Conceptual structure of spacetimes, and category of concept algebras

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Abstract

In the first part of the talk, we explore the first-order logic conceptual structure of special relativistic spacetime: We describe the algebra of explicitly definable relations of Minkowski-spacetime, and we draw conclusions such as "the concept of lightlike-separability can be defined from that of timelike-separability by using 4 variables but not by using three variables", or "no non-trivial equivalence relation can be defined in Minkowski-spacetime", or "there are no interpretations between the classical (Newtonian) and the relativistic spacetimes, in either direction".
In the second part of the talk, we generalize the notion of a concept algebra from first-order language/logic to any language. A duality between algebra and category theory emerges here quite nicely. Namely, category theoretic properties of the category of all concept algebras shed light on definability properties of the language. For example, "all implicitly definable concepts are explicitly definable (Beth definability property) if and only if epimorphisms are surjective in the category of concept algebras", or "all existence-requiringly implicitly definable concepts are explicitly definable (weak Beth definability property) if and only if there is no proper epi-reflective subcategory of the category of concept algebras that contain the so-called full concept algebras". Connections with category theoretic injectivity logic will be pointed out.

Text for the slides. This text is intended to be readable without looking at the slides. However, looking at the slides, especially at the drawings in slides 17 – 19, may be illuminating. The numbering of the items below coincides with that of the slides.

2. When we were already working deep in algebraic logic, Leon Henkin suggested us (Hajnal and Istvan) to apply our knowledge to some real-life scientific theory. At that time, this task seemed to us futuristic. But now, we are doing this with Judit and Gergely, we report on the findings in the first part of our talk. Donald Monk also suggested this task:

[J.D. Monk, An introduction to cylindric set algebras, IGPL, 2000, p.455] writes: "The corresponding facit of the theory of cylindric algebras is to describe the cylindric set algebras for important models. This amounts to looking at complete theories only, which is customary in model theory. It is somewhat surprising that this aspect of the theory of cylindric algebras has been almost entirely neglected."

In the second part of the talk, we depart from first-order logic, and we repeat all what we have done for any other logic. For example, for second-order logic. By some miracle, category theory will emerge here.

3. **PART I**

The concept algebra of a concrete physical theory: special relativistic spacetime SR

4. What is SR spacetime? There is a great variety for how to define spacetimes. We will consider a spacetime as telling what inertial motions are possible. In this spirit, SR is the system of timelike straight lines, and this can be coded with the ternary relation of *timelike collinearity*. This choice seems to be quite a minimalistic, slim one. However, we will see that from timelike collinearity, the whole full-fledged scale-invariant SR can be defined.
What is the concept algebra of SR? The algebra of concepts of a theory or of a structure is one of the main players in algebraic logic. Alfred Tarski invented them for FOL, he called them cylindric algebras because of their geometric aspect. The work of Tarski is centered around definability theory. He saw things through algebra, logic, and geometry, and these three aspects show up very nicely in algebraic logic.

5. The concept algebra of SR
In the first part, we work inside first-order logic FOL. When we say "formula", "expressible", etc. we have in mind FOL as our language. In the second part, we will generalize what we do to arbitrary languages (satisfying some conditions that we will spell out).
What is a concept? For example, we can take "red" to be a concept. We can think of it both as the open FOL-formula "Red(x)", or as the set of all red things (in a model). It does not matter which one we choose, but we have to choose. In this talk, we choose the second one: We think of a concept as being the extension of an open formula. The operations of the concept algebra are "the" natural ones on concepts.

6. Classical non-relativistic spacetime NT
You will get a package of "two in one". Two-in-one, because it is best to investigate what relativistic spacetime is like when investigating what a non-relativistic spacetime is like. You will get a better understanding. (For the definition of NT see the next item.)
Just for comparison, here is SR again. You can see that we concentrated on the speed-limit aspect of SR.

7. Summing up:
We will call our two spacetimes just Newton and Einstein, in place of non-relativistic and relativistic, or Newtonian and Einstenian spacetimes.
Newton is defined as the structure with 4-dimensional real space as universe, i.e., the points are 4-tuples of real numbers, and with a ternary relation representing all lines that are not horisontal: three points are in this Col relation if and only if they lie on a slanted (that is, not horisontal) straight line.
Einstein is the same except that the ternary relation represents fewer lines: only those with slope less than 1 (i.e., lines that are more vertical than horisontal).
Intuition: we think of points as events, the vertical direction represents time, thus slanted lines represent motion, the more slanted a line is, the faster motion it represents. Newton: no speed limit for moving bodies, Einstein: 1 as speed limit. So, we concentrate on the speed-limit aspect of relativity theory.

