \succ_q k$.

The following axiom gives the promised connection between the world-views of the inertial and the accelerated observers:

AxAcc**:** At any point on the self-line of any observer, there is a co-moving inertial observer:

$$\forall k \in \mathrm{Ob} \ \ \forall q \in tr_k(k) \ \exists m \in \mathrm{IOb} \quad m \succ_q k.$$

Let AccRel_0 be the collection of the axioms introduced so far:

$$\mathsf{AccRel}_0 := \{\mathsf{AxSelf}^-, \mathsf{AxPh}, \mathsf{AxEv}, \mathsf{AxSym}, \mathsf{AxAcc}, \mathsf{AxFrame}\}.$$

Let $\mathfrak{R}$ denote the ordered field of the real numbers. Accelerated clocks behave as expected in models of AccRel_0 when the ordered field reduct of the model is $\mathfrak{R}$ (cf. Theorems 3.1, 3.2, and in more detail Prop.5.2, Rem.5.3); but not otherwise (cf. Theorems 3.5, 3.6). Thus to prove properties of accelerated clocks (and observers), we need more properties of the field reducts than their being Euclidean ordered fields. As it turns out, adding all FOL-formulas valid in the field of the reals does not suffice (cf. Theorem 3.5). The additional property of $\mathfrak{R}$ we need is that in $\mathfrak{R}$ every bounded non-empty set has a **supremum**, i.e. a least upper bound. This is a second-order logic property which we cannot use in a FOL axiom system. Instead, we will use a kind of "induction" axiom schema. It will state that every non-empty, bounded subset of the ordered field reduct which can be defined by a FOL-formula using the extra part of the model, e.g., using the world-view relation, has a supremum. We now begin to formulate our FOL induction axiom schema.[4]

To abbreviate formulas of FOL we often omit parentheses according to the following convention. Quantifiers bind as long as they can, and $\land$ binds stronger than $\Longrightarrow$. E.g., $\forall x\ \varphi \land \psi \Longrightarrow \exists y\ \delta \land \eta$ means $\forall x\big((\varphi \land \psi) \Longrightarrow \exists y(\delta \land \eta)\big)$. Instead of curly brackets we sometimes write square brackets in formulas, e.g., we may write $\forall x\ (\varphi \land \psi \Longrightarrow [\exists y \delta] \land \eta)$.

If φ is a formula and x is a variable, then we say that x is a **free variable** of φ iff x does not occur under the scope of either $\exists x$ or $\forall x$.

Let φ be a formula; and let $x, y_1, \ldots, y_n$ be all the free variables of φ. Let $\mathfrak{M} = \langle U; \mathrm{B}, \ldots \rangle$ be a model. Whether φ is true or false in $\mathfrak{M}$ depends on how we associate elements of U to these free variables. When we associate $d, a_1, \ldots, a_n \in U$ to $x, y_1, \ldots, y_n$, respectively, then $\varphi(d, a_1, \ldots, a_n)$ denotes this truth-value, thus $\varphi(d, a_1, \ldots, a_n)$ is either true or false in $\mathfrak{M}$. For example, if φ is $x \le y_1 + \ldots + y_n$, then $\varphi(0, 1, \ldots, 1)$ is true in $\mathfrak{R}$ while $\varphi(1, 0, \ldots, 0)$ is false in $\mathfrak{R}$. φ is said to be **true** in $\mathfrak{M}$ if φ is true in $\mathfrak{M}$ no matter how we associate elements to the free variables. We say that a subset H of F is **definable by** φ iff there are $a_1, \ldots, a_n \in U$ such that $H = \{d \in \mathrm{F} : \varphi(d, a_1, \ldots, a_n) \text{ is true in } \mathfrak{M}\}$.

AxSup_φ**:** Every subset of F definable by φ has a supremum if it is non-empty and **bounded**:

$$\forall y_1, \ldots, y_n \quad [\exists x \in \mathrm{F} \quad \varphi] \ \land\ [\exists b \in \mathrm{F} \quad (\forall x \in \mathrm{F} \quad \varphi \Longrightarrow x \le b)] \ \Longrightarrow$$
$$\exists s \in \mathrm{F}\ \ \forall b \in \mathrm{F} \quad (\forall x \in \mathrm{F} \quad \varphi \Longrightarrow x \le b) \iff s \le b.$$

We say that a subset of F is **definable** iff it is definable by a formula. Our axiom scheme IND below says that every non-empty bounded and definable subset of F has a supremum.

$$\mathsf{IND} := \{\mathsf{AxSup}_\varphi : \varphi \text{ is a FOL-formula of our language } \}.$$

Notice that IND is true in any model whose ordered field reduct is $\mathfrak{R}$. Let us add IND to AccRel_0 and call it AccRel:

$$\mathsf{AccRel} := \mathsf{AccRel}_0 \cup \mathsf{IND}.$$

AccRel is a countable set of FOL-formulas. We note that there are non-trivial models of AccRel, cf. e.g., Remark 5.3 way below. Furthermore, we note that the construction in Misner-Thorne-Wheeler [14, Chapter 6 entitled "The local coordinate system of an accelerated observer", especially pp. 172-173 and Chapter 13.6 entitled "The proper reference frame of an accelerated observer" on pp. 327-332] can be used for constructing models of AccRel. Models of AccRel are discussed to some detail in [19]. Theorems 3.1 and 3.2 (and also Prop.5.2, Rem.5.3) below show that AccRel already implies properties of accelerated clocks, e.g., it implies the Twin Paradox.

[4] This way of imitating a second-order formula by a FOL-formula schema comes from the methodology of approximating second-order theories by FOL ones, examples are Tarski's replacement of Hilbert's second-order geometry axiom by a FOL schema or Peano's FOL induction schema replacing second-order logic induction.

3. Main results: Accelerated clocks and the Twin Paradox in our FOL axiomatic setting

Twin Paradox (TP) concerns two twin siblings whom we shall call Ann and Ian. ("A" and "I" stand for accelerated and for inertial, respectively). Ann travels in a spaceship to some distant star while Ian remains at home. TP states that when Ann returns home she will be *younger* than her *twin brother* Ian. We now formulate TP in our FOL language.

The **segment** between $p \in \mathrm{F}^d$ and $q \in \mathrm{F}^d$ is defined as:

$$[pq] := \{\lambda p + (1-\lambda)q : \lambda \in \mathrm{F} \ \land \ 0 \le \lambda \le 1\}.$$

We say that observer k is in **twin-paradox relation** with observer m iff whenever k leaves m between two meetings, k measures less time between the two meetings than m:

$$\forall p,q \in tr_k(k) \ \ \forall p',q' \in tr_m(m)$$
$$\langle p,p'\rangle, \langle q,q'\rangle \in f_m^k \ \land \ [pq] \subseteq tr_k(k) \ \land \ [p'q'] \not\subseteq tr_m(k) \quad \Longrightarrow \quad \big|q_t - p_t\big| < \big|q'_t - p'_t\big|,$$

cf. Figure 2. In this case we write $\mathsf{Tp}(k < m)$.

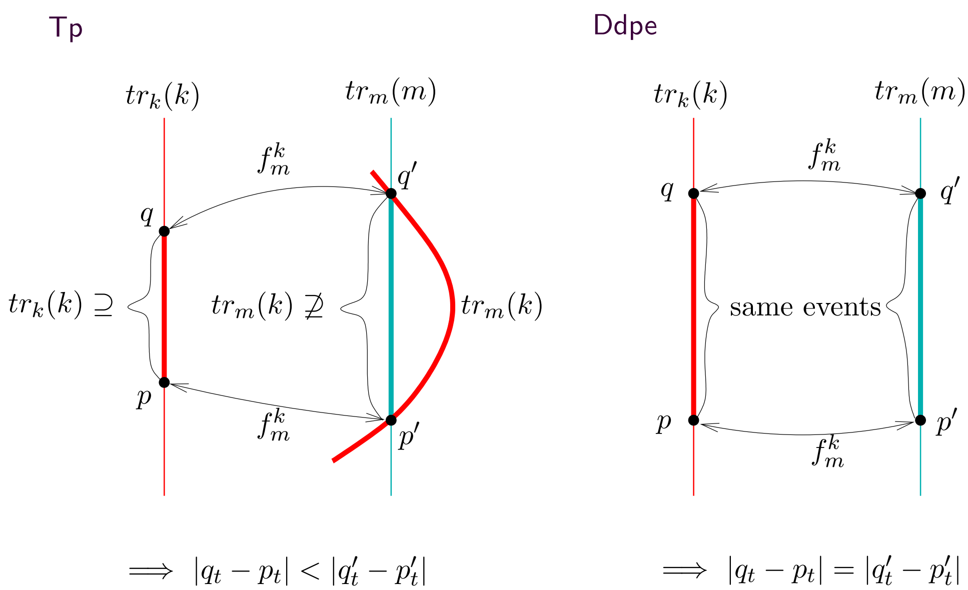

Figure 2. for Tp and Ddpe.

Tp**:** Every observer is in twin-paradox relation with every inertial observer:

$$\forall k \in \mathrm{Ob} \ \ \forall m \in \mathrm{IOb} \quad \mathsf{Tp}(k < m).$$

Let φ be a formula and Σ be a set of formulas. $\Sigma \models \varphi$ denotes that φ is true in all models of Σ. Gödel's completeness theorem for FOL implies that whenever $\Sigma \models \varphi$, there is a (syntactic) derivation of φ from Σ via the commonly used derivation rules of FOL. Hence the next theorem states that the formula Tp formulating the Twin Paradox is provable from the axiom system AccRel.

Theorem 3.1. $\mathsf{AccRel} \models \mathsf{Tp}$ if $d > 2$.

The proof of the theorem is in Section 5.

Now we turn to formulating a phenomenon which we call Duration Determining Property of Events.

