Non-representable algebras of relations

András Simon

Kandidátusi értekezés

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Contents

Introduction

On the language of this work. Most results in algebraic logic have two equivalent forms, one algebraic, and one logical. To keep the present work short, here we present and discuss the results only in their algebraic form. An exception is the present Introduction, in which we do discuss the logical corollaries of our results, too. The main body of the dissertation, however, is in purely algebraic form. That is, in the main body we stick to the language and methodology of algebra and will present the results as if they did not have equivalent purely logical meanings, too. The logical corollaries of the algebraic results can be obtained by applying the methods described e.g. in Andréka-Németi-Sain [ANS97], [ANS94] or Henkin-Monk-Tarski [HMT85, §4.3].

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We understand *algebras of relations* the following way: (i) the elements of our algebras are relations, and (ii) the operations of our algebras are those operations on relations which arise naturally from their being relations (i.e., sets of sequences). Sometimes we say algebras of relations of higher ranks to emphasise that our relations may be binary, ternary, four-ary, etc.

Algebras of relations in this sense have been investigated since the middle of the last century, beginning with works of De Morgan, Peirce, Macfarlane, Schröder and Löwenheim. E.g., Peirce [Pei33] studies n-ary relations for *arbitrary* n. There has always been an emphasis on investigating the equations axiomatizing the class(es) of algebras of relations. In the last century, such equations were called "laws" (e.g., "distributive law", "De Morgan law", "Peircian law").

The motivation for the theory of algebras of relations comes from at least three areas: (i) Logic (predicate logics, propositional multi-modal logics, dynamic logics, etc.). We will come back to this later. (ii) Algebra: the algebraic theory of these algebras is of interest in its on right, e.g., these algebras served as a starting point for the (by now quite successful) theory of discriminator varieties. (iii) The theories of relations: relations are of interest in themselves.

The most well-known algebras of relations are represented by the classes RRA, RCA_n and RPEA_n of representable relation algebras, representable cylindric algebras and representable polyadic equality algebras, respectively. The latter two are algebras of n-ary relations. RRA goes back to the middle of the last century, as was indicated, RCA_n is due to Tarski, while RPEA_n is due to Halmos (from around 1940 and 1950, respectively).

The theory of these classes of algebras was developed in a fashion parallel with that of Boolean algebras (BA's for short). For example, the class BA of Boolean algebras is a variety (equational class), and indeed, RRA, RCA_n and RPEA_n are varieties, too (cf. e.g. [HMT85]). However, there is a point of divergence between BA-theory and the theory of algebras relations of higher ranks. Namely, while BA is axiomatizable with finitely many equations (by Stone's theorem), it turned out that RRA, RCA_n and RPEA_n are not finitely axiomatizable (Lyndon, Monk, Jónsson, Andréka) assuming $n > 2$. Therefore finitely axiomatizable approximations $\mathsf{RA} \supseteq \mathsf{RRA}$, $\mathsf{CA}_n \supseteq \mathsf{RCA}_n$ and $\mathsf{PEA}_n \supseteq \mathsf{RPEA}_n$

(relation algebras, cylindric algebras and polyadic equality algebras, respectively) were introduced and studied. Let us take the example of RCA_n. When defining the approximating class CA_n for RCA_n, Tarski kept two conflicting goals in mind: (i) to keep the axiomatization of CA_n simple, and (ii) to make CA_n as close to RCA_n as possible.

Indeed, the equations defining CA_n are simple, and all the known equations separating RCA_n from CA_n are relatively complicated. The situation for RA and PEA_n is analogous.

By a CA_n we understand a BA $\mathfrak{B}$ with extra unary operations c_i (the *cylindrifications*) and constants d_{ij} (the *diagonals*) for $i, j < n$. An axiomatization of CA_n is recalled in Chapter 1. Intuitively, the axioms say the following: for all $i, j < n$,

(i) c_i is an additive complemented closure operator,
(ii) $\mathsf{c}_i\mathsf{c}_j x = \mathsf{c}_j\mathsf{c}_i x$, i.e., the cylindrifications commute,
(iii) the diagonals satisfy the usual axioms of equality, such as e.g. $\mathsf{d}_{ij} \cap \mathsf{d}_{jk} \leq \mathsf{d}_{ik}$ (transitivity),
(iv) $\mathsf{d}_{ij} \cap \mathsf{c}_i x \leq x$ for all $x \leq \mathsf{d}_{ij}$.

All these axioms are easily translated to equations.

To understand the intuition behind (iii), (iv), we note that d_{ij} is the algebraic counterpart of the formula $v_i = v_j$ (of predicate logic), or, equivalently, the n-ary relation $\{s \in {}^nU : s_i = s_j\}$, and c_i corresponds to existential quantification (of the ith variable v_i). So, axiom (iv) is an algebraic version of Leibniz's law of equality, saying that equal objects are indistinguishable by formulas, i.e., that

$$\varphi \to v_i = v_j \vdash (v_i = v_j \wedge \exists v_i \varphi) \to \varphi.$$

Let us turn to discussing briefly what the CA_n-RCA_n relationship means when applying CA-theory to logic. RCA_n corresponds to the n-variable fragment L_n of first order logic. CA_n corresponds to the natural axiomatization (or inference system) $\vdash_n$ of L_n (cf. [HMT85, §4.3]). The fact that $\mathsf{CA}_n \supset \mathsf{RCA}_n$ means, in logic, that $\vdash_n$ is sound but not complete for L_n. The quoted non-finitizability results (of Monk, Andréka and others) mean that $\vdash_n$ cannot be expanded to a strongly sound and complete axiomatization of L_n. A CA_n $\mathfrak{C}$ is called *representable*, iff $\mathfrak{C} \in \mathsf{RCA}_n$. The study of non-representable CA_n's (i.e. those CA_n's which do not belong to RCA_n) corresponds to studying validities of L_n which are not $\vdash_n$-provable. This is so, because the mentioned L_n-formulas correspond to those equations which are valid in RCA_n but not in CA_n. Among others, we will study such equations, as well as implications (like "$(\mathsf{CA}_n + e_i) \models e_j$") between them. Some of these results are analogous to independence results of set theory. Indeed, the logical form of such a result says that some formula φ is a potential extra axiom for L_n, in the sense that $\models \varphi$ (i.e., φ is valid) but $\nvdash_n \varphi$; and perhaps that the relationship between two such potential axioms φ and ψ is that $\varphi \nvdash_n \psi$. (Of course, the interest of such independence results depend on what the actual formulas φ and ψ look like.) Results of this kind are found in Chapters 4 and 5.

The early negative results motivated Henkin (around 1960) to raise the question whether CA_n is a really good finitary approximation of RCA_n in some concrete sense (to be specified later). A positive answer would mean that all CA_n's can be regarded as algebras of "real" relations (i.e., algebras, whose elements are really sets of sequences). Henkin specified how much "distortion" he would permit on the original algebras (RCA_n's) of relations in this problem. These distortions are called relativization and twisting. Relativization means that we permit the greatest relation V in our algebra to be a subset of a full Cartesian space $U \times \cdots \times U$ in place of $V = U \times \cdots \times U$. The elements of our algebras remain, however, just as concrete relations as before. Twisting is a similarly natural "distortion" of the original algebra of relations (again leaving the elements of our algebras concrete sets of sequences, i.e., real relations). In Chapter 3 we will prove that, roughly speaking, every CA_3

is representable by a twisted and relativized RCA_3, solving Henkin's problem positively for the $n = 3$ case.[1]

The logical counterparts of our twisted-relativized RCA_n's are the so-called general assignment models (general models for short) for $\vdash_n$ studied in Andréka-van Benthem-Németi [AvBN97], Németi [Ném96], van Lambalgen-Simon [LS94].[2] An important feature of general models is that the key ingredients for such models remain a *classical* model $\mathbf{M} = \langle M, R_i \rangle_{i \in I}$ and a set V of *usual* evaluations (or assignments) of the individual variables into M. The truth values of formulas remain sets of usual evaluations. The representation theorem in Chapter 3 means in terms of logic that $\vdash_3$ is a strongly complete inference system for such general assignment models (obtained by twisting and relativization).

As we mentioned earlier, traditionally, algebras of relations come in three main variants, RA's, CA's and PEA's. These three theories are closely linked, and their connections have been studied in detail (the first thorough study was probably Monk [Mon61]). For the study of these connections, in the present work we restrict our attention to the $n = 3$ case. For the representable algebras, these connections are fairly well understood (being simply the connections between RRA, RCA_3 and RPEA_3). By contrast, for the axiomatic classes (i.e., for RA, CA and PEA) the connections are much more subtle, and more useful for understanding the interaction of the three theories of relations (and logics).

On the "logical aspect" of these connections: The logical counterpart of RA (and of its semi-associative variant SA) is a propositional bi-modal logic in which the book Tarski-Givant [TG87] builds up set theory.[3] Variants of this logic are studied under the name *arrow logics* by the Amsterdam-London school, cf. e.g. [MPM96], van Benthem [Ben91], [BM97]. Arrow logic is also the core of both dynamic logic and many susbstructural logics, cf. e.g. Pratt [Pra94], Marx [Mar95], Mikulás [Mik96], Marx-Venema [MV96]. The logical counterpart of PEA_3 is L_3^+, obtained from L_3 by adding simultaneous substitutions $[v_i := v_j, v_j := v_i]$ as explicit new modalities, cf. Venema [Ven94]. The logical counterpart of the CA_3-RA-PEA_3 connections consists of the proof theoretic connections between L_3, arrow logic, and L_3^+. (For completeness we note that L_3 is what is called restricted L_3 in [HMT85]. In L_3, $Rv_0v_1v_2$ is an atomic formula, but $Rv_1v_0v_2$ is not. The difference between L_3 and L_3^+ is that $Rv_1v_0v_2$ is expressible in L_3^+ but not in L_3.)

In Chapters 4 and 5 we study the CA_3-RA-PEA_3 connections, solving a problem from Monk [Mon61]. (We note that the problems stated in [Mon61] were further investigated in Maddux [Mad78], [Mad91], the problem session paper in [AMN91], Henkin-Monk-Tarski [HMT85] and elsewhere.)

Finally, in Chapter 6 we solve a related problem of Don Pigozzi raised in [Pig69] and [Pig71], by using the methods developed in earlier chapters. Pigozzi asks whether CA_α has the embedding property (EP for short) if α is infinite. (In the second sentence of item 2.4.3 (p. 347) of [Pig71] he writes, roughly, that this is the most interesting question formulated in these two works.) For finite $\alpha \geq 3$, neither CA_α, nor RCA_α has EP, but for infinite α's RCA_α does have *EP* (these results are due to Pigozzi).

[1] The first results in this direction were obtained by Diane Resek [Res75] and then by Resek and Thompson [RT91] (cf. also Andréka-Thompson [AT88]), who proved that by adding extra axiom schemas to CA_n, its members become representable by relativized RCA_n's.

[2] To be precise, we note that twisting adds an extra dimension of generality to these models, which, we think, was not explicitly discussed in the quoted papers.

[3] [TG87] builds up set theory in RA. For SA, this task was carried through in Németi [Ném86] (cf. also Németi [Ném85]).

The logical counterpart of CA_ω is L_ω, which is the "structural" version of standard predicate logic, cf. e.g. Blok-Pigozzi [BP89, Appendix] or [HMT85, §4.3]. Here again, the emphasis is on the axiomatic (or proof-theoretic) version of L_ω. (This logic is equivalent to the infinite-dimensional multi-modal logic of Venema [Ven92], Marx-Venema [MV96].[4]) The logical counterpart of Pigozzi's question amounts to asking whether L_ω has the so-called relevance property[5]. The latter means that whenever the vocabularies (i.e. the non-logical symbols) of φ and ψ are disjoint, $\varphi \vdash_\omega \psi$ can hold only the trivial way (that is, either $\vdash_\omega \psi$ or $\vdash_\omega \neg\varphi$).

In Chapter 6 it will be shown that CA_ω fails to have the EP, hence L_ω does not have the relevance property (this result is joint with Judit Madarász). So the answer to the first open question in [Pig71, 2.4.3] is in the negative. For completeness, we note that the answer to the second open question there is in the positive. That is, the class RCA_ω has the strong embedding property. This will be proved in a joint work with Madarász (in preparation).

Brief outline of the dissertation. Chapter 2 introduces the tools, dilation and twisting, that we will be using throughout. In Chapter 3 we solve the first part of Problem 11 in the Problem Session paper of [AMN91] (Theorem 3.3), exhibit an essentially infinite non-representable CA_3 (the first of its kind) and show that the class of RA-reducts of CA_3's is not closed under taking subalgebras. The main result of the chapter (Theorem 3.11) solves a problem posed by Henkin [Hen67, p. 48]: roughly, it says that every CA_3 is isomorphic to a "relativized-twisted" representable CA_3. At the end of the chapter (Corollary 3.15) we show that to obtain this result, one needs to use the extension of the method of twisting we introduce in section 2.3.

In chapter 4 we investigate the interconnections between the RA-axioms within CA_3, and show that only four implications hold between them (one of which was proved earlier by Monk). As corollaries, we obtain a solution to Problem 4 in Monk [Mon61], and we get a complete characterization of those subsets of the standard set of axioms for RA which axiomatize RA over the RA-reducts of CA_3's; and the same for the class SA of semi-associative relation algebras.

In Chapter 5 we characterize the class of RA-reducts of PEA_3's and find that it is exactly the class SA of semi-associative relation algebras. We introduce a natural CA-theoretic equation MGR^+, a strengthening of the well-known Merry-Go-Round equation MGR of CA-theory. We show that for CA_3's, MGR^+ is equivalent to the RA-reduct being an SA, and that MGR^+ implies that the RA-reduct determines the algebra itself, while MGR is not sufficient for either of these to hold. (The negative part of this is shown in Theorem 4.10 in Chapter 4.) Further, we get that a CA_3 is a subreduct of a polyadic equality algebra iff MGR^+ holds in it.

In Chapter 6, which is joint work with Judit Madarász, we show that CA_ω does not have the embedding property. This result solves a problem raised by Don Pigozzi [Pig69], [Pig71].

We consider Theorems 3.7, 3.11, 4.11, 5.16 and 6.2 the main results of this dissertation.

