Non-Turing computations via Malament–Hogarth space-times

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Abstract

We investigate the Church–Kalmár–Kreisel–Turing Theses concerning theoretical (necessary) limitations of future computers and of deductive sciences, in view of recent results of classical general relativity theory. We argue that (i) there are several distinguished Church–Turing-type Theses (not only one) and (ii) validity of some of these theses depend on the background physical theory we choose to use. In particular, if we choose classical general relativity theory as our background theory, then the above mentioned limitations (predicted by these Theses) become no more necessary, hence certain forms of the Church–Turing Thesis cease to be valid (in general relativity). (For other choices of the background theory the answer might be different.)

We also look at various "obstacles" to computing a non-recursive function (by relying on relativistic phenomena) published in the literature and show that they can be avoided (by improving the "design" of our future computer). We also ask ourselves, how all this reflects on the arithmetical hierarchy and the analytical hierarchy of uncomputable functions.

Keywords: *Church–Turing Thesis, Computers, Black Holes*

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1 Introduction

Certain variants of the so-called Church–Turing Thesis play a basic role in the foundations of theoretical computer science, logic, meta-mathematics and the so-called "fundamentals of deductive sciences". The Thesis is a well-reasoned, well-motivated "conjecture" [3]. The Thesis was formulated before "black hole physics" was developed. We will recall the Thesis and some of its variants in detail in Section 2.

Roughly speaking, the variant we are interested in concerns inherent limitations of possible future computing devices. These limitations deal with *idealized* computers, therefore they do not involve particular data like the size of our Universe, etc. i.e. these limitations are supposed to be necessarily (i.e. theoretically) true in some sense. On the other hand they do involve some physical theory about time, space, motion and things like that as we will argue in Section 2. Clearly, if we do not presuppose a consistent theory about time, space, motion etc. then it is impossible to formulate Theses of the kind we are referring to. Very roughly, these variants of the Church–Turing Thesis conjecture that if a mathematical function f will be realizable at least in principle by an arbitrary future "artificial computing system" then f must be Turing-computable[4]. Here again, the future "artificial computing system" is understood as being *idealized* and f is realized by the system if *in theory* it is realized by the theoretical description (design) of the system.

In passing we note that this Thesis has many important consequences. One consequence says that "paper-and-pencil-computability" coincides (and will always coincide) with machine-computability. Here by paper-and-pencil-computability we understand realizability by an *algorithm* in the mathematical sense (where we note that the mathematical notion of the algorithm goes back to ancient Greeks, in some sense).

Remark. Some authors, e.g. Pitowsky [25] argue that the Thesis we are interested in is not really Church's Thesis but Wolfram's Thesis (cf. [Wolfram 1985] from the references in [25]). The argument states that Church was not interested in computers, but instead he was interested in the "purely mathematical" notion of an algorithm. We would like to pose the following counter-arguments to this objection collected in items (1)-(5) below.

(1) It is exactly this subtlety because of which we refer to those variants of the Thesis we want to discuss here as *Church–Kalmár–Turing Theses*[5] (instead of calling it Church's or Church–Turing Theses). Perhaps Church himself was not interested in computers but Kalmár and Turing were, and they did take part (emphatically) in refining, publishing etc. of the "abstract, idealized,

[3] We mean the kind of conjecture which cannot be proved but can, in principle, be refuted.

[4] This is only a first, incomplete approximation of a part of the Church–Kalmár–Turing Theses, however. We refer to Theses 2-4' in Section 2 for a better illustration of what the theses we want to discuss here are about.

[5] Actually we should call them as Church–Kalmár–Kreisler–Turing Theses but for simplicity we will write Church–Kalmár–Turing Theses.

theoretical, future-computer-oriented" version of the Thesis[6].

(2) Independently of Church's original motivation, if we look into the literature of our natural sciences today, we find that in the branches listed in the beginning of this introduction (e.g. theoretical computer science, artificial intelligence, cognitive science, etc.) the "abstract computer-oriented" version of the Thesis is being used essentially under the name "Church's Thesis", cf. Odifreddi [19], I.8. (pp. 101-122), Gandy [8], Kreisel [14]. Consequently we think that it is completely justified to investigate the question of under what assumptions these "incarnations" of the Thesis are valid and it is reasonable then to refer to these incarnations as (variants of) Church's Thesis.

(3) We quote from the textbook of Odifreddi [19], p. 51: "Turing machines are theoretical devices, but have been *designed with an eye on physical limitations*". Hence, if we are talking about the Church–Turing Thesis (as is quite customary) then we cannot agree with Pitowsky's and others claim that the Thesis would be only about the purely mathematical notion of algorithms and would have nothing to do with the limitations of idealized physical computers. (Actually, Gandy [8] investigated in some detail the "idealized physical computer" aspect of the Church's Thesis.)

(4) The issue whether the Church–Turing Thesis is only about the pure mathematical notion of an algorithm or whether it also concerns the theoretical limitations of idealized, abstract computing devices (based on some physical theory), has been discussed extensively in the literature of theoretical computer science, logic and related fields. E.g. a special issue of the Notre Dame Journal of Formal Logic [18] is devoted to the subject. We cannot quote all the relevant references here but many of them can be found in [19], I.8, scattered over pp. 101-123.

The general conclusion is that the Church–Turing Thesis is not one Thesis but a collection of *several* Theses[7] some of which deal with the purely mathematical concept of algorithms while other (just as respectable) ones concern (among others) the theoretical limitations of idealized future computing systems which subject will be further elaborated in a more unambiguous manner in Section 2 below.

(5) In [19], p. 103 one can read that in meta-mathematics Church's Thesis is used to prove "absolute unsolvability". To our minds this clearly points in the direction we want to go; namely if a problem is decidable by performing a thought-experiment (consistent, say, with the classical general relativity) then the problem is not absolutely unsolvable (nevertheless it may remain unsolvable for various reasons like lack of resources). We finished our remarks concerning Pitowsky's objection.

The notion of "computable function" splits up into at least three notions. These are

[6]One of the present authors (I.N.) knew Kalmár in person, and he remembers that it was the present variants (cf. Theses 2-4' below) of the Church–Turing Thesis Kalmár was most interested in.

[7]cf. e.g. [19] p. 123.

(i) computability by a pure mathematical algorithm (in the purely mathematical sense);

(ii) computability by some idealized, future computing device based on *some* physical theory (like classical general relativity or quantum mechanics, etc.);

(iii) computability by some computing device based on our present physical world-view i.e. taking into account *all* of our present day physical, cosmological, etc. knowledge on the Universe we are living in.

We would like to illustrate that distinctions between (i)-(iii) are reasonable and not trivial by the following.

In connection with the distinction between (i) and (ii) we note that if we want to define paper-and-pencil-computability done by a group of mathematicians, the question comes up whether we allow one of the mathematicians to take an air trip during the course of their computations (or take a trip by a space-ship to a rotating black hole); if we say yes we need to select a physical theory to control these motions.

Our intended, main distinction between (ii) and (iii) above is that in (ii) physical theories are considered as sets of consistent physical laws without initial data in contrast with (iii) where particular initial data are also taken into account (and the most general known physical theory is used). Furthermore we emphasize that "selection of a particular physical theory" in (ii) is acceptable from science-historical viewpoint only, i.e. without taking into account the particular development of physical sciences we have no reason to choose a certain physical theory, we always should use the whole present physical world-view. Note also that by lack of "monotonity" of the development of physical theories, by selecting a certain theory, we have to face the fact that our statements within the framework of the chosen theory may not continue to hold in a more general (future) theory[8].

We will call the ways of computability listed in (i)-(iii) as *computability of the first kind*, etc., respectively. In the present work we want to show (among other things) that computability of the first kind and second kind are *not* necessarily equivalent[9].

Further, we note that computability of the third kind does not fit smoothly with present day computability theory in the sense that most Turing-computable functions are not computable of the third kind (e.g. by lack of enough time for a huge calculation if the Universe has finitely long future only). Hence in the present work we do not want to discuss computability of the

[8]E.g. in *classical electrodynamics* one deduces that electrons must emit electromagnetic radiation while orbiting around nuclei; this statement is not true in a more general theory, called *quantum mechanics*.

[9]In principle this non-equivalence could be attacked by the approach of the school of Pour-El et al [27] but here we are "more ambitious" in the sense that we want to keep our computers "programmable and logic oriented" (i.e. "digital" as opposed to "analog") explained more clearly in Section 3. We will show the non-equivalence by describing idealized future computing devices (e.g. in Proposition 1) which realize functions not Turing computable. We think computability of the first kind cannot be too different from Turing computability (and our second kind computable functions in Section 3 are rather far from being Turing computable).

third kind, while we acknowledge that it is a potentially interesting subject. We note that in our opinion the most emphatically used obstacle in [25] applies only to computability of the third kind, hence it does not apply to the main subject of this paper which is computability of the second kind. (We also note that the famous classical theorists of the field e.g. Kalmár, Kreisler, Turing, were more interested in computability of the second kind than in the third kind, in our opinion).

Our paper is organized as follows. In Section 2 we will recall and discuss the above mentioned variants or incarnations of the Church–Turing Thesis (called Church–Kalmár–Turing Theses). Then, in Section 3, we will raise the question whether within the framework of classical general relativity theory some forms of the Church–Kalmár–Turing Theses admit a counterexample. We will find that, very probably, such a counterexample is possible, at least in theory. Both in [4] and in [25] there are some obstacles to the possibility of such kinds of counterexamples. We will look at these obstacles one by one in Section 4 and will argue that they can be avoided in the case of a certain thought-experiment (i.e. a certain "design of the idealized future computing device"). E.g. we will argue that the observer who will find out the solution of an "unsolvable problem"[10] does not have to pay with his destruction for accessing this piece of knowledge.