Ddpe: If each of two observers observes the very same (non-empty) events in a segment of their self-lines, they measure the same time between the end points of these two segments:

$$\forall k, m \in \mathrm{Ob}\ \forall p, q \in tr_k(k)\ \forall p', q' \in tr_m(m)$$
$$\emptyset \notin \{ev_k(r) : r \in [pq]\} = \{ev_m(r') : r' \in [p'q']\} \quad \Longrightarrow \quad |q_t - p_t| = |q'_t - p'_t|,$$

see the right hand side of Figure 2.

The next theorem states that Ddpe also can be proved from our FOL axiom system AccRel.

Theorem 3.2. $\mathsf{AccRel} \models \mathsf{Ddpe}$ if $d > 2$.

The proof of the theorem is in Section 5.

Remark 3.3. The assumption $d > 2$ cannot be omitted from Theorem 3.1. However, Theorems 3.1 and 3.2 remain true if we omit the assumption $d > 2$ and assume auxiliary axioms AxIOb and AxLine below, i.e.

$$\mathsf{AccRel} \cup \{\mathsf{AxIOb}, \mathsf{AxLine}\} \models \mathsf{Tp}\ \wedge\ \mathsf{Ddpe}$$

holds for $d = 2$, too. A proof for the latter statement can be obtained from the proofs of Theorems 3.1 and 3.2 by [19, items 4.3.1, 4.2.4, 4.2.5] and [1, Theorem 1.4(ii)].

AxIOb: In every inertial observer's coordinate system, every line of slope less than 1 is the life-line of an inertial observer:

$$\forall m \in \mathrm{IOb} \quad \{tr_m(k) : k \in \mathrm{IOb}\} \supseteq \{l \in \mathit{Lines} : \mathit{slope}(l) < 1\}.$$

AxLine: Traces of inertial observers are lines as observed by inertial observers:

$$\forall m, k \in \mathrm{IOb} \quad tr_m(k) \in \mathit{Lines}.$$

$\triangleleft$

Question 3.4. Can the assumption $d > 2$ be omitted from Theorem 3.2, i.e. does $\mathsf{AccRel} \models \mathsf{Ddpe}$ hold for $d = 2$?

The following theorem says that Theorems 3.1 and 3.2 do not remain true if we omit the axiom scheme IND from AccRel. If a formula φ is not true in a model $\mathfrak{M}$, we write $\mathfrak{M} \not\models \varphi$.

Theorem 3.5. For every Euclidean ordered field $\mathfrak{F}$ different from $\mathfrak{R}$, there is a model $\mathfrak{M}$ of AccRel_0 such that $\mathfrak{M} \not\models \mathsf{Tp}$, $\mathfrak{M} \not\models \mathsf{Ddpe}$ and the ordered field reduct of $\mathfrak{M}$ is $\mathfrak{F}$.

The proof of the theorem is in Section 5.

An ordered field is called **non-Archimedean** if it has an element a such that, for every positive integer n, $-1 < \underbrace{a + \ldots + a}_{n} < 1$. We call these elements **infinitesimally small**.

The following theorem says that, for countable or non-Archimedean Euclidean ordered fields, there are quite sophisticated models of AccRel_0 in which Tp and Ddpe are false.

Theorem 3.6. For every Euclidean ordered field $\mathfrak{F}$ which is countable or non-Archimedean, there is a model $\mathfrak{M}$ of AccRel_0 such that $\mathfrak{M} \not\models \mathsf{Tp}$, $\mathfrak{M} \not\models \mathsf{Ddpe}$, the ordered field reduct of $\mathfrak{M}$ is $\mathfrak{F}$ and (i)–(iv) below also hold in $\mathfrak{M}$.

(i) Every observer uses the whole coordinate system for coordinate-domain:

$$\forall m \in \mathrm{Ob} \quad Cd(m) = \mathrm{F}^d.$$

(ii) At any point in F^d, there is a co-moving inertial observer of any observer:

$$\forall k \in \mathrm{Ob}\ \forall q \in \mathrm{F}^d\ \exists m \in \mathrm{IOb} \quad m \succ_q k.$$

(iii) All observers observe the same set of events:

$$\forall m, k \in \mathrm{Ob}\ \forall p \in \mathrm{F}^d\ \exists q \in \mathrm{F}^d \quad ev_m(p) = ev_k(q).$$

(iv) Every observer observes every event only once:

$$\forall m \in \mathrm{Ob}\ \forall p, q \in \mathrm{F}^d \quad ev_m(p) = ev_m(q) \implies p = q.$$

$\lhd$

The proof of the theorem is in Section 5.

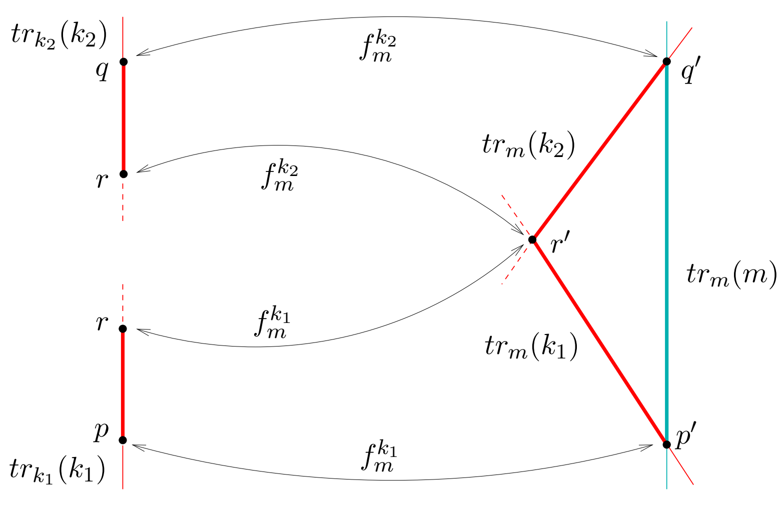

FIGURE 3. for $\mathsf{AxTp}^{\mathsf{in}}$.

Finally we formulate a question. To this end we introduce the inertial version of the twin paradox and some auxiliary axioms. In the inertial version of the twin paradox, we use the common trick of the literature to talk about the twin paradox without talking about accelerated observers. We replace the accelerated twin with two inertial ones, a leaving and an approaching one.

We say that observers k_1 and k_2 are in **inertial twin-paradox relation** with observer m if the following holds:

$$\forall p, q, r \in tr_{k_1}(k_1) \cap tr_{k_2}(k_2) \ \ \forall p', q' \in tr_m(m) \ \ \forall r' \in \mathrm{F}^d$$
$$\langle p, p'\rangle, \langle r, r'\rangle \in f_m^{k_1} \ \wedge \ \langle r, r'\rangle, \langle q, q'\rangle \in f_m^{k_2} \ \wedge \ p'_t < r'_t < q'_t \ \wedge \ r' \notin [p'q']$$
$$\Longrightarrow \quad |q'_t - p'_t| > |q_t - r_t| + |r_t - p_t|,$$

cf. Figure 3. In this case we write $\mathsf{Tp}(k_1k_2 < m)$.

$\mathsf{AxTp^{in}}$**:** Every three inertial observers are in inertial twin-paradox relation:

$$\forall m, k_1, k_2 \in \mathrm{IOb} \quad \mathsf{Tp}(k_1k_2 < m).$$

AxTrn**:** To every inertial observer m and coordinate point p there is an inertial observer k such that the world-view transformation between m and k is the translation by vector p:

$$\forall m \in \mathrm{IOb} \ \ \forall p \in \mathrm{F}^d \ \exists k \in \mathrm{IOb} \ \ \forall q \in \mathrm{F}^d \quad f_k^m(q) = q + p.$$

AxLt**:** The world-view transformation between inertial observers m and k is a linear transformation if $f_k^m(o) = o$:

$$\forall m, k \in \mathrm{IOb} \ \ \forall p, q \in \mathrm{F}^d \ \ \forall \lambda \in \mathrm{F}$$
$$f_k^m(o) = o \quad \Longrightarrow \quad [f_k^m(p+q) = f_k^m(p) + f_k^m(q) \ \wedge \ f_k^m(\lambda p) = \lambda f_k^m(p)].$$

Question 3.7. Does Theorem 3.1 remain true if we replace AxSym in AccRel with the inertial version of the twin paradox and some auxiliary axioms, e.g., is

$$\{\mathsf{AxSelf^-}, \mathsf{AxPh}, \mathsf{AxEv}, \mathsf{AxTp^{in}}, \mathsf{AxAcc}, \mathsf{AxLt}, \mathsf{AxIOb}, \mathsf{AxTrn}, \mathsf{AxFrame}\} \cup \mathsf{IND} \models \mathsf{Tp}$$

true if $d > 2$? Cf. Question 5.6. We note that $\mathsf{AxTp^{in}}$ and AxLt are true in the models of $\mathsf{Specrel}$ in the case when $d > 2$, cf. [1, Theorem 1.2], [13, Theorem 2.8.28] and [19, §3]. $\lhd$

4. Pieces from non-standard analysis: some tools from real analysis captured in FOL

In this section we gather the statements (and proofs from AccRel) of the facts we will need from analysis. The point is in formulating these statements in FOL and for an arbitrary ordered field in place of using the second-order language of the ordered field $\mathfrak{R}$ of reals.

In the present section $\mathsf{AxFrame}$ is assumed without any further mentioning.

Let $a, b, c \in \mathrm{F}$. We say that b is **between** a and c iff $a < b < c$ or $a > b > c$. We use the following notation: $[a, b] := \{x \in \mathrm{F} : a \le x \le b\}$ and $(a, b) := \{x \in \mathrm{F} : a < x < b\}$.

Convention 4.1. Whenever we write $[a, b]$, we assume that $a, b \in \mathrm{F}$ and $a \le b$. We also use this convention for (a, b).

