Theses for PhD-dissertation

Logic and Relativity
(in the light of definability theory)

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1. Introduction

The combined investigation of mathematical logic and relativity theory is not at all new, as follows. Direction (1): Already at the beginnings, i.e. around 1920, Einstein's friend Reichenbach set to himself the goal of building up relativity as a logical theory, i.e. a theory purely in first-order logic, cf. [30]. Similarly, Carnap [8, 9] pursued and advocated the same goal using a more sophisticated version of mathematical logic. Definability theory, one of the most important branches of modern logic, was brought to existence in the relativity book [30] because of the special needs of relativity theory. (This field reached maturity via extensive work by Tarski on definability as indicated in the dissertation.) Direction (2): The foundations of logic presuppose a kind of world-view (Weltanschauung) which has an effect on the "structure" of the theory of logic. In this sense the latest developments of relativity and related areas (e.g. Gödel space-time, Kerr space-time) provide a new world-view which has an effect on the development (and directions) of logic, cf. e.g. Hogarth [17, 18], Etesi-Németi [12].

In connection with Direction (1) above, besides Reichenbach, Carnap, we also have to mention pioneering work by Tarski, Gödel, Suppes, Goldblatt and others. The common feature is that we wish to build up relativity theory (and its variants) as a theory of possibly many-sorted first-order logic. Obtaining such a logical theory is not the end-result of our investigation, but rather the starting point. E.g. one of our aims is the analysis of the logical structure of relativity theory. Clearly, we can start pursuing this aim only after having obtained a version of relativity as a purely logical theory but obtaining this theory is only "step zero", the real fun, the real use of logic begins only after this step. The goals of our approach are summarized in [2, §1.1], here we mention only one. By using decomposability of a logical theory into axioms, we want to obtain a version of relativity theory which is "modular" or is of a "lego-toy" character. This means that if for some unforeseen reason we want to change our theory a little, then we can pull out the "module" we want to change and replace it with another. Here, the role of "modules" is played by the axioms, or refinements of the axioms. This provides our theory with a kind of flexibility which is usually not available in non-logical theories. This flexibility makes our theory (i) adaptable to change, (ii) capable for formulating and answering the why-type questions. We do not mean the causal version of "why" but rather the deductive-nomological version, cf. e.g. Lewis [19, p.231], or even better we mean the "why" for which Gödel was called "Herr Warum" in his childhood [11]. We are interested in those why-questions which seek explanation for regularities (i.e. for law-like phenomena) cf. e.g. Ernest Nagel [28].

To pursue the above aims, first we had to extend definability theory (in the direction needed for relativity). The new direction of definability we have in mind can be traced back to early works of Michael Makkai, and to more recent works of Shelah, Pillay and others [15, 29]. The essentially new element is that now we want to allow definition of new universes with brand new elements, while in the classical version of definability only new relations on old elements were allowed.

2. Definability theory

Let $\mathfrak{M} = \langle U_0, \ldots, U_n; R_1, \ldots, R_k \rangle$ be a (many-sorted) model (the U_i's are the universes or sorts, while the R_j's are the relations [or functions].) Let $\mathfrak{M}^+ = \langle \mathfrak{M}, U^+, \bar{R}^+ \rangle$ be an expansion of $\mathfrak{M}$ (here $\bar{R}^+ = \langle R_1^+, \ldots, R_r^+ \rangle$). The set Σ of formulas implicitly defines $\mathfrak{M}^+$ over $\mathfrak{M}$ iff for any expansion $\mathfrak{M}' = \langle \mathfrak{M}, U', \bar{R}' \rangle \in \mathsf{Mod}(\Sigma)$ there is a unique isomorphism $h : \mathfrak{M}^+ \rightarrowtail\!\!\twoheadrightarrow \mathfrak{M}'$ which is the identity on $\mathfrak{M}$. (Intuitively, Σ describes $U^+, \bar{R}^+$ over $\mathfrak{M}$ up to a unique isomorphism leaving the elements of $\mathfrak{M}$ fixed.) Notice that for the special case $U^+ = \emptyset$, this agrees with the classical notion of implicit definition of new relations $\bar{R}^+$.

Let K and K^+ be classes of models. We say that Σ implicitly defines K^+ over the class K exactly when the natural generalization of the above condition holds (the generalization goes in the usual spirit of model theory).