The basic ideas elaborated in this paper have been around for a while. For example in the academic year 1987/88 at the University of Iowa in Ames (USA) one of the present authors gave a course in which these ideas were discussed[11], see also [1] [2]; in 1990 Pitowsky in [25] considered such ideas in a slightly more pessimistic spirit, and in 1995 Earman in [4] examined such ideas under the name of constructibility or possibility of Plato machines ([4] pp. 101-123). However, the emphasis in Earman's book and other works like [5], [6] and [7] is more on "supertasks" rather than on the Church–Kalmár–Turing Theses. Other related work we mention is Grünbaum's [9]. This list of references is far from being complete, e.g. we should have mentioned the important paper of Hogarth [12] which will be essential in our considerations. Hamkins and Lewis [10] is a recent paper in this direction. In view of the above, the purposes of the present paper are the following: (i) put the emphasis on the Church–Kalmár–Turing Theses (instead of e.g. supertasks) in a thorough, systematic way; (ii) formulate exactly which versions of the Church–Kalmár–Turing Theses we want to investigate (and what do they mean); (iii) formulate carefully what we understand under a counterexample for these variants; (iv) see if the apparent obstacles e.g. listed in earlier works can be avoided (at least in theory).

[10] For instance the consistency of ZFC set theory can be such a problem.

[11] Iowa State University, Department of Mathematics. Ph. D. course during the academic year 1987/88. Subject: "On logic, relativity, and the limitations of human knowledge." Lecturer: I. Németi.

2 The Church–Kalmár–Turing Theses

In this section we formulate some variants of the Church–Turing Thesis based on the hierarchy of definable functions $f : \mathbb{N} \to \mathbb{N}$. We follow notation and definitions of [19]. Thesis 1 below is only the first approximation of the Church–Kalmár–Turing Theses we want to investigate; therefore beyond Thesis 1 we will use more unambiguous, more carefully specified, more tangible formulations-variants of the Theses. These will be Thesis 2-2' and 3 (Theses 4-4' are for completeness only).

Let X be a finite set and denote by X^* the set of finite sequences over X. For sake of convenience we choose $X := \{0, 1\}$.

Definition 1. We call a function $f : X^* \to X^*$ *Turing-computable* if there is a Turing machine which realizes f. $\diamond$

For the definition of a Turing machine, see Definition I.4.1 while the realization of a function by a Turing machine is formulated in Definition I.4.2 of [19].

As it is well-known, the set of natural numbers, $\mathbb{N} = \{0, 1, 2, \ldots\}$, can be represented as X^* i.e. there is a bijection $\mathbb{N} \cong X^*$ which is effectively computable in the intuitive sense. Consequently the notion of a Turing-computable number-theoretic function $f : \mathbb{N} \to \mathbb{N}$ is well-defined i.e. Turing-computability of these functions is independent of the representation of $\mathbb{N}$ as an X^* [12].

Introducing the notation

$$\mathbb{N}^k := \underbrace{\mathbb{N} \times \ldots \times \mathbb{N}}_{k} \qquad \text{for } k \in \mathbb{N}^+,$$

where $\mathbb{N}^+ = \{1, 2, \ldots\}$ denotes the set of positive integers, we can see that $\mathbb{N}^k$ can also be regarded as a subset of Y^* where Y contains some extra element in comparison with X, for example $Y := \{0, 1, -\} = X \cup \{-\}$. As an example, $101 - 11 \in Y^*$ corresponds to the pair $(5, 3) \in \mathbb{N}^2$ in this notation. In this way we can talk about the Turing-computability of a function $f : \mathbb{N}^k \to \mathbb{N}^m$ for each $k, m \in \mathbb{N}^+$.

Definition 2. A subset $R \subseteq \mathbb{N}^m$ is called an *(m-ary) relation.*

(i) A relation $R \subseteq \mathbb{N}^m$ is called *decidable* if its characteristic function $\chi_R : \mathbb{N}^m \to \{0, 1\}$, given by

$$\chi_R(x_1, \ldots, x_m) := \begin{cases} 1 & \text{if } (x_1, \ldots, x_m) \in R \\ 0 & \text{if } (x_1, \ldots, x_m) \notin R, \end{cases}$$

[12] We could introduce the notion of a *recursive function* $f : \mathbb{N} \to \mathbb{N}$ as well (see the various definitions in Chapter I of [19]). But according to a theorem of Turing (e.g. Theorem I.4.3 of [19]) a function $f : \mathbb{N} \to \mathbb{N}$ *is Turing-computable if and only if it is recursive*, hence we will use the term "Turing-computable" systematically throughout this paper.

is Turing-computable;

(ii) A relation $R \subseteq \mathbb{N}^m$ is called *recursively enumerable* if there is a Turing-computable function $f_R : \mathbb{N} \to \mathbb{N}^m$ such that $\mathrm{im} f_R = R$ where in general

$$\mathrm{im}\, f := \{(y_1, \dots, y_m) \mid \exists\, x \;\; f(x) = (y_1, \dots, y_m)\}$$

is the *image* of a function $f : \mathbb{N} \to \mathbb{N}^m$. $\diamond$

In this way we have defined decidable and recursively enumerable m-ary relations for all $m \in \mathbb{N}^+$. Next we introduce a natural hierarchy from the computability viewpoint on the set of relations.

Definition 3. Let $R \subseteq \mathbb{N}^m$ be a relation.

(i) We say that the relation $R \subseteq \mathbb{N}^m$ is a Σ_1*-relation* i.e. $R \in \Sigma_1$ if R is recursively enumerable;

(ii) We say that the relation $R \subseteq \mathbb{N}^m$ is a Π_1*-relation* i.e. $R \in \Pi_1$ if $\overline{R} \in \Sigma_1$. Here $\overline{R} := \mathbb{N}^m \setminus R$ is the complement of R with respect to $\mathbb{N}^m$;

(iii) In general, we say that a relation $R \subseteq \mathbb{N}^m$ *is a* Σ_n*-relation* i.e. $R \in \Sigma_n$ $(n \in \mathbb{N}, n \geq 2)$ if there is a $k \in \mathbb{N}$ and a Π_{n-1}-relation $S \subset \mathbb{N}^{m+k}$ such that

$$R = \{(x_1, \dots, x_m) | \,\exists\, (x_{m+1}, \dots, x_{m+k}) \in \mathbb{N}^k,\; (x_1, \dots, x_{m+k}) \in S\}.$$

(iv) In general, we say that a relation $R \subseteq \mathbb{N}^m$ *is a* Π_n*-relation* i.e. $R \in \Pi_n$ if $\overline{R} \in \Sigma_n$. $\diamond$

We will use Σ_n also as the set of all Σ_n-relations, and similarly for Π_n. Thus e.g. $R \in \Sigma_2 \setminus (\Sigma_1 \cup \Pi_1)$ means that $R \in \Sigma_2$ but $R \notin \Sigma_1$ and $R \notin \Pi_1$, i.e. $R \in \Sigma_2$ and neither R nor its complement is recursively enumerable.

Notice that every function $f : \mathbb{N}^k \to \mathbb{N}^m$ may be considered as a relation

$$R_f := \{(x_1, \dots, x_k, y_1, \dots, y_m) \mid f(x_1, \dots, x_k) = (y_1, \dots, y_m)\} \subset \mathbb{N}^{m+k}.$$

R_f is called the *graph* of f. We will say that a function $f : \mathbb{N}^k \to \mathbb{N}^m$ is a Σ_n*-function* (resp. Π_n*-function*) if and only if its graph R_f is a Σ_n- (resp. Π_n-) relation.

By keeping in mind the definition of Turing machines, one can easily show the following (see e.g. [19]):

(i) A function f is Turing-computable if and only if its graph R_f is recursively enumerable, i.e. if $R_f \in \Sigma_1$.

(ii) A relation R is decidable if and only if both R and its complement are recursively enumerable, i.e. if and only if $R \in \Sigma_1 \cap \Pi_1$.

Thus, $R \in \Sigma_1 \setminus \Pi_1$ means that R is recursively enumerable but R is not decidable. As an example, one may consider the relation D_e defined by a Diophantine equation $e(x, y, a)$ as follows

$$D_e := \{(x, y) \mid \exists\, a \;\; e(x, y, a)\} \subset \mathbb{N}^2$$

which is clearly Σ_1 but not necessarily Π_1 i.e. it is not necessarily decidable although recursively enumerable. Indeed, there are choices of the equation $e(x, y, a)$ for which D_e is undecidable. One can see that there are relations in Σ_2 which are not recursively enumerable because of using existential quantifications in their definitions. In general, there are Σ_n-relations which are not Σ_{n-1}-relations, and, intuitively, the Σ_n-relations are "harder to compute" than the Σ_{n-1}-relations. The sets Σ_n, Π_n measure the degree of *non-computability* of a relation by means of Turing machines i.e. algorithms. For details see Chapter IV of [19].

The hierarchy $\Sigma_1, \Pi_1, \dots, \Sigma_n, \Pi_n, \dots$ $(n \in \mathbb{N}^+)$ is called *arithmetical hierarchy.* It provides us subsets $R \subset \mathbb{N}^m$ which are further and further away from being computable. Beyond the arithmetical hierarchy comes the so-called *analytical hierarchy.* We note that the first order logic theory $Th(\langle \mathbb{N}, +, * \rangle)$ of arithmetics is at the bottom of the analytical hierarchy.