Let $H \subseteq \mathrm{F}^n$. $p \in \mathrm{F}^n$ is said to be an **accumulation point** of H if for all $\varepsilon \in \mathrm{F}^+$, $B_\varepsilon(p) \cap H$ has an element different from p. H is called **open** if for all $p \in H$, there is an $\varepsilon \in \mathrm{F}^+$ such that $B_\varepsilon(p) \subseteq H$. Let $R \subseteq A \times B$ and $S \subseteq B \times C$ be binary relations. The **composition** of R and S is defined as: $R \circ S := \{\langle a, c\rangle \in A \times C : \exists b \in B \ \langle a, b\rangle \in R \ \wedge \ \langle b, c\rangle \in S\}$. The **domain** and the **range** of R are denoted by $Dom(R) := \{a \in A : \exists b \in B \ \langle a, b\rangle \in R\}$ and $Rng(R) := \{b \in B : \exists a \in A \ \langle a, b\rangle \in R\}$, respectively. R^{-1} denotes the **inverse** of R, i.e. $R^{-1} := \{\langle b, a\rangle \in B \times A : \langle a, b\rangle \in R\}$. We think of a **function** as a special binary relation. Notice that if f, g are functions, then $f \circ g(x) = g\big(f(x)\big)$ for all $x \in Dom(f \circ g)$. $f : A \to B$ denotes that f is a function from A to B, i.e. $Dom(f) = A$ and $Rng(f) \subseteq B$. Notation $f : A \xrightarrow{\circ} B$ denotes that f is a **partial function** from A to B; this means that f is a function, $Dom(f) \subseteq A$ and $Rng(f) \subseteq B$. Let $f : \mathrm{F} \xrightarrow{\circ} \mathrm{F}^n$. We call f **continuous** at x if $x \in Dom(f)$, x is an accumulation point of $Dom(f)$ and the usual formula of continuity holds for f and x, i.e.

$$\forall \varepsilon \in \mathrm{F}^+ \ \exists \delta \in \mathrm{F}^+ \ \forall y \in Dom(f) \quad |y - x| < \delta \implies |f(y) - f(x)| < \varepsilon.$$

We call f **differentiable** at x if $x \in Dom(f)$, x is an accumulation point of $Dom(f)$ and there is an $a \in \mathrm{F}^n$ such that

$$\forall \varepsilon \in \mathrm{F}^+ \ \exists \delta \in \mathrm{F}^+ \ \forall y \in Dom(f) \quad |y - x| < \delta \implies |f(y) - f(x) - (y - x)a| \le \varepsilon |y - x|.$$

This a is unique. We call this a the **derivate** of f at x and we denote it by $f'(x)$. f is said to be continuous (differentiable) on $H \subseteq \mathrm{F}$ iff $H \subseteq Dom(f)$ and f is continuous (differentiable) at every $x \in H$. We note that the basic properties of the differentiability remain true since their proofs use only the ordered field properties of $\mathfrak{R}$, cf. Propositions 4.2, 4.3 and 4.4 below.

Let $f, g : \mathrm{F} \xrightarrow{\circ} \mathrm{F}^n$ and $\lambda \in \mathrm{F}$. Then $\lambda f : \mathrm{F} \xrightarrow{\circ} \mathrm{F}^n$ and $f + g : \mathrm{F} \xrightarrow{\circ} \mathrm{F}^n$ are defined as $\lambda f := \big\{\langle x, \lambda f(x)\rangle : x \in Dom(f)\big\}$ and $f + g := \big\{\langle x, f(x) + g(x)\rangle : x \in Dom(f) \cap Dom(g)\big\}$. Let $h : \mathrm{F} \xrightarrow{\circ} \mathrm{F}$. h is said to be **increasing** on H iff $H \subseteq Dom(h)$ and for all $x, y \in H$, $h(x) < h(y)$ if $x < y$, and h is said to be **decreasing** on H iff $H \subseteq Dom(h)$ and for all $x, y \in H$, $h(x) > h(y)$ if $x < y$.

Proposition 4.2. Let $f, g : \mathrm{F} \xrightarrow{\circ} \mathrm{F}^n$ and $h : \mathrm{F} \xrightarrow{\circ} \mathrm{F}$. Then (i)–(v) below hold.

(i) If f is differentiable at x then it is also continuous at x.

(ii) Let $\lambda \in \mathrm{F}$. If f is differentiable at x, then λf is also differentiable at x and $(\lambda f)'(x) = \lambda f'(x)$.

(iii) If f and g are differentiable at x and x is an accumulation point of $Dom(f) \cap Dom(g)$, then $f + g$ is differentiable at x and $(f + g)'(x) = f'(x) + g'(x)$.

(iv) If h is differentiable at x, g is differentiable at $h(x)$ and x is an accumulation point of $Dom(h \circ g)$, then $h \circ g$ is differentiable at x and $(h \circ g)'(x) = h'(x) g'\big(h(x)\big)$.

(v) If h is increasing (or decreasing) on (a, b), differentiable at $x \in (a, b)$ and $h'(x) \neq 0$, then h^{-1} is differentiable at $h(x)$.

on the proof. Since the proofs of the statements are based on the same calculations and ideas as in real analysis, we omit the proof, cf. [16, Theorems 28.2, 28.3, 28.4 and 29.9]. ■

Let $i \leq n$. $\pi_i : \mathrm{F}^n \to \mathrm{F}$ denotes the i-th projection function, i.e. $\pi_i : p \mapsto p_i$. Let $f : \mathrm{F} \xrightarrow{\circ} \mathrm{F}^n$. We denote the i-th coordinate function of f by f_i, i.e. $f_i := f \circ \pi_i$. We also denote f_1 by f_t. A function $A : \mathrm{F}^n \to \mathrm{F}^j$ is said to be an **affine map** if it is a linear map composed by a translation.[5]

The following proposition says that the derivate of a function f composed by an affine map A at a point x is the image of the derivate $f'(x)$ taken by the linear part of A.

Proposition 4.3. Let $f : \mathrm{F} \xrightarrow{\circ} \mathrm{F}^n$ be differentiable at x and let $A : \mathrm{F}^n \to \mathrm{F}^j$ be an affine map. Then $f \circ A$ is differentiable at x and $(f \circ A)'(x) = A\big(f'(x)\big) - A(o)$. In particular, $f'(x) = \langle f_1'(x), \dots, f_n'(x)\rangle$, i.e. $f_i'(x) = f'(x)_i$.

on the proof. The statement is straightforward from the definitions. ∎

$f : \mathrm{F} \xrightarrow{\circ} \mathrm{F}$ is said to be **locally maximal** at x iff $x \in Dom(f)$ and there is a $\delta \in \mathrm{F}^+$ such that $f(y) \leq f(x)$ for all $y \in B_\delta(x) \cap Dom(f)$. The **local minimality** is defined analogously.

Proposition 4.4. If $f : \mathrm{F} \xrightarrow{\circ} \mathrm{F}$ is differentiable on (a, b) and locally maximal or minimal at $x \in (a, b)$, then its derivate is 0 at x, i.e. $f'(x) = 0$.

on the proof. The proof is the same as in real analysis, cf. e.g., [17, Theorem 5.8]. ∎

Let $\mathfrak{M} = \langle U; \dots\rangle$ be a model. An n-ary relation $R \subseteq \mathrm{F}^n$ is said to be **definable** iff there is a formula φ with only free variables $x_1, \dots, x_n, y_1, \dots, y_i$ and there are $a_1, \dots, a_i \in U$ such that $R = \{\langle p_1, \dots, p_n\rangle \in \mathrm{F}^n : \varphi(p_1, \dots, p_n, a_1, \dots, a_i) \text{ is true in } \mathfrak{M}\}$. Recall that IND says that every non-empty, bounded and definable subset of F has a supremum.

Theorem 4.5 (Bolzano's Theorem)**.** Assume IND. Let $f : \mathrm{F} \xrightarrow{\circ} \mathrm{F}$ be definable and continuous on $[a, b]$. If c is between $f(a)$ and $f(b)$, then there is an $s \in [a, b]$ such that $f(s) = c$.

proof. Let c be between $f(a)$ and $f(b)$. We can assume that $f(a) < f(b)$. Let $H := \{x \in [a, b] : f(x) < c\}$. Then H is definable, bounded and non-empty. Thus, by IND, it has a supremum, say s. Both $\{x \in (a, b) : f(x) < c\}$ and $\{x \in (a, b) : f(x) > c\}$ are non-empty open sets since f is continuous on $[a, b]$. Thus $f(s)$ cannot be less than c since s is an upper bound of H and cannot be greater than c since s is the smallest upper bound. Hence $f(s) = c$ as desired. ∎

Theorem 4.6. Assume IND. Let $f : \mathrm{F} \xrightarrow{\circ} \mathrm{F}$ be definable and continuous on $[a, b]$. Then the supremum s of $\{f(x) : x \in [a, b]\}$ exists and there is an $y \in [a, b]$ such that $f(y) = s$.

proof. The set $H := \{y \in [a, b] : \exists c \in \mathrm{F}\ \forall x \in [a, y] \quad f(x) < c\}$ has a supremum by IND since H is definable, non-empty and bounded. This supremum has to be b and $b \in H$ since f is continuous on $[a, b]$. Thus $Ran(f) := \{f(x) : x \in [a, b]\}$ is bounded. Thus, by IND, it has a supremum, say s, since it is definable and non-empty. We can assume that

[5] I.e. A is an affine map if there are $L : \mathrm{F}^n \to \mathrm{F}^j$ and $a \in \mathrm{F}^j$ such that $A(p) = L(p) + a$, $L(p + q) = L(p) + L(q)$ and $L(\lambda p) = \lambda L(p)$ for all $p, q \in \mathrm{F}^n$ and $\lambda \in \mathrm{F}$.

$f(a) \neq s$. Let $A := \{y \in [a,b] : \exists c \in \mathrm{F}\ \forall x \in [a,y] \quad f(x) < c < s\}$. By IND, A has a supremum. At this supremum, f cannot be less than s since f is continuous on $[a,b]$ and s is the supremum of $Ran(f)$. ■

Throughout this work $Id : \mathrm{F} \to \mathrm{F}$ denotes the identity function, i.e. $Id : x \mapsto x$.