[4]More precisely, we are referring to what is called the orthodox version of these logics in these works.

[5][Pig71] calls this the restricted interpolation property.

CHAPTER 1

Algebras of relations

In this chapter we introduce the classes of algebras that will be our main concern in the rest of the dissertation, and establish some basic properties that will be used throughout.

Notation. We use lower case Greek letters (α, β, …) from the beginning of the alphabet to denote arbitrary ordinals, and lower case Roman letters (m, n, …) to denote finite ordinals (i.e. natural numbers). We will use $i, j, k, \ldots$ as subscripts of ordinally indexed operations. Each ordinal is assumed to be the set of its predecessors. Hence $\alpha \in \beta$ iff $\alpha < \beta$. ω is the first infinite ordinal (the set of natural numbers). $\subset$ denotes "proper subset", and $\mathcal{P}(X)$ and ${}^X Y$ denote the powerset of X and the set of functions from X into Y, respectively. The composition $\{\langle x, y\rangle : \exists z(x\,R\,z \wedge z\,S\,y)\}$ of binary relations R and S will be denoted by $R\,|\,S$ and sometimes by $R \circ S$; and if R is an $n+1$-ary relation and $X_0, X_1, \ldots, X_{n-1}$ are sets then $R^*(X_0, \ldots, X_{n-1})$ stands for the R-image $\{y : (\exists x_0 \in X_0) \ldots (\exists x_{n-1} \in X_{n-1}) R x_0 \ldots x_{n-1} y\}$ of $X_0, X_1, \ldots, X_{n-1}$. The equivalence class of an element x in the domain of the equivalence relation E will be denoted by x/E. Composition of functions is denoted by juxtaposition, and $fg \stackrel{\text{def}}{=} g\,|\,f$, i.e., $fg(x) = f(gx)$.

Upper case German letters $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, … denote algebras (or first order structures), and the corresponding Roman letters A, B, C, … denote their universes, except that permutation groups will be identified with their universes. Unless otherwise stated, by a first order (or relational) structure we always mean a set with relations on it (i.e., no functions or distinguished elements).

BA denotes the variety of Boolean algebras; for a set U, we let $\mathfrak{P}(U)$ denote the Boolean algebra of all subsets of U. If $\mathfrak{A}$ is a Boolean algebra possibly endowed with extra operations, then $\mathfrak{Bl}\mathfrak{A}$ denotes the Boolean reduct of $\mathfrak{A}$.

For a class K of algebras, $\mathbf{I}K$, $\mathbf{H}K$, $\mathbf{S}K$, $\mathbf{P}K$ and $\mathbf{Up}K$ denote the classes of isomorphic copies, homomorphic images, subalgebras, products and ultraproducts of members of K. We understand products and ultrapoducts to be defined up to isomorphism only, so that e.g. $\mathbf{IP}K = \mathbf{P}K$. If X is a subset of the universe of the algebra $\mathfrak{A}$, then $\mathsf{Sg}^{(\mathfrak{A})}X$ denotes the subalgebra of $\mathfrak{A}$ generated by X. Other operations on classes of algebras will be introduced when needed.

For a set Φ of first-order formulas, $\mathrm{Mod}\,\Phi$ denotes the class of models of Φ. E.g., if Φ consists of the axioms for groups, then $\mathrm{Mod}\,\Phi$ is the class of all groups.

1.1. Cylindric Algebras

Cylindric algebras can be defined in several ways: as Boolean algebras with commuting, self-conjugate closure operators together with certain distinguished elements (i.e., 0-ary operations) and a few equations that describe their interaction (cf. Lemma 1.6); or, alternatively, as Boolean algebras with unary and 0-ary operators satisfying certain equations (cf. Section 2.1 below). The definition below from [HMT71, 1.1.1] reflects the logical origin of cylindric algebras.

DEFINITION 1.1. *By a **cylindric algebra of dimension** α, where α is an ordinal, we mean an algebra*

$$\mathfrak{A} = \langle A, +, \cdot, -, 0, 1, \mathsf{c}_i, \mathsf{d}_{ij} \rangle_{i,j<\alpha}$$

such that

(C0) $\langle A, +, \cdot, -, 0, 1 \rangle$ *is a Boolean algebra,*

d_{ij} *is a constant and* c_i *is a unary operation for each* $i, j < \alpha$, *and the following postulates are satisfied for all* $x, y \in A$ *and all* $i, j, k < \alpha$:

(C1) $\mathsf{c}_i 0 = 0$;
(C2) $x \leq \mathsf{c}_i x$;
(C3) $\mathsf{c}_i(x \cdot \mathsf{c}_i y) = \mathsf{c}_i x \cdot \mathsf{c}_i y$;
(C4) $\mathsf{c}_i \mathsf{c}_j x = \mathsf{c}_j \mathsf{c}_i x$;
(C5) $\mathsf{d}_{ii} = 1$;
(C6) $\mathsf{d}_{jk} = \mathsf{c}_i(\mathsf{d}_{ji} \cdot \mathsf{d}_{ik})$ *whenever* $i \neq j, k$;
(C7) $\mathsf{c}_i(\mathsf{d}_{ij} \cdot x) \cdot \mathsf{c}_i(\mathsf{d}_{ij} \cdot -x) = 0$ *whenever* $i \neq j$.

The class of all cylindric algebras of dimension α *is denoted by* CA_α. *The elements* d_{ij} *are called **diagonal elements**, and the operations* c_i *are called **cylindrifications**.*

We will use the notation "CA_α" as an abbreviation for the phrase "cylindric algebra of dimension α"; a similar remark applies to the classes of algebras introduced later.

REMARK 1.2. It would be possible to relax the condition that the dimension set of a CA be an ordinal; clearly, any set would do. One reason why we stick to the definition given above is that most of the results can be more compactly phrased in this setting. There are, however, a few cases, especially in the treatment of (neat) reducts, when this restriction proves inconvenient. In those cases we simply lift the restriction, and leave it to the reader to verify that this does not lead to confusion.

The motivating examples of cylindric algebras are the following algebras of α-ary relations. Note that the elements of $\mathcal{P}({}^\alpha U)$ are α-ary relations on U.

DEFINITION 1.3. *Let* U *be a set and* α *an ordinal. The **full cylindric set algebra of dimension** α **with base** U is the algebra*

$$\langle \mathcal{P}({}^\alpha U), \cup, \cap, \sim, \emptyset, {}^\alpha U, \mathsf{C}_i, \mathsf{D}_{ij} \rangle_{i,j<\alpha},$$

where $\cup$, $\cap$ *and* $\sim$ *are union, intersection and complementation (wrt the top element* ${}^\alpha U$*),*

$$\mathsf{D}_{ij} = \{ s \in {}^\alpha U : s_i = s_j \},$$

and for a subset X *of* ${}^\alpha U$,

$$\mathsf{C}_i X = \{ s \in {}^\alpha U : \exists z \in X \forall j < \alpha(j \neq i \rightarrow s_j = z_j) \}.$$

By a full cylindric set algebra we mean a full cylindric set algebra with base U, *for some set* U; *and a cylindric set algebra (with base* U*) is a subalgebra of a full cylindric set algebra (with base* U*). The class of cylindric set algebras of dimension* α *is denoted by* Cs_α. *A* CA_α *is **representable** if it is isomorphic to a subdirect product of* Cs_α*'s.* RCA_α *is the class of representable* CA_α*'s.*

By a result of Tarski, RCA_α is a variety; in case α is finite, RCA_α is a discriminator variety whose simple members are, up to isomorphism, the Cs_α's with non-empty base set. It is easy to check that RCA_α validates all axioms of CA_α, and it is natural to ask, whether the reverse inclusion holds, that is, whether every CA_α is representable. The answer is "yes" for $\alpha \leq 1$[1] (cf. 3.2.54, 55 in [HMT85]), and "no" for $\alpha > 1$. The real cutting point, however, is 2, as shown by the following results of Henkin and Monk:

THEOREM 1.4 (Henkin). RCA_2 *is axiomatizable by finitely many equations.*

The equations that characterize RCA_2 are the CA_2 axioms together with the equation (known as the *Henkin equation*) that appears in Example 2.15 in Section 2.2 below.

THEOREM 1.5 (Monk [Mon69]). RCA_α *is not axiomatizable by finitely many equations if* $\alpha > 2$.

This result, together with its subsequent improvements by Andréka [And96], Bíró [jsl], Hodkinson [Hod96], Maddux [Mad89], Venema [Ven97] and others, shows that narrowing the class of CA_α's by adding finitely many equations, or even reasonably simple schemes of equations, will not be sufficient to obtain a characterization of the representable algebras.

The next lemma collects some elementary properties of CA's.

LEMMA 1.6. *The following hold in every* CA_α:

(i) $\mathsf{c}_i\mathsf{c}_i x = \mathsf{c}_i x$;
(ii) $x \cdot \mathsf{c}_i y = 0$ *iff* $y \cdot \mathsf{c}_i x = 0$;
(iii) $\mathsf{c}_i(x+y) = \mathsf{c}_i x + \mathsf{c}_i y$; *more generally, if* $\sum_{j\in J} x_j$ *exists, then so does* $\sum_{j\in J} \mathsf{c}_i x_j$, *and* $\mathsf{c}_i \sum_{j\in J} x = \sum_{j\in J} \mathsf{c}_i x_j$;
(iv) $\mathsf{c}_i - \mathsf{c}_i x = -\mathsf{c}_i x$
(v) $\mathsf{c}_i \mathsf{d}_{ij} = 1$;
(vi) $\mathsf{d}_{ij} \cdot \mathsf{c}_i x = 0$ *iff* $x = 0$;
(vii) *if* $i \neq j$, *then* $\mathsf{d}_{ij} \cdot \mathsf{c}_i(\mathsf{d}_{ij} \cdot x) = \mathsf{d}_{ij} \cdot x$;
(viii) *if* $i \neq j$, *then* $\mathsf{c}_i x = \mathsf{c}_i(\mathsf{d}_{ij} \cdot \mathsf{c}_i x)$;
(ix) *if* x *is an atom, then so is* $\mathsf{d}_{ij} \cdot \mathsf{c}_i x$, *provided* $i \neq j$.

PROOF. (i)–(vii) are 1.2.3, 1.2.5, 1.2.6, 1.2.11, 1.3.2, 1.3.8 and 1.3.9 of [HMT71]. Applying c_i to both sides of (vii) and using (C3) we get $\mathsf{c}_i \mathsf{d}_{ij} \cdot \mathsf{c}_i(\mathsf{d}_{ij} \cdot \mathsf{c}_i x) = \mathsf{c}_i \mathsf{d}_{ij} \cdot \mathsf{c}_i x$, and from this, (viii) follows by (v).

To see that (ix) holds, first note that if x is an atom, then $\mathsf{d}_{ij} \cdot \mathsf{c}_i x \neq 0$ by (vi). Hence if $\mathsf{d}_{ij} \cdot \mathsf{c}_i x$ is not an atom, then, for some y, both $y \cdot \mathsf{d}_{ij} \cdot \mathsf{c}_i x$ and $-y \cdot \mathsf{d}_{ij} \cdot \mathsf{c}_i x$ are non-zero . But then, using (C2) and (C3), we get $0 \neq \mathsf{c}_i(y \cdot \mathsf{d}_{ij} \cdot \mathsf{c}_i x) = \mathsf{c}_i x \cdot \mathsf{c}_i(\mathsf{d}_{ij} \cdot y)$ and thus $x \leq \mathsf{c}_i(\mathsf{d}_{ij} \cdot y)$ by (ii) and since x is an atom. The same argument gives $x \leq \mathsf{c}_i(\mathsf{d}_{ij} \cdot -y)$. Hence $x = 0$ by (C7), a contradiction. □

If $\mathfrak{C}$ is a Cs_α with base U, then for any $\beta < \alpha$ with $\alpha \setminus \beta$ finite, the elements of $\mathfrak{C}$ that are fixed by all C_i, $i \geq \beta$, can be thought of as (representations of) β-ary relations on U. In fact, if we keep only these elements and those operations whose indices are all in β, then the resulting algebra is obviously isomorphic to a Cs_β (and in fact to one with base U).

This observation carries over to cylindric algebras in general, as follows.

DEFINITION 1.7 ([HMT71, 2.6.28]). *Let* $\mathfrak{C} \in \mathsf{CA}_\alpha$, $\beta \leq \alpha$. *The* ***neat*** $\boldsymbol{\beta}$***-reduct of*** $\mathfrak{C}$ *is the algebra*

$$\mathfrak{Nr}_\beta \mathfrak{C} \overset{\text{def}}{=} \langle \mathrm{Nr}_\beta \mathfrak{C}, +, \cdot, -, 0, 1, \mathsf{c}_i, \mathsf{d}_{ij} \rangle_{i,j\in\beta},$$

where

$$\mathrm{Nr}_\beta \mathfrak{C} \overset{\text{def}}{=} \{x \in C : \mathsf{c}_i x = x \quad \textit{whenever } \beta \leq i < \alpha\},$$

[1]For $\alpha = 0$, this is just a restatement of Stone's representation theorem for Boolean algebras.

and the operations are the restrictions of the corresponding operations of $\mathfrak{C}$. For a class $K \subseteq \mathsf{CA}_\alpha$, $\mathbf{Nr}_\beta K \stackrel{\text{def}}{=} \mathbf{I}\{\mathfrak{Nr}_\beta \mathfrak{C} : \mathfrak{C} \in K\}$.

It is not hard to see that $\mathrm{Nr}_\beta \mathfrak{C}$ is closed under the indicated operations (see e.g. [HMT71, 2.6.27]), so the definition of $\mathfrak{Nr}_\beta \mathfrak{C}$ is correct. It is also easy to see that $\mathfrak{Nr}_\beta \mathfrak{C} \in \mathsf{CA}_\beta$. We note that neat reducts are subreducts (subalgebras of reducts) in the universal algebraic sense: when forming the neat β-reduct of a CA_α, we throw away not only operations, but also elements of the algebra.

REMARK 1.8. By an abuse of notation we will sometimes write $\mathrm{Nr}_I \mathfrak{A}$ or $\mathfrak{Nr}_I \mathfrak{A}$ (for $\mathfrak{A} \in \mathsf{CA}_\alpha$), where I is an arbitrary (i.e., not necessarily an initial) subset of α . For example, $\mathrm{Nr}_I \mathfrak{A}$ is the set of elements of $\mathfrak{A}$ that are fixed by all c_i, $i \in \alpha \setminus I$. Cf. Remark 1.2.