8. The most important difference between the two spacetimes is that the grey part in Newton spacetime is "thin" (3-dimensional), while in Einstein it is "thick" (4-dimensional). This difference is emphasized in many relativity books, e.g., in the Introduction of the Wald General relativity book.
The gray part belonging to a concrete event – the black dot in the picture -- is the set of events that cannot be reached from that event by blue lines. In Newton, a gray part belonging to a concrete event is a hyperplane (called the simultaneity of that event), and these grey parts form an equivalence relation. This is a highly nontrivial equivalence relation, called foliation of the spacetime.
The grey parts in Einstein do not form an equivalence relation. Moreover, the equivalence relation generated by them is the largest one, with one equivalence block. Even more can be proved: in Einstein spacetime no nontrivial equivalence relation can be defined on the events. This will imply that there is no interpretation from Newton to Einstein.
Let's see how we can prove such a theorem.

9. **Theorem 1.** No nontrivial equivalence relation can be defined in SR .
Note: We think of timelike, lightlike, and spacelike connectedness as disjoint from the equality (e.g., p,q are timelike connected implies that p and q are distinct). [In place of timelike connected we sometimes say timelike separated, etc, they mean the same.]

Proofidea: First, show that any two timelike connected pairs of events can be taken to each other by an automorphism of SR. Here use Lorenz transformations, which are key players in relativity theory. Do the same for spacelike and lightlike connected pairs of events. Then show that the transitive closures of each of these relations have one block. QED

We use double standards in this talk: when proving that something is not definable, we use automorphisms in the proof. Then the statement is true for all languages (because a language is required to be automorphism-invariant). When we claim that something is definable, then we will give a concrete FOL-formula that defines it.

10. **Corollary 2**. NT cannot be interpreted in SR.
Proofidea:
Logic perspective: Assume that there is an interpretation from NT to SR. Let ϕ(v0,v1) be the formula expressing in Newton that v0 and v1 are simultaneous (i.e, that they are in "grey relation"). Let the interpretation from Newton to Einstein take the formula ϕ(v0,v1) to the formula Φ(v0,v1). Then Φ(v0,v1) defines a non-trivial equivalence relation in SR, because ϕ(v0,v1) defines such in NT, and the property of defining a nontrivial equivalence relation can be formulated in FOL, hence it is preserved by any interpretation. This contradicts Theorem 1 on the previous page. QED

Algebra perspective: The reason why Newton cannot be interpreted in Einstein is that it can be "seen" in the structure of concepts that a concept is a nontrivial equivalence relation or not.
In more detail: There is an equation e(x) such that e(x) holds for a "two-dimensional" element x of the concept algebra if and only if "x is an equivalence relation". Namely, e(x) is, roughly, "d_{01}+x^+x.x is contained in x", where d_{01} is the concept "v0=v1" (this is a constant in the concept algebras), x^ denotes the "converse of x" and x.x denotes "x composed with x", we

will see soon that these can be expressed with three variables in FOL, hence with terms of the 3-dimensional concept algebras. Interpretations are homomorphisms between the concept algebras, and homomorphisms preserve equations.
Note: equations about concept algebras talk about all defined concepts, equivalently, they correspond to formula-schemes of FOL. More on this can be read in [Andreka-Nemeti: How many varieties of cylindric algebras are there, Transactions of the AMS, 2017].

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A few words about definitions. Definitions = FOL formulas with free variables. We give a name, a shorthand for a long formula, for ease of talk. At any time we can eliminate these shorthands, or we can introduce them. Adding them to the language does not alter the structure, it only may unfold, may make visible some structure. They are tools for convenience.
Interpretations live between theories of structures or are homomorphisms between concept-algebras of structures.
What is an *interpretation*? Interpretation of one structure in another means what one would think it means: imitate one in the other. In terms of mathematical logic, it is a usual interpretation from the theory of the first structure into the theory of the second. In algebraic terms, an interpretation means a homomorphism from the *concept algebra* of the first structure into the concept algebra of the second one.
The concept algebra of a structure is a natural algebra of its defined concepts. A *defined concept* in a structure is just a definable relation on the universe of the structure, we define relations by using open first-order formulas. For example, we can define the unary relation “tall” as “of bigger height than Peter”. After this, we can use “tall” whenever we want, or we can eliminate the word “tall” whenever we want. Definitions are abbreviations, their essence is that they highlight structure, but they do not introduce any new information, they do not change theory.