At this point one may raise the question if there is a hypothetical *extended Turing machine* such that all the elements of Σ_1 would become decidable by this machine. Such an extended Turing machine should possess only one extra property compared to the ordinary Turing machines. Indeed, it should be able to answer the following question in finite time: does a given ordinary Turing machine stop with a given input y or not? Such an extended Turing machine certainly exists as an abstract mathematical object but it may or may not be realized physically.

It is possible to show that, by using this one extra ability, all elements of Σ_1 become decidable (in the extended sense) while elements of Σ_2 would become recursively enumerable (in the extended sense). This means that, by using these extended Turing machines, every relation would become "less non-computable with one unit".

The concept of a Turing machine is an extraction, idealization or an abstract formulation of our experience with physical computers. By a *physical computer (in the narrow sense)* we mean a discrete physical system together with a physical theory for its behaviour (see [19], p. 104). Hence one may ask if the above mentioned extended Turing machine can be realized as a physical computer. We will say that a function $f : \mathbb{N}^k \to \mathbb{N}^m$ is *effectively computable* if there is a physical computer realizing it. Here by "realization by a physical computer"we mean the following.

Let P be a physical computer, and $f : \mathbb{N}^k \to \mathbb{N}^m$ a (mathematical) function. Then we say that P *realizes* f if an imaginary observer O can do the following with P. Assume O receives an arbitrary element $(x_1, \dots, x_k) \in \mathbb{N}^k$ from, say, his "opponent". Then O can "start" the computer P with $(x_1, \dots, x_k)$ as an input and then sometime later (according to O's internal clock) O "receives" data $(y_1, \dots, y_m) \in \mathbb{N}^m$ from P as an output such that $(y_1, \dots, y_m)$ coincides with the value $f(x_1, \ldots, x_k)$ of the function f at input $(x_1, \ldots, x_k)$. The reason why we wrote "start" and "receives" in quotation marks is that we do not want to specify how O can start P etc., these can be specified by the designer of the computer P. The essential idea is that O can use P as a device for computing f. The difficulty which we have to circumnavigate (when defining what we mean by saying "P realizes f") is that f is an infinite object. The solution is that we postulate that for *any* permitted choice of the input data $(x_1, \ldots, x_k)$, computer P will produce an output $(y_1, \ldots, y_m)$ and in addition, this output will coincide with $f(x_1, \ldots, x_k)$. We emphasize that this definition does not require repeated activations of P , instead it says that whatever inputvalue $(x_1, \ldots, x_k)$ we would choose, P will produce an output coinciding with $f(x_1, \ldots, x_k)$.

In this context we may quote the original form of the Church–Turing Thesis ([19] p. 102):

Thesis 1 (Church–Turing). *Every effectively computable function $f : \mathbb{N}^k \to \mathbb{N}^m$ gives rise to a relation $R_f \in \Sigma_1$ i.e. every effectively computable function is Turing-computable.*

In light of Thesis 1 above, our extended Turing machines cannot be regarded as physical computers in the narrow sense, since they are able to realize elements of Σ_2.

By using ideas of László Kalmár, let us try to formulate a more tangible (and somewhat stronger) version of the above Thesis. Of the many roles Turing machines play in scientific thinking, let us concentrate on the following one. Turing machines provide *idealized*, abstract "approximation" of *artificial computing systems*[13]. The next version of the Thesis will say that arbitrary future artificial computing systems will realize only such functions $f : \mathbb{N}^k \to \mathbb{N}^m$ which are Turing-computable (i.e. recursive). To make the meaning of the next version of the Thesis clear, we ask ourselves what artificial computing systems are. The answer is the following.

Any such system presupposes that we fix a physical theory (which is consistent with our present day knowledge) and on the basis of this theory we design an artificial system which is capable to associate natural numbers to natural numbers in some well-defined way. (Here well-defined means that, in terms of the chosen physical theory, it is clearly explained how to give an "input" to this system and how to interpret whether it gave an "output" and what this output is.)

But what is an artificial system? Does it have to fit into a box, for example? If yes, what are the limits of the size of the box? (What happens if the system uses a futuristic version of, say, Internet? What if this net grows *during* the course of computation in question?) If we do not want to be "short-sighted" we should not suppose that the system fits into a box (or anything like that).

In view of the above considerations, for the purposes of the present paper, we propose to identify an artificial computing system G with what we call here a *thought-experiment*[14]. The

[13] Here one can think of a "futuristic" notion of computer.

[14] Assume a physical theory is fixed. Then by a *thought-experiment relative to the fixed physical theory* we mean

definition of when we say that a mathematical *function f is realized by a fixed artificial computing system G* (or thought experiment) follows the same pattern as we defined earlier the concept of when a physical computer (in the narrow sense) realizes function f. Therefore we do not repeat that definition.

Definition 4. We regard the above considerations as the definition of when a mathematical function $f : \mathbb{N}^k \to \mathbb{N}^m$ is *realized by an artificial computing system G*. $\diamond$

We would like to clarify a bit the sense in which we use the expression "thought-experiment" in Definition 4 above. If G is a thought-experiment (i.e. artificial computing system), then there is a fixed physical theory Th associated to G such that using theory Th one can specify precisely how the thought experiment G should be carried out. If using Th together with the specification of G someone can prove that G realizes f, then we conclude that indeed G realizes f. We note that this does not mean that using Th and the specification of G we could compute with pencil and paper what the answer of G will be to a certain input, say 3. We only know that $G(3) = f(3)$ holds[15].

Trivially, the class of artificial computing systems, defined in this way, includes the class of physical computers (in the narrow sense) used to formulate Thesis 1 (cf. [19] p. 104). Moreover, the question naturally arises whether extended Turing machines introduced above exist in the class of artificial computing systems or not? Notice also that a function f which is realizable by an artificial computing system is computable of the second kind according to the terminology developed in Section 1.

Now we are ready to formulate a sharper version of Thesis 1.

Thesis 2 (Church–Kalmár–Turing). *Every function $f : \mathbb{N}^k \to \mathbb{N}^m$ realizable by an artificial computing system gives rise to a relation $R_f \in \Sigma_1$ i.e. every function realizable by an artificial computing system is Turing-computable.*

Or, trivially re-formulated, we can state:

Thesis 2' (Church–Kalmár–Turing). *Every function $f : \mathbb{N}^k \to \mathbb{N}^m$ realizable by a thought-experiment gives rise to a relation $R_f \in \Sigma_1$ i.e. every function realizable by a thought-experiment is Turing-computable.*

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a theoretically possible experiment, i.e. an experiment which can be carefully designed, specified, etc. according to the rules of the physical theory but for the actual realization of it we might not have the necessary sources, technical level, enough time, etc.

To illustrate the idea: if the physical theory in question is classical mechanics then we conjecture that there are no thought-experiments which would realize a function f which is not Turing-computable.

[15] On the other hand, if in a "universe" U, the theory Th was true and someone had the resources for carrying the thought-experiment through, then at the end he would find out the value of f for any given prespecified input.

Clearly, all versions of the Thesis (i.e. 1-2') presuppose some physical theory as a background. We will argue that the truth of Theses 2-2' can actually depend on the choice of our background physical theory.

A kind of corollary of the Thesis taken together with Gödel's Second Incompleteness Theorem is the following.

Thesis 3. *Assume ZFC set theory is consistent. Then, necessarily, Humankind, or its Successors, can never prove or become certain that this is so.*

The above form is a kind of common meta-mathematical interpretation of Gödel's Second Incompleteness Theorem. We will argue that the refutability (or provability) of Thesis 3 can also depend on the choice of our background physical theory.

For completeness, below we will formulate a further version of the Thesis which goes off in a different angle called sometimes "limitations of human knowledge". This will be Thesis 4-4'. We may formulate Thesis 4 as follows. If we suppose that the "input-output aspect" of each single human problem solving activity is nothing but a finite answer to a finite question formulated in a language fixed in advance, then one may declare:

Thesis 4 (Church–Descartes–Turing). *Every mental activity of human beings realizes Turing-computable functions.*

This idea can be traced back to *Descartes*. If we accept psychological materialism in the form that every mental product of a human being is completely determined by his brain the above Thesis can be re-formulated as

Thesis 4' (Church–Descartes–Turing). *The human brain realizes Turing-computable functions.*

We included Theses 4-4' only for completeness, in our investigations we will concentrate on Theses 2-3. Our reason for formulating so many versions of the Thesis is that for *each one* of Theses 2-3 we will argue that they admit counterexamples if we work in classical general relativity theory[16]. So, if the reader is interested in *any* one of Theses 2-3 then he can read the rest of this paper with that version of the Thesis in mind. For definiteness, we will always formulate our statements to attack Thesis 3.

In the following section we try to construct an artificial computing system based on the

[16]For most of these versions our main point is not so much refuting the Thesis but instead is showing that the Thesis is not independent of the background physical theory. Our main message is that the theories of computability and meta-mathematics can be better developed if we take into account the current state of theoretical physics. To be more blunt: we would like to show that it is not "healthy" to regard and develop these theories as being completely disjoint and isolated from theoretical physics. In other words what we are arguing for is the "unity of science".

ordinary theory of Turing machines and classical general relativity which is supposed to be able to realize non-Turing-computable i.e. non-recursive functions. These machines are also counterexamples for versions 2-3 of the Church–Kalmár–Turing Theses formulated above. The basic idea is essentially the same as that of Malament, Hogarth [12] and Pitowsky [25]; it is summarized by Earman (see Chapter 4 of [4]). Moreover we will see that, our thought-experiment, which is a modified version of the one constructed in [4], is free of the problems listed by Earman and Pitowsky.