LEMMA 1.9. *Let α be an ordinal, $\emptyset \neq I \subseteq \alpha$ with $|\alpha \setminus I|$ finite, $\mathfrak{C} \in \mathsf{CA}_\alpha$, and $\mathfrak{A} = \mathfrak{Nr}_I \mathfrak{C}$. Then the set of atoms of $\mathfrak{A}$ is $\{\mathsf{c}_{k_1} \dots \mathsf{c}_{k_n} a : a$ is an atom of $\mathfrak{C}\}$, where $\{k_1, \dots, k_n\} = \alpha \setminus I$. If $\mathfrak{C}$ is atomic, then so is $\mathfrak{A}$.*

PROOF. It is easy to see that $\mathfrak{Nr}_I \mathfrak{C} = \mathfrak{Nr}_I \mathfrak{Nr}_{I \cup \{k\}} \mathfrak{C}$ when $\mathfrak{C} \in \mathsf{CA}_\alpha$ and $I \cup \{k\} \subseteq \alpha$ (cf. [HMT71, 2.6.31]). Hence it suffices to prove both parts of the lemma under the additional assumption that $|\alpha \setminus I|$ is a singleton, say $\alpha \setminus I = \{k\}$.

If a is an atom of $\mathfrak{C}$ and $x \in \mathfrak{Nr}_I \mathfrak{C}$, then, by Lemma 1.6(ii), $x \cdot \mathsf{c}_k a \neq 0$ implies $a \cdot x = a \cdot \mathsf{c}_k x \neq 0$, whence $a \leq \mathsf{c}_k x$ and thus $\mathsf{c}_k a \leq \mathsf{c}_k x = x$. This, together with the fact that $\mathsf{c}_k a$ is non-zero by (C2), shows that $\mathsf{c}_k a$ is an atom of $\mathfrak{A}$. Now suppose that x is an atom of $\mathfrak{A}$. Fix $i \in I$. To see that $x = \mathsf{c}_k a$ for some atom a in $\mathfrak{C}$, it suffices to show that $\mathsf{d}_{ik} \cdot x$ is an atom of $\mathfrak{C}$. This is because $x = \mathsf{c}_k(\mathsf{d}_{ik} \cdot x)$ by Lemma 1.6(viii). Now the argument we used to prove Lemma 1.6(ix) can be repeated here to show that $\mathsf{d}_{ik} \cdot x$ is an atom of $\mathfrak{C}$. In more detail: $\mathsf{d}_{ik} \cdot x \neq 0$ by Lemma 1.6(vi), and if there were an element $y \in C$ such that $\mathsf{d}_{ik} \cdot x \cdot y$ and $\mathsf{d}_{ik} \cdot x \cdot -y$ are both non-zero, then, using (C2) and (C3), we would get $0 \neq \mathsf{c}_k(\mathsf{d}_{ik} \cdot x \cdot y) = x \cdot \mathsf{c}_k(\mathsf{d}_{ik} \cdot y)$ and thus $x \leq \mathsf{c}_k(\mathsf{d}_{ik} \cdot y)$ since x is an atom in $\mathfrak{A}$, and, similarly, $x \leq \mathsf{c}_k(\mathsf{d}_{ik} \cdot -y)$. But then $x = 0$ by (C7), a contradiction.

Suppose now that $\mathfrak{C}$ is atomic, and let $0 < x \in A$. By atomicity of $\mathfrak{C}$, there is an atom $a \in C$ below x. We have $\mathsf{c}_k a \leq \mathsf{c}_k x = x$ by 1.6(iii), and since $\mathsf{c}_k a$ is an atom of $\mathfrak{A}$ by the first part of the lemma, this shows that $\mathfrak{A}$ is atomic. ☐

DEFINITION 1.10. *For $i, j, k < \alpha$, we set*

$$\mathsf{s}^i_j x \stackrel{\text{def}}{=} \begin{cases} x & \text{if } i = j \\ \mathsf{c}_i(\mathsf{d}_{ij} \cdot x) & \text{if } i \neq j \end{cases} \qquad \text{and} \qquad {}_k\mathsf{s}(i,j)x \stackrel{\text{def}}{=} \mathsf{s}^k_i \, \mathsf{s}^i_j \, \mathsf{s}^j_k x.$$

The operations s^i_j are called *substitutions*. In representable algebras, s^i_j copies the ith argument of a c_j-closed relation to its jth coordinate. E.g., if x is an element of a Cs_3 with base U, then $\mathsf{s}^0_1 x = \{\langle u, v, w\rangle \in {}^3U : \langle v, v, w\rangle \in x\}$; hence if x is c_1-closed, i.e., if it is a binary relation between the zeroth and second coordinates, then $\mathsf{s}^0_1 x$ is the same binary relation, but between the first and second coordinates. Consequently, in representable algebras, ${}_k\mathsf{s}(i,j)$ interchanges the the ith and jth coordinate of a c_k-closed relation, using the kth coordinate as a "place holder".

Chapter 1.5 of [HMT71] contains a thorough discussion of these operations. Some of their properties are summarized in the following lemma.

LEMMA 1.11. *The following hold in every $\mathfrak{A} \in \mathsf{CA}_\alpha$:*

(i) $\mathsf{s}^k_l \mathsf{c}_k x = \mathsf{c}_k x$;

(ii) *if* $m \neq k, l$ *then* $\mathsf{s}^k_l \mathsf{c}_m x = \mathsf{c}_m \mathsf{s}^k_l x$;
(iii) *if* $k \neq l$, *then* $\mathsf{c}_k \mathsf{s}^k_l x = \mathsf{s}^k_l x$;
(iv) $\mathsf{s}^k_l \mathsf{s}^l_k x = \mathsf{s}^k_l x$;
(v) s^l_k *and* ${}_k\mathsf{s}(l,m)$ *are complete endomorphisms of* $\mathfrak{B}\mathfrak{l}\mathfrak{A}$;
(vi) *if* $k \neq m$, *then* $\mathsf{s}^k_l \mathsf{d}_{km} = \mathsf{d}_{lm}$;
(vii) $\mathsf{s}^k_l \mathsf{s}^m_k \mathsf{c}_k x = \mathsf{s}^m_l \mathsf{c}_k x$;
(viii) $\mathsf{s}^k_l \mathsf{s}^l_k \mathsf{c}_k x = \mathsf{c}_k x$;
(ix) *if* $k \neq m$ *or* $l = m$ *then* ${}_k\mathsf{s}(l,m) \mathsf{s}^m_l \mathsf{c}_k x = \mathsf{s}^l_m \mathsf{c}_k x$;
(x) ${}_m\mathsf{s}(k,l) \mathsf{s}^m_k x = \mathsf{s}^m_k {}_k\mathsf{s}(l,m) x$;
(xi) ${}_k\mathsf{s}(l,m) \mathsf{s}^k_m \mathsf{c}_m x = \mathsf{s}^k_m {}_m\mathsf{s}(k,l) \mathsf{c}_m x$;
(xii) $\mathsf{s}^k_l \restriction \mathrm{Nr}_{\alpha\setminus\{l\}}\mathfrak{A}$ *is an isomorphism from* $\mathfrak{B}\mathfrak{l}\mathfrak{Nr}_{\alpha\setminus\{l\}}\mathfrak{A}$ *to* $\mathfrak{B}\mathfrak{l}\mathfrak{Nr}_{\alpha\setminus\{k\}}\mathfrak{A}$ *with* s^l_k *as its inverse*;
(xiii) ${}_k\mathsf{s}(l,m) {}_k\mathsf{s}(m,l) \mathsf{c}_k x = \mathsf{c}_k x$;
(xiv) *if* $|\{k,l,m\}| = 3$ *then* ${}_k\mathsf{s}(l,m) \restriction \mathrm{Nr}_{\alpha\setminus\{k\}}\mathfrak{A}$ *is an automorphism of* $\mathfrak{B}\mathfrak{l}\mathfrak{Nr}_{\alpha\setminus\{k\}}\mathfrak{A}$ *with* ${}_k\mathsf{s}(m,l)$ *as its inverse.*

PROOF. (i)–(vii) are proved in [HMT71], Theorems 1.5.8(i),(ii), 1.5.9(ii), 1.5.10(v), 1.5.3, 1.5.16, 1.5.4(i), and 1.5.11(i).

(viii) Follows from (iv) and (i).

(ix) If $l = m$ then ${}_k\mathsf{s}(l,m) \mathsf{s}^m_l \mathsf{c}_k x = \mathsf{s}^k_m \mathsf{s}^m_k \mathsf{c}_k x = \mathsf{c}_k x = \mathsf{s}^l_m \mathsf{c}_k x$ by (viii). Hence it suffices to show that the equation holds when $k \neq m \neq l$. Under these assumptions we have

$$
\begin{aligned}
{}_k\mathsf{s}(l,m) \mathsf{s}^m_l \mathsf{c}_k x &= \mathsf{s}^k_l \mathsf{s}^l_m \mathsf{s}^m_k \mathsf{s}^m_l \mathsf{c}_k x \\
&= \mathsf{s}^k_l \mathsf{s}^l_m \mathsf{s}^m_l \mathsf{c}_k x && \text{by (iii), (i) and } m \neq l \\
&= \mathsf{s}^k_l \mathsf{s}^l_m \mathsf{c}_k x && \text{by (iv)} \\
&= \mathsf{s}^l_m \mathsf{c}_k x
\end{aligned}
$$

if $k = l$; and if $k \neq l$ then $\mathsf{s}^k_l \mathsf{s}^l_m \mathsf{c}_k x =$

$$
\begin{aligned}
&= \mathsf{s}^k_l \mathsf{c}_k \mathsf{s}^l_m x && \text{by (ii) and } k \notin \{l,m\} \\
&= \mathsf{c}_k \mathsf{s}^l_m x && \text{by (i)} \\
&= \mathsf{s}^l_m \mathsf{c}_k x && \text{by (ii) and } k \notin \{l,m\}
\end{aligned}
$$

as desired.

(x) By the definition of ${}_k\mathsf{s}(l,m)$ and ${}_k\mathsf{s}(m,l)$.

(xi) If $k = l$ then this follows from the definitions. Otherwise we have

$$
\begin{aligned}
{}_k\mathsf{s}(l,m) \mathsf{s}^k_m \mathsf{c}_m x &= \mathsf{s}^k_l \mathsf{s}^l_m \mathsf{s}^m_k \mathsf{s}^k_m \mathsf{c}_m x \\
&= \mathsf{s}^k_l \mathsf{s}^l_m \mathsf{c}_m x && \text{by (viii)} \\
&= \mathsf{s}^k_m \mathsf{s}^m_k \mathsf{s}^k_l \mathsf{s}^l_m \mathsf{c}_m x && \text{by } k \neq l\text{, (iii), (viii)} \\
&= \mathsf{s}^k_m {}_m\mathsf{s}(k,l) \mathsf{c}_m x.
\end{aligned}
$$

(xii) By (v), (iii) and (viii).

(xiii) If $l \notin \{k, m\}$ then

$$
\begin{aligned}
{}_k\mathsf{s}(l,m)\,{}_k\mathsf{s}(m,l)\,\mathsf{c}_k &= \mathsf{s}^k_l\,\mathsf{s}^l_m\,\mathsf{s}^m_k\,\mathsf{s}^k_m\,\mathsf{s}^m_l\,\mathsf{s}^l_k\,\mathsf{c}_k x \\
&= \mathsf{s}^k_l\,\mathsf{s}^l_m\,\mathsf{s}^m_l\,\mathsf{s}^l_k\,\mathsf{c}_k x && \text{by } m \neq l, \text{(iii), (viii)} \\
&= \mathsf{s}^k_l\,\mathsf{s}^l_k\,\mathsf{c}_k x && \text{by } k \neq l, \text{(iii), (viii)} \\
&= \mathsf{c}_k x, && \text{by (viii)}
\end{aligned}
$$

and if $l \in \{k, m\}$, then

$$
\begin{aligned}
{}_k\mathsf{s}(l,m)\,{}_k\mathsf{s}(m,l)\,\mathsf{c}_k &= \mathsf{s}^k_m\,\mathsf{s}^m_k\,\mathsf{s}^k_m\,\mathsf{s}^m_k\,\mathsf{c}_k x \\
&= \mathsf{s}^k_m\,\mathsf{s}^m_k\,\mathsf{c}_k x && \text{by (viii)} \\
&= \mathsf{c}_k x && \text{by (viii)}
\end{aligned}
$$

(xiv) By (v), (iii) and (xiii). □

Next we define a set of CA_3 equations (known as the *merry-go-round equations*). We restrict ourselves to the 3-dimensional case (but see Remark 1.14 below) because this is what we need in later chapters.

DEFINITION 1.12. *Let* $\{k, l, m\} = 3$. *MGR_k is the equation*

$$ {}_k\mathsf{s}(l,m)\,\mathsf{c}_k x = {}_k\mathsf{s}(m,l)\,\mathsf{c}_k x. $$

By Lemma 1.13 below, MGR_k says that interchanging the ith and jth coordinate of a c_k-closed element twice (using the kth coordinate as a place holder) leaves it fixed. While obviously valid in representable CA's, the *MGR* equations do not follow from the CA axioms; this was first shown by Henkin. In later chapters we will meet many examples of CA's in which they fail. The importance of these equations was recognized when Diane Resek [Res75] proved that every CA in which *MGR* holds, is representable as a relativized set algebra. We note that the present form of *MGR* is due to R. Thompson. We will quote this result more precisely in Chapter 3.

LEMMA 1.13. *Let* $\mathfrak{A} \in \mathsf{CA}_3$ *and* $\{k, l, m\} = 3$. *Then*

(i) $\mathfrak{A} \models MGR_k$ *iff* $\mathfrak{A} \models {}_k\mathsf{s}(l,m)\,{}_k\mathsf{s}(l,m)\,\mathsf{c}_k x = \mathsf{c}_k x$;

(ii) $\mathfrak{A} \models MGR_k$ *iff* $\mathfrak{A} \models MGR_l$.