11. **Theorem 3.** SR cannot be interpreted in NT, either.
Theorem 1 may suggest that fewer types of relations can be defined in SR than in NT. Just the opposite is true!
The reason for why Einstein cannot be interpreted in Newton is, briefly, the following. We notice that only finitely many binary relations can be defined in Einstein, in fact the definable binay relations of Einstein form a 4-atom Boolean algebra. The same is true in Newton, but there the definable binary relations form only a 3-atom Boolean algebra (BA). These BAs actually have unary functions on them: those corresponding to the existential quantifiers (domain and range of binary relations). This additional structure makes them simple in the algebraic sense: they have no congruence relations, i.e., any (nontrivial) homomorphism from them must be one-to-one. This gives a proof sketch that Einstein cannot be interpreted into Newton. QED

Remark: There is the “modern” notion of interpretability, where one can define new sorts, too. That is a weaker notion than the one we are talking about here. In this weaker sense, both NT can be interpreted in SR and SR can be interpreted in NT, but they are not definitionally equivalent in this weaker sense, either, because their automorphism groups are not isomorphic. However, in this weaker sense, SR is definitionally equivalent to its automorphism group, to the Poincare group. Similarly, NT is definitionally equivalent to its automorphism group. These results are relevant to Klein’s Erlangen program. These results will be contained in the sequel to the present paper.

12. Structure of binary defined relations of SR and NT
We have seen, in the proof of Theorem 1, that only two unary relations can be defined in SR: the empty one and the whole universe. The same is true for NT.
How many binary relations can be defined in SR? We can define the identity relation (v0=v1), and its complement, the diversity relation (v0 not= v1). Below the diversity relation, we can define timelike connectedness from timelike-collinearity, and its complement in the diversity relation. We have already three relations. In NT, we cannot define more (atomic) relations, but in SR the complement of timelike connectedness has a "border", namely lightlike connectedness, and this can be defined from timelike connectedness (we will see soon). This is a difference between NT and SR: in NT the complement of timelike connectedness is an atom (no nontrivial subrelation of it can be defined in NT), while in SR it can be cut into two definable pieces: spacelike and lightlike connectedness. This is a consequence of the main difference between NT and SR, namely that the grey relation in NT is "thin" while it is "thick" in SR.
Timelike, lighlike, and spacelike connectedness are all atoms (no nontrivial subset can be defined), we saw this in the proof of Theorem 1, by using automorphisms.
This will prove that SR cannot be interpreted into NT, as soon as we have shown that lightlike and spacelike connectedness can be defined from timelike connectedness. Lets begin to do that.

13. **Theorem 4.** Lightlike connectedness can be defined from timelike connectedness in SR. The definition uses 4 variables.
Proofidea: "a and b are in distinct half-cones of c" can be defined from T(imelike connectedness) as "for all d [dTc --> (dTa or dTb)]". This uses 4 variables. Using this, it is relatively easy to define lightlike connectedness from timelike one.
(See Andreka-Madarasz-Nemeti: Logic of space-time and relativity theory. Section 2.6, p.654. https://old.renyi.hu/pub/algebraic-logic/Logicofspacetime.pdf) QED
A very similar proof shows that lightlike connectedness can be defined from spacelike connectedness.
Corollary: timelike connectedness can be defined from causal connectedness, etc.

14. **Theorem 5.** Lightlike connectedness cannot be defined from timelike connectedness in SR *by using only three variables*.
Proofidea: In the *relation algebra* of the binary definable relations, timelike connectedness does not generate lightlike connectedness (reason is given below). By a theorem from algebraic logic, this implies that lightlike connectedness cannot be defined with a FOL-formula using only 3 variables (see e.g., the Tarsi-Givant book cited below).
Timelike connectedness does not generate lightlike connectedness in the relation algebra of binary concepts: The operations of this relation algebra are converse, and relation composition besides the Booleans. In the relation algebra of binary concepts of SR, converse is identity on the algebra, and composition of any two, not necessarily distinct, diversity relations is 1. (See Andreka-Madarasz-Nemeti: Logic of space-time and relativity theory. Section 2.6, p.655. https://old.renyi.hu/pub/algebraic-logic/Logicofspacetime.pdf) QED

Remark on the importance of relation algebras: Mathematics could have been based on the equational theory of relation algebras in place of the FOL Zermelo-Frankel set theory. This is the subject of the Tarski-Givant book Formalizing set theory without variables.