It would be interesting to see which level of the arithmetical hierarchy can be made "computable" by using classical general relativity theory; and what is the "price" of going further up in the hierarchy. That is, what extra assumptions do we need to make (if any) if we want to make a higher level of the hierarchy to become "computable". The complexity classes in the analytical hierarchy are denoted by Σ^k_n and Π^k_n $(k, n \in \mathbb{N})$. For any of these functions the question whether it can be made "computable" admits a precise, unambiguous formulation *because* all these functions (in Σ^k_n etc.) are *definable* in the language of set theory (and even in the higher-order logic language of arithmetics $\langle \mathbb{N}, 0, 1, +, *\rangle$). So, one can write up the arithmetical definition of the function f and one can ask if there is a thought-experiment realizing precisely this function.

We note, however, that non-computable functions necessarily remain even if one uses relativistic (or other) powerful phenomena to compute more and more complicated functions. The reason for this is a simple cardinality argument: any thought-experiment can be expressed as a finite sequence of (English) sentences, therefore there are countably many thought-experiments only. It follows, that only countably many functions can be realized by a thought-experiment. On the other hand, the cardinality of $\mathbb{N} \to \mathbb{N}$ type functions is the continuum. Therefore, there must exist a function, which cannot be realized by a thought-experiment.

3 Computers in the Kerr space-time

In this section we will follow the notation and terminology of [4] (see also [11], [28]). By a space-time we mean a pair (M, g) where M is a smooth, oriented and time-oriented four-manifold while g is a smooth Lorentzian metric on M which is a solution to the Einstein's equations with respect to a physically reasonable matter field represented by a smooth stress-energy tensor T on M (i.e. T satisfies one of the standard energy conditions). For the notions concerning general relativity we refer to [11],[28]. The length of an at least once continuously differentiable time-like curve $\gamma : \mathbb{R} \to M$ is the integral

$$\|\gamma\| = \int\limits_{\gamma} \mathrm{d}\gamma = \int\limits_{\mathbb{R}} \sqrt{-g(\dot{\gamma}(\tau), \dot{\gamma}(\tau))}\, \mathrm{d}\tau.$$

As usual, we interpret a future-directed, time-like, at least once continuously differentiable curve $\gamma : \mathbb{R} \to M$ as a "world-line" of an observer moving in (M, g), i.e. $\mathrm{im}\gamma \subset M$ is the collection of those events in M which the observer meets throughout its existence. Moreover $\|\gamma\|$, the length of the world-line, is thought of as the proper time measured by the observer γ from its beginning of existence to its end. This can be finite or infinite depending on the curve and the geometrical structure of the space-time characterized by the metric g. Now we introduce an important class of space-times related with our subject. Consider a point (event) $q \in M$. The set of all points

$$J^-(q) := \{x \in M \mid \text{there is a future-directed non-space-like continuous curve joining } x \text{ with } q\}$$

is called the *causal past* of the event q (the causal future is defined similarly). Intuitively, $J^-(q)$ consists of those events $x \in M$ from which one can "travel" to q without exceeding locally the speed of light i.e. by an "allowed" motion.

Definition 5. A space-time (M, g) is called a *Malament–Hogarth space-time* if there is a future-directed time-like half-curve $\gamma_P : \mathbb{R}^+ \to M$ such that $\|\gamma_P\| = \infty$ and there is a point $p \in M$ satisfying $\mathrm{im}\gamma_P \subset J^-(p)$. The event $p \in M$ is called a *Malament–Hogarth event.* $\diamond$

Note that if (M, g) is a Malament–Hogarth space-time, then there is a future-directed time-like curve $\gamma_O : [a, b] \to M$ from a point $q \in J^-(p)$ to p satisfying $\|\gamma_O\| < \infty$. The point $q \in M$ can be chosen to lie in the causal future of the past endpoint of γ_P. Below we will discuss if such space-times are physically reasonable or not.

Consider a Turing machine realized by a physical computer P moving along the curve γ_P of *infinite* proper time. Hence the physical computer (identified with γ_P) can perform arbitrarily long calculations. Being (M, g) a Malament–Hogarth space-time, there is an observer following the curve γ_O (hence denoted by γ_O) of *finite* proper time such that he touches the Malament–Hogarth event $p \in M$ in finite proper time. But by definition $\mathrm{im}\gamma_P \subset J^-(p)$ hence in p he can receive the answer for a *yes or no question* as the result of an *arbitrarily long* calculation carried out by the physical computer γ_P since it can send a light beam to γ_O at arbitrarily late proper time. Clearly the pair (γ_P, γ_O) is an *artificial computing system* G with respect to classical general relativity theory since it is a correct thought-experiment within the framework of this theory. Hence $G := (\gamma_P, \gamma_O)$ carries out a *computation of the second kind.* In this moment, it is not clear what kind of space-time M is and what time-like curves γ_P and γ_O are. For instance, it is possible that the acceleration along one curve is unbounded making the idea physically unreasonable [25]. A very concrete, physically reasonable realization of this device in the case of the Kerr space-time will be explained soon.

Imagine the following situation, as an example. γ_P is asked to check all theorems of our usual set theory (ZFC) in order to check consistency of mathematics. This task can be carried out by γ_P since its world line has infinite proper time. If γ_P finds a contradiction, it can send a message (for example a light beam) to γ_O. Hence if γ_O receives a signal from γ_P *before* the Malament–Hogarth event p he can be sure that ZFC set theory is not consistent. On the other hand, if γ_O does not receive a signal before p then, *after* p, γ_O can conclude that ZFC set theory is consistent. Note that γ_O having finite proper time between the events $\gamma_O(a) = q$ (starting with the experiment) and $\gamma_O(b) = p$ (touching the Malament–Hogarth event), he can be sure about the consistency of ZFC set theory in finite (possibly very short) time. This contradicts Thesis 3 above.

At this point we may ask if Malament–Hogarth space-times are reasonable physically or not. Most examples are very artificial but it is quite surprising that among these space-times one can recognize the *anti-de Sitter space-time*, which is a solution to the vacuum Einstein's equations with negative cosmological constant and is in the focus of recent investigations in theoretical physics; the *Reissner–Nordström space-time* describing a spherically symmetric black hole of small electric charge and the *Kerr–Newman space-time* representing a slowly rotating black hole of small electric charge. For a description of these space-times see [11] as a standard reference. In what follows we are going to focus our attention to the Kerr space-time because in light of the celebrated black hole uniqueness theorem (see [11], or for an overview [28] while a short new proof was presented by Mazur [15]) this space-time is the only candidate for the late-time evolution of a collapsed rotating star. Hence existence of Kerr black holes in the Universe is physically very reasonable even in our neighbourhood. For instance, a candidate for such a black hole is the supermassive compact object in the center of the Milky Way; this question can be decided in the next few decades [16]. In this context it is remarkable that this space-time possesses the Malament–Hogarth property.

Now we would like to construct the artificial computing system $G = (\gamma_P, \gamma_O)$ as a correct thought-experiment in the case of the vacuum Kerr space-time (M, g). This means that we have to describe the time-like curves γ_O and γ_P around a slowly rotating black hole of zero electric charge. To do this, we will follow [20]. Using Boyer–Lindquist coordinates $(t, r, \vartheta, \varphi)$, the Kerr metric g with parameters $m > 0$ (mass) and a (angular momentum per unit mass) locally takes the shape (see [11][20][28])

$$\mathrm{d}s^2 = -\left(1 - \frac{2mr}{\Sigma}\right)\mathrm{d}t^2 - \frac{2mra\sin^2\vartheta}{\Sigma}\mathrm{d}t\mathrm{d}\varphi + \frac{\Sigma}{\Delta}\mathrm{d}r^2 + \Sigma\mathrm{d}\vartheta^2 + \left(r^2 + a^2 + \frac{2mra^2\sin^2\vartheta}{\Sigma}\right)\sin^2\vartheta\mathrm{d}\varphi^2$$

where $\Sigma(r, \vartheta) = r^2 + a^2\cos^2\vartheta$ and $\Delta(r) = r^2 - 2mr + a^2$. We choose the underlying manifold M to be a smooth four-manifold which can carry the maximal analytical extension of the metric[17]. This metric possesses two Killing fields, namely $\partial/\partial t$ and $\partial/\partial\varphi$ corresponding to time-translations and rotations around the "axis" of the black hole, respectively. The singularity is given by the equation $\Sigma(r, \vartheta) = 0$ and has ring-shape while the event horizons are characterized by the real

[17] This determines the range of the values of t, r, ϑ, φ, see [11][20].

roots $r_\pm$ to the equation $\Delta(r)=0$:

$$r_\pm = m \pm \sqrt{m^2-a^2}.$$

Note that this equation has real roots only if $|a|\leq m$, i.e. in the case of "slowly rotating" black holes. We restrict ourself to the non-extremal case $|a|<m$.