Let K, L be two classes of models (their languages may be different). K and L are said to be definitionally equivalent, in symbols $\mathsf{K} \equiv_\Delta \mathsf{L}$, iff they admit a common expansion K^+ definable over both of them. $Fm(\mathsf{K})$ denotes the set of formulas in the language of K. Hence $\mathsf{Th}(\mathsf{K}) \subseteq Fm(\mathsf{K})$ where $\mathsf{Th}(\mathsf{K})$ is the theory of K.

Theorem 1 (4.3.45, 4.3.48) *Assume* $\mathsf{K} \equiv_\Delta \mathsf{L}$ *and* K, L *are axiomatizable classes. Then there are two translation mappings (or "interpretations")* $\mathsf{tr}_1 : Fm(\mathsf{K}) \longrightarrow Fm(\mathsf{L})$ *and* $\mathsf{tr}_2 : Fm(\mathsf{L}) \longrightarrow Fm(\mathsf{K})$ *such that* $\mathsf{tr}_1, \mathsf{tr}_2$ *are "truth-preserving"*[1] *meaning preserving and inverses of each other* ($\mathsf{K} \models \varphi \leftrightarrow \mathsf{tr}_1(\mathsf{tr}_2(\varphi))$*, similarly for* L*, moreover*
$\mathsf{K}^+ \models \{\varphi \leftrightarrow \mathsf{tr}_1(\varphi), \psi \leftrightarrow \mathsf{tr}_2(\psi) : \varphi \in Fm(\mathsf{K}), \psi \in Fm(\mathsf{L})\}$).

Intuitively, by the above theorem, if $\mathsf{K} \equiv_\Delta \mathsf{L}$, then $\mathsf{Th}(\mathsf{K})$ and $\mathsf{Th}(\mathsf{L})$ are really the same single theory with two different choices of basic symbols. Similarly, if K, L are axiomatizable then $\mathsf{K} \equiv_\Delta \mathsf{L}$ means that in some sense they are the same single class of models with two different choices of basic symbols. Classical examples for this are K =(Boolean algebras), L =(Boolean rings with a unit), or $\mathsf{Th}(\mathsf{K})$ =(Peano's arithmetic) and $\mathsf{Th}(\mathsf{L})$ =(finite set theory).

The conclusion of Theorem 1 above does not imply $\mathsf{K} \equiv_\Delta \mathsf{L}$ while it did this in the case of classical definability theory.

The above notion of implicit definability is, clearly, a natural extension of the classical notion (cf. e.g. Hodges [15]). Analogously, the generalization of explicit definability to the new situation is an extension of the classical notion (used e.g. in Beth's Theorem) elaborated in §4.3.2 (p.236) of the dissertation. We do not recall this from the dissertation, but we note that, analogously to the classical case, an explicit definition of a new universe involves a concrete (and uniform) construction of the new elements (and the new universe formed by them) from the old ones. E.g. if U, V are old universes, then $U \times V$ is explicitly definable over the old model $\langle U, V, \ldots \rangle$.

Our next result is a generalization of Beth's celebrated theorem.

Theorem 2 (4.3.31, 4.3.49) *Assume* K^+ *is an axiomatizable expansion of* K*. Then*
K^+ *is explicitly definable over* K *iff*
K^+ *is implicitly definable over* K.

Intuitively, by the above theorem, explicit definability and implicit definability coincide. Hence, as the slogan of Beth's classical theorem says, implicitly defined symbols can be eliminated (from a theory). Cf. also Theorem 1 above.

[1] $\mathsf{tr}_1(\mathsf{Th}(\mathsf{K})) = \mathsf{Th}(\mathsf{L})$ and similarly for tr_2.

We note that in Hodges-Hodkinson-Macpherson [16], at the bottom of the first page, the authors mention the question of generalizing Beth's theorem to the case where besides new relations also new elements are allowed. Cf. also Hodges [14].

Beth's theorem has interested people for a long time: In 1900 Padoa announced it without a proof, in 1926 Tarski proved it for the special case of the logic of Whitehead and Russel, Beth proved its classical form in 1953, after which Michael Makkai, C. C. Chang, and many of the model theorists mentioned in the present text elsewhere worked on generalizing it further.

At this point we mention that in connection with definability theory, the present author's refereed publications include e.g. [20, 22, 24, 23, 21, 25].