15. **Theorem 6.** NT is not definitionally equivalent to any structure that has only binary relations.
There is another interesting difference between Newton and Einstein. In Einstein, you can define lightlike connectedness, and from lightlike connectedness you can define back the whole structure. This means that Einstein is definitionally equivalent to the structure where the only relation is lightlike connectedness Ph. In Einstein, we can introduce Ph as a shorthand for a specific definable relation (namely for lightlike connectedness), with this we did not change the structure we only added notation for ease of use, then we can forget the original collinearity relation because from Ph it can be defined back, we again did not change the structure. This is the essence of definitional equivalence: we can introduce definable relations and we can eliminate relations that are definable from the rest.
The same way, Einstein can be defined from spacelike connectedness, or from timelike connectedness. All these four structures are definitionally equivalent.
On the other hand, Newton is not definitionally equivalent to any structure that contains at most binary relations, because the concept algebra of Newton is not generated by any set of binary relations, in Newton there is no definable binary relation from which the whole structure could be defined back. You can see this by checking, one by one, that from the 8 definable binary relations depicted on p.12, you cannot define timelike connectedness (hint: use automorphisms).

16. Structure of ternary defined relations of SR and NT
We have seen that 2 unary relations, and 16 binary relations can be defined in SR. How many ternary relations can be defined in SR? How many ternary relations can be defined in NT?
Claim: Collinearity can be defined in both. In both, Euclidean equidistance can be defined on lines, but it cannot be defined on triangles (i.e., it cannot be defined for arbitrary p,q,r that the Euclidean distances of pq and qr are the same). Minkowski equidistance can be defined in SR. Thus, rational ratio between three points on a timelike line can be defined in both, this is already infinitely many definable ternary relations!
More ratio can be defined in SR: e.g., the ratio of square-root 2 can be defined, by constructing diagonals of squares and using the equi-distance relation. Similarly, all constructible number ratio can be defined. Third-root of 2 is not constructible. Can third-root of 2 ratio be defined in SR?

Until now we did not use that our field is the field of reals, everything we did so far can be done in arbitrary arithmetic fields (fields in which a linear order can be defined). From now on we rely on the fact that we have the field of reals (and most of what we do uses Tarski's elimination of quantifiers theorem for this field).

Exactly the algebraic number ratios can be defined. (A number is algebraic if it is the root of a polynomial, these are exactly the definable numbers in the field of reals.) Thus, third-root of 2 ratio can be defined in SR.
Here is a summary of what we know so far of the algebra of ternary definable relations of SR: It is atomic and we can describe the atoms. It has non-trivial subalgebras (in contrast, the algebra of binary definable relations of SR does not have such). [Here, a trivial subalgebra is a subalgebra of the "minimal one", which is the algebra of concepts that can be defined by using only the identity as primitive relation.]

Instead of proceeding to the algebra of quaternary defined relations of SR and NT, in the next three items we give a glimpse of the geometric aspect of concept algebras.

17. Converse can be defined with 3 variables

The importance of converse and relation composition of binary relations have already come up. Here is how the converse of a binary relation can be defined with 3 variables in FOL. Instead of dealing with formulas, we will show geometrically how we can construct the converse of a binary relation in 3-dimensional space.

A binary relation of SR is a set of pairs of points of the 4-dimensional real space R^4. Let us put R^4 on the axes 0,1 of 2-dimensional geometry. Then a binary relation in SR is a set of points of this 2-dimensional space. Converse of a relation is then just the mirror-image of this subset wrt the diagonal. For example, let R denote the red line in the 01-plane, then its converse is the yellow line in the 01 plane.