Theorem 4.7 (Mean Value Theorem)**.** Assume IND. Let $f : \mathrm{F} \xrightarrow{\circ} \mathrm{F}$ be definable, differentiable on $[a,b]$. If $a \neq b$, then there is an $s \in (a,b)$ such that $f'(s) = \frac{f(b)-f(a)}{b-a}$.

proof. Assume $a \neq b$. Let $h := \big(f(b) - f(a)\big)Id - (b-a)f$. Then h is differentiable on $[a,b]$ and $h'(x) = f(b) - f(a) - (b-a)f'(x)$ for all $x \in [a,b]$ by (ii) and (iii) of Proposition 4.2 since Id is differentiable on $[a,b]$ and its derivate is 1 for all $x \in [a,b]$. If h is constant on $[a,b]$,[6] then $h'(s) = 0$ for all $s \in (a,b)$. Otherwise, by Theorem 4.6, there is a maximum or minimum of h different from $h(a) = f(b)a - bf(a) = h(b)$ at an $s \in (a,b)$. Hence $h'(s) = 0$ by Proposition 4.4. This completes the proof since $a \neq b$ and $h'(s) = f(b) - f(a) - (b-a)f'(s)$. ■

Corollary 4.8 (Rolle's Theorem)**.** Assume IND. Let $f : \mathrm{F} \xrightarrow{\circ} \mathrm{F}$ be definable and differentiable on $[a,b]$. If $f(a) = f(b)$ and $a \neq b$, then there is an $s \in (a,b)$ such that $f'(s) = 0$. ■

Proposition 4.9. Assume IND. Let $f, g : \mathrm{F} \xrightarrow{\circ} \mathrm{F}$ be definable and differentiable on (a,b). If $f'(x) = g'(x)$ for all $x \in (a,b)$, then there is a $c \in \mathrm{F}$ such that $f(x) = g(x) + c$ for all $x \in (a,b)$.

proof. Assume that $f'(x) = g'(x)$ for all $x \in (a,b)$. Let $h := f - g$. Then $h'(x) = f'(x) - g'(x) = 0$ for all $x \in (a,b)$ by (ii) and (iii) of Proposition 4.2. If there are $y, z \in (a,b)$ such that $h(y) \neq h(z)$ and $y \neq z$, then, by the Mean Value Theorem, there is an x between y and z such that $h'(x) = \frac{h(z)-h(y)}{z-y} \neq 0$ and this contradicts $h'(x) = 0$. Thus $h(y) = h(z)$ for all $y, z \in (a,b)$. Hence there is a $c \in \mathrm{F}$ such that $h(x) = c$ for all $x \in (a,b)$. ■

5. Proofs of the main results

In the present section AxFrame is assumed without any further mentioning.

Let $\widehat{\ } : \mathrm{F} \to \mathrm{F}^d$ be the natural embedding defined as $\widehat{\ } : x \mapsto \langle x, 0, \ldots, 0\rangle$. We define the **life-curve** of observer k as seen by observer m as $Tr^k_m := \widehat{\ } \circ f^k_m$. Throughout this work we denote $\widehat{\ }(x)$ by $\widehat{x}$, for $x \in \mathrm{F}$. Thus $Tr^k_m(t)$ is the coordinate point where m observes the event "k's wristwatch shows t", i.e. $Tr^k_m(t) = p$ iff $ev_m(p) = ev_k(\langle t, 0, \ldots, 0\rangle) = ev_k(\widehat{t}\,)$.

In the following proposition, we list several easy but useful consequences of some of our axioms.

Proposition 5.1. Let $m \in \mathrm{IOb}$ and $k \in \mathrm{Ob}$. Then (i)–(viii) below hold.

(i) Assume AxPh. Then $Cd(m) = \mathrm{F}^d$ and for all distinct $p, q \in \mathrm{F}^d$, $ev_m(p) \neq ev_m(q)$.
(ii) Assume AxPh and AxSelf$^-$. Then $tr_m(m) = \bar{t}$.
(iii) Assume AxPh. Then $f^k_m : \mathrm{F}^d \xrightarrow{\circ} \mathrm{F}^d$ and $Tr^k_m : \mathrm{F} \xrightarrow{\circ} \mathrm{F}^d$.

[6], i.e. if there is a $c \in \mathrm{F}$ such that $h(x) = c$ for all $x \in [a,b]$,

(iv) Assume AxPh and AxEv. If $k \in \mathrm{IOb}$, then $f^k_m : \mathrm{F}^d \to \mathrm{F}^d$ is a bijection and $Tr^k_m : \mathrm{F} \to \mathrm{F}^d$ is an injection.
(v) Assume AxEv. Let $h \in \mathrm{IOb}$. Then $f^k_h = f^k_m \circ f^m_h$ and $Tr^k_h = Tr^k_m \circ f^m_h$.
(vi) Assume AxAcc and AxEv. Then $tr_k(k) \subseteq Dom(f^k_m)$.
(vii) Assume AxSelf$^-$. Then $\{\widehat{x} : x \in Dom(Tr^k_m)\} \subseteq tr_k(k)$ and $Rng(Tr^k_m) \subseteq tr_m(k)$.
(viii) Assume AxAcc, AxEv and AxSelf$^-$. Then $\{\widehat{x} : x \in Dom(Tr^k_m)\} = tr_k(k)$.

proof. To prove (i), let $p, q \in \mathrm{F}^d$ be distinct points. Then there is a line of slope 1 that contains p but does not contain q. By AxPh, this line is the trace of a photon. For such a photon ph, we have $ph \in ev_m(p)$ and $ph \notin ev_m(q)$. Hence $ev_m(p) \neq ev_m(q)$ and $ev_m(p) \neq \emptyset$. Thus (i) holds.

(ii) follows from (i) since $tr_m(m) = Cd(m) \cap \bar{t}$ by AxSelf$^-$.

(iii) and (iv) follow from (i) by the definitions of the world-view transformation and the life-curve.

To prove (v), let $\langle p, q\rangle \in f^k_h$. Then $ev_k(p) = ev_h(q) \neq \emptyset$. Since, by AxEv, h and m observe the same set of events, there is an $r \in \mathrm{F}^d$ such that $ev_m(r) = ev_h(q)$. But then $\langle p, r\rangle \in f^k_m$ and $\langle r, q\rangle \in f^m_h$. Hence $\langle p, q\rangle \in f^k_m \circ f^m_h$. Thus $f^k_h \subseteq f^k_m \circ f^m_h$. The other inclusion follows from the definition of the world-view transformation. Thus $f^k_h = f^k_m \circ f^m_h$ and $Tr^k_h = \widehat{\ } \circ f^k_h = \widehat{\ } \circ f^k_m \circ f^m_h = Tr^k_m \circ f^m_h$.

To prove (vi), let $q \in tr_k(k)$. By AxAcc, there is an $h \in \mathrm{IOb}$ such that h is a co-moving observer of k at q. For such an h, we have $f^k_h(q) = q$ and, by (v), $Dom(f^k_h) \subseteq Dom(f^k_m)$. Thus $q \in Dom(f^k_m)$.

To prove (vii), let $\langle x, q\rangle \in Tr^k_m$. Then $\langle \widehat{x}, q\rangle \in f^k_m$. But then $ev_k(\widehat{x}) = ev_m(q) \neq \emptyset$. Thus $\widehat{x} \in Cd(k) \cap \bar{t}$. By AxSelf$^-$, $\widehat{x} \in tr_k(k)$; and this proves the first part of (vii). By $\widehat{x} \in tr_k(k)$, we have $k \in ev_k(\widehat{x}) = ev_m(q)$. Thus $q \in tr_m(k)$ and this proves the second part of (vii).

The "$\subseteq$ part" of (viii) follows from (vii). To prove the other inclusion, let $p \in tr_k(k)$. Then, by AxSelf$^-$ and (vi), $p \in \bar{t} \cap Dom(f^k_m)$. Thus there are $x \in \mathrm{F}$ and $q \in \mathrm{F}^d$ such that $\widehat{x} = p$ and $\langle p, q\rangle \in f^k_m$. But then $\langle x, q\rangle \in \widehat{\ } \circ f^k_m = Tr^k_m$. Hence $x \in Dom(Tr^k_m)$. ■

We say that f is **well-parametrized** iff $f : \mathrm{F} \xrightarrow{\circ} \mathrm{F}^d$ and the following holds: if $x \in Dom(f)$ is an accumulation point of $Dom(f)$, then f is differentiable at x and its derivate at x is of Minkowski-length 1, i.e. $\mu\big(f'(x)\big) = 1$. Assume $\mathfrak{F} = \mathfrak{R}$. Then the curve f is well-parametrized iff f is parametrized according to Minkowski-length, i.e. for all $x, y \in \mathrm{F}$, if $[x, y] \subseteq Dom(f)$, the Minkowski-length of f restricted to $[x, y]$ is $y - x$. (By Minkowski-length of a curve we mean length according to Minkowski-metric, e.g., in the sense of Wald [22, p.43, (3.3.7)]). **Proper time** or **wristwatch time** is defined as the Minkowski-length of a time-like curve, cf. e.g., Wald [22, p.44, (3.3.8)], Taylor-Wheeler [20, 1-1-2] or d'Inverno [5, p.112, (8.14)]. Thus a curve defined on a subset of $\mathfrak{R}$ is well-parametrized iff it is parametrized according to proper time, or wristwatch-time. (Cf. e.g., [5, p.112, (8.16)].)

The next proposition states that life-curves of accelerated observers in models of AccRel$_0$ are well-parametrized. This implies that accelerated clocks behave as expected in models of AccRel$_0$. Remark 5.3 after the proposition will state a kind of "completeness theorem" for life-curves of accelerated observers, much in the spirit of Remark 2.3.