3. Logical analysis of relativity theory.

A typical model of our simplest kind of relativity theories is a two-sorted structure

$$\mathfrak{M} = \langle B, \mathfrak{F}; Obs, Ph, Ib, W\rangle$$

where $\mathfrak{F}$ is an ordered field, the universes (or sorts) of $\mathfrak{M}$ are B and F, while its relations are $Obs, Ph, Ib \subseteq B$ and $W \subseteq B \times {}^nF \times B$ (together with the ones of $\mathfrak{F}$). B is the set of bodies (whose motion we are interested in), ${}^nF$ is the biggest coordinate system our observers $m \in Obs$ use for coordinatizing the set $Mn \subseteq \mathcal{P}(B)$ of events (m may use a proper subset of ${}^nF$ for coordinatizing events, cf. $\mathbf{Par}(Th)$ at the end of §3), Ib, Ph and Obs are the sets of inertial bodies, photons and observers respectively. For every $m \in Obs$, our W induces a world-view map $w_m : {}^nF \longrightarrow \mathcal{P}(B)$ defined by $w_m(p) = \{b \in B : \langle m, p, b\rangle \in W\}$. Here n is the number of space-time's dimensions, the most important case is $n = 4$. $\mathcal{P}(B)$ is the powerset of B.

We strive for using a small number of transparent, natural, and intuitively convincing axioms and at the same time proving strong theorems from them. Our most flexible axiom systems are local in the sense of general relativity, cf. e.g. $\mathbf{Par}(\mathbf{Bax}^{-\oplus})$ in §3 and on pp.A-28 – A-29. These admit the manifolds with CTC's constructed in Charette et al. [10]. We do not recall these theories.

One of our theories is the following.

$$\mathbf{Pax} \stackrel{\text{def}}{=} \{\mathbf{Ax2}, \mathbf{Ax3_0}, \mathbf{Ax4}, \mathbf{Ax5_{Obs}}^{--}, \mathbf{Ax6}_{00}, \mathbf{Ax6}_{01}\},$$

where

Ax2 $Obs \cup Ph \subseteq Ib$
(i.e. all observers and photons are inertial bodies[2])

Ax3$_0$ Every observer m "sees" via w_m the life-lines of inertial bodies as straight lines (assuming he sees them). Throughout, "seeing" means coordinatizing via w_m, hence m sees event e means $e \in Rng(w_m)$.

Ax4 Every observer m sees its own life-line on the time-axis of ${}^nF$
(i.e. $W(m, p, m) \Leftrightarrow p = \langle p_0, 0, 0, \ldots\rangle$).

[2]Later, in the theory of accelerated observers, we will omit the $Obs \subseteq Ib$ part of **Ax2**

Ax5$_{\mathbf{Obs}}$$^{--}$ Intuitively, it is possible to move in any direction (at least slowly). Formally: For any observer m and direction d there is $\lambda \in {}^{+}F$ such that m "sees" all speeds $< \lambda$ realized in direction d by some observer, through each point $p \in {}^{n}F$.

Ax6$_{00}$ Any event in which m "sees" k participating is also "seen" by k, for $m, k \in Obs$. I.e. $k \in e \in Rng(w_m) \Rightarrow e \in Rng(w_k)$.

Definition: $\mathsf{f}_{mk} \stackrel{\text{def}}{=} w_m \circ (w_k)^{-1}$, for $m, k \in Obs$. Intuitively, f_{mk} transforms the world-view of m to that of k.

Ax6$_{01}$ $Dom(\mathsf{f}_{mk})$ is an open subset of ${}^{n}F$, for $m, k \in Obs$.

Later, in generalizations in the direction of general relativity, we will suitably modify the above axioms, e.g. we will use geodesics instead of straight lines (cf. §4.7 "Geodesics"), we will replace the axioms with their local versions mentioned way above.

Theorem 3 (3.2.6)

(i) *Assume* **Pax**. *Then the world-view transformations* f_{mk} *are bijective collineations of* ${}^{n}F$, *if* m *sees observer* k.

(ii) *If* $\mathfrak{F}$ *has no nontrivial automorphisms, then the* f_{mk}*'s are affine transformations, assuming* **Pax**.

In usual presentations of relativity, the conclusion of the above theorem is assumed as a basic postulate. By the above theorem, this is superfluous, since this postulate is a logical consequence of a small fraction of the remaining usual postulates. It turns out in the dissertation that **Pax** is so weak that it is a common part of the following theories (of kinematics): (i) Newton's kinematics, (ii) Einstein's special relativity, (iii) Lorenzian spec. rel., (iv) Reichenbach-Grünbaumian spec. rel. In [2] strongly related to the dissertation we throroughly investigate the logical interconnections between theories (i)-(iv), e.g. we elaborate a so called transfer principle by which every model of (iv) can be constructed from one of (i) and vice versa.