Binary relations are treated as ternary ones, we just add a dummy third place. Then R as a ternary relation is the "2-cylinder" over the red line in the 01-plane, the vertical pink striped rectangle in the drawing. The three diagonals of this cube are denoted by Id_{01}, Id_{02}, Id_{12}. The above process (of defining converse) consists of three similar steps: from c_2R we define $c_1(Id_{12} . c_2R)$, the blue 1-cylinder over the blue line. From this 1-cylinder we define the yellow 0-cylinder similarly, and from this we define the 2-cylinder over the yellow lines yet by another similar step.

What are these steps? A step corresponds to substitution of variables in logic. Tarski noticed the following. Assume that we have a formula $\phi(x)$, with one free variable x. Let $\phi(y)$ denote the same formula such that we change this free variable x to y everywhere in it (dealing with clashes of bound variables). Then, semantically $\phi(y)$ is equivalent to "exists x(x=y and $\phi(x)$)". This is just the scheme of steps we did in defining the converse above! (See item 19.)

Computational aspect: the above defining formula also specifies a program for how to interchange the contents of two registers, by using a third auxiliary register.

18. Converse cannot be defined with 2 variables

Here is a geometrical proof that " inverse of R" cannot be defined with a 2-variable FOL formula. The atoms of the algebra that R generates are the colored squares and the 4 line-segments within them. The four line-segments are the red line, and the three segments of the diagonal. All the relations that can be defined from R with a 2-variable FOL-formula are among the unions of these. Finally, notice that the mirror image of R, its converse, is not among these unions.

19. Some natural properties of converse cannot be proved with 3 variables

This last drawing illustrates a beautiful idea of Leon Henkin, and shows the unity of the three aspects (algebraic, logical and geometric) of concept algebras.

Imagine that the usual 3-dimensional Euclidean space is distorted in the front left corner as shown in the drawing. This distortion can be detected if you look at the points p and q separately (because commutativity of the cylindrifications fail for these), but it cannot be detected if you look only the subsets generated by {p,q} (because one can check that all the cylindric algebraic equations remain true). Leon invented this method for showing that some natural properties of three-dimensional space cannot be proved with three variables. The method is called "twisting".

Literature on twisting: [Henkin-Monk-Tarski: Cylindric algebras Part II, North-Holland, 1985. Construction 3.2.71, pp.89-91] and [Simon, A., Non-representable algebras of relations, PhD Diss., 1997. sections 2,3. https://old.renyi.hu/pub/algebraic-logic/simthes.ps].
See also [Monk: An introduction to cylindric set algebras, IGPL 2000, p.488].

The drawing shows that it cannot be proved with 3 variables that it does not matter in which order we define converse by using substitutions: first we save the content of x in z and then use x as auxiliary variable in the rest, and in the final step restore the content of x by using z, or doing the same but saving the content of y in z. It does not matter which process we use, the end-result will be the same: we interchanged the contents of the variables x and y, by using z as an auxiliary variable. The drawing shows that this semantically true statement that contains only three variables x,y,z, cannot be proved by the usual FOL proof system restricted to three variables. (This statement can be proved with 4 variables, i.e., with two auxiliary registers to save the initial values.) We use that the usual FOL proof system translates in algebra to the cylindric algebra equations.

20. Concept algebras have geometric aspects. Concept algebras have algebraic aspects. Concept algebras have logical aspects. They have categorical aspects, too!

21. **PART II**
Duality between algebra and category theory
If this blue drop of water in the yin yang symbol represents algebra, then we will talk about the little red disc inside of it that represents category theory emerging inside algebra. To our minds, it is nice to work completely inside algebra, and it is nice to work completely inside category theory, but it is definitely fun to work with the duality of the two.

22. We depart from first-order logic: we deal with "any" logic, second-order logic SOL, many-sorted logic, modal logics, etc. So, what IS a logic? There is a branch of algebraic logic that is called abstract algebraic logic. Within this, there is a branch that concentrates on logics which have quantification, this is more or less a generalization of Tarskian algebraic logic. Universal algebra, a beautiful part of mathematics (the science of systems!), plays an important role in this subject, and that's why we gave the name Universal Algebraic Logic.

23. Category of concept algebras.
We are going to define concept algebras for an arbitrary language in the framework of general language theory. (Logic and language are more or less synonims from now on.)

Definition: (F, M, mng) is an algebraizable language if
1. F=W(P,Cn) is a context-free language.
2. Compositionality: the meaning of a compound term depends only on the meanings of the compounds.