Assume a future-directed time-like geodesic $\gamma:\mathbb{R}^+\to M$ is given, describing the free motion of a point-like particle of unit mass. In the above coordinate system this curve locally is given by the four functions $\gamma(\tau)=(t(\tau),r(\tau),\vartheta(\tau),\varphi(\tau))$ satisfying the well-known second order geodesic equations. We can identify such a curve uniquely by fixing the initial position and velocity $(\gamma(0),\dot\gamma(0))$ where dot means differentiation with respect to the affine parameter $\tau\in\mathbb{R}^+$. However, if $\gamma(0)$ is not on the axis of the black hole, then by Lemma 4.2.5 of [20] we can use the data $(\gamma(0),\operatorname{sgn}\dot r(0),\operatorname{sgn}\dot\vartheta(0),q,E,L,Q)$ to fix the geodesic γ as well[18]. The quantities (q,E,L,Q) are the "first integrals" of the geodesic motion, i.e. these quantities are constant along the geodesic curve. Here $q:=g(\dot\gamma,\dot\gamma)$ is equal to -1 since γ is time-like and the point particle is of unit mass. $E:=-g(\dot\gamma,\partial/\partial t)$ is the total energy of the particle measured by a distant observer, $L:=g(\dot\gamma,\partial/\partial\varphi)$ is the angular momentum of the particle with respect to the "axis" of the black hole given by points satisfying $\vartheta=0,\pi$. The constant Q is called the *Carter-constant* and is characterized by the system of ordinary differential equations (see Section 4.2 of [20])

$$\Sigma^4(r,\vartheta)\dot r^2 = -\Delta(r)(r^2+Q+(L-aE)^2)+(r^2+a^2)E-aL,$$

$$\Sigma^4(r\,\vartheta)\dot\vartheta^2 = Q+(L-aE)^2-a^2\cos^2\vartheta-\frac{L}{\sin^2\vartheta}-aE.$$

A remarkable observation of Carter shows that Q is constant along a Kerr-geodesic (see Theorem 4.2.2 of [20]) and characterizes Kerr-geodesics in a simple way whether they hit or not the ring singularity.

First, we consider the freely falling observer $\gamma_O:[0,\tau_-]\to M$. Choose a particular point $q\in M$ somewhere "outside" the black hole, not lying on the axis and let $\gamma_O(0):=q$. Let $\operatorname{sgn}\dot r_O(0)=-1$, while $\operatorname{sgn}\dot\vartheta_O(0)=\pm1$ arbitrary and take $0<E_O$, $|L_O|<2mE_Or_+/a$. These data provide for a (particle-like) observer moving along γ_O to enter the Kerr black hole, i.e. to cross the outer event horizon. Moreover, if we take $Q_O\neq0$ then by Corollary 4.5.1 of [20] γ_O does not hit the singularity $\Sigma=0$ of the black hole. Furthermore, if we fix $E_O^2\geq1$ then the passenger has enough energy to escape some infinite, asymptotically flat region of M again (see Proposition 4.8.1 of [20]) particularly he crosses the inner horizon as well. Finally, if we choose the angular momentum L_O of the geodesic γ_O carefully, namely

$$\frac{2mE_Or_-}{a}<L_O<\frac{2mE_Or_+}{a}$$

[18] Here sgn is the sign of a real number.

(in particular this gives $0 < L_O$, showing γ_O cannot be an axial geodesic since in that case $L = 0$), then γ_O hits the inner horizon in a Malament–Hogarth event (see Fig. 4.19 of [20]). It is worth mentioning at this point that such an orbit does not exist for non-rotating ($a = 0$) i.e. Schwarzschild black holes. The above type of geodesics are called "time-like long flyby orbits of type B" and are examined on pp. 245–247 of [20]. The Malament–Hogarth event is characterized by the equation $r_O(\tau_-) = r_-$. Clearly, τ_- is finite since γ_O reaches the inner horizon under the above conditions, hence

$$\|\gamma_O\| = \int\limits_0^{\tau_-} \sqrt{-g(\dot\gamma_O(\tau), \dot\gamma_O(\tau))}\, \mathrm{d}\tau = \int\limits_0^{\tau_-} \mathrm{d}\tau = \tau_- < \infty.$$

The case of the physical computer is very simple. We may assume the initial data are $\gamma_P(0) = \gamma_O(0) = q$ (the observer γ_O and the computer γ_P start from the same point) and take $\gamma_P : \mathbb{R}^+ \to M$ to be a geodesic corresponding to a stable circular orbit in the equatorial plane of the Kerr black hole. This implies $\operatorname{sgn} \dot r_P(0) = 0$, $\operatorname{sgn} \dot\vartheta_P(0) = 0$, $Q_P = 0$ and $E_P > 0$, $E_P^2 < 1$. We can calculate the radius of the circular orbit of γ_P by Lemma 4.14.9 while the corresponding angular momentum L_P can be determined via Corollary 4.14.8 of [20] (the concrete values are not interesting for us in this moment). Trivially, $\|\gamma_P\| = \infty$.

This arrangement shows that, since both γ_P and γ_O move along geodesics, their acceleration is constantly zero, i.e. remains bounded throughout their existence. A three-dimensional picture of the machine is shown on Figure 1.

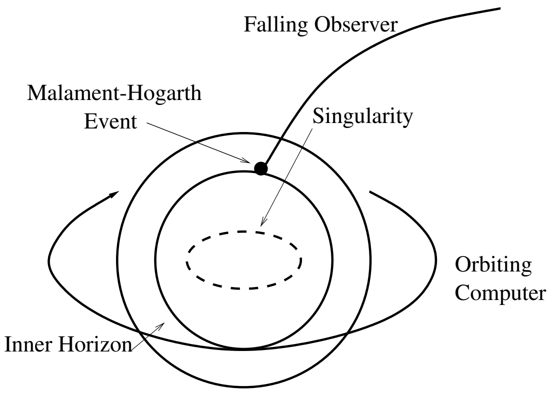

Figure 1. The three-dimensional picture of the device $G = (\gamma_P, \gamma_O)$.

It is worth presenting a four-dimensional space-time diagram of $G = (\gamma_P, \gamma_O)$ as well. Such diagrams are called Penrose diagrams and show the whole development of the system.

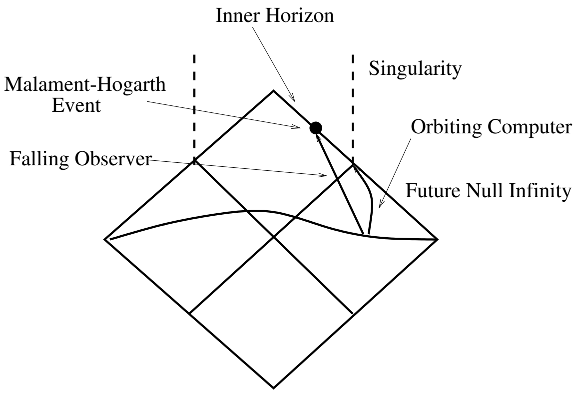

Figure 2. The Penrose diagram picture of the device $G = (\gamma_P, \gamma_O)$.

We can see that in the case of Kerr space-time the Malament–Hogarth event appears for γ_O when he touches the inner horizon of the Kerr black hole (in a finite proper time, of course). As it is well known [11][28] the inner horizon of the Kerr black hole is a Cauchy horizon for outer observers showing that this space-time fails to be globally hyperbolic. Later we will see that this is a general property of Malament–Hogarth space-times. Although after crossing the inner horizon the predictability of the fate of γ_O breaks down, it seems he can avoid the encounter with the final destroying singularity in the stomach of the Kerr black hole as a consequence of the ring-like shape of the singularity.

Now that the Kerr-orbits of the falling traveler γ_O and the orbiting computer γ_P are determined, let us turn our attention to the communication between them by fixing a simple coding system. For sake of definiteness, assume we want to attack Thesis 3. Consequently we have to derive all the theorems $\phi_1, \phi_2, \ldots$ of ZFC set theory and check if there exists a theorem, say ϕ_i, which coincides with the formula FALSE or equivalently with $x \neq x$. Then γ_O and γ_P choose a Turing machine T which enumerates all the theorems of ZFC. In this way T realizes a function $f_T : \mathbb{N} \to \{\text{Formulas of ZFC}\}$ such that $\mathrm{im} f_T$ is exactly the set of theorems in ZFC (it is easy to find such a T). Now, γ_O and γ_P agree on using the same choice of T. Then γ_O departs for the Kerr black hole (taking a copy of T with him) while γ_P keeps on executing the following simple algorithm.

A. $i := 0$
B. Derive theorem $f_T(i)$ from ZFC set theory
C. Check if $f_T(i) =$ FALSE
D. If yes, send a signal to γ_O
E. If no, let $i := i + 1$ and go to B

Suppose that ZFC is inconsistent. Then γ_P will find the first $i \in \mathbb{N}$ for which $f_T(i)$ =FALSE. Suppose the proper time needed for γ_P to find this i was τ_P^i (the experiment started at $\tau_O =$

$\tau_P = 0$). Let us mention that for anyone who has a copy of T and knows the speed of γ_P's implementation of T, the number i is computable from τ_P^i.

Since γ_P knows when it is sending the signal and it knows γ_O's plans, γ_P can compute how much time γ_O will have for receiving the coded signal and can also compute the expectable blueshift of the signal (see Section 4). So γ_P can make compensations for these effects (to the extent theoretically possible).

Now, γ_P sends off a signal. γ_O receives it before the Malament–Hogarth event p and measures the time τ_O^i (according to his own clock) when the signal arrived (we will return soon to the question of measurement of this signal). By knowing the time τ_O^i and by using general relativity theory, γ_O can compute the time τ_P^i hence the number i. Then γ_O computes $f_T(i)$ and checks if it is the formula FALSE. If yes, he knows that ZFC is inconsistent. If not, then γ_O received a fake signal[19].