The theory **Pax** does not involve photons or anything related to electrical phenomena. In this connection: (i) The author expanded **Pax** to a natural, convincing and transparent axiom system **Relnoph** (p.A-30) which still does not involve anything related to electricity (like photons) and from which all predictions[3] of special relativity are derivable. (ii) Below we expand **Pax** with a minimum amount of very weak assumptions on photons.

Ax5$_{\mathbf{Ph}}$ All observers see photons moving in all (spatial) directions with positive speeds, in every $p \in {}^{n}F$.

AxP1 Photons do not move like bullets, i.e. they do not "race" with each other. Formally: If m sees photons ph_1, ph_2 moving in the same direction at the same point $p \in {}^{n}F$, then m will see them move with the same speed.

[3] of course only those which do not involve photons

$\mathbf{Ax5_{Obs}}$ In axiom $\mathbf{Ax5_{Obs}}^{--}$ way above, the value λ is the speed of a photon moving in the direction d in question (according to m, of course).

$$\mathbf{Bax}^{-} \stackrel{\text{def}}{=} \mathbf{Pax} \cup \{\mathbf{Ax5_{Ph}}, \mathbf{AxP1}, \mathbf{Ax5_{Obs}}\}.$$

For any theory Th we let $Th^{\oplus} \stackrel{\text{def}}{=} Th +$ *"(all photons have finite speeds for all observers)"*.

Theorem 4 (3.2.13, 3.2.14) *Assume $n > 2$ and that square-roots of positive elements exist in $\mathfrak{F}$. Then:*

(i) $\mathbf{Bax}^{-\oplus} \models$ *"there are no faster-than-light (FTL) observers".*[4]

(ii) $\mathbf{Bax}^{-}$ *does not imply the conclusion of (i).*

Beginning with the above theorem, in the dissertation we try to find the "turning points" i.e. the weakest theories proving the no FTL theorem. This gives a hint of answer to the question "why is FTL travel impossible?". (Cf. the introduction about why-type questions.)

In the usual presentations or "derivations" of relativity, the conclusion of the above theorem is usually assumed as an axiom. Then the justification of this "no FTL axiom" uses arguments the correctness or acceptability of which is still intensively debated in the literature. By Theorem 4 above, this axiom is superfluous since it follows from a small fragment of the remaining axioms which are always assumed anyway. The author's publications about the above subject include [1, 2, 6].

4. Relativistic geometry, duality theory.

In the literature of relativity, the distinction between "observer oriented" concepts and "theoretical" concepts has always been important, cf. e.g. Friedman [13].[5] A theoretical concept counts as acceptable if it is definable from observational ones. This is what motivated Reichenbach to initiate the logical theory of definability already around 1920, cf. Reichenbach [30].

In the present work, the models $\mathfrak{M} \in \mathsf{Mod}(\mathbf{Pax})$ correspond to the observation oriented worlds of relativity. In Def.4.4.1 of the dissertation, over each model $\mathfrak{M}$ we define a geometric structure

$$\mathfrak{G}_{\mathfrak{M}} \stackrel{\text{def}}{=} \langle Mn, \mathsf{Lines}; \in Bw, eq, \perp, \prec, g \rangle.$$

Here $Mn \subseteq \mathcal{P}(B)$ is the set of events, $\in$ is the incidence relation hence $\langle Mn, \mathsf{Lines}; \in \rangle$ is a familiar kind of geometry. The definition of $\mathfrak{G}_{\mathfrak{M}}$ over $\mathfrak{M}$ goes in the following spirit. If $e, e_1 \in Mn$ are on the life-line of observer m, then $g(e, e_1)$ is the time m measures elapsing between events e and e_1. More precisely, since there may be more than one such observers, $g(e, e_1)$ is the minimum of such time measurements, assuming it exists. Otherwise it is undefined. This

[4] No m thinks that observer k is FTL.