Proposition 5.2. Assume AccRel_0 and $d > 2$. Let $m \in \mathrm{IOb}$ and $k \in \mathrm{Ob}$. Then Tr^k_m is well-parametrized and definable.

proof. Let $m \in \mathrm{IOb}$, $k \in \mathrm{Ob}$. Then Tr^k_m is definable by its definition. Furthermore, $f^k_m : \mathrm{F}^d \xrightarrow{\circ} \mathrm{F}^d$ and $Tr^k_m : \mathrm{F} \xrightarrow{\circ} \mathrm{F}^d$ by (iii) of Proposition 5.1. Let $x \in Dom(Tr^k_m)$ be an accumulation point of $Dom(Tr^k_m)$. We would like to prove that Tr^k_m is differentiable at x and its derivate at x is of Minkowski-length 1. $\widehat{x} \in tr_k(k)$ by (vii) of Proposition 5.1. Thus, by AxAcc, there is a co-moving inertial observer of k at $\widehat{x}$. By Proposition 4.3, we can assume that m is a co-moving inertial observer of k at $\widehat{x}$, i.e. $m \succ_{\widehat{x}} k$, because of the following three statements. By (v) of Proposition 5.1, for every $h \in \mathrm{IOb}$, either of Tr^k_m and Tr^k_h can be obtained from the other by composing the other by a world-view transformation between inertial observers. By Theorem 2.2, world-view transformations between inertial observers are Poincaré-transformations. Poincaré-transformations are affine and preserve the Minkowski-distance.

Now, assume that m is a co-moving inertial observer of k at $\widehat{x}$. Then $f^k_m(\widehat{x}) = \widehat{x}$, $z\widehat{1} = \widehat{z}$ and $Tr^k_m(z) = f^k_m(\widehat{z})$ for every $z \in Dom(Tr^k_m)$. Therefore

$$\forall y \in Dom(Tr^k_m) \quad \left|Tr^k_m(y) - Tr^k_m(x) - (y-x)\widehat{1}\right| = \left|f^k_m(\widehat{y}) - \widehat{y}\right|. \tag{1}$$

Since $Dom(f^k_m) \subseteq Cd(k)$ and $\widehat{y} \in Dom(f^k_m)$ if $y \in Dom(Tr^k_m)$, we have that for all $\delta \in \mathrm{F}^+$,

$$\forall y \in Dom(Tr^k_m) \quad |y-x| < \delta \implies \widehat{y} \in B_\delta(\widehat{x}) \cap Cd(k). \tag{2}$$

Let $\varepsilon \in \mathrm{F}^+$ be fixed. Since $m \succ_{\widehat{x}} k$ and $f^k_m : \mathrm{F}^d \xrightarrow{\circ} \mathrm{F}^d$, there is a $\delta \in \mathrm{F}^+$ such that

$$\forall p \in B_\delta(\widehat{x}) \cap Cd(k) \quad \left|p - f^k_m(p)\right| \le \varepsilon|p - \widehat{x}|. \tag{3}$$

Let such a δ be fixed. By (2), (3) and the fact that $|\widehat{y} - \widehat{x}| = |y - x|$, we have that

$$\forall y \in Dom(Tr^k_m) \quad |y-x| < \delta \implies \left|\widehat{y} - f^k_m(\widehat{y})\right| \le \varepsilon|y-x|.$$

By this and (1), we have

$$\forall y \in Dom(Tr^k_m) \quad |y-x| < \delta \implies \left|Tr^k_m(y) - Tr^k_m(x) - (y-x)\widehat{1}\right| \le \varepsilon|y-x|.$$

Thus $(Tr^k_m)'(x) = \widehat{1}$. This completes the proof since $\mu(\widehat{1}) = 1$. ■

Remark 5.3. Well parametrized curves are exactly the life-curves of accelerated observers, in models of AccRel_0, as follows. Let $\mathfrak{F}$ be an Euclidean ordered field and let $f : \mathrm{F} \xrightarrow{\circ} \mathrm{F}^d$ be well-parametrized. Then there are a model $\mathfrak{M}$ of AccRel_0, $m \in \mathrm{IOb}$ and $k \in \mathrm{Ob}$ such that $Tr^k_m = f$ and the ordered field reduct of $\mathfrak{M}$ is $\mathfrak{F}$. Recall that if $\mathfrak{F} = \mathfrak{R}$, then this $\mathfrak{M}$ is a model of AccRel. This is not difficult to prove by using the methods of the present paper. ◁

We say that $p \in \mathrm{F}^d$ is **vertical** iff $p \in \bar{t}$.

Lemma 5.4. Let $f : \mathrm{F} \xrightarrow{\circ} \mathrm{F}^d$ be well-parametrized. Then (i) and (ii) below hold.

(i) Let $x \in Dom(f)$ be an accumulation point of $Dom(f)$. Then f_t is differentiable at x and $|f_t'(x)| \ge 1$. Furthermore, $|f_t'(x)| = 1$ iff $f'(x)$ is vertical.

(ii) Assume IND and that f is definable. Let $[a,b] \subseteq Dom(f)$. Then f_t is increasing or decreasing on $[a,b]$. If f_t is increasing on $[a,b]$ and $a \neq b$, then $f_t'(x) \geq 1$ for all $x \in [a,b]$.

proof. Let f be well-parametrized.

To prove (i), let $x \in Dom(f)$ be an accumulation point of $Dom(f)$. Then $f'(x)$ is of Minkowski-length 1. By Proposition 4.3, f_t is differentiable at x and $f_t'(x) = f'(x)_t$. Now, (i) follows from the fact that the absolute value of the time component of a vector of Minkowski-length 1 is always greater than 1 and it is 1 iff the vector is vertical.

To prove (ii), assume IND and that f is definable. Let $[a,b] \subseteq Dom(f)$. From (i), we have $f_t'(x) \neq 0$ for all $x \in [a,b]$. Thus, by Rolle's theorem, f_t is injective on $[a,b]$. Thus, by Bolzano's theorem, f_t is increasing or decreasing on $[a,b]$ since f_t is continuous and injective on $[a,b]$. Assume that f_t is increasing on $[a,b]$ and $a \neq b$. Then $f_t'(x) \geq 0$ for all $x \in [a,b]$ by the definition of the derivate. Hence, by (i), $f_t'(x) \geq 1$ for all $x \in [a,b]$. ■

Theorem 5.5. Assume IND. Let $f : \mathrm{F} \xrightarrow{\circ} \mathrm{F}^d$ be definable, well-parametrized and $[a,b] \subseteq Dom(f)$. Then (i) and (ii) below hold.

(i) $b - a \leq |f_t(b) - f_t(a)|$.
(ii) If $f(x)_s \neq f(a)_s$ for an $x \in [a,b]$, then $b - a < \big|f_t(b) - f_t(a)\big|$.

proof. Let $f : \mathrm{F} \xrightarrow{\circ} \mathrm{F}^d$ be definable, well-parametrized and $[a,b] \subseteq Dom(f)$. We can assume that $a \neq b$. For every $i \leq d$, f_i is definable and differentiable on $[a,b]$ by Proposition 4.3. Then, by the Main Value Theorem, there is an $s \in (a,b)$ such that $f_t'(s) = \frac{f_t(b)-f_t(a)}{b-a}$. By (i) of Lemma 5.4, we have $1 \leq |f_t'(s)|$. But then, $b - a \leq \big|f_t(b) - f_t(a)\big|$. This completes the proof of (i).

To prove (ii), let $x \in [a,b]$ be such that $f(x)_s \neq f(a)_s$. Let $1 < i \leq d$ be such that $f_i(x) \neq f_i(a)$. Then, by the Main Value Theorem, there is an $y \in (a,b)$ such that $f_i'(y) = \frac{f_i(x)-f_i(a)}{x-a} \neq 0$. Thus $f'(y)$ is not vertical. Therefore, by (i) of Lemma 5.4, we have $1 < |f_t'(y)|$. Thus, by the definition of the derivate, there is a $z \in (y,b)$ such that $1 < \frac{|f_t(z)-f_t(y)|}{z-y}$. Hence we have

$$z - y < |f_t(z) - f_t(y)|.$$

Let us note that $a < y < z < b$. By applying (i) to $[a,y]$ and $[z,b]$, respectively, we get

$$y - a \leq \big|f_t(y) - f_t(a)\big| \quad \text{and} \quad b - z \leq \big|f_t(b) - f_t(z)\big|.$$

f_t is increasing or decreasing on $[a,b]$ by (ii) of Lemma 5.4. Thus $f_t(a) < f_t(y) < f_t(z) < f_t(b)$ or $f_t(a) > f_t(y) > f_t(z) > f_t(b)$. Now, by adding up the last three inequalities, we get $b - a < \big|f_t(b) - f_t(a)\big|$. ■

Let $a \in \mathrm{F}^d$. For convenience, we introduce the following notation: $a^+ := a$ if $a_t \geq 0$ and $a^+ := -a$ if $a_t < 0$. A set $H \subseteq \mathrm{F}^d$ is called **twin-paradoxical** iff $\widehat{1} \in H$, $o \notin H$, $slope(p) < 1$ if $p \in H$, for all $p \in \mathrm{F}^d$ if $slope(p) < 1$, then there is a $\lambda \in \mathrm{F}$ such that $\lambda p \in H$, and for all distinct $p, q, r \in H$ and for all $\lambda, \mu \in \mathrm{F}^+$, $r^+ = \lambda p^+ + \mu q^+$ implies that $\lambda + \mu < 1$.

A positive answer to the following question would also provide a positive answer to Question 3.7, cf. [19, §3].

Question 5.6. Assume IND. Let $f : \mathrm{F} \xrightarrow{\circ} \mathrm{F}^d$ be definable such that f is differentiable on $[a,b]$ and $f(a), f(b) \in \bar{t}$. Furthermore, let the set $\{f'(x) : x \in [a,b]\}$ be a subset of a twin-paradoxical set. Are then (i) and (ii) below true?