To keep the number of possible fake signals at minimum, we may assume that γ_P will not send a simple light beam only but uses some modulation or coding (some Morse-type sequence of "long" and "short" impulses, for instance) to make its signal much more unique. We emphasize that this modulation, or coding is also fixed once and for all in advance between γ_O and γ_P.

If γ_O does receive a signal before the Malament–Hogarth event, then he checks if the relevant theorems in ZFC are consistent or not. If no, then γ_O concludes that what he received was a fake signal.

If he did not receive any other signal by p, then he concludes that ZFC is consistent. If he received the prearranged coded signal at some different time, say τ_O', too, then he goes through the above checking procedure for deciding whether this second signal is fake or not. We assume that γ_O and γ_P agree on a sufficiently complicated and long code to minimize the chance for fake signals. Further, by the nature of the possible origin (or cause) of a fake signal and by taking into account that on γ_O's clock only finite time goes by between γ_O's departure and his arrival at the Malament–Hogarth event p, we can expect that γ_O will receive only finitely many fake signals (before reaching p, of course). Consequently γ_O has to check only a finite number of signals and after that he will know whether or not ZFC is consistent.

Let us briefly return to the possible imprecision of γ_O's measuring τ_O^i. Suppose γ_O knows only that the signal arrived between τ_O^i and $\tau_O^i + \varepsilon_O$ (with $\tau_O^i + \varepsilon_O$ being before the Malament–Hogarth event). Then he will calculate that it was sent between τ_P^i and $\tau_P^i + \varepsilon_P$. But only finitely many theorems were checked by γ_P within this interval, consequently γ_O can corrigate this uncertainty with finite calculations only (i.e. by checking the falsity of finitely many theorems from ZFC only).

[19] As γ_O approaches the Malament–Hogarth event which lies on the inner horizon of the Kerr black hole i.e. on a Cauchy horizon of the Kerr space-time, it is more and more difficult to decide whether a light beam came from γ_P or a possible past singularity, see [4], p. 118. Consequently receiving fake signals cannot be *a priori* excluded.

Hence, by assuming the ability of time measurement of arbitrary accuracy (which is always possible in classical physics, but see remarks in Section 4), the arrangement G provides a thought-experiment, consistent with classical general relativity, contradicting Thesis 3. Having designed an artificial computing system which checks consistency of ZFC, we now turn to seeing what other jobs can be done with similar artificial computing systems. Let us return in general to Theses 2-3 formulated in Section 2. As we said in that section, first we have to assume a physical theory. Let this theory be the classical general relativity. Next, let us suppose that the observer γ_O wants to decide a Σ_1-set of $\mathbb{N}$ which is not Π_1 i.e. recursively enumerable but non-decidable. The above considerations can be used by γ_O for designing a thought-experiment that is, an artificial computing system $G = (\gamma_P, \gamma_O)$ which will help him to decide such a set.

Definition 6. Let $R \subseteq \mathbb{N}^m$ be a relation. We say that an artificial computing system (or thought-experiment) G *decides* R if and only if G realizes the characteristic function $\chi_R : \mathbb{N}^m \to \{0, 1\}$. $\diamond$

From now on, we will call $G = (\gamma_O, \gamma_P)$ a *relativistic computer*, indicating that this is a special artificial computing system i.e. thought-experiment. Now we are in a position to state:

Proposition 1. (i) *Let $R \in \Sigma_1$ be a relation with possibly $R \notin \Pi_1$ i.e. recursively enumerable but possibly non-decidable. Then there is a relativistic computer $G = (\gamma_P, \gamma_O)$ which decides R.*
(ii) *There exist infinitely many relations $R \in \Sigma_1 \setminus \Pi_1$. Hence there are infinitely many Turing-undecidable relations which are decidable by some $G = (\gamma_P, \gamma_O)$ as in* (i) *above.*

Proof. (ii) This is well known (c.f. [19]). An example is if we take R to be the set of valid theorems of first-order logic.

(i) Let $R \in \Sigma_1$. Then R is recursively enumerable i.e. there is a Turing machine T which enumerates R. (In other words, T realizes a surjective function $f_T : \mathbb{N} \to R$ with $\mathrm{im} f_T = R$.)

Now, we design the relativistic computer G which, we claim, can decide R. To test this claim, the "opponent" chooses a random element $(x_1, \dots, x_k) \in \mathbb{N}^k$ and gives it to G for deciding whether or not $(x_1, \dots x_k) \in R$. In the initial state of their computation, γ_O and γ_P are sitting together, making plans about how to decide this question. γ_P receives the task of using T to enumerate the elements of R and checking whether $(x_1, \dots, x_k) \in \mathbb{N}^k$ shows up during this enumeration. That is, γ_P executes the program

A. $i := 0$
B. If $f_T(i) = (x_1, \dots, x_k)$ then send a signal to γ_O and go to D
C. $i := i + 1$ and go to B
D. Make sure that the signal for γ_O is adequately coded, prepared etc.
Make other planned actions to ensure that γ_O receives the signal. STOP

The rest of the preparations γ_P and γ_O make are exactly the same as was the case of the relativistic computer G described above Proposition 1 for refuting Thesis 3. (So here again they rely on precise measurment of time to rule out fake signals, and again γ_O takes the Turing machine T with him such that he can compute $f_T(i)$ for any fixed i).

After the Malament–Hogarth event p, γ_O will be able to decide whether the input $(x_1,\dots,x_k)$ received from the "opponent" is in R because if he received a signal (before p) and he (successfully) checked the signal for correctness in the above outlined way, then he knows $(x_1,\dots,x_k)\in R$. Otherwise he knows $(x_1,\dots,x_k)\notin R$. We finished the proof. $\diamond$

Corollary 1. *There are infinitely many functions $f:\mathbb{N}\to\mathbb{N}$ such that*
(i) *f is realized by a relativistic computer $G=(\gamma_P,\gamma_O)$;*
(ii) *f is non-Turing computable.*

Proof. Let $R\in\Sigma_1\setminus\Pi_1$. It is known that there are infinitely many such sets, cf. e.g. [19]. Let $f:=\chi_R$ be the characteristic function of R. Then $f:\mathbb{N}\to\{0,1\}$ is non-Turing computable because R is undecidable by $R\notin\Pi_1$. Let $G:=(\gamma_P,\gamma_O)$ be the relativistic computer deciding R. This exists by Proposition 1. Let G' be the same but instead of "yes" or "no" let it give as an output 1 or 0. Then G' realizes f. $\diamond$

Below we will prove stronger theorems. By Proposition 1, relativistic computers can decide any undecidable but recursively enumerable relations $R\in\Sigma_1\setminus\Pi_1$. It is natural to ask whether harder sets of natural numbers become decidable if we switch to relativistic computers. The next proposition says that the answer is in the affirmative.

Proposition 2. *Let $n>0$. There are infinitely many relations $R\in\Sigma_2\setminus(\Sigma_1\cup\Pi_1)$, $R\subseteq\mathbb{N}^n$ such that some relativistic computer decides R.*

Proof. Let $H\in\Sigma_1\setminus\Pi_1$ be arbitrary. Define

$$R:=\{(x,1)\mid x\in H\}\cup\{(y,0)\mid y\in\overline{H}\},$$

where $\overline{H}=\mathbb{N}^n\setminus H$. That is,

$$R=(H\times\{1\})\cup(\overline{H}\times\{0\}).$$

(i) $R\notin\Sigma_1$ because we cannot enumerate its second part $\overline{H}\times\{0\}$ and $R\notin\Pi_1$ because we cannot enumerate the complement of its first part $H\times\{1\}$. (Hint: $R\in\Sigma_1\Rightarrow$ we can enumerate $R\Rightarrow$ we can enumerate those elements of R which end with $0\Rightarrow$ we can enumerate $\overline{H}\times\{0\}\Rightarrow$ we can enumerate $\overline{H}$.) It can be seen that $R\in\Sigma_2$ (by $\overline{H}\in\Pi_1$).

(ii) By Proposition 1 there is a relativistic computer G deciding H.

The new G' deciding R does the following: If it receives an input (x,k) and if $k>1$, then G' answers "no". Assume $k\le 1$. Then G' asks $G=(\gamma_P,\gamma_O)$ to decide whether $x\in H$. If $k=1$ then G' prints out the same answer as G. If $k=0$ then G' prints out the negation of the answer of G.

Clearly, G' is a relativistic computer deciding $R \in \Sigma_2 \setminus (\Sigma_1 \cup \Pi_1)$. $\diamond$

Since $R \notin \Sigma_1 \cup \Pi_1$, our new computer G' constructed in the proof of Proposition 2 decides sets harder than *recursively enumerable sets* and *complements of recursively enumerable ones.* This means that we can "climb higher" with one extra degree of unsolvability with Proposition 2. We have the following corollary, immediate from Proposition 2.

Corollary 2. *There are infinitely many $\Sigma_2 \setminus (\Sigma_1 \cup \Pi_1)$ functions $f : \mathbb{N} \to \mathbb{N}$ realizable by relativistic computers. (Of course these functions are non-Turing computable).* $\diamond$

We note that the simplest examples of $R \in \Sigma_2 \setminus \Sigma_1$ relations are the characteristic functions χ_H of relations $H \in \Sigma_1 \setminus \Pi_1$. We claim that relations decided relativistically by Proposition 2 cannot be obtained in this way. Therefore by Proposition 2 we can decide Σ_2-relations which are strictly more complex (i.e. harder) than the simplest examples for $R \in \Sigma_2 \setminus \Sigma_1$.