PROOF. (i) By Lemma 1.11(iii),(xiv).

(ii) If MGR_k holds in $\mathfrak{A}$, then

$$
\begin{aligned}
& {}_k\mathsf{s}(l,m)\,\mathsf{s}^k_l\,\mathsf{c}_l x = {}_k\mathsf{s}(m,l)\,\mathsf{s}^k_l\,\mathsf{c}_l x && \text{by 1.11(iii) and } MGR_k \\
\Rightarrow\quad & \mathsf{s}^k_l\,{}_l\mathsf{s}(m,k)\,\mathsf{c}_l x = \mathsf{s}^k_l\,{}_l\mathsf{s}(k,m)\,\mathsf{c}_l x && \text{by 1.11(x), (xi)} \\
\Rightarrow\quad & \mathsf{s}^l_k\,\mathsf{s}^k_l\,{}_l\mathsf{s}(m,k)\,\mathsf{c}_l x = \mathsf{s}^l_k\,\mathsf{s}^k_l\,{}_l\mathsf{s}(k,m)\,\mathsf{c}_l x \\
\Rightarrow\quad & {}_l\mathsf{s}(m,k)\,\mathsf{c}_l x = {}_l\mathsf{s}(k,m)\,\mathsf{c}_l x, && \text{by 1.11(iii), (viii)}
\end{aligned}
$$

as desired. □

By Lemma 1.13(ii), we write *MGR* for any of the equations MGR_k ($k < 3$).

REMARK 1.14. If $\alpha > 3$, then *MGR* consists of two schemes of equations,

($_3MGR$) $\quad {}_k\mathsf{s}(l,m)\,\mathsf{c}_k x = {}_k\mathsf{s}(m,l)\,\mathsf{c}_k x \quad$ when $|\{k, l, m\}| = 3$, and

($_4MGR$) $\quad {}_k\mathsf{s}(l,m)\,{}_k\mathsf{s}(m,n)\,\mathsf{c}_k x = {}_k\mathsf{s}(n,l)\,{}_k\mathsf{s}(l,m)\,\mathsf{c}_k x \quad$ when $|\{k, l, m, n\}| = 4$.

The proof of Lemma 1.13(ii) shows that all instances of ${}_3MGR$ involving a fixed three-element subset of α are equivalent.

This completes our introductory survey of cylindric algebras. Other, more special, properties will be proved when needed.

1.2. Relation Algebras

The subject of relation algebras grew out of attempts by De Morgan, Schröder and Pierce in the last century to provide an abstract algebraic framework for the study of binary relations. Its modern day development have begun with Tarski [Tar41]. The axioms below are from [HMT85, 5.3.1].

DEFINITION 1.15. *A **relation algebra** is an algebra*

$$\mathfrak{A} = \langle A, +, \cdot, -, 0, 1, ;, \breve{}, 1' \rangle$$

such that

(R0) $\langle A, +, \cdot, -, 0, 1 \rangle$ *is a Boolean algebra,*

the operations ; *and* $\breve{}$ *of **composition** and **conversion** are, respectively, binary and unary,* 1' *is a distinguished constant called the **identity element**, and the following postulates are satisfied for all* $x, y, z \in A$*:*

(R1) $x ; (y ; z) = (x ; y) ; z$;
(R2) $(x + y) ; z = (x ; z) + (y ; z)$;
(R3) $x ; 1' = x$;
(R4) $x\breve{}\breve{} = x$;
(R5) $(x + y)\breve{} = x\breve{} + y\breve{}$;
(R6) $(x ; y)\breve{} = y\breve{} ; x\breve{}$;
(R7) $x\breve{} ; -(x ; y) \leq -y$.

The class of all relation algebras is denoted by RA.

Alternatively, one may define RA as the class of algebras satisfying axioms (R0), (R1), (R3) and the quasi-equations

$$(x ; y) \cdot z = 0 \leftrightarrow (x\breve{} ; z) \cdot y = 0 \leftrightarrow (z ; y\breve{}) \cdot x = 0,$$

the *cycle law*. An other possibility would be to say that an algebra $\mathfrak{A}$ (as in 1.15) is an RA if $\langle A, +, \cdot, -, 0, 1 \rangle$ is a Boolean algebra, $\langle A, ;, \breve{}, 1' \rangle$ is an involuted monoid, the operations ; and $\breve{}$ distribute over +, and $\mathfrak{A}$ satisfies (R7). See e.g. Jónsson [Jón82] and Maddux [Mad91] for surveys of relation algebras.

The definition of the class RRA of *representable relation algebras* (the principal examples of RA's) is analogous to the definition of RCA's. Its members are subdirect products of algebras whose elements are binary relations on some set U, and whose operations are the Booleans, relation composition, converse and the identity relation on U. See [HMT85, 5.3.2] and Németi [Ném91] for details. Most of what we said about CA's, RCA's and their relationship could be repeated with obvious changes for RA's and RRA's. The basic non-finitizability result here is

THEOREM 1.16 (Monk [Mon64]). RRA *is not axiomatizable by finitely many equations.*

Monk's result was strengthened, among others, by Andréka, Jónsson and Maddux.

Just as it is natural to look for CA's of smaller dimension in a CA, for $\alpha \geq 3$ we can find RA-like algebras in CA_α's.

DEFINITION 1.17 ([HMT85, 5.3.7]). *Let* $\mathfrak{C} \in \mathsf{CA}_\alpha$, $\alpha \geq 3$. $\mathfrak{Ra}\,\mathfrak{C}$, *the* RA***-reduct of*** $\mathfrak{C}$ *is the algebra*

$$\langle \mathrm{Nr}_2\,\mathfrak{C}, +, \cdot, -, 0, 1, ;, \breve{}, 1' \rangle,$$

where $x\,;y = \mathsf{c}_2(\mathsf{s}_2^1 x \cdot \mathsf{s}_2^0 y)$, $x\breve{} = {}_2\mathsf{s}(0,1)x$ *and* $1' = \mathsf{d}_{01}$. *For a class* K *of* CA_α*'s we define* $\mathfrak{Ra}^*\,K \stackrel{\text{def}}{=} \mathbf{I}\{\,\mathfrak{Ra}\,\mathfrak{C} : \mathfrak{C} \in K\,\}$.

REMARK 1.18. When $\alpha > 3$, it is equally natural to take as the universe of $\mathfrak{Ra}\,\mathfrak{C}$, not $\mathrm{Nr}_2\,\mathfrak{C}$, but the set $\{\,x \in \mathfrak{C} : x = \mathsf{c}_2 x\,\}$, cf. Monk [Mon61].

From now on, whenever convenient, we will pretend that $\breve{}$ and ; are operations of every CA_α (if $\alpha > 2$). We note that the RA-reduct of a CA_3 is not necessarily an RA: in fact, it is known that none of the axioms (R1), (R4), (R6) and (R7) holds in $\mathfrak{Ra}^*\,\mathsf{CA}_3$. Chapter 4 gives a complete description of those subsets of the RA-axioms that can fail in the RA-reduct of a CA_3.

Motivated by the failure of the associative law (R1) in $\mathfrak{Ra}^*\,\mathsf{CA}_3$, Maddux [Mad78], [Mad82] introduced the weaker axiom

(R1$_0$) $(x\,;1)\,;1 = x\,;1$,

called the *semiassociative law*. Roughly speaking, (R1$_0$) is how much of the associative law (R1) can be proved in CA_3. Maddux defined the class SA of *semiassociative relation algebras* as the class of RA-type algebras which satisfy (R1$_0$), and all the RA-axioms, except (R1).

The following lemma is a special case of Lemma 1.9.

LEMMA 1.19. *Let* $\alpha > 2$, $\mathfrak{C} \in \mathsf{CA}_\alpha$, *and let* $\mathfrak{A}$ *be the* RA*-reduct of* $\mathfrak{C}$. *Then the set of atoms of* $\mathfrak{A}$ *is* $\{\,\mathsf{c}_2\,a : a$ *is an atom of* $\mathfrak{C}\,\}$. *If* $\mathfrak{C}$ *is atomic, then so is* $\mathfrak{A}$.

LEMMA 1.20. *Let* $\mathfrak{A} \in \mathbf{S}\mathfrak{Ra}^*\,\mathsf{CA}_3$. *Then*

(i) *the operation* $\breve{}$ *is a 1-1 Boolean endomorphism in* $\mathfrak{A}$ *(and a Boolean automorphism if* $\mathfrak{A} \in \mathfrak{Ra}^*\,\mathsf{CA}_3$*); in particular,* $x = 0$ *iff* $x\breve{} = 0$ *for all* $x \in A$;

(ii) $\mathfrak{A} \models \forall x(x \leq x\breve{}\breve{})$ *iff* $\mathfrak{A} \models \forall x(x \geq x\breve{}\breve{})$ *iff* $\mathfrak{A} \models \forall x(x = x\breve{}\breve{})$;

(iii) $\mathfrak{A} \models x \leq 1' \to x\breve{} = x$.

PROOF. (i) By assumption, $\mathfrak{A} \subseteq \mathfrak{A}' = \mathfrak{Ra}\,\mathfrak{C}$ for some $\mathfrak{C} \in \mathsf{CA}_3$. Now $\breve{}$ is a Boolean automorphism of $\mathfrak{A}'$ by Lemma 1.11(xiv); the rest follows from this since $\mathfrak{A}$ is closed under $\breve{}$.

(ii) We only prove the implication

$$\mathfrak{A} \models \forall x(x \leq x\breve{}\breve{}) \Rightarrow \mathfrak{A} \models \forall x(x \geq x\breve{}\breve{}),$$

since the proof of the reverse implication is analogous. Let $\mathfrak{A} \in \mathbf{S}\mathfrak{Ra}^*\,\mathsf{CA}_3$, and assume that $\mathfrak{A} \models \forall x(x\breve{}\breve{} \leq x)$. Then, in particular, $(-x)\breve{}\breve{} \leq -x$ holds in $\mathfrak{A}$, and from this we get $-x\breve{}\breve{} \leq -x$ and hence $x \leq x\breve{}\breve{}$ by (i).

(iii) Suppose $\mathfrak{A} \subseteq \mathfrak{Ra}\,\mathfrak{C}$ for some $\mathfrak{C} \in \mathsf{CA}_3$, and let $\mathsf{d}_{012} = \mathsf{d}_{01} \cdot \mathsf{d}_{12}$ (in $\mathfrak{C}$). Then

$$\mathfrak{C} \models x \leq \mathsf{d}_{012} \to \mathsf{d}_{ij} \cdot \mathsf{c}_i x = x \tag{1}$$

for all distinct $i, j < 3$. The inclusion $\geq$ is obvious, since $\mathsf{c}_i x \geq x$ and $\mathsf{d}_{ij} \geq \mathsf{d}_{012} \geq x$. The reverse direction holds because

$$\mathsf{d}_{ij} \cdot \mathsf{c}_i x \cdot -x \leq \mathsf{c}_i(\mathsf{d}_{ij} \cdot -x) \cdot \mathsf{c}_i(\mathsf{d}_{ij} \cdot x) = 0$$

by (C2) and (C7). Now if $x \in \mathrm{Nr}_2\,\mathfrak{C}$ and $x \le 1' = \mathsf{d}_{01}$, then $\mathsf{d}_{12} \cdot x \le \mathsf{d}_{012}$. Hence

$$\begin{aligned}
x^{\smile} &= {}_2\mathsf{s}(0,1)x \\
&= \mathsf{s}_0^2\,\mathsf{c}_0\big(\mathsf{d}_{01}\cdot\mathsf{c}_1(\mathsf{d}_{12}\cdot x)\big) \\
&= \mathsf{s}_0^2\,\mathsf{c}_0(\mathsf{d}_{12}\cdot x) && \text{by (1)} \\
&= \mathsf{c}_2\big(\mathsf{d}_{02}\cdot\mathsf{c}_0(\mathsf{d}_{12}\cdot x)\big) \\
&= \mathsf{c}_2(\mathsf{d}_{12}\cdot x) && \text{by (1)} \\
&= \mathsf{c}_2\,\mathsf{d}_{12}\cdot x && \text{by (C3), since } x = \mathsf{c}_2\,x \\
&= x && \text{by 1.6(v)}
\end{aligned}$$

as desired. □

In Theorem 3.6 we will see that 1.20(ii) cannot be improved to say that $\mathfrak{Ra}^* \models x^{\smile\smile} \le x \to x^{\smile\smile} = x$. And Theorem 3.7 gives an example of an algebra in $\mathbf{S}\,\mathfrak{Ra}^*\,\mathsf{CA}_3$ in which $^{\smile}$ is not a Boolean automorphism. The next chapter introduces the tools we will use to construct algebras with such anomalous behaviour.

CHAPTER 2

Non-representable algebras

The purpose of this chapter is to recall methods for constructing non-representable cylindric algebras: *dilation*, *twisting* and *relativization*, and to establish their basic properties.

Relativization, historically the first (and perhaps the simplest) of these contructions, is a straightforward generalization of the corresponding Boolean algebraic notion. Dilation and twisting, on the other hand, do not have BA theoretic counterparts; they were devised to construct algebras of relations of higher ranks. We note that dilation and twisting, as well as the use of relativization in constructing non-representable CA's, originate with Henkin.

2.1. Atom structures

In both dilation and twisting one starts out with a complete and atomic CA, adjoins new elements and/or changes the operations to get a new, complete and atomic CA with certain prescribed properties. In CA's of this kind, and actually in all complete and atomic normal Boolean algebras with operators (BAO's) where the extra-Boolean operations distribute over arbitrary joins, the operators are determined by their behaviour on the atoms. Hence it is possible, and often simpler, to work with the *atom structure* of such an algebra instead of the algebra itself.

In the first part of this section we recall the notion of atom structure of an algebra and of the *complex algebra* of a relational structure. Since there is nothing in these constructions that would require the strong axioms of CA's, and because we will also be interested in atom structures of certain definitional extensions of CA's, we present these notions in the context of BAO's.