Examples: FOL, SOL, modal logic, propositional logic, …
Non-examples: equational logic, injectivity logic, …
The two conditions (1), (2) are exactly what is needed for forming concept algebras!

In (1), P is called the vocabulary, its elements are called primitive formulas, and Cn is called the set of connectives. (2) is Frege's principle of compositionality.
The non-examples above are for the violation of (1). Only rather artificial languages violate (2). One example is the following. Let P consist of two letters, A and B, and let Cn consist of one binary connective that we will call concatenation. Then F consists of all words written up by using the letters A and B, indicating the order we concatenated. (This is non-associative, e.g., (AA)B is a word distinct from A(AB).) Let us have two meanings, "right" and "wrong" and let the meaning of a word be "right" if its bracket-free version does not contain two identical letters side-by-side, "wrong" otherwise. Then mng(A)=mng(AB)="right" while mng[A(AB)]="wrong". Also, mng(A)=mng(BA)="right" while mng[A(BA)]="right". This second example satisfies (1) but violates (2).

24. Algebraic version of L

Definition: $\mathscr{F}$= (W(P,Cn), c : c in Cn) is the *word-algebra*, where c(w1,w2) = cw1w2.
The *concept algebra* of M is CA(M) = (Ca(M), c : c in Cn), where Ca(M) = { mng(φ, M) : φ in F } is the set of meanings of M, and the operation c is defined by c(mng(φ, M), mng(ψ, M)) = mng(cφψ, M).
Alg_m(L) = { CA(M) : M in M}, is called the class of concept algebras.
Alg(L) = I{$\mathscr{F}$/ ≈K : K is a subclass of M}, where [φ ≈K ψ iff mng(φ, M)=mng(ψ, M) for all M in K], the class of Lindenbaum-Tarski *formula algebras*, up to isomorphisms.

The above is a generalization for the concept algebra of SR that we used in the first part of this talk, i.e., when the language is FOL. But now we can do the same for second-order logic SOL. The SOL-concept algebra of SR will contain concepts about subsets of SR, too. For example, a SOL-concept for SR is "X is a timelike line", or "X is a subset of a timelike line, but not the whole one".
Remark: When forming the concept algebra, we do not mark which concepts are "primary", i.e., which concepts are the meanings of elements of P. This is common with category theoretical logic. This feature makes it possible to "solve for the language": it makes possible to use concept algebras for deciding which concepts are best to use as primary ones, depending on the task at hand. This is one of the core problems in definability theory.

25. Introducing truth: L = (F, M, mng, ⊨), language becomes logic

Definition: (F, M, mng, ⊨) is an algebraizable logic if besides (1),(2) above we have

(3) We can "code" mng(φ)=mng(ψ) with formulas, by using derived connectives ↔, ↑ as
M ⊨ φ ↔ ψ iff mng(φ, M)=mng(ψ, M), and
M ⊨ φ iff M ⊨ φ ↔ ↑ .

Algebraic logic is bridge between Logic and Algebra: formula schemes of L correspond to equations of Alg(L), the tautologies of L correspond to the equational theory of Alg(L), L is complete iff Alg(L) is axiomatizable by finitely many equational implications, L is compact iff Alg(L) is closed under ultraproducts, etc. We are going to see that definability properties of L correspond to category theoretic properties of Alg(L).

26. **Universal Algebraic Logic**
Universal algebraic logic (a generalization of Tarski's cylindric algebra theory) is the mathematical theory of concept algebras. It belongs to algebraic logic, which is part of both algebra and logic. In algebraic logic, we elaborate a bridge between algebra and logic. In particular, we devise an algebraic form of logic, and then we investigate what logical properties correspond to in algebra, and vice versa, we investigate what algebraic properties correspond to in logic. In most cases we find that important logical properties and important algebraic properties correspond to each other. That vindicates, in some sense, the algebraic logic approach.

A book is in preparation about Universal Algebraic Logic with authors Andreka-Gyenis-Nemeti-Sain. Some literature:
[Henkin-Monk-Tarski: Cylindric Algebras Part II, North-Holland, 1985. Chapters 4.3 and 5.6],
[Andreka-Nemeti-Sain: Algebraic Logic. In the Handbook of Philosphical Logic, 2001, https://old.renyi.hu/pub/algebraic-logic/handbook.pdf],
[Nemeti: Algebraizations of quantifier logics, an introductory overview. Studia Logica 50 (1991), 485-569, https://old.renyi.hu/pub/algebraic-logic/survey.ps],
[Andréka-Kurucz-Németi-Sain: Applying Algebraic Logic; a General Methodology. https://old.renyi.hu/pub/algebraic-logic/meth.pdf].