Let us ask if we can decide even harder sets than in Proposition 2. Each relation decided by Proposition 2 can be regarded as a disjoint union of a Σ_1-set and a Π_1-set. In the next proposition we will decide relations in $\Sigma_2 \setminus (\Sigma_1 \cup \Pi_1)$ which cannot be obtained as such disjoint unions. In some sense this means that we can decide even broader spectrum of hard relations.

Proposition 3. *Let $n > 0$. There are infinitely many relations $R \in \Sigma_2 \setminus (\Sigma_1 \cup \Pi_1)$, $R \subseteq \mathbb{N}^n$ such that*

(i) *R cannot be obtained as a disjoint union of finitely many Σ_1 and Π_1 relations;*

(ii) *R is decidable by a relativistic computer $G = (\gamma_O, \gamma_P)$.*

Proof. Let $H \in \Sigma_1 \setminus \Pi_1$ be arbitrary, $n > 0$. Define

$$X_H := \{(a, b) \mid a \in H \text{ and } b \in \overline{H}\}.$$

Let $\chi_{X_H} =: f : \mathbb{N}^{2n} \to \{0, 1\}$ be the characteristic function of X_H. Then $R_f \subseteq \mathbb{N}^{2n} \times \{0, 1\} \subset \mathbb{N}^{2n+1}$ is a $2n + 1$-ary relation.

(i) To decide R_f, our $G = (\gamma_O, \gamma_P)$ is similar to the one that was before but now γ_P can send *two* different kinds of signals to γ_O, say S_a and S_b. The input for G is of the form $(a, b, k) = ((a, b), k)$ (where k refers to the $(2n + 1)$-th component of R_f). The case distinction between $k > 1$ and $k \le 1$ is similar to that in the proof of Proposition 2. If $k > 1$ then γ_O automatically prints "no".

Assume $k = 1$. γ_P does the following: It starts searching for a in H. If it finds $a \in H$ then sends out S_a to γ_O and starts a search for $b \in H$. If it finds b, then sends S_b. Now γ_O does the following: If he receives no signal, then prints out "no". If he receives S_a and no S_b then prints "yes". If he receives S_b and no S_a then prints "no". Finally, if he receives both an S_a and S_b, then he prints "no".

Assume $k = 0$. Then γ_P starts two parallel processes P_a and P_b. If P_a finds $a \in H$ it sends off a S_a while if P_b finds $b \in H$ it sends S_b. If γ_O receives no signal, he prints "yes". If he receives S_a but no S_b then prints "no". If he receives an S_b then prints "yes" (independently of other possibly received signals).

(ii) In connection with R_f not being a disjoint union of a Σ_1 and a Π_1 set, we note only the following. Let

$$R_i := \{(a, b, i) \mid (a, b, i) \in R_f\}.$$

Then $R_1 = R \times \{1\}$ with $R = T_H$. So, in some sense, the "complexity" of R_1 is determined by the "complexity" of R. But R was of the form $R = \{(a, b) \mid a \in H \text{ and } b \notin H\}$ with $H \in \Sigma_1 \setminus \Pi_1$. So clearly $R \notin \Sigma_1$ because of the "b part" and $R \notin \Pi_1$ because of the "a part". To save space, we omit the rest of the proof, since the present proposition is not of a central importance. $\diamond$

By Proposition 3 above, relativistic computers can solve problems much harder than the non-Turing computable problem of deciding an undecidable but recursively enumerable (i.e. $\Sigma_1 \setminus \Pi_1$) relation.

The next proposition shows that the *extended Turing machine*, we discussed between Definition 3 and Definition 4 in Section 2—which for any Turing machine T and possible input $(x_1, \ldots, x_k)$ decides whether T terminates—is also realizable by a relativistic computer G.

Proposition 4. *There is a relativistic computer $G = (\gamma_P, \gamma_O)$ which takes as input a program $\mathrm{pr}(T)$ for a Turing machine T and a possible input $(x_1, \ldots, x_k)$ for T. Then G yields output "diverges" if T diverges for $(x_1, \ldots, x_k)$ or else "converges with output $(y_1, \ldots, y_l)$" if T indeed converges for input $(x_1, \ldots, x_k)$ with output $(y_1, \ldots, y_l)$.*

Proof. G is of the form (γ_P, γ_O) as usual. γ_P and γ_O, sitting together, receive as input a program $\mathrm{pr}(T)$ for some T and a possible input $(x_1, \ldots, x_k)$ for T. For simplicity we will write "T" for $\mathrm{pr}(T)$.

Then γ_O takes a copy of T and $(x_1, \ldots, x_k)$ with himself and starts his journey "toward the Malament–Hogarth event" $p \in M$. Then γ_P starts executing T with input $(x_1, \ldots, x_k)$. If T terminates, γ_P sends a signal to γ_O. At (or after) the Malament–Hogarth event, γ_O does the following. If no signal arrived then prints "diverges". If he received a signal from γ_P then γ_O knows that T converges with $(x_1, \ldots, x_k)$. Consequently γ_O can safely start executing T with $(x_1, \ldots, x_k)$ and he knows that T will terminate in finite time. When T terminates, then γ_O prints out whatever output T yielded. $\diamond$

In light of Propositions 1-4 and Corollaries 1-2 above we can decide general Σ_1-sets and are able to realize hard Σ_2-sets by our relativistic computer G contradicting Theses 2-2'. In our opinion, the above considerations point in the direction that if we choose classical general relativity as the background physical theory then Theses 2-3 turn out to be false because they deal with computability of the second kind. The reason why Thesis 1 cannot be attacked is the difference between an artificial computing system (i.e. a thought-experiment in a consistent physical theory) and a physical computer in the narrow sense: it is possible that our artificial computing system cannot be realized as a physical computer, although we remark that the almost-sure existence of large rotating black holes in galactic nuclei and properties of these black holes (see below) point towards the effective realizability of our thought-experiment, i.e. towards the possible violation of Thesis 1, too.

Remark. A reader who is not a specialist of general relativity theory, may ask the following question. Why do we need (something as "fancy" as e.g.) rotating black holes, why is a "simple" Schwarzschild black hole (of sufficiently big mass) not enough for our thought-experiment[20]? (Instead of rotating ones, electrically charged black holes would do the job just as well, but this is not the issue here, since the question is why do we need something more complex than the most "classical" Schwarzschild holes.) The answer is the following.

For the sake of argument, let us use Schwarzschild coordinates for describing the spacetime outside the non-rotating black hole. Let γ_O and γ_P behave as in the thought-experiment described above (involving Kerr black holes). Now, it is true that from the point of view of γ_P, the clocks of γ_O slow down so much that when γ_O receives a yes (or no) answer from its computer, then according to γ_P's coordinate system, γ_O is still outside of the event horizon. The problem is that to send the answer to γ_O such that he receives it still before hitting the singularity (which event is impossible to avoid in this case), γ_P would need to use so called tachyons (FTL-signals). Indeed, if γ_P finishes the computation in a large enough but finite time, then the light γ_P sends after γ_O will converge to γ_O in a similar rate as γ_O converges to the singularity but will not reach γ_O before γ_O crosses it[21]. The problem is alleviated e.g. by using rotating black holes, very roughly as follows. In a rotating black hole behind the event horizon we just discussed, there is a second inner event horizon which is a Cauchy horizon as well. If γ_O approaches the black hole along the orbit considered above, then not later than γ_O reaches the *second* horizon, he will meet the signal sent by γ_P. As we indicated, making our black hole rotate is only one of the possible solutions but this choice is strongly supported by the "naturality" of rotating black holes i.e. their very possible real existence.

4 On the physical reality of the model

To understand if the above model is realistic from the physical point of view, we collect properties of Malament–Hogarth space-times using results from recent physical literature. First we summarize two important general characteristics of Malament–Hogarth space-times. We can state

[20]Cf. e.g. footnote 5 on p.83 of [25].

[21]This can be seen by looking at the Penrose diagram of the extended Schwarzschild space-time.

(see Lemma 4.1. and Lemma 4.3. of [4]):

Proposition 5. *Let (M, g) be a Malament–Hogarth space-time with a Malament–Hogarth event $p \in M$. Then M is not globally hyperbolic.*

Moreover, choose any connected space-like hypersurface $S \subset M$ satisfying $\mathrm{im}\gamma_P \subset D^+(S)$. Then either $p \in H^+(S)$ i.e. p lies on the future Cauchy horizon of S or $p \notin D^+(S)$ i.e. does not belong to the future Cauchy development of S. $\diamond$

The meaning of Proposition 5 is the following. A very important property of globally hyperbolic space-times is that they possess a so-called initial data surface (called Cauchy surface) i.e. fixing data of physical fields along the Cauchy surface only (which is a three-dimensional submanifold of M), one can determine the values of these fields over the whole space-time via the corresponding field equations. The above theorem shows that a Malament–Hogarth space-time does not possess such an initial data surface i.e. always contains events $q \in M$ which are unpredictable even fixing initial data on arbitrary large subsets of M, for example on a space-like submanifold $S \subset M$. Especially, the Malament–Hogarth event $p \in M$ is such an event. We met this phenomenon in the special case of the Kerr space-time already. The difficulties caused by this fact will be discussed soon.