DEFINITION 2.1. *An algebra* $\mathfrak{A} = \langle \mathfrak{A}_0, f_i \rangle_{i\in I}$ *is a* ***Boolean algebra with operators****, or* BAO*, if* $\mathfrak{A}_0 \in$ BA*, and the extra-Boolean operations* f_i $(i \in I)$ *distribute over the Boolean join in each of their arguments. The operations* f_i *are called* ***operators****. An operator is* ***normal*** *if its value is* 0 *whenever one of its argument is* 0*. A* BAO *is normal if all its operators are normal. A* BAO *is said to be* ***complete*** *if its Boolean reduct is complete and its operators distribute over arbitrary joins.*

Examples of normal BAO's include almost all known algebras of relations. E.g., CA's are normal BAO's, and complete CA's are complete BAO's by (C1) and Lemma 1.6(iii).

DEFINITION 2.2. *Let* $\mathfrak{A} = \langle \mathfrak{A}_0, f_i \rangle_{i\in I}$ *be a normal, atomic* BAO*. The* ***atom structure*** $\mathfrak{At}\mathfrak{A}$ *of* $\mathfrak{A}$ *is the structure* $\langle \mathrm{At}\mathfrak{A}, R_i \rangle_{i\in I}$*, where* $\mathrm{At}\mathfrak{A}$ *is the set of atoms of* $\mathfrak{A}$*, and if* f_i *is an n-ary operator, then* R_i *is the* $n+1$*-ary relation* $\{\langle x_1,\dots,x_n,y\rangle : y \le f_i(x_1,\dots,x_n)\}$.

The relation R_i is sometimes referred to as the *accessibility relation* of f_i.

When speaking of atom structures of CA's, the relations corresponding to cylindrifications and diagonals will always be denoted by (variants of) T_i and E_{ij}, respectively. Note that by the definition above, in a complete and atomic CA, E_{ij} is the set of atoms below d_{ij}.

DEFINITION 2.3. *Let $\mathfrak{B} = \langle B, R_i \rangle_{i \in I}$ be a relational structure. The* ***complex algebra*** $\mathfrak{Cm}\,\mathfrak{B}$ *of* $\mathfrak{B}$ *is the algebra $\langle \mathcal{P}(B), \cup, \cap, \sim, \emptyset, B, f_i \rangle_{i \in I}$, where $f_i = R_i^*$ if R_i is of rank > 1, and $f_i = R_i$ otherwise (i.e., if $R_i \subseteq B$).*

In particular, if $\mathfrak{B}$ is a structure $\langle B, T_i, E_{ij} \rangle_{i,j<\alpha}$, the relations T_i are binary and E_{ij} are unary, then $\mathfrak{Cm}\,\mathfrak{B}$ is the algebra $\langle \mathcal{P}(B), \cup, \cap, \sim, \emptyset, B, T_i^*, E_{ij} \rangle_{i,j<\alpha}$, similar to CA_α's.

THEOREM 2.4 ([JT51]). *If $\mathfrak{A}$ is a normal, complete and atomic* BAO, *then the map $h : x \mapsto \{a : a \leq x, a \in \mathrm{At}\mathfrak{A}\}$ is an isomorphism from $\mathfrak{A}$ to $\mathfrak{Cm}\,\mathfrak{At}\,\mathfrak{A}$. If $\mathfrak{B}$ is a relational structure, then $\mathfrak{Cm}\,\mathfrak{B}$ is a normal, complete and atomic* BAO, *and the map $g : b \mapsto \{b\}$ is an isomorphism from $\mathfrak{B}$ to $\mathfrak{At}\,\mathfrak{Cm}\,\mathfrak{B}$.*

REMARK 2.5. Because of Theorem 2.4, we may identify normal, complete and atomic BAO's with the complex algebra of their atom structures, and relational structures with the atom structure of their complex algebras. We will do so without explicitly referring to 2.4.

We note that there exists a natural duality between (not necessarily complete and atomic) BAO's and relational structures, an extension of Stone duality. On a few occasions (e.g., in the proof of Theorem 4.4) we will be able to use results of its well-developed theory.

Now we recall the first order characterization of atom structures of CA's.

DEFINITION 2.6. *Let α be an ordinal. A structure*

$$\mathfrak{B} = \langle B, T_i, E_{ij} \rangle_{i,j<\alpha}$$

with binary relations T_i and unary relations E_{ij} is a ***cylindric atom structure of dimension*** α *iff the following conditions hold for all i, j, $k < \alpha$:*

(i) *T_i is an equivalence relation on B;*
(ii) $T_i \,|\, T_j = T_j \,|\, T_i$;
(iii) $E_{ii} = B$;
(iv) *$E_{ij} = T_k{}^*(E_{ik} \cap E_{kj})$ if $k \notin \{i, j\}$;*
(v) *$T_j \cap (E_{ij} \times E_{ij}) \subseteq Id$ if $i \neq j$.*

Ca_α is the class of cylindric atom structures of dimension α.

The commuting equivalence relations T_i in a cylindric atom structure correspond to cylindrifications, and the unary relations E_{ij} to the diagonals. We note that e.g. axiom (v) says that each block of T_j has at most one representative that belongs to E_{ij}, provided $i \neq j$. (Using the rest of the axioms, one can actually show that in each block of T_j there is exactly one such representative, cf. Lemma 2.10(i).)

THEOREM 2.7 ([HMT71, 2.7.38, 2.7.40]). *Ca_α is the class of atom structures of complete and atomic CA_α's, and a structure $\mathfrak{B}$ of the appropriate type is in Ca_α iff its complex algebra $\mathfrak{Cm}\,\mathfrak{B} \in \mathsf{CA}_\alpha$.*

PROOF. By [HMT71, 2.7.38, 2.7.39, 2.7.40]. □

DEFINITION 2.8. *Let $\alpha \geq 3$ be an ordinal. For any structure $\mathfrak{B} = \langle B, T_i, E_{ij} \rangle_{i,j<\alpha}$ with binary relations T_i and unary relations E_{ij} we introduce the binary relations ${}_kK_{ij}^{\mathfrak{B}}$ (for i, j, $k < \alpha$) and the ternary relation $C^{\mathfrak{B}}$ defined by the following formulas (we will always omit the superscript when this* *cannot lead to confusion):*

(1) $$a\ {}_kK_{ij}\ b \overset{\text{def}}{\iff} a \in E_{jk} \wedge (\exists c \in E_{ij})(\exists d \in E_{ik}) a\ T_j\ c\ T_i\ d\ T_k\ b$$

(2) $$C(a,b,c) \overset{\text{def}}{\iff} a \in E_{12} \wedge b \in E_{02} \wedge \exists d(a\ T_1\ d\ T_0\ b \wedge d\ T_2\ c),$$

and $K_{ij} \overset{\text{def}}{=} {}_kK_{ij}$*, where k is the smallest ordinal distinct from i and j.*

LEMMA 2.9. *Let* $\alpha \geq 3$, $\mathfrak{B} \in \mathsf{Ca}_\alpha$ *and let* $x, y \in C$ *for some* $\mathfrak{C} \subseteq \mathfrak{Cm}\,\mathfrak{B}$. *Then* ${}_kK_{ij}^*x = {}_k\mathsf{s}(i,j)x$, *and* $C^*(x,y) = \mathsf{c}_2(\mathsf{s}_2^1 x \cdot \mathsf{s}_2^0 y)$.

So the relations $K_{01} = {}_2K_{01}$ and C reflect the derived operations $\breve{}$ and ; (cf. Definition 1.17) just as T_i and E_{ij} reflect the operations c_i and d_{ij} on the level of atoms. I.e., K_{01} and C are the accessibility relations of $\breve{}$ and ;.

PROOF. Both statements are straightforward consequences of the definitions involved. As an example, we prove the second. Let $x, y \subseteq B$ and $c \in B$; then $c \in C^*(x,y)$ iff there are $a \in x \cap E_{12}$ and $b \in y \cap E_{02}$ such that $(\exists d \in B) a\ T_1\ d\ T_0\ b \wedge d\ T_2\ c$ (by 2.8), iff there are atoms a, b and d of $\mathfrak{Cm}\,\mathfrak{B}$ such that $a \leq x \cdot \mathsf{d}_{12}$, $b \leq y \cdot \mathsf{d}_{02}$, $d \leq \mathsf{c}_1\, a \cdot \mathsf{c}_0\, b$ and $c \leq \mathsf{c}_2\, d$ (by 2.3), iff $c \leq \mathsf{c}_2(\mathsf{c}_1(x \cdot \mathsf{d}_{12}) \cdot \mathsf{c}_0(y \cdot \mathsf{d}_{02})) = \mathsf{c}_2(\mathsf{s}_2^1 x \cdot \mathsf{s}_2^0 y)$ (by 1.6(iii)). Note that it is the convention introduced in 2.5 that lets us treat elements of $\mathfrak{B}$ and atoms of $\mathfrak{Cm}\,\mathfrak{B}$ interchangeably; so that we can e.g. write $\mathsf{c}_1\, a$ in place of the more cumbersome $\mathsf{c}_1\{a\}$. □

LEMMA 2.10. *Let* $\mathfrak{B} \in \mathsf{Ca}_\alpha$, $\mathfrak{C} = \mathfrak{Cm}\,\mathfrak{B}$, *and let* i, j, $k < \alpha$ *be distinct. Then*

(i) *for every* $a \in B$, $\mathsf{d}_{ij} \cdot \mathsf{c}_i\, a$ *is the unique element in* E_{ij} *that is* T_i*-related to* a;
(ii) ${T_k}^* E_{ij} = E_{ij}$;
(iii) *the map* $f : x \mapsto \mathsf{d}_{jk} \cdot x$ *is an isomorphism from* $\langle \mathrm{At}\,\mathfrak{Nr}_{\alpha\setminus\{k\}}\,\mathfrak{C},\ {}_k\mathsf{s}(i,j)\rangle$ *onto* $\langle E_{jk},\ {}_kK_{ij} \cap {}^2E_{jk}\rangle$, *with* $g : x \mapsto \mathsf{c}_k\, x$ *as its inverse*;
(iv) ${}_kK_{ij} \cap {}^2E_{jk}$ *is a permutation of* E_{jk}.

PROOF. (i) Because of Lemma 1.6(ix), $\mathsf{d}_{ij} \cdot \mathsf{c}_i\, a$ is an atom; it is in E_{ij} and is T_i related to a by definition. Uniqueness follows from 2.6(v).

(ii) By (iv) and (i) of 2.6 we have

$$E_{ij} = {T_k}^*(E_{ik} \cap E_{kj}) = (T_k \mid T_k)^*(E_{ik} \cap E_{kj}) = {T_k}^*\left({T_k}^*(E_{ik} \cap E_{kj})\right) = {T_k}^* E_{ij}.$$

(iii) If $x \in \mathrm{At}\,\mathfrak{Nr}_{\alpha\setminus\{k\}}\,\mathfrak{C}$, then $x = \mathsf{c}_k\, a$ for some $a \in B$ by Lemma 1.9, so $fx = \mathsf{d}_{jk} \cdot x = \mathsf{d}_{jk} \cdot \mathsf{c}_k\, a \in E_{jk}$ by (i), and $g(fx) = \mathsf{c}_k(\mathsf{d}_{jk} \cdot \mathsf{c}_k\, a) = \mathsf{c}_k\, a = x$ by Lemma 1.6(viii). If $a \in E_{jk}$, then $\mathsf{c}_k\, a \in \mathrm{At}\,\mathfrak{Nr}_{\alpha\setminus\{k\}}\,\mathfrak{C}$ by Lemma 1.9, and $f(ga) = \mathsf{d}_{jk} \cdot \mathsf{c}_k\, a = a$ by (i) and $a \in E_{jk}$. This shows that f is one-to-one and onto. To complete the proof we must show that

$$a\ {}_kK_{ij}\ b \iff {}_k\mathsf{s}(i,j)\,\mathsf{c}_k\, a = \mathsf{c}_k\, b \quad \text{for all } a, b \in E_{jk}.$$

So let $a, b \in E_{jk}$, and assume first that $a\ {}_kK_{ij}\ b$. Then $b \in {}_kK_{ij}^*\{a\} = {}_k\mathsf{s}(i,j)a \leq {}_k\mathsf{s}(i,j)\,\mathsf{c}_k\, a$ by 2.9, and hence $\mathsf{c}_k\, b \leq {}_k\mathsf{s}(i,j)\,\mathsf{c}_k\, a$ by 1.6(i), and from this, $\mathsf{c}_k\, b = {}_k\mathsf{s}(i,j)\,\mathsf{c}_k\, a$ follows, since ${}_k\mathsf{s}(i,j)\,\mathsf{c}_k\, a$ is an atom of $\mathfrak{Nr}_{\alpha\setminus\{k\}}\,\mathfrak{C}$ by Lemma 1.11(xiv). Conversely, if ${}_k\mathsf{s}(i,j)\,\mathsf{c}_k\, a = \mathsf{c}_k\, b$, then $b \in {}_kK_{ij}^*(a/T_k) = {}_kK_{ij}^*\{a\}$ by 2.9, the definition of ${}_kK_{ij}$ and 2.6(v), and hence $a\ {}_kK_{ij}\ b$.

(iv) ${}_k\mathsf{s}(i,j)$ is an automorphism of $\mathfrak{Bl}\,\mathfrak{Nr}_{\alpha\setminus\{k\}}\,\mathfrak{C}$ by Lemma 1.11(xiv), and from this it follows by an easy Boolean argument that it is a permutation of the atoms of $\mathfrak{Nr}_{\alpha\setminus\{k\}}\,\mathfrak{C}$. From this the statement follows by (iii). □

The following lemma illustrates how certain properties of a complete and atomic algebra are reflected in its atom structure. The *MGR* equations were defined in 1.12 in Chapter 1.

LEMMA 2.11. *Let* $\mathfrak{B} \in \mathsf{Ca}_3$. *Then*

$$\mathfrak{Cm}\,\mathfrak{B} \models MGR_k \quad \textit{iff} \quad \mathfrak{B} \models T_k \,|\, {}_kK_{ij} = T_k \,|\, {}_kK_{ji}$$

provided $\{i, j, k\} = 3$.