(i) $b - a \leq \big|f_t(b) - f_t(a)\big|$.
(ii) If $f(x)_s \neq f(a)_s$ for an $x \in [a,b]$, then $b - a < \big|f_t(b) - f_t(a)\big|$. ◁

Theorem 5.7. Assume IND. Let $f, g : \mathrm{F} \xrightarrow{\circ} \mathrm{F}^d$ be definable and well-parametrized. Let $[a,b] \subseteq Dom(f)$ and $[a',b'] \subseteq Dom(g)$ be such that $\{f(r) : r \in [a,b]\} = \{g(r') : r' \in [a',b']\}$. Then $b - a = b' - a'$.

proof. By (ii) of Lemma 5.4, f_t is increasing or decreasing on $[a,b]$ and so is g_t on $[a',b']$. We can assume that $Dom(f) = [a,b]$, $Dom(g) = [a',b']$ and that f_t and g_t are increasing on $[a,b]$ and $[a',b']$, respectively.[7] Then $Rng(f) = Rng(g)$. Furthermore, f and g are injective since f_t and g_t are such. Since $Rng(f) = Rng(g)$ and g_t is injective, $f \circ g^{-1} = f_t \circ g_t^{-1}$. Let $h := f \circ g^{-1} = f_t \circ g_t^{-1}$. Since $Rng(f_t) = Rng(g_t)$ and f_t and g_t are increasing, h is an increasing bijection between $[a,b]$ and $[a',b']$. Hence $h(a) = a'$ and $h(b) = b'$. We are going to prove that $b - a = b' - a'$ by proving that there is a $c \in \mathrm{F}$ such that $h(x) = x + c$ for all $x \in [a,b]$. We can assume that $a \neq b$ and $a' \neq b'$. By Lemma 5.4, f_t and g_t are differentiable on $[a,b]$ and $[a',b']$, respectively, and $f_t'(x) > 0$ for all $x \in [a,b]$ and $g_t'(x') > 0$ for all $x' \in [a',b']$. By (iv) and (v) of Proposition 4.2, $h = f_t \circ g_t^{-1}$ is also differentiable on (a,b). By $h = f \circ g^{-1}$, we have $f = h \circ g$. Thus $f'(x) = h'(x)g'\big(h(x)\big)$ for all $x \in (a,b)$ by (iv) of Proposition 4.2. Since both $f'(x)$ and $g'\big(h(x)\big)$ are of Minkowski-length 1 and their time-components are positive[8] for all $x \in (a,b)$, we conclude that $h'(x) = 1$ for all $x \in (a,b)$. By Proposition 4.9, we get that there is a $c \in \mathrm{F}$ such that $h(x) = x + c$ for all $x \in (a,b)$ and thus for all $x \in [a,b]$ since h is an increasing bijection between $[a,b]$ and $[a',b']$. ■

proof of Theorem 3.1. Assume AccRel and $d > 2$. Let $m \in \mathrm{IOb}$ and $k \in \mathrm{Ob}$. Let $p, q \in tr_k(k)$, $p', q' \in tr_m(m)$ be such that $\langle p, p'\rangle, \langle q, q'\rangle \in f^k_m$, $[pq] \subseteq tr_k(k)$ and $[p'q'] \not\subseteq tr_m(k)$, cf. Figure 2. Let us abbreviate Tr^k_m by Tr. We are going to prove that $|q_t - p_t| < |q'_t - p'_t|$ by applying Theorem 5.5 to Tr and $[p_t, q_t]$. By Proposition 5.2,

$$Tr : \mathrm{F} \xrightarrow{\circ} \mathrm{F}^d \text{ is well-parametrized and definable.} \tag{4}$$

By AxSelf$^-$, $p, q, p', q' \in \bar{t}$. By $\widehat{p_t} = p$, by $\widehat{q_t} = q$, by $\langle p, p'\rangle, \langle q, q'\rangle \in f^k_m$ and by $Tr = \widehat{\ } \circ f^k_m$,

$$Tr(p_t) = p' \quad \text{and} \quad Tr(q_t) = q'. \tag{5}$$

[7] It can be assumed that f_t is increasing on $[a,b]$ because the assumptions of the theorem remain true when f and $[a,b]$ are replaced by $-Id \circ f$ and $[-b,-a]$, respectively, and f_t is decreasing on $[a,b]$ iff $(-Id \circ f)_t$ is increasing on $[-b,-a]$.

[8] i.e. $f_t'(x) > 0$ and $g_t'\big(h(x)\big) > 0$

By $p, q \in tr_k(k)$ and $\langle p, p'\rangle, \langle q, q'\rangle \in f^k_m$, we have that $p', q' \in tr_m(k)$. Thus, by $[p'q'] \not\subseteq tr_m(k)$, we have that $p' \neq q'$. Hence, by (5), $p_t \neq q_t$. We can assume that $p_t < q_t$. By (viii) of Proposition 5.1, $\{\widehat{x} : x \in Dom(Tr)\} = tr_k(k)$. Since $[pq] \subseteq tr_k(k)$,

$$[p_t, q_t] \subseteq Dom(Tr). \tag{6}$$

By (i) of Lemma 5.4, (4) and (6), we have that Tr_t is differentiable on $[p_t, q_t]$, thus it is continuous on $[p_t, q_t]$. Let $x' \in [p'q'] \subseteq \bar{t}$ be such that $x' \notin tr_m(k)$. By Bolzano's theorem and (5), there is an $x \in [p_t, q_t]$ such that $Tr_t(x) = x'_t$. Let such an x be fixed. $Tr(x) \in tr_m(k)$ since $Rng(Tr) \subseteq tr_m(k)$ by (vii) of Proposition 5.1. But then $Tr(x) \neq x'$. Hence $Tr(x) \notin \bar{t}$. Thus

$$x \in [p_t, q_t] \quad \text{and} \quad Tr(x)_s \neq Tr(p_t)_s \tag{7}$$

since $Tr(p_t) = p' \in \bar{t}$. Now, by (4)–(7) above, we can apply (ii) of Theorem 5.5 to Tr and $[p_t, q_t]$, and we get that $|q_t - p_t| < |Tr_t(q_t) - Tr_t(p_t)| = |q'_t - p'_t|$. ∎

proof of Theorem 3.2. Assume AccRel and $d > 2$. Let k and m be observers. Let $p, q \in tr_k(k)$, $p', q' \in tr_m(m)$ be such that $\emptyset \notin \{ev_k(r) : r \in [pq]\} = \{ev_m(r') : r' \in [p'q']\}$, cf. the right hand side of Figure 2. Thus $[pq] \subseteq Cd(k)$ and $[p'q'] \subseteq Cd(m)$. By AxSelf$^-$, $tr_k(k) = Cd(k) \cap \bar{t}$ and $tr_m(m) = Cd(m) \cap \bar{t}$. Therefore $[pq] \subseteq tr_k(k) \subseteq \bar{t}$ and $[p'q'] \subseteq tr_m(m) \subseteq \bar{t}$. We can assume that $p_t \leq q_t$ and $p'_t \leq q'_t$. Let $h \in$ IOb. We are going to prove that $\big|q_t - p_t\big| = \big|q'_t - p'_t\big|$, by applying Theorem 5.7 as follows: let $[a, b] := [p_t, q_t]$, $[a', b'] := [p'_t, q'_t]$, $f := Tr^k_h$ and $g := Tr^m_h$. By (viii) of Proposition 5.1, by $[pq] \subseteq tr_k(k)$ and by $[p'q'] \subseteq tr_m(m)$, we conclude that $[a, b] \subseteq Dom(f)$ and $[a', b'] \subseteq Dom(g)$. By Proposition 5.2, f and g are well-parametrized and definable. We have $\{f(r) : r \in [a, b]\} = \{g(r') : r' \in [a', b']\}$ since $\{ev_k(r) : r \in [pq]\} = \{ev_m(r') : r' \in [p'q']\}$. Thus, by Theorem 5.7, we conclude that $b - a = b' - a'$. Thus $\big|q_t - p_t\big| = \big|q'_t - p'_t\big|$ and this is what we wanted to prove. ∎

proofs of Theorems 3.5 and 3.6. We will construct three models. Let $\mathfrak{F} = \langle \mathrm{F}; +, \cdot, \leq\rangle$ be an Euclidean ordered field different from $\mathfrak{R}$. For every $p \in \mathrm{F}^d$, let $m_p : \mathrm{F}^d \to \mathrm{F}^d$ denote the translation by vector p, i.e. $m_p : q \mapsto q + p$. $f : \mathrm{F}^d \to \mathrm{F}^d$ is called **translation-like** iff for all $q \in \mathrm{F}^d$, there is a $\delta \in \mathrm{F}^+$ such that for all $p \in B_\delta(q)$, $f(p) = m_{f(q)-q}(p)$ and for all $p, q \in \mathrm{F}^d$, $f(p) = f(q)$ and $p \in \bar{t}$ imply that $q \in \bar{t}$. Let $k : \mathrm{F}^d \to \mathrm{F}^d$ be translation-like. First we construct a model $\mathfrak{M}_{(\mathfrak{F},k)}$ of AccRel$_0$ and (i) and (ii) of Theorem 3.6 for $\mathfrak{F}$ and k, which will be a model of (iii) and (iv) of Theorem 3.6 if k is a bijection. We will show that Tp is false in $\mathfrak{M}_{(\mathfrak{F},k)}$. Then we will choose $\mathfrak{F}$ and k appropriately to get the desired models in which Ddpe is false, too.

Let the ordered field reduct of $\mathfrak{M}_{(\mathfrak{F},k)}$ be $\mathfrak{F}$. Let $\{I_1, I_2, I_3, I_4, I_5\}$ be a partition[9] of F such that every I_i is open, $x \in I_2 \iff x + 1 \in I_3 \iff x + 2 \in I_4$ and for all $y \in I_i$ and

[9]I.e. I_i's are disjoint and $\mathrm{F} = I_1 \cup I_2 \cup I_3 \cup I_4 \cup I_5$.