Another very important property of Malament–Hogarth space-times is the "infinite blueshift effect". Roughly speaking, as a consequence of the infinite time contraction seen by the observer γ_O approaching the Malament–Hogarth event $p \in M$, all signals of finite energy or frequency will hit γ_O at $p \in M$ by an infinite amount of energy, i.e. Malament–Hogarth space-times act as unbounded gravitational amplifiers near $p \in M$. More precisely, the following theorem holds (Lemma 4.2. of [4]):

Proposition 6. *Let (M, g) be a Malament–Hogarth space-time with time-like curves γ_P and γ_O as in Definition 5. Suppose that the family of null-geodesics connecting γ_P with γ_O forms a two dimensional integral submanifold in M in which the order of emission from γ_P matches the order of absorption by γ_O. If the photon frequency ω_P is constant measured by γ_P (i.e. γ_P does not stop sending signals to γ_O) then the time-integrated photon frequency*

$$\int\limits_{\gamma_O} \omega_O \mathrm{d}\gamma_O = \int\limits_a^b \omega(\gamma_O(\tau))\sqrt{-g(\dot{\gamma}_O(\tau), \dot{\gamma}_O(\tau))}\,\mathrm{d}\tau$$

received by γ_O is divergent. $\diamond$

This theorem is a trivial consequence of the assumption that the original observer γ_P sent an infinite amount of energy to γ_O, since it sends signals of constant frequency ω_P throughout its infinite existence.

Now we wish to discuss the consequences of these properties of Malament–Hogarth space-times in the special case of the Kerr black hole against building relativistic computers constructed in Section 3.

(1) First we are going to study the effects of the infinite blueshift, the problem formulated in Proposition 6 above. We consider first whether γ_O can survive the encounter with the inner event horizon or not. A similar but more detailed consideration like Proposition 6 shows that near its inner horizon, the Kerr black hole amplifies every, arbitrarily small deviation from the original vacuum space-time structure in an unbounded amount, yielding that this horizon rather looks like a real curvature singularity (i.e. not a pure "coordinate singularity"). This phenomenon is known as the "infinite mass-inflation" in the physics literature and appears if one calculates the effect of the infinitely amplified absorbed energy on the metric near the inner horizon. At first look, in the case of perturbations of the metric by a scalar field, this singularity turns out to be a scalar curvature divergence on the inner horizon [26]. This fact is usually interpreted as the instability of the (vacuum) Kerr space-time. Hence, after realizing the mass-inflation phenomenon, physicists supposed the non-traversability of the Kerr black hole.

A more careful analysis of the situation was carried out by Ori [21][22] in the case of the Reissner–Nordström black hole and partially in the case of the Kerr–Newman black hole, however. In accordance with his calculations (accepting the validity of certain technical assumptions) it seems that despite the existence of the scalar curvature divergence, the tidal forces remain finite moreover negligible in the case of realistic Kerr black holes when crossing the inner horizon. Hence although the inner horizon (which contains the Malament–Hogarth event) is a real curvature singularity it is only a so-called *weak singularity* because the tidal forces still remain finite on it. As an example [22], for a Kerr black hole of mass $M = 10^7 m_\odot$ ($m_\odot$ refers to solar mass) and age $T = 10^6$ years (more precisely this is the age of the initial perturbation of the Kerr black hole) the relative distortion of an object of typical size l crossing the inner horizon is

$$\frac{\Delta l}{l} \leq 10^{-55}.$$

In summary, although the Malament–Hogarth event is situated in a "dangerous" region of the Kerr–Newman space-time, in theory at least, it can be approached by the observer γ_O.

Next we may ask about the strong (electromagnetic) radiation absorbed by γ_O during the course of crossing the inner event horizon, as another consequence of the blueshift effect. This problem was studied by Burko and Ori [3]. They conclude that these effects remain also finite making it theoretically possible to survive such an encounter for γ_O although one may worry about the intensive pair creation induced by the extremely high energy photons [3]. Of course these considerations require more detailed analysis in the future, see also [24].

Summing up, we can conclude that accepting a very rough, classical picture for the inner horizon of the Reissner–Nordström and Kerr–Newman black holes, their traversability is reasonable.

In our artificial computing system $G = (\gamma_P, \gamma_O)$, the physical computer γ_P sends a modulated light beam to γ_O. Proposition 6 above suggests that even the (energetically) mildest answer of γ_P will simply destroy γ_O by receiving an infinite amount of energy. This is valid only if γ_P sends electromagnetic signals of constant frequency through an infinite proper time (measured by γ_P), hence Proposition 6 is not surprising. If γ_P sends a finite signal answering a simple "yes" or "no", the received energy by γ_O remains finite in light of results of Burko and Ori. Hence, the pessimistic consequences of Proposition 6 are ruled out for the relativistic computer designed in the present paper.

(2) The stability of the circular orbit around the Kerr black hole required for γ_P was also studied by Kennefick and Ori [13] and Ori [23]. They studied the effect of the gravitational radiation of the Kerr black hole on the evolution of a point particle moving on an initially circular orbit around the Kerr black hole. The answer is also encouraging, the perturbation seems to be negligible yielding the stability of stationary circular orbits, hence the computer γ_P can orbit around the black hole for long time (hence with little effort for ever).

(3) We mention at this point again, that both curves γ_P and γ_O are geodesics in the Kerr space-time i.e. the acceleration along them is zero i.e. bounded. Hence in this situation one does not have to worry about the negative consequences of a possibly infinite acceleration, see [25].

(4) Next we turn our attention to the consequences of Proposition 5. The essence of this theorem is that the Malament–Hogarth event $p \in M$ cannot be predicted even fixing initial data on the whole spatial surface $S \subset M$ which is a Cauchy surface for the outer observer γ_P. This fact is interpreted by Earman in [4], p. 118 by saying that the observer γ_O, while crossing $p \in M$, is able to decide whether a signal came from γ_P or from a possibly past singularity if and only if he is able to perform an infinitely precise discrimination in spatial directions which is physically unreasonable (but although theoretically it is allowed in classical physics). As suggested also by Earman this problem is apparently solved by using a coding system between γ_P and γ_O because in this case the information of the result of the calculation γ_P just completed is not carried simply by the direction of the light beam. But this solution is also rejected by Earman by another "infinitely precise discrimination" argument ([4] p. 118) essentially based on the assumption that γ_P wishes to send a possibly unbounded amount of information to γ_O. But as we clarified in the beginning of this section for our purposes we need to answer *yes or no questions* only, using a previously fixed code. Hence the length of the message sent by γ_P is bounded uniformly hence Earman's infinite discrimination argument is not valid in this moment.

But apparently, as we have seen, γ_O must be able to perform infinitely precise measurement of time because in our model the detection time carries a lot of information. Notice however that this assumption is not an extra one; it is already assumed by accepting that γ_O, before crossing the Malament–Hogarth event $p \in M$, is always able to detect signals from γ_P. Namely, the length of the signal received by γ_O tends to zero, hence very close to $p \in M$ γ_O must be able to detect arbitrary short signals; consequently if he can do it certainly can measure its detection time arbitrary accurately, too; we soon discuss how to deal with this problem (of "infinite precision").

Summing up, we went through all the major possible *classical* obstacles published so far against building the artificial computing system $G = (\gamma_P, \gamma_O)$ performing computability of the second kind, listed in [4], [25] and references therein. We found that these obstructions can be removed at the classical level (hence in classical general relativity) i.e. they do not kill the idea of designing a thought-experiment suitable for deciding Σ_1-sets of natural numbers (because the quoted objections do not destroy the idea of the relativistic computer we designed in the present paper).

As a final remark we have to emphasize again that we have omitted all the *quantum effects* in our model. In general, the accuracy of time measurement, which is required for γ_O, is not a problem in classical physics while in quantum physics it is constrained by quantum fluctuations. In this moment we do not possess a satisfying theory to describe the consequences of these quantum fluctuations in the presence of strong gravitational fields. Of course this is because a satisfactory theory of quantum gravity has not been formulated yet. We can do only naive considerations taking into account the basic principles of quantum mechanics and general relativity. Using results of Ng [17] we can say the following about the accuracy of time measurment. Assume we have a clock with total running time ΔT over which it can remain accurate and is capable for a time measurment of accuracy Δt. Then one can derive an inequality

$$\Delta t \geq (\Delta T\, t_P^2)^{1/3}$$

where $t_P = \sqrt{\hbar G/c^5} \approx 10^{-43}$ *sec* is the Planck-time. In our case we require a time measurment with an unboundedly increasing accuracy from γ_O's clock *till the Malament–Hogarth event.* Consequently, without the violation of the above inequality, in principle the observer γ_O can constantly "tune" his clock to be more and more accurate ($\Delta t \to 0$) for shorter and shorter times as he approaches the Malament–Hogarth event ($\Delta T \to 0$). Possibly this clock cannot be used *after* the Malament–Hogarth event but this is not a problem. But notice that interpreting the Planck-time t_P as the fundamental smallest time unit then accuracy beyond t_P is meaningless. This might destroy the realizability of our thought-experiment in a quantum framework. This means that if we use quantum gravity in place of classical general relativity as our background theory, then we should design our artificial computing system differently. However, since the theory of quantum gravity does not exist yet, it seems pointless to try to elaborate the details nowadays.

Moreover a generally accepted quantum gravitational phenomenon is the black hole evaporation. Finally this may cause that every (Kerr) black hole will evaporate in finite time making it impossible for γ_P to send signals to γ_O in very late times. Hence these and other, yet unknown, quantum phenomena occurring in strong gravitational fields can eventually also annihilate our considerations.

5 Concluding Remarks

In this paper we have studied the physical reality of performing an infinitude of calculations in finite time in order to answer very interesting questions.

One of the present authors (I.N.) had discussed the various Theses formulated in Section 2 with one of their originators László Kalmár, and he feels that Kalmár would be pleased by the kind of approach taken in the present paper.

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