PROOF. This is a simple consequence of the fact that elements of $\mathfrak{Cm}\,\mathfrak{B}$ are subsets of B; and for a subset x of B, $\mathsf{c}_k x = T_k{}^* x$ and ${}_k\mathsf{s}(i, j)x = {}_kK^*_{ij} x$ by Definition 2.3 and Lemma 2.9. Hence $\mathfrak{Cm}\,\mathfrak{B} \models {}_k\mathsf{s}(i, j)x = {}_k\mathsf{s}(j, i)x$ iff ${}_kK^*_{ij} T_k{}^* x = {}_kK^*_{ji} T_k{}^* x$ for all $x \subseteq B$, that is, iff $T_k \,|\, {}_kK_{ij} = T_k \,|\, {}_kK_{ji}$. □

REMARK 2.12. An advantage of working with atom structures is that they can be drawn. One possibility is to draw T_i-classes as lines (or segments) and elements of a class as points on them. As an illustration, consider once again the *MGR*-equations. Keeping in mind that each T_i-class has exactly one member that belongs to E_{ij} (when $j \neq i$), we obtain the following picture of the equation ${}_k\mathsf{s}(i, j)\,\mathsf{c}_k x = {}_k\mathsf{s}(j, i)\,\mathsf{c}_k x$ (see Figure 1). The interesting thing about this picture is that one can

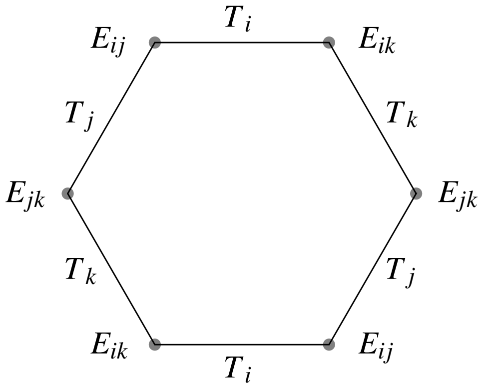


FIGURE 1. Merry-Go-Round

practically read off from it both parts of Lemma 1.13.

We close our brief survey of atom structures by recalling from [HMT85, 3.2.68] a simple method for constructing atomic Cs's. Let α be an ordinal, and let $\mathfrak{A}$ be the full Cs_α on a set U. If G is a group of permutations of U, then each element g of G induces a permutation $s \mapsto g \circ s$ of U-sequences of a fixed length. Call two U-sequences s and z G-equivalent (written $s \equiv_G z$) if $z = g \circ s$ for some $g \in G$. If s is a U-sequence, $s^- \stackrel{\text{def}}{=} s/{\equiv_G} = \{\, g \circ s : g \in G \,\}$ denotes the orbit of s under the permutations induced by elements of G. It is easy to see that the set[1] $Fx_G\mathfrak{A} \stackrel{\text{def}}{=} \big\{ X \subseteq {}^\alpha U : (\forall g \in G) X = \{\, g \circ s : s \in X \,\} \big\}$ is a subuniverse of $\mathfrak{A}$. (This is true by virtue of the fact that the operations of a Cs are *permutation invariant* in the sense of Tarski–Givant [TG87] or Németi [Ném91].) We denote by $\mathfrak{Fr}_G\mathfrak{A}$ the subalgebra of $\mathfrak{A}$ with universe $Fx_G\mathfrak{A}$. Clearly, $\mathfrak{Fr}_G\mathfrak{A}$ is atomic, and its atoms are exactly the relations s^-, where $s \in {}^\alpha U$. Further, two atoms s^- and z^- are T_i-related in the atom structure of $\mathfrak{Fr}_G\mathfrak{A}$ iff $s \restriction (\alpha \setminus \{i\}) \equiv_G z \restriction (\alpha \setminus \{i\})$, and $s^- \in E_{ij}$ iff $s_i = s_j$.

[1]This notation is slightly different from that used in [HMT85].

2.2. Dilation

The idea behind dilation may be expressed vaguely as follows: in a CA, if it is not outright impossible to have an atom in a certain position, then insert a new atom there. Here "impossible" means that the existence of the new atom would contradict the CA-axioms; and the "position" of an atom is how it is related to other atoms (and itself) via the relations T_i and E_{ij}. The definition (recalled from [HMT85, 3.2.69]) below is an attempt to formalize this idea. It is, however, a compromise, in that it sacrifices certain possibilities for the sake of simplicity: namely, condition (D1) could be replaced by a weaker but more involved condition.

DEFINITION 2.13. *Let $\mathfrak{B} = \langle B, T_i, E_{ij} \rangle_{i,j<\alpha} \in \mathsf{Ca}_\alpha$, let $\xi \in {}^\alpha B$ and suppose that*

(D1) $$\xi_i \notin E_{jk} \quad \textit{if } |\{i,j,k\}| = 3, \textit{ and}$$

(D2) $$\xi_i \, T_i \,|\, T_j \, \xi_j \quad \textit{for all } i, j < \alpha.$$

Then the result of dilating $\mathfrak{B}$ with ξ is $\mathfrak{B}^\xi = \langle B^\xi, T_i^\xi, E_{ij}^\xi \rangle_{i,j<\alpha}$, where $B^\xi = B \cup \{\nu\}$ for some $\nu \notin B$, $E^\xi_{ij} = E_{ij}$ if $i \neq j$, $E^\xi_{ii} = B^\xi$, and for all $i < \alpha$, T_i^ξ is the smallest equivalence relation on B^ξ extending T_i such that $\langle \xi_i, \nu \rangle \in T_i^\xi$ (i.e., $T_i^\xi \cap {}^2B = T_i$, and $b \, T_i^\xi \, \nu$ iff $b \, T_i \, \xi$, for all $b \in B$).

Although we defined dilation for atom structures, we will also apply it to atomic algebras. When $\mathfrak{C}$ is an atomic CA_α with atom structure $\mathfrak{B}$, and $\xi \in {}^\alpha B$ satisfies the conditions of 2.13 in $\mathfrak{B}$, then by the result of dilating $\mathfrak{C}$ with ξ we mean (any isomorphic copy of) $\mathfrak{Cm}\,\mathfrak{B}^\xi$.

When dilating an atomic CA (or its atom structure) with a sequence of atoms ξ, we will usually distinguish the "dilated" operations (i.e., those affected by dilation, such as cylindrifications and converse, but not the diagonals or the Boolean join) by the superscript ξ. So, for example, $\mathsf{c}_i^\xi x$ will denote the ith cylindrification of x, and $x^{\smile\xi}$ the converse of x in the dilated algebra. Similarly, we write K^ξ_{ij} and C^ξ, cf. Definition 2.8.

The next theorem shows that dilation does not lead out of CA.

THEOREM 2.14 ([HMT85, 3.2.69]). *Let $\mathfrak{B} \in \mathsf{Ca}_\alpha$ and suppose that $\xi \in {}^\alpha B$ satisfies conditions* (D1) *and* (D2) *of 2.13. Then $\mathfrak{B}^\xi$, the result of dilating $\mathfrak{B}$ with ξ, is a Ca_α.*

EXAMPLE 2.15. The drawing in Figure 2 shows a simple (but already useful) example of dilation. The atom structure on the left hand side shows the atoms of the full Cs_2 on a 2-element set; the

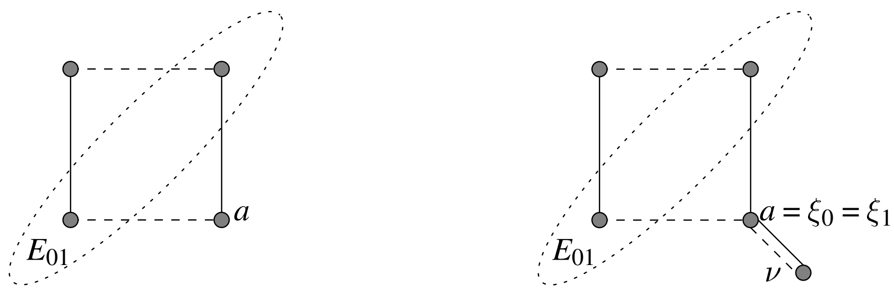


FIGURE 2. Dilation

dashed lines represent T_0 and the solid lines represent T_1. It is obvious that the sequence $\langle a, a \rangle$ satisfies conditions (D1) and (D2). The atom structure on the right hand side is what we get by dilating with

$\langle a,a\rangle$. This example was used by Henkin to show that the equation[2] $\mathsf{d}_{01}\cdot\mathsf{c}_1(-x\cdot\mathsf{c}_0 x)\cdot-\mathsf{c}_0(\mathsf{c}_1 x\cdot-\mathsf{d}_{01})=0$, true in RCA_2, is not valid in CA_2 (it fails when $x=a$).

The following lemma relates the accessibility relations corresponding to converse and composition in $\mathfrak{B}^\xi$ to those of $\mathfrak{B}$.

LEMMA 2.16. *Let $\mathfrak{B}\in\mathsf{Ca}_\alpha$, $\alpha>2$, and suppose that $\xi\in{}^\alpha B$ satisfies the dilating conditions* (D1) *and* (D2). *If $i,j<\alpha$ are distinct, and k is the smallest element of $\alpha\setminus\{i,j\}$, then*

(i) $K^\xi_{ij}=K_{ij}\,|\,T_k{}^\xi=K_{ij}\cup\big\{\langle a,\nu\rangle : a\,K_{ij}\,\xi_k\big\}$;
(ii) $(\mathsf{c}^\xi_2 a)^{\smile\xi}=\mathsf{c}^\xi_2\big((\mathsf{c}_2 a)^\smile\big)$ *for all $a\in B$*;
(iii) $C^\xi=C\cup\big\{\langle a,b,\nu\rangle : C(a,b,\xi_2)\big\}\cup\big\{\langle a,b,c\rangle : \xi_1\,T_1\,a\in E_{12},\xi_0\,T_0\,b\in E_{02}$ *and* $\xi_2\,T^\xi_2\,c\big\}$;
(iv)

$$\mathsf{c}^\xi_2 a\,;^\xi\,\mathsf{c}^\xi_2 b=\mathsf{c}^\xi_2(\mathsf{c}_2 a\,;\mathsf{c}_2 b)+\begin{cases}\mathsf{c}^\xi_2\xi_2 & \textit{if } \mathsf{d}_{12}\cdot\mathsf{c}_1\xi_1\le\mathsf{c}_2 a \textit{ and } \mathsf{d}_{02}\cdot\mathsf{c}_0\xi_0\le\mathsf{c}_2 b\\ 0 & \textit{otherwise}\end{cases}$$

for all $a,b\in B$.

PROOF. (i) First we note that

$$K_{ij}=K^\xi_{ij}\cap{}^2B; \tag{1}$$

by 2.8, the inclusion $\subseteq$ is obvious, and $\supseteq$ holds since $E^\xi_{ij}\cup E^\xi_{ik}\subseteq B$ and $T^\xi_l\cap{}^2B\subseteq T_l$ for all $l\in\{i,j,k\}$.

Suppose now that $a\,K^\xi_{ij}\,b$. Then $a\in E_{jk}\subseteq B$; hence if $b\in B$, then $\langle a,b\rangle\in K_{ij}\subseteq K_{ij}\,|\,T_k{}^\xi$ by (1). If $b\notin B$ (i.e., if $b=\nu$), then $\langle a,\xi_k\rangle\in(K^\xi_{ij}\,|\,T_k{}^\xi)\cap{}^2B\subseteq K^\xi_{ij}\cap{}^2B\subseteq K_{ij}$ by (1), and hence $\langle a,b\rangle\in K_{ij}\,|\,T_k{}^\xi$.

Next, to see that $K_{ij}\,|\,T_k{}^\xi\subseteq K_{ij}\cup\big\{\langle a,\nu\rangle : a\,K_{ij}\,\xi_k\big\}$, suppose that $\langle a,b\rangle\in(K_{ij}\,|\,T_k{}^\xi)\setminus K_{ij}$. Then again, $a\in B$ and $a\,K_{ij}\,c\,T^\xi_k\,b$ for some $c\in B$. Since $\langle a,b\rangle\notin K_{ij}=K_{ij}|\,T_k$, we must have $b=\nu$ and $c\,T_k\,\xi_k$, and the latter, together with $a\,K_{ij}\,c$, implies $a\,K_{ij}\,\xi_k$, as desired.

Finally, the inclusion $K_{ij}\cup\big\{\langle a,\nu\rangle : a\,K_{ij}\,\xi_k\big\}\subseteq K^\xi_{ij}$ is an obvious consequence of (1), $\xi_k\,T^\xi_k\,\nu$ and $K_{ij}\,|\,T^\xi_k\subseteq K^\xi_{ij}$.

(ii) If $a\in B$, then

$$\begin{aligned}(\mathsf{c}^\xi_2 a)^{\smile\xi} &=K^{\xi*}_{01}T^{\xi*}_2\{a\} && \text{by 2.9}\\ &=K^{\xi*}_{01}T^{*}_2\{a\} && \text{since } \{a\}\cup\operatorname{dom}K^\xi_{01}\subseteq B\\ &=(K_{01}\,|\,T^\xi_2)^*T^*_2\{a\} && \text{by (i)}\\ &=T^{\xi*}_2K^*_{01}T^*_2\{a\}\\ &=\mathsf{c}^\xi_2\big((\mathsf{c}_2 a)^\smile\big) && \text{by 2.9}\end{aligned}$$

(iii) First we note that C^ξ is not a "conservative" extension of C, that is, we have nothing analogous to (1) above for C.

As in (i), the inclusion $\supseteq$ follows easily from the definitions: clearly, $C\subseteq C^\xi$, $C(a,b,\xi_2)$ implies $C^\xi(a,b,\nu)$ because $\xi_2T^\xi_2\nu$, and if $\xi_1\,T_1\,a\in E_{12}$, $\xi_0\,T_0\,b\in E_{02}$ and $\xi_2\,T^\xi_2\,c$, then ν "witnesses" $C^\xi(a,b,c)$.

[2]This is actually a simplified (but equivalent) form of Henkin's equation, due to de Rijke–Venema [RV95].

Now suppose that $C^\xi(a,b,c)$ holds. Then by definition, $a \in E_{12} \subseteq B$, $b \in E_{02} \subseteq B$ and there is a $d \in B^\xi$ such that

$$a \, T_1^\xi \, d \, T_0^\xi \, b \wedge d \, T_2^\xi \, c. \tag{2}$$

Suppose first that $d \in B$ and hence $a \, T_1 \, d \, T_0 \, b$. Then either $c \in B$, too, and then $C(a,b,c)$ holds, or $c = \nu$, and then $C(a,b,\xi_2)$ holds, since $d \, T_2^\xi \, \nu$ implies $d \, T_2 \, \xi_2$. On the other hand, if $d = \nu$, then (2) gives $\xi_1 \, T_1 \, a \in E_{12}$, $\xi_0 \, T_0 \, b \in E_{02}$ and $c \, T_2^\xi \, \xi_2$, as desired.