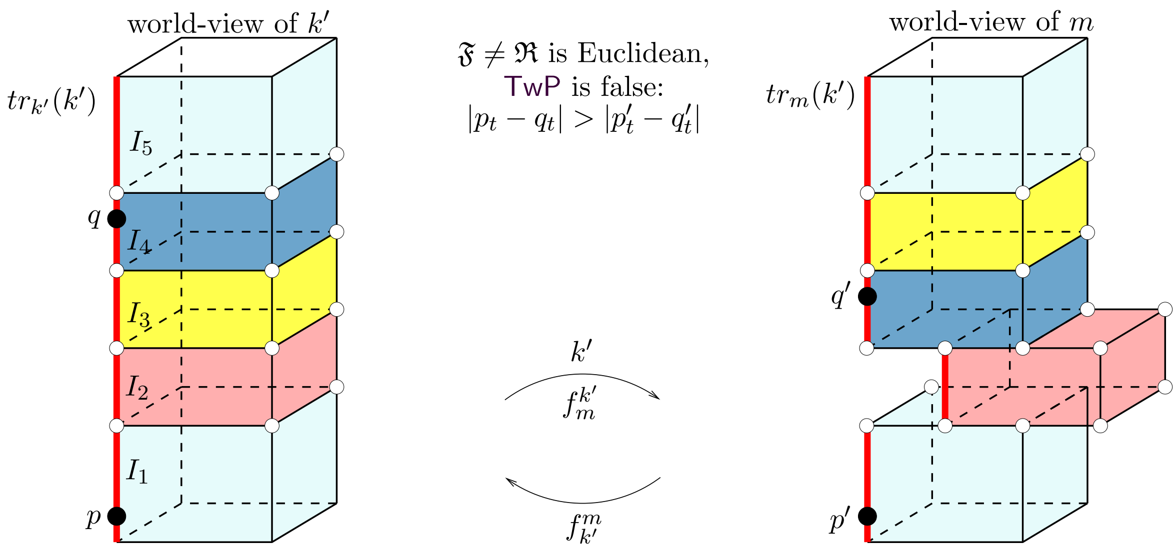

FIGURE 4. for the proofs of Theorems 3.5 and 3.6.

$z \in I_j$, $y \le z \iff i \le j$. Such a partition can easily be constructed.[10] Let

$$k'(p) := \begin{cases} p & \text{if } p_t \in I_1 \cup I_5 \\ p - \widehat{1} & \text{if } p_t \in I_4 \\ p + \widehat{1} & \text{if } p_t \in I_3 \\ p + \langle 0, 1, 0, \dots, 0\rangle & \text{if } p_t \in I_2 \end{cases}$$

for every $p \in \mathrm{F}^d$, cf. Figure 4. It is easy to see that k' is a translation-like bijection. Let $\mathrm{IOb} := \{m_p : p \in \mathrm{F}^d\}$, $\mathrm{Ob} := \mathrm{IOb} \cup \{k, k'\}$, $\mathrm{Ph} := \{l \in \mathit{Lines} : \mathit{slope}(l) = 1\}$ and $\mathrm{B} := \mathrm{Ob} \cup \mathrm{Ph}$. Recall that $o := \langle 0, \dots, 0\rangle$ is the origin. First we give the world-view of m_o then we give the world-view of an arbitrary observer h by giving the world-view transformation between h and m_o. Let $tr_{m_o}(ph) := ph$ and $tr_{m_o}(h) := \{h(x) : x \in \bar{t}\}$ for all $ph \in \mathrm{Ph}$ and $h \in \mathrm{Ob}$. And let $ev_{m_o}(p) := \{b \in \mathrm{B} : p \in tr_{m_o}(b)\}$ for all $p \in \mathrm{F}^d$. Let $f^h_{m_o} := h$ for all $h \in \mathrm{Ob}$. From these world-view transformations, we can obtain the world-view of each observer h in the following way: $ev_h(p) := ev_{m_o}\big(h(p)\big)$ for all $p \in \mathrm{F}^d$. And from the world-views, we can obtain the W relation as follows: for all $h \in \mathrm{Ob}$, $b \in \mathrm{B}$ and $p \in \mathrm{F}^d$, let $\mathrm{W}(h, b, p)$ iff $b \in ev_h(p)$. Thus we are given the model $\mathfrak{M}_{(\mathfrak{F},k)}$. We note that $f^m_h = m \circ h^{-1}$ and $m_{h(q)-q} \succ_q h$ for all $m, h \in \mathrm{Ob}$ and $q \in \mathrm{F}^d$. It is easy to check that the axioms of $\mathsf{AccRel_0}$ and (i) and (ii) of Theorem 3.6 are true in $\mathfrak{M}_{(\mathfrak{F},k)}$ and that if k is a bijection, then (iii) and (iv) of Theorem 3.6 are also true in $\mathfrak{M}_{(\mathfrak{F},k)}$. Let $p, q \in \bar{t}$ be such that $p_t \in I_1$, $q_t \in I_4$; and let $p' := k'(p) = p$, $q' := k'(q) = q - \widehat{1}$ and $m := m_o$. It is easy to

[10] Let $H \subset \mathrm{F}$ be a non-empty bounded set that does not have a supremum. Let $I_1 := \{x \in \mathrm{F} : \exists h \in H \quad x < h\}$, $I_2 := \{x + 1 \in \mathrm{F} : x \in I_1\} \setminus I_1$, $I_3 := \{x + 1 \in \mathrm{F} : x \in I_2\}$, $I_4 := \{x + 1 \in \mathrm{F} : x \in I_3\}$ and $I_5 := \mathrm{F} \setminus (I_1 \cup I_2 \cup I_3 \cup I_4)$.

check that Tp is false in $\mathfrak{M}_{(\mathfrak{F},k)}$ for k', m, p, q, p' and q', i.e. $p, q \in tr_{k'}(k')$, $p', q' \in tr_m(m)$, $\langle p, p'\rangle, \langle q, q'\rangle \in f_m^{k'}$, $[pq] \subseteq tr_{k'}(k')$, $[p'q'] \not\subseteq tr_m(k')$ and $\left|q_t - p_t\right| \not< \left|q'_t - p'_t\right|$, cf. Figure 4.

To construct the first model, let $\mathfrak{F}$ be an arbitrary Euclidean ordered field different from $\mathfrak{R}$ and let $\{I_1, I_2\}$ be a partition of F such that for all $x \in I_1$ and $y \in I_2$, $x < y$. Let

$$k(p) := \begin{cases} p & \text{if } p_t \in I_1 \\ p - \widehat{1} & \text{if } p_t \in I_2 \end{cases}$$

for every $p \in \mathrm{F}^d$, cf. Figure 5. It is easy to see that k is translation-like. Let $p, q \in \bar{t}$ be such that $p_t, p_t + 1 \in I_1$ and $q_t, q_t - 1 \in I_2$; and let $p' := k(p) = p$, $q' := k(q) = q - \widehat{1}$ and $m := m_o$. It is also easy to check that Ddpe is false in $\mathfrak{M}_{(\mathfrak{F},k)}$ for k, m, p, q, p' and q', i.e. $p, q \in tr_k(k)$, $p', q' \in tr_m(m)$, $\emptyset \notin \{ev_k(r) : r \in [pq]\} = \{ev_m(r') : r' \in [p'q']\}$ and $\left|q_t - p_t\right| \neq \left|q'_t - p'_t\right|$, cf. Figure 5. This completes the proof of Theorem 3.5.

To construct the second model, let $\mathfrak{F}$ be an arbitrary non-Archimedean Euclidean ordered field. Let $a \sim b$ if $a, b \in \mathrm{F}$ and $a - b$ is infinitesimally small. It is easy to see that $\sim$ is an equivalence relation. Let us choose an element from every equivalence class of $\sim$ and let $\tilde{a}$ denote the chosen element equivalent with $a \in \mathrm{F}$. Let $k(p) := \langle p_t + \tilde{p}_t, p_s\rangle$ for every $p \in \mathrm{F}^d$, cf. Figure 5. It is easy to see that k is a translation-like bijection. Let $p := o$, $q := \widehat{1}$, $p' := k(p) = \langle \tilde{0}, 0, \ldots, 0\rangle$, $q' := k(q) = \langle 1 + \tilde{1}, 0, \ldots, 0\rangle$ and $m := m_o$. It is also easy to check that Ddpe is false in $\mathfrak{M}_{(\mathfrak{F},k)}$ for k, m, p, q, p' and q', cf. Figure 5.

To construct the third model, let $\mathfrak{F}$ be an arbitrary countable Archimedean Euclidean ordered field and let $k(p) = \langle f(p_t), p_s\rangle$ for every $p \in \mathrm{F}^d$ where $f : \mathrm{F} \to \mathrm{F}$ is constructed as follows, cf. Figures 5, 6. We can assume that $\mathfrak{F}$ is a subfield of $\mathfrak{R}$ by [11, Theorem 1 in §VIII]. Let a be a real number that is not an element of F. Let us enumerate the elements of $[a, a+2] \cap \mathrm{F}$ and denote the i-th element with r_i. First we cover $[a, a+2] \cap \mathrm{F}$ with infinitely many disjoint subintervals of $[a, a+2]$ such that the sum of their length is 1, the length of each interval is in F and the distance of the left endpoint of each interval from a is also in F. We are going to construct this covering by recursion. In the i-th step, we will use only finitely many new intervals such that the sum of their length is $1/2^i$. In the first step, we cover r_1 with an interval of length $1/2$. Suppose that we have covered r_i for each $i < n$. Since we have used only finitely many intervals yet, we can cover r_n with an interval that is not longer than $1/2^n$. Since $\sum_{i=1}^{n} 1/2^i < 1$, it is easy to see that we can choose finitely many other subintervals of $[a, a+2]$ to be added to this interval such that the sum of their length is $1/2^n$. We are given the covering of $[a, a+2]$. Let us enumerate these intervals. Let I_i be the i-th interval, d_i be the length of I_i, $d_0 := 0$ and $a_i \geq 0$ the distance of a and the left endpoint of I_i. $\sum_{i=1}^{\infty} d_i = 1$ since $\sum_{i=1}^{\infty} 1/2^i = 1$. Let

$$f(x) := \begin{cases} x & \text{if } x < a \\ x - 1 & \text{if } a + 2 \leq x \\ x - a_n + \sum\limits_{i=0}^{n-1} d_i & \text{if } x \in I_n \end{cases}$$

for all $x \in \mathrm{F}$, cf. Figure 6. It is easy to see that k is a translation-like bijection. Let $p, q \in \mathrm{F}^d$ be such that $p_t < a$ and $a + 2 < q_t$; and let $p' := k(p) = p$, $q' := k(q) = q - \widehat{1}$ and