The work of Tarski was centered around definability theory, this or that way. Definability theory also played a central role in relativity theory, from the beginning (due to the Vienna circle.) Tarski devised cylindric algebras, in analogy with Boolean algebras. In our present terminology, the class of cylindric algebras is the algebraic version of FOL, while that of Boolean algebras is the algebraic version of classical propositional logic.

27. Bridges are important

28. Definability theory of general languages and category of concept algebras
In definability theory, we need to vary the vocabulary/signature P, this motivates the definition of a general logic.

Definition: A general logic is $L = (L^P : P \text{ in } \mathscr{P})$ where $L^P = (F^P, M^P, mng^P, \models^P)$ satisfies (1)-(3) for all P in $\mathscr{P}$, and also

(4) Cn, ↔, ↑ are the same for all L^P, P in $\mathscr{P}$.

(5) Some conditions between L^P and L^Q for P,Q in $\mathscr{P}$ such as

----- there are arbitrary large signatures in $\mathscr{P}$

----- If P is a subset of Q then L^P is the natural restriction of L^Q to L^P

----- all meanings can be chosen to be the meanings of atomic formulas in P for some P in $\mathscr{P}$.

FOL, SOL, modal logic, propositional logic are general logics.

29. Algebraic version of a general logic
Let $L = (L^P : P \text{ in } \mathscr{P})$ be a general logic. Then $Alg_m(L)$ is the union of the $Alg_m(L^P)$ for P in $\mathscr{P}$, and Alg(L) is the union of the $Alg(L^P)$ for P in $\mathscr{P}$.

30. Category of Alg(L)
The objects of this category are the elements of Alg(L), and the morphisms are the homomorphisms f between members A,B of Alg(L) together with their domains and co-domains as (A,f,B) . (Thus, a homomorphism here is not just a set of pairs.) In this category, the objects correspond to theories of L, and the morphisms correspond to interpretations between these theories. However, in the following we shall concentrate on them as just being algebras and homomorphisms between these algebras.

31. Categories of Algebras I
When we have a category consisting of *all* algebras of a given similarity type, there is a nice correspondence between algebraic and category theoretic notions. Let us call these internal and external properties. Internal properties are the ones that are determined by the concrete, set theoretic structure of the algebras and functions. External properties are the ones that are determined solely by the context. (Analogous in automata theory to automata as a black box specified only by its behaviour, or automata as given by a set of states and a state-transition function.)
In a category of all algebras of a given similarity type, the onto (surjective) mappings are exactly the epimorphisms (i.e., left cancellative morphisms), the one-to-one (i.e., injective) mappings are exactly the monomorphisms (i.e., right cancellative morphisms), and the direct products of algebras are, up to isomorphisms, the sources of certain natural universal (smallest) cones constructed from them. These are examples of coincidence of internal and external properties. The above internal and external properties are equivalent when the context is the class of all algebras of a given similarity type. They cease to be equivalent when we vary the context, e.g., when the context is only a subclass of all algebras of a similarity type.

32. Categories of Algebras II
Let us have a full subcategory of one as above. Thus, the objects are some algebras of a similarity type, and the morphisms are all the homomorphisms between these algebras. Internal properties remain the same, independent from context. An algebra takes that with itself as in a suitcase. In contrast, external properties are rather sensitive to context!
For example, in many well-behaved classes of algebras, there are epimorphisms that are not surjective.

It is a well investigated question in algebra to determine in which classes of algebras are epimorphisms surjective. This is considered a purely algebraic question naturally arising inside algebra. We are going to see that this algebraic question corresponds to Beth definability property of a logic.

The category of algebras of a similarity type and of models is investigated in the monograph [Adamek-Herrlich-Strecker: Abstract and concrete categories. 2004]

33. Beth definability property of general logics
Let $L = (L^P : P \text{ in } \mathscr{P})$ be a general logic. Let P, Q be in $\mathscr{P}$, and $R = Q - P$. Let Σ be a subset of L^Q, that is Σ is a set of formulas in the bigger language.