(iv) follows from (iii) the way (ii) follows from (i). □

As a first application of the above lemmas we show that dilation does not destroy the *MGR* equations.

THEOREM 2.17. *Let $\mathfrak{B} \in \mathsf{Ca}_3$ and let $\mathfrak{B}^\xi$ be the result of dilating $\mathfrak{B}$ with ξ. Then $\mathfrak{Cm}\,\mathfrak{B}^\xi \models MGR$ provided $\mathfrak{Cm}\,\mathfrak{B} \models MGR$.*

PROOF. Let $\mathfrak{B} = \langle B, T_i, E_{ij} \rangle_{i,j<3}$ and $\mathfrak{B}^\xi = \langle B^\xi, T_i^\xi, E_{ij}^\xi \rangle_{i,j<3}$ as before. By Lemma 2.11 it is enough to check that

$$T_k \,|\, {}_k K_{ij} \subseteq T_k \,|\, {}_k K_{ji} \quad \text{implies} \quad T_k^\xi \,|\, {}_k K_{ij}^\xi \subseteq T_k^\xi \,|\, {}_k K_{ji}^\xi$$

when $\{i,j,k\} = 3$. So assume that $T_k \,|\, {}_k K_{ij} \subseteq T_k \,|\, {}_k K_{ji}$. Then

$$\begin{aligned}
T_k^\xi \,|\, {}_k K_{ij}^\xi &= T_k \,|\, {}_k K_{ij}^\xi && \text{since } \mathrm{dom}_k K_{ij}^\xi \subseteq B \\
&= T_k \,|\, {}_k K_{ij} \,|\, T_k^\xi && \text{by 2.16(i)} \\
&\subseteq T_k \,|\, {}_k K_{ji} \,|\, T_k^\xi && \text{by assumption} \\
&= T_k \,|\, {}_k K_{ij}^\xi && \text{by 2.16(i)} \\
&= T_k^\xi \,|\, {}_k K_{ij}^\xi && \text{since } \mathrm{dom}_k K_{ij}^\xi \subseteq B.
\end{aligned}$$

□

There are special properties (i.e., properties that, unlike validity of *MGR*, do not apply to all representable cylindric algebras) which are or can be preserved when dilating a CA_3. Lemma 2.19 below illustrates this.

DEFINITION 2.18. *Let $\mathfrak{A}$ be an RA-type algebra, and let $x, y \in A$. We say that x is* ***symmetric*** *if $x = x^\smile$, and x and y* ***commute*** *if $x \,;\, y = y \,;\, x$. $\mathfrak{A}$ is symmetric if all elements of $\mathfrak{A}$ are symmetric. $\mathfrak{A}$ is commutative if all pairs of elements in $\mathfrak{A}$ commute.*

LEMMA 2.19. *Let $\mathfrak{B} \in \mathsf{Ca}_3$, $\mathfrak{C} = \mathfrak{Cm}\,\mathfrak{B}$, $a, b \in B$ and suppose that $\xi \in {}^3B$ satisfies* (D1) *and* (D2) *in 2.13. Then*

(i) *if $\mathsf{c}_2 a$ is symmetric in $\mathfrak{Ra}\,\mathfrak{C}$, then $\mathsf{c}_2^\xi a$ is symmetric in $\mathfrak{Ra}\,\mathfrak{C}^\xi$;*

(ii) *if $\mathfrak{B}$ satisfies the formula*

$$(\mathsf{d}_{02} \cdot \mathsf{c}_0 \, \xi_0) \; T_2 \; (\mathsf{d}_{12} \cdot \mathsf{c}_1 \, \xi_1) \tag{1}$$

and $\mathsf{c}_2 a, \mathsf{c}_2 b$ commute in $\mathfrak{Ra}\,\mathfrak{C}$, then $\mathsf{c}_2^\xi a, \mathsf{c}_2^\xi b$ commute in $\mathfrak{Ra}\,\mathfrak{C}^\xi$.

If $\mathfrak{Ra}\,\mathfrak{C}$ is symmetric, then so is $\mathfrak{Ra}\,\mathfrak{C}^\xi$, and if $\mathfrak{Ra}\,\mathfrak{C}$ is commutative and $\mathfrak{B}$ satisfies (1)*, then $\mathfrak{Ra}\,\mathfrak{C}^\xi$ is also commutative.*

PROOF. First of all, the last sentence of the lemma follows from parts (i) and (ii), because $\mathfrak{Ra}\,\mathfrak{C}^\xi$ is atomic and its atoms are all of the form $\mathsf{c}_2^\xi a$ for some $a \in B$ (by 1.19 and since $\mathsf{c}_2^\xi \nu = \mathsf{c}_2^\xi \xi_2$).

If $\mathsf{c}_2 a$ is symmetric in $\mathfrak{Ra}\,\mathfrak{C}$, that is, $(\mathsf{c}_2 a)^{\smile} = \mathsf{c}_2 a$, then

$$\begin{aligned}(\mathsf{c}_2^\xi a)^{\smile\xi} &= \mathsf{c}_2^\xi\big((\mathsf{c}_2 a)^{\smile}\big) && \text{by 2.16(ii)}\\ &= \mathsf{c}_2^\xi(\mathsf{c}_2 a)\\ &= \mathsf{c}_2^\xi a,\end{aligned}$$

completing the proof of (i).

To prove (ii) it suffices to show that if $\mathfrak{B}$ satisfies (1) and $\mathsf{c}_2 a \,;\mathsf{c}_2 b \le \mathsf{c}_2 b\,;\mathsf{c}_2 a$, then $\mathsf{c}_2^\xi a \,;^\xi \mathsf{c}_2^\xi b \le \mathsf{c}_2^\xi b \,;^\xi \mathsf{c}_2^\xi a$. By the assumption $\mathsf{c}_2 a \,;\mathsf{c}_2 b \le \mathsf{c}_2 b\,;\mathsf{c}_2 a$ we have

$$\mathsf{c}_2^\xi(\mathsf{c}_2 a\,;\mathsf{c}_2 b) \le \mathsf{c}_2^\xi(\mathsf{c}_2 b\,;\mathsf{c}_2 a)$$

and hence by Lemma 2.16(iv) we are done if we can show that

$$\mathsf{d}_{12}\cdot \mathsf{c}_1\xi_1 \le \mathsf{c}_2 a \quad \text{implies} \quad \mathsf{d}_{02}\cdot \mathsf{c}_0\xi_0 \le \mathsf{c}_2 a$$

and

$$\mathsf{d}_{02}\cdot \mathsf{c}_0\xi_0 \le \mathsf{c}_2 b \quad \text{implies} \quad \mathsf{d}_{12}\cdot \mathsf{c}_1\xi_1 \le \mathsf{c}_2 a.$$

But these implications are immediate consequences of (1). □

REMARK 2.20. The following simple example shows that the extra assumption (1) in (ii) and the very last part of the lemma cannot be omitted. Let G be the group of permutations of 3 generated by the successor function mod 3, and let $\mathfrak{C} = \mathfrak{Fr}_G\langle \mathfrak{P}({}^33), \mathsf{C}_i, \mathsf{D}_{ij}\rangle_{i,j<3}$. It is easy to see that $\mathfrak{Ra}\,\mathfrak{C}$ is commutative. (The reason is that the ;-reduct $\mathfrak{Ra}\,\mathfrak{C}$ is isomorphic to G, the cyclic group of order 3.) Let $\xi_0 = \langle 0,2,0\rangle^-$ and $\xi_1 = \xi_2 = \langle 0,1,1\rangle^-$. It is clear that the sequence ξ satisfies condition (D1). Since G is transitive on 3, $\mathsf{c}_i\mathsf{c}_j\xi_i = 1$ whenever $i \ne j < 3$ and from this it follows that ξ satisfies (D2). But in the RA-reduct of the dilated algebra $\mathfrak{C}^\xi$ we have

$$\begin{aligned}\mathsf{c}_2^\xi\xi_0 \,;^\xi \mathsf{c}_2^\xi\xi_1 = \mathsf{c}_2^\xi(\mathsf{c}_2\xi_0\,;\mathsf{c}_2\xi_1) = \mathsf{c}_2^\xi\mathsf{D}_{01} = \mathsf{D}_{01} \ne \mathsf{D}_{01} + \mathsf{c}_2^\xi\xi_2 = \\ \mathsf{c}_2^\xi\mathsf{D}_{01} + \mathsf{c}_2^\xi\xi_2 = \mathsf{c}_2^\xi(\mathsf{c}_2\xi_1\,;\mathsf{c}_2\xi_0) + \mathsf{c}_2^\xi\xi_2 = \mathsf{c}_2^\xi\xi_1\,;^\xi \mathsf{c}_2^\xi\xi_0\end{aligned}$$

by Lemma 2.16(iv).

The next lemma describes what happens if we iterate dilation.

LEMMA 2.21. *Let $\mathfrak{B} \in \mathsf{Ca}_\alpha$ and let $\xi, \zeta \in {}^\alpha B$ satisfy the dilating conditions* (D1) *and* (D2). *Let ν be the new element corresponding to ξ and μ be the new element corresponding to ζ. Then*

(i) *ζ satisfies conditions* (D1) *and* (D2) *in $\mathfrak{B}^\xi$, ξ satisfies conditions* (D1) *and* (D2) *in $\mathfrak{B}^\zeta$, and*

$$\mathfrak{B}^{\xi\zeta} = \mathfrak{B}^{\zeta\xi},$$

that is, dilating with ξ and ζ commute;

(ii)

$$T_i^{\xi\zeta} = T_i^\xi \cup T_i^\zeta \cup \begin{cases}\{\langle\mu,\nu\rangle,\langle\nu,\mu\rangle\} & \text{if } \langle\xi_i,\zeta_i\rangle \in T_i\\ \emptyset & \text{otherwise};\end{cases}$$

(iii) $K^{\xi\zeta}_{ij}{=}K^{\xi}_{ij}\cup K^{\zeta}_{ij}$ *for all* $i\neq j<3$;
(iv) *if* $\langle\xi_2,\zeta_2\rangle\notin T_2$ *then* $C^{\xi\zeta}=C^{\xi}\cup C^{\zeta}$.

PROOF. The first part of (i) is true because $E^{\xi}_{ij}=E_{ij}=E^{\zeta}_{ij}$ when $i\neq j$, and $T^{\xi}_i,T^{\zeta}_i\supseteq T_i$. Since $B^{\xi\zeta}=B\cup\{\mu,\nu\}=B^{\zeta\xi}$ and $E^{\xi\zeta}_{ij}=E^{\zeta\xi}_{ij}$ for all $i,j<\alpha$, it remains to show that $T^{\xi\zeta}_i=T^{\zeta\xi}_i$. If $a,b\in B$ then

$$a\;T^{\xi\zeta}_i\;b\leftrightarrow a\;T^{\xi}_i\;b\leftrightarrow a\;T_i\;b\leftrightarrow a\;T^{\zeta}_i\;b\leftrightarrow a\;T^{\zeta\xi}_i\;b,$$
$$a\;T^{\xi\zeta}_i\;\nu\leftrightarrow a\;T^{\xi}_i\;\nu\leftrightarrow a\;T_i\;\xi_i\leftrightarrow a\;T^{\zeta}_i\;\xi_i\leftrightarrow a\;T^{\zeta\xi}_i\;\nu,$$
$$a\;T^{\xi\zeta}_i\;\mu\leftrightarrow a\;T^{\xi}_i\;\zeta_i\leftrightarrow a\;T_i\;\zeta_i\leftrightarrow a\;T^{\zeta}_i\;\mu\leftrightarrow a\;T^{\zeta\xi}_i\;\mu$$

and

$$\nu\;T^{\xi\zeta}_i\;\mu\leftrightarrow \nu\;T^{\xi}_i\;\zeta_i\leftrightarrow \xi_i\;T_i\;\zeta_i\leftrightarrow \xi_i\;T^{\zeta}_i\;\mu\leftrightarrow \nu\;T^{\zeta\xi}_i\;\mu.$$

The claim now follows because T_i, T^{ξ}_i, $T^{\xi\zeta}_i$ etc. are all reflexive and symmetric.

Before proving (ii), (iii) and (iv) we note that in all three cases the inclusion $\supseteq$ follows from the definitions and (i). In the rest of this proof we will use (i) without explicitly referring to it.

(ii) Let $a,b\in B^{\xi\zeta}$ with $a\;T^{\xi\zeta}_i\;b$. If $\{a,b\}\subseteq B^{\xi}$ then $a\;T^{\xi}_i\;b$, and if $\{a,b\}\subseteq B^{\zeta}$ then $a\;T^{\zeta}_i\;b$. If $\{a,b\}\not\subseteq B^{\xi}$ and $\{a,b\}\not\subseteq B^{\zeta}$, then $\{a,b\}=\{\nu,\mu\}$, and this implies $\xi_i\;T_i\;\zeta_i$ as in the proof of (i). This completes the proof of (ii).

(iii) Let $a,b\in B^{\xi\zeta}$ and suppose that $\langle a,b\rangle\in K^{\xi\zeta}_{ij}\setminus K^{\xi}_{ij}$ for some $i\neq j<3$. Then by Lemma 2.16(i) $b=\mu$ and $a\;K^{\xi}_{ij}\;\zeta_k$ (where k is the smallest element in $\alpha\setminus\{i,j\}$). But then $a\;K_{ij}\;\zeta_k$ (since $a,\zeta_k\in B$), and hence $a\;K^{\zeta}_{ij}\;b$ by Lemma 2.16(i) again.