[5] E.g. Einstein rejected the concept of aether because it is not an observational concept nor is it definable from observational concepts.

definition of g corresponds to the spirit of Reichenbach and e.g. Busemann [7]. This definition can be extended by saying that $g(e, e_1)$ is defined if either e, e_1 are on the life-line of a photon (and then $g(e, e_1) = 0$) or there is $m \in Obs$ who sees e, e_1 on one of its coordinate axes. Then m can measure the distance between e, e_1 (along that axis) and $g(e, e_1)$ is the minimum of the values so obtained. So, g is a kind of pseudo-metric derived from how the observers "see" things. The rest of $\mathfrak{G}_{\mathfrak{M}}$ is derived from the behaviour of observers in the same spirit. E.g. $e \prec e_1$ if some observer sees e, e_1 on its life-line and thinks that e happened earlier. Similarly, intervals e, e_1 and e_2, e_3 are congruent according to eq_0 if some observers see both intervals on some coordinate axis and measure them as of equal length. Further, $eq \subseteq {}^2Mn \times {}^2Mn$ is the smallest equivalence relation containing eq_0.

Returning to the "observational/theoretical" distinction, in our approach $\mathfrak{G}_{\mathfrak{M}}$ contains the theoretical concepts associated to $\mathfrak{M}$. For a relativity theory Th (like $\mathbf{Bax}^{-\oplus}$ above), $\mathsf{Mod}(Th)$ denotes the class of models of Th while $\mathsf{Ge}(Th) = \mathbf{I}\{\mathfrak{G}_{\mathfrak{M}} : \mathfrak{M} \models Th\}$. So $\mathsf{Ge}(Th)$ is the class of relativistic geometries associated to Th, up to isomorphism.

We obtain $\mathbf{Bax}^\star$ from $\mathbf{Bax}^{-\oplus}$ by assuming that the speed of light does not depend on its direction and by adding some extremely natural auxiliary assumptions like the existence of square-roots of positive elements in $\mathfrak{F}$, or that "if the life-lines of k and m coincide, then f_{mk} is an isometry".

Theorem 5 (4.3.38)

$$\mathsf{Mod}(Th) \equiv_\Delta \mathsf{Ge}(Th),$$

assuming $Th \models \mathbf{Bax}^\star$, *and* $n > 2$.

The above theorem says that not only is the world $\mathfrak{G}_{\mathfrak{M}}$ of theoretical concepts explicitly definable over $\mathfrak{M}$ (in accordance with the rules of definability theory), but also, in the other direction, the whole of the original "observational world" $\mathfrak{M}$ is recoverable (explicitly definable) from the theoretical world $\mathfrak{G}_{\mathfrak{M}}$, under some assumptions. Applying this theorem to the literature, it implies that the so-called "bottom-up" developments of relativity are logically equivalent with the "top-down" ones, under assuming $\mathbf{Bax}^\star$. E.g. by Theorem 1 there are two translating mappings $\mathsf{tr}_1, \mathsf{tr}_2$ which translate these two theories (bottom-up, top-down) to each other in a meaning preserving way and $\mathsf{tr}_1, \mathsf{tr}_2$ have many desirable properties. By Theorem 2, these two worlds $\mathsf{Mod}(Th)$ and $\mathsf{Ge}(Th)$ are explicitly definable over each other, under some assumptions.

Next, we ask ourselves in the dissertation what happens with this nice equivalence if we do not want to assume the whole of $\mathbf{Bax}^\star$, or if we do not want to use all parts of the geometry $\mathfrak{G}_{\mathfrak{M}}$. These questions lead to results in the dissertation which generalize Theorem 5 to the world of duality theories. E.g. under weaker assumptions than those of Theorem 5 we still obtain a pair of functors

$$\mathcal{G} : \mathsf{Mod}(Th) \longrightarrow \mathsf{Ge}(Th), \quad \text{and} \quad \mathcal{M} : \mathsf{Ge}(Th) \longrightarrow \mathsf{Mod}(Th)$$

where $\mathcal{G}$ and $\mathcal{M}$ are so called weak inverses of each other, or in other words, form a duality theory. This means something in the direction that there is still a strong and useful two-way "bridge" between the worlds $\mathsf{Mod}(Th)$ and $\mathsf{Ge}(Th)$ through which problems in one world can be translated to the other, solved there etc. This notion of weak inverse or duality theory is a concept used throughout mathematics and mathematical physics under different names. Examples for such names are "adjoint functor pairs", representation theorems (e.g. in algebraic logic), or residuation, Galois theories, to mention a few. The author's refereed publications in the above line include [4, 5, 27, 26, 22, 24, 23, 25].

Preparations in the direction of general relativity are the section on geodesics (§4.7), the local variants **Par**$(-)$ (end of §3, p.A-27– A-28), theory of accelerated observers, e.g. [3] and the Reichenbachian versions of our theories (e.g. [2, §4.5]).

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