Intuitively, we look at Σ as a description of the elements of R in terms of elements of P. We say that Σ defines R implicitly in Q iff Σ contains enough information to pin down the meanings of the R's. We formalize this as: in each P-model, there is at most one way to find elements of R that satisfy Σ (or, in other words, each P-model can be extended in at most one way to a Σ-model). We say that Σ defines R explicitly in Q iff each element of R can be in fact defined by a P-formula, on the basis of our "knowledge", or information, Σ.
The logic L has the Beth definability property, iff each implicit definition as above is also an explicit one.

34. Categorical characterization of Beth definability property

Theorem 7. Let L be a general logic with patchwork property for models. (FOL, SOL, … have it.). Then (i) and (ii) below are equivalent.
(i) L has Beth definability property
(ii) Epimorphisms are surjective in Alg(L).

Proofidea: Morphisms correspond to definitions. Epimorphisms correspond to implicit definitions. Surjections correspond to explicit definitions. QED

35. Weak Beth definability property of general logics
Σ defines R strongly implicitly in Q if, in addition to the above, each P-model can indeed be extended to a model of Σ. Intuitively, this means that Σ contains enough information not only for uniqueness of the defined concepts, but also it ensures the existence of the defined concepts.
The logic L has the weak Beth definability property, iff each strong implicit definition as above is an explicit one.

We are going to see that this weaker property also has categorical characterizations, and they even give a more colorful landscape. We need some definitions for the characterizations.

36. K-injective morphism, full model
Let C be a category, K be a subcategory and f a morphism in C. Then, f is K–injective iff all morphisms from the domain of f into an object of K factor through f. (This is the same as validity in injectivity logic.)
Let L be a general logic. We call a concept algebra CA(M) in $Alg_m(L)$ maximal iff it is not a proper subalgebra of any member of $Alg_m(L)$. Here we rely strongly on the fact that $Alg_m(L)$ is not required to be closed under isomorphisms. Full(L) denotes the class of all maximal members of $Alg_m(L)$ in this sense.

37. Categorical characterization of weak Beth definability property

Theorem 8. Let L be a general logic in which each model can be expanded to a maximal one (FOL, SOL, … are like this). Then (i) - (iii) below are equivalent, and perhaps with (iv) also.
(i) L has weak Beth definability property
(ii) Full(L)-injective epimorphisms are surjective in Alg(L).
(iii) Alg(L) has no proper reflective subcategory containing Full(L).
?(iv) Alg(L) has no proper limit-closed subcategory containing Full(L).

Equivalence of (i) and (iv) may be independent of set theory, but it cannot
be false!

The reason is the following. If we assume the existence of large cardinals, a limit-closed subcategory is reflective. So, there cannot be a counterexample for the equivalence of (i) and (iv) that is independent of set theory. Moreover, assuming the negation of Vopenka-s principle (which negation is consistent with set theory), we can construct a category C and a subcategory of C which is limit-closed but not reflective in C.

38. Conclusion
What have we learnt?
What have we learnt about the world?
What have we learnt about the physical world?
The algebra of concepts of special relativistic spacetime SR is interesting, especially when compared with the algebra of concepts of classical Newtonian spacetime NT. The conceptual structure of SR is richer than that of NT, but neither can be interpreted in the other (in the classical Tarskian sense). In SR, each of timelike, spacelike, or lightlike connectedness have the same information as scale-invariant Minkowski metric. One can think about concept algebras purely in geometrical way, or purely in logical way, and of course, purely in algebraic way. Concerning methodology of science, there is a duality between algebra and category theory. Algebraic (set-theoretic) properties are built in the objects, they are context-independent, while category theoretic properties reflect context, and thus they are sensitive to context. This duality shows up in investigating definability theory in Universal Algebraic Logic.

39. Upcoming conference
Logic, categories and philosophy of mathematics. Budapest 20 June - 21 June 2019
Michael Makkai is turning 80 in 2019. We are pleased to announce that the Alfréd Rényi Institute of Mathematics, the Department of Logic, Institute of Philosophy, Eötvös University, and the Faculty of Science, Eötvös University are organizing a conference celebrating this occasion.
The main topics of the conference are logic, category theory, model theory, philosophy of mathematics.

40. Thank you for your attention!