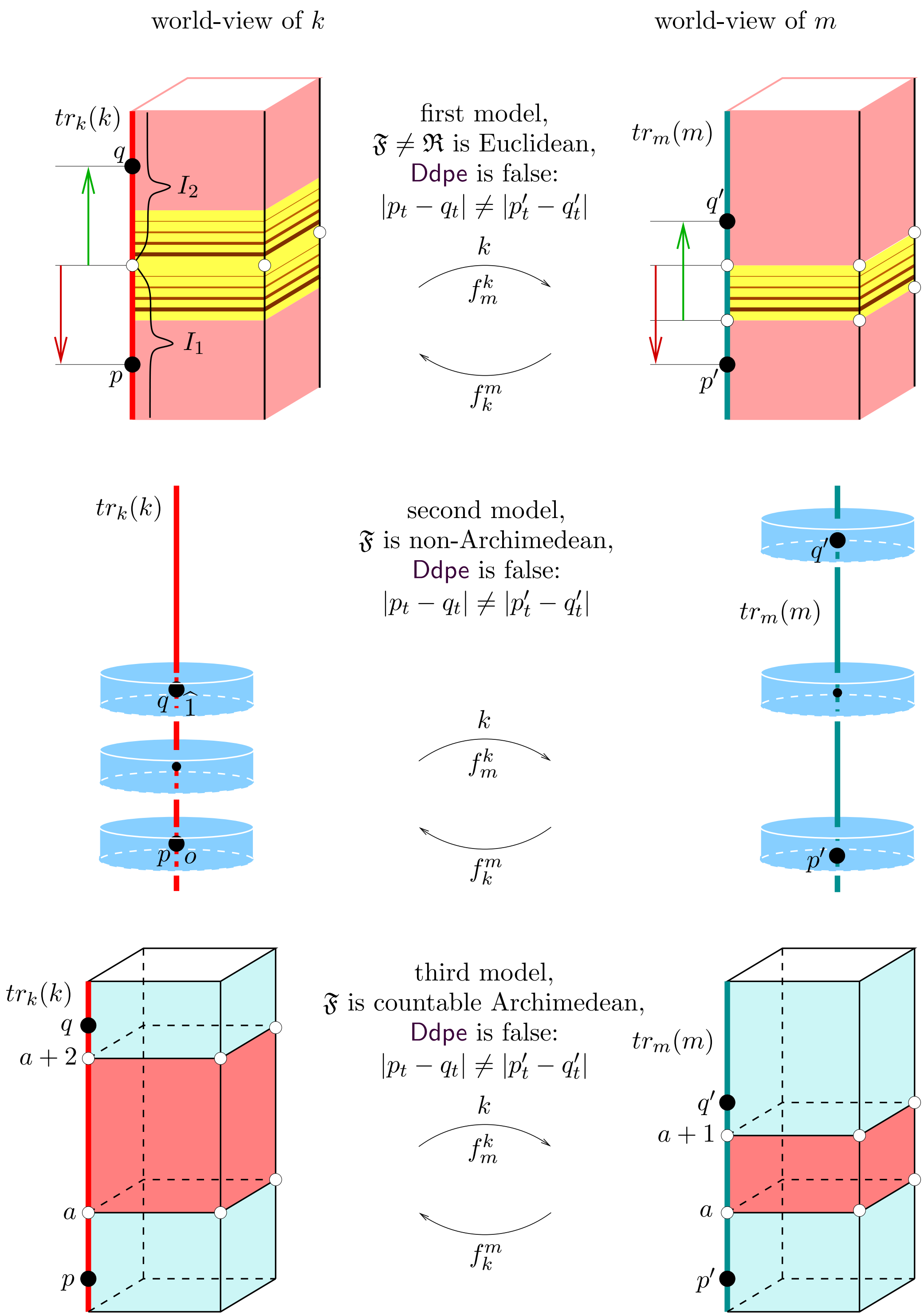

FIGURE 5. for the proofs of Theorems 3.5 and 3.6.

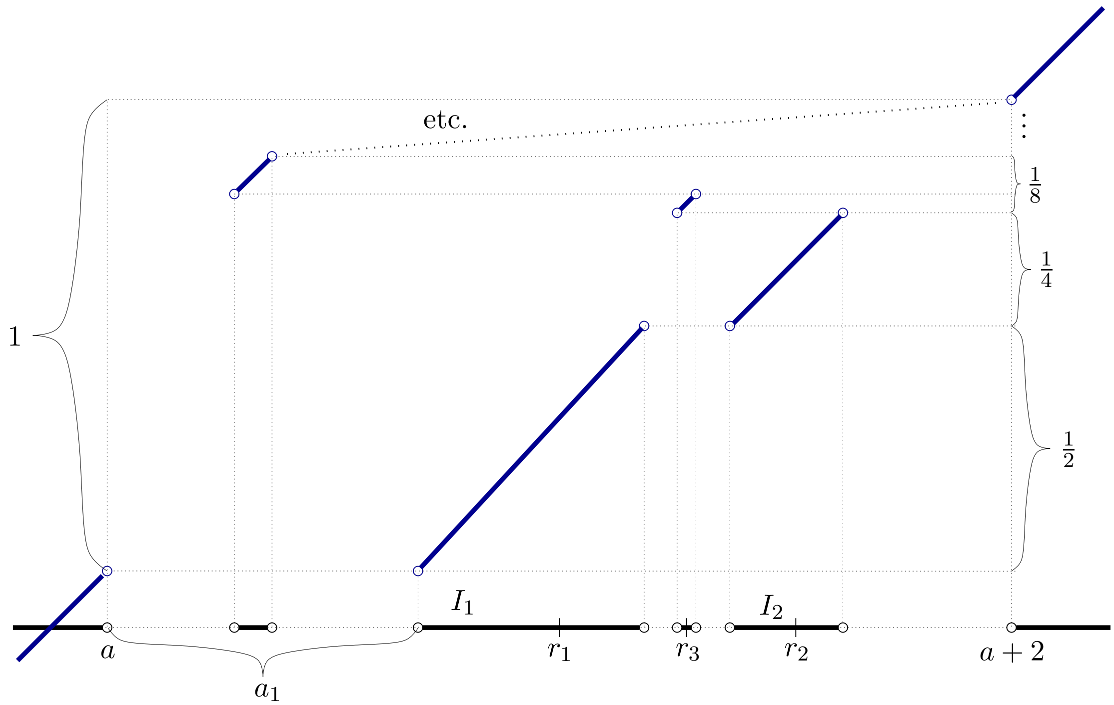

FIGURE 6. for the proofs of Theorems 3.5 and 3.6.

$m := m_o$. It is also easy to check that Ddpe is false in $\mathfrak{M}_{(\mathfrak{F},k)}$ for k, m, p, q, p' and q', cf. Figure 5. ∎

In a subsequent paper, we will discuss how the present methods and in particular AccRel and IND can be used for introducing gravity via Einstein's equivalence principle and for proving that gravity "causes time run slow" (known as the Tower Paradox). In this connection we would like to point out that it is explained in Misner et al. [14] that the theory of accelerated observers (in flat space-time!) is a rather useful first step in building up general relativity by using the methods of that book.

APPENDIX

A FOL-formula expressing AxSelf$^-$ is:

$$\forall m\, \forall p \quad \mathrm{Ob}(m) \wedge \mathrm{F}(p_1) \wedge \ldots \wedge \mathrm{F}(p_d) \quad \Longrightarrow$$
$$\Big[\mathrm{W}(m,m,p) \iff \big(\exists b\ \mathrm{B}(b) \wedge \mathrm{W}(m,b,p) \wedge p_2 = 0 \wedge \ldots \wedge p_d = 0\big)\Big].$$

A FOL-formula expressing AxPh is:

$$\forall m\, \forall p\, \forall q \quad \mathrm{IOb}(m) \wedge \mathrm{F}(p_1) \wedge \mathrm{F}(q_1) \wedge \ldots \wedge \mathrm{F}(p_d) \wedge \mathrm{F}(q_d) \quad \Longrightarrow$$
$$\Big[(p_1 - q_1)^2 = (p_2 - q_2)^2 + \ldots + (p_d - q_d)^2 \iff \exists ph\ \mathrm{Ph}(ph) \wedge \mathrm{W}(m, ph, p) \wedge \mathrm{W}(m, ph, q)\Big] \wedge$$
$$\Big[\forall ph\, \forall \lambda \quad \mathrm{Ph}(ph) \wedge \mathrm{F}(\lambda) \wedge \mathrm{W}(m, ph, p) \wedge \mathrm{W}(m, ph, q) \implies \mathrm{W}\big(m, ph, q + \lambda(p-q)\big)\Big].$$

A FOL-formula expressing AxEv is:

$$\forall m\, \forall k\, \forall p \quad \mathrm{IOb}(m) \wedge \mathrm{IOb}(k) \wedge \mathrm{F}(p_1) \wedge \ldots \wedge \mathrm{F}(p_d) \quad \Longrightarrow$$
$$\exists q \quad \mathrm{F}(q_1) \wedge \ldots \wedge \mathrm{F}(q_d) \wedge \big(\forall b \quad \mathrm{B}(b) \implies [\mathrm{W}(m, b, p) \iff \mathrm{W}(k, b, q)]\big).$$

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