(iv) Let $a,b,c\in B^{\xi\zeta}$ and suppose that $C^{\xi\zeta}(a,b,c)$. Then

$$a\in E_{12}\wedge b\in E_{02}\wedge\exists d(a\;T^{\xi\zeta}_1\;d\;T^{\xi\zeta}_0\;b\wedge d\;T^{\xi\zeta}_2\;c).$$

By the assumption $\langle\xi_2,\zeta_2\rangle\notin T_2$ and (ii) we have either $\{c,d\}\subseteq B^{\xi}$ and $d\;T^{\xi}_2\;c$ or $\{c,d\}\subseteq B^{\zeta}$ and $d\;T^{\zeta}_2\;c$; and, of course, $\{a,b\}\subseteq B$. Hence, by (i), we have $C^{\xi}(a,b,c)$ in the former case and $C^{\zeta}(a,b,c)$ in the latter. $\square$

COROLLARY 2.22. *Let* $\mathfrak{B}\in\mathsf{Ca}_{\alpha}$, $a,b\in B$ *and let* $\xi,\zeta\in{}^{\alpha}B$ *satisfy conditions* (D1) *and* (D2). *Then*

(i) $\mathsf{c}^{\xi\zeta}_i\,a=\mathsf{c}^{\xi}_i\,a+\mathsf{c}^{\zeta}_i\,a$;
(ii) $(\mathsf{c}^{\xi\zeta}_2\,a)^{\smile\xi\zeta}=(\mathsf{c}^{\xi}_2\,a)^{\smile\xi}+(\mathsf{c}^{\zeta}_2\,a)^{\smile\zeta}$;
(iii) *if* $\langle\xi_2,\zeta_2\rangle\notin T_2$ *then* $(\mathsf{c}^{\xi\zeta}_2\,a)\,;^{\xi\zeta}(\mathsf{c}^{\xi\zeta}_2\,b)=[(\mathsf{c}^{\xi}_2\,a)\,;^{\xi}(\mathsf{c}^{\xi}_2\,b)]+[(\mathsf{c}^{\zeta}_2\,a)\,;^{\zeta}(\mathsf{c}^{\zeta}_2\,b)]$.

PROOF. In all three equations, the inclusion $\geq$ follows from the fact that all the extra-Boolean operations are monotone (in fact, they distribute over $+$) and that for every sequence ξ and every $R\in\{T_i,K_{ij},C\}$, we have $R^{\xi}\supseteq R$, either by definition or by Lemma 2.16.

The inclusion $\leq$ holds in (i) because of Lemma 2.21(ii) and since $a\notin\{\nu,\mu\}$.

Next, if $b\in(\mathsf{c}^{\xi\zeta}_2\,a)^{\smile\xi\zeta}$, then $a\;T^{\xi\zeta}_2\;a'\;K^{\xi\zeta}_{01}\;b$ for some $a'\in B^{\xi\zeta}$; but, since $\{a\}\cup\mathrm{dom}\,K^{\xi\zeta}_{01}\subseteq B$, we have in fact $a\;T_2\;a'\;K^{\xi\zeta}_{01}\;b$. By Lemma 2.21(iii), either $a'\;K^{\xi}_{01}\;b$ or $a'\;K^{\zeta}_{01}\;b$, and hence either $b\in(\mathsf{c}_2\,a)^{\smile\xi}\leq(\mathsf{c}^{\xi}_2\,a)^{\smile\xi}$ or $b\in(\mathsf{c}_2\,a)^{\smile\zeta}\leq(\mathsf{c}^{\zeta}_2\,a)^{\smile\zeta}$, as desired.

Finally, to see that $\leq$ holds in (iii), suppose that $c \in (\mathsf{c}_2^{\xi\zeta} a) ;^{\xi\zeta} (\mathsf{c}_2^{\xi\zeta} b)$. Then there are $a', b' \in B$ with $a\, T_2^{\xi\zeta}\, a'$ and $b\, T_2^{\xi\zeta}\, b'$ such that $C^{\xi\zeta}(a', b', c)$, so by Lemma 2.21(iv) either $C^{\xi}(a', b', c)$ or $C^{\zeta}(a', b', c)$ holds. But this means that we have either $c \in (\mathsf{c}_2 a) ;^{\xi} (\mathsf{c}_2 b) \leq (\mathsf{c}_2^{\xi} a) ;^{\xi} (\mathsf{c}_2^{\xi} b)$ or $c \in (\mathsf{c}_2 a) ;^{\zeta} (\mathsf{c}_2 b) \leq (\mathsf{c}_2^{\zeta} a) ;^{\zeta} (\mathsf{c}_2^{\zeta} b)$. □

2.3. Twisting

The second method of constructing non-representable cylindric algebras from representable ones is called *twisting*. It is due to Henkin, but here we will extend his construction (described in [HMT85, 3.2.71]) in order to be able to produce CA's that do not arise as the result of Henkin's method. For example, by using our extended version of twisting we will be able to construct a CA_3 in which the quasi-equation $x^{\smile\smile} \leq x \rightarrow x^{\smile\smile} = x$ fails (see Theorem 3.6 below), and we will show that this is impossible using Henkin's construction[3].

Just like dilation, twisting is used to "distort" atomstructures. First, we describe the construction and show that it indeed produces Ca_α's from Ca_α's.

DEFINITION 2.23. *Let $\mathfrak{B} = \langle B, T_i, E_{ij} \rangle_{i,j<\alpha} \in \mathsf{Ca}_\alpha$, $t \in \alpha$ and $\xi \in {}^I B$ for some set I, and suppose that*

(T1) $$\langle \xi_i, \xi_j \rangle \notin T_t \quad \text{for all distinct } i, j \in I$$

(T2) $$\xi_i \notin E_{jk} \quad \text{for all } i \in I \text{ and all distinct } j, k < \alpha \text{ such that } t \notin \{j, k\}.$$

For $i \in I$, let Ξ_i denote the T_t-class of ξ_i, let π be a permutation of I, and for all $i \in I$, let Ξ_i be partitioned into Ξ'_i and Ξ''_i. Assume that for all $i \in I$ and $j < \alpha$, $j \neq t$,

(T3) $$\begin{aligned} &\operatorname{dom}\big(T_j \cap (\Xi'_i \times \Xi'_{\pi i})\big) \supseteq \Xi'_i, && \operatorname{ran}\big(T_j \cap (\Xi'_i \times \Xi'_{\pi i})\big) \supseteq \Xi'_{\pi i}, \\ &\operatorname{dom}\big(T_j \cap (\Xi''_i \times \Xi''_{\pi i})\big) \supseteq \Xi''_i, && \operatorname{ran}\big(T_j \cap (\Xi''_i \times \Xi''_{\pi i})\big) \supseteq \Xi''_{\pi i}. \end{aligned}$$

Then we form a new relational structure $\mathfrak{B}' = \langle B, T'_i, E_{ij} \rangle_{i,j<\alpha}$ by letting T'_t be the equivalence relation on B with equivalence classes x/T_t for $x \in B \setminus \bigcup_{i\in I} \Xi_i$, together with the classes $\Xi'_i \cup \Xi''_{\pi i}$ ($i \in I$), and $T'_i \stackrel{\text{def}}{=} T_i$ if $i \neq t$. We say that $\mathfrak{B}'$ ($\mathfrak{Cm}\,\mathfrak{B}'$) is a twisted version of $\mathfrak{B}$ ($\mathfrak{Cm}\,\mathfrak{B}$). For a class K of CA_α's,

$$\mathbf{Tw} K \stackrel{\text{def}}{=} \mathbf{I} K \cup \mathbf{I}\{\mathfrak{A} : \mathfrak{A} \text{ is a twisted version of some } \mathfrak{C} \in K\}.$$

We note that by Lemma 2.10(ii), (T2) is equivalent with the condition

$$E_{jk} \cap \bigcup_{i\in I} \Xi_i = \emptyset \quad \text{for all distinct } j, k < \alpha \text{ such that } t \notin \{j, k\}.$$

The difference between Henkin's version of twisting and ours is that he only allowed two T_t-classes to be "twisted" (i.e., in his construction, $|I| = 2$ always[4]). Figure 3 gives a schematic picture of twisting. The solid vertical lines represent T_t-classes before twisting, and the dashed lines illustrate the T'_t classes. The arrows on the top represent the action of the permutation π. More concrete examples of twisting will come up in Chapters 3, 4 and 6 below.

THEOREM 2.24. *Let $\alpha > 2$ and let $\mathfrak{B}'$ be the result of twisting $\mathfrak{B} \in \mathsf{Ca}_\alpha$. Then $\mathfrak{B}' \in \mathsf{Ca}_\alpha$.*

[3] finitely many times

[4] The possibility of twisting with an infinite set seems to have been first considered by Diane Resek in [Res75].

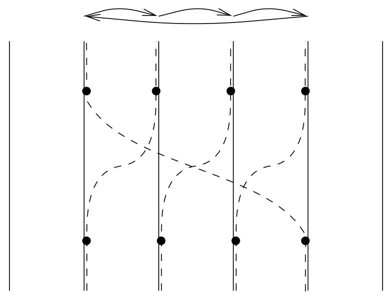


FIGURE 3. Twisting

PROOF. We use the notation introduced above. We check that conditions (i) through (v) in 2.6 hold in $\mathfrak{B}'$. (i) and (iii) are obvious. To show that (ii) holds, it is enough to see that $T'_t \,|\, T_j = T_j \,|\, T'_t$ for every $j < \alpha$, $j \neq t$. First we claim that

$$T_j \,|\, T_t = T_j \,|\, T'_t \quad \text{for every } j < \alpha,\ j \neq t. \tag{1}$$

Indeed, if $\langle a,b\rangle \in T_j \,|\, T_t$, then there is a $c \in B$ such that $a\ T_j\ c\ T_t\ b$, and then either $c\ T'_t\ b$, and hence $\langle a,b\rangle \in T_j \,|\, T'_t$, or there is an $i \in I$ with either $c \in \Xi'_i$ and $b \in \Xi''_i$ or $c \in \Xi''_i$ and $b \in \Xi'_i$. By condition (T3), in the first case there is a $c' \in \Xi'_{\pi^{-1}i}$ such that $c'\ T_j\ c$, and in the second there is a $c' \in \Xi''_{\pi i}$ such that $c'\ T_j\ c$. Hence in both cases we have $a\ T_j\ c'\ T'_t\ b$, that is, $\langle a,b\rangle \in T_j \,|\, T'_t$. This establishes the inclusion $\subseteq$ in (1). Conversely, if $\langle a,b\rangle \in T_j \,|\, T'_t$, then there is a $c \in B$ such that $a\ T_j\ c\ T'_t\ b$, and then either $c\ T_t\ b$, and hence $\langle a,b\rangle \in T_j \,|\, T_t$, or there is an $i \in I$ with either $c \in \Xi'_i$ and $b \in \Xi''_{\pi i}$ or $c \in \Xi''_{\pi i}$ and $b \in \Xi'_i$. Again by (T3), in the first case we find a $c' \in \Xi'_{\pi i}$ with $c'\ T_j\ c$, and in the second a $c' \in \Xi''_i$ with $c'\ T_j\ c$, and hence in both cases we have $a\ T_j\ c'\ T_t\ b$, that is, $\langle a,b\rangle \in T_j \,|\, T_t$. This completes the proof of (1). Taking converses of both sides of (1) and using the fact that T_t, T'_t and T_j are symmetric, we get

$$T_t \,|\, T_j = T'_t \,|\, T_j, \quad \text{for every } j < \alpha,\ j \neq t \tag{2}$$

and from (1) and (2), using the fact that 2.6(ii) holds in $\mathfrak{B}$, we get

$$T_j \,|\, T'_t = T_j \,|\, T_t = T_t \,|\, T_j = T'_t \,|\, T_j$$

as desired.

To prove (iv), it is enough to show that $E_{lm} = T'^*_t(E_{lt} \cap E_{tm})$ if $t \notin \{l,m\}$. Assume first that $l = m$. We have to show that $B \subseteq T'^*_t E_{lt}$. Let $k \in \alpha \setminus \{l,t\}$. Then

$$\begin{aligned}
B &\subseteq T^*_t E_{lt} && \text{(iv) holds in } \mathfrak{B}\\
&\subseteq (T_t \,|\, T_k)^* E_{lt} && T_t \subseteq T_t \,|\, T_k\\
&= (T'_t \,|\, T_k)^* E_{lt} && \text{by (2) above}\\
&= T'^*_t(T^*_k E_{lt}) &&\\
&\subseteq T'^*_t E_{lt} && \text{by 2.10(ii).}
\end{aligned}$$

Assume now that $l \neq m$. Since (iv) holds in $\mathfrak{B}$, if $a \in E_{lm}$, then there is a $b \in E_{lt} \cap E_{tm}$ with $a\ T_t\ b$. By condition (T2), $a \notin \bigcup_{i\in I} \Xi_i$, whence $a\ T'_t\ b$. Conversely, if $b \in E_{lt} \cap E_{tm}$, then $b \in T_t{}^*(E_{lt} \cap E_{tm}) = E_{lm}$, whence $b \notin \bigcup_{i\in I} \Xi_i$ by (T2). Hence if $a\ T'_t\ b$, then $a\ T_t\ b$ and from this $a \in E_{lm}$ follows by Lemma 2.10(ii).

Finally, to prove that condition (v) holds in $\mathfrak{B}'$ it suffices to show that $t \neq l < \alpha$, $a\ T_t'\ b$ and $a, b \in E_{lt}$ imply $a = b$. If $a\ T_t\ b$, then we are done by (v) in $\mathfrak{B}$. Otherwise, by symmetry we may assume that $a \in \Xi_i'$ and $b \in \Xi_{\pi i}''$ for some $i \in I$. Let $m \in \alpha \setminus \{l, t\}$. By condition (T3), there is an $a' \in \Xi_{\pi i}'$ that is T_m-related to a. From Lemma 2.10(ii) we get that $a' \in E_{lt}$, and since $a', b \in \Xi_{\pi i}$ we have $a'\ T_t\ b$ and hence $a' = b$ (by (v) in $\mathfrak{B}$), a contradiction, since $\Xi_{\pi i}'$ and $\Xi_{\pi i}''$ are disjoint. ☐

The next lemma is used to simplify computations in a twisted algebra. It says that the effect of twisting the tth cylinder of an algebra is invisible if c_t is buried deep enough in a term. (That cylindrifications other than c_t and the diagonals are not affected by twisting dimension t is clear from the definition.) We use the notation of Definition 2.23.

LEMMA 2.25. *Let $\alpha > 2$ and let $\mathfrak{B}'$ be the result of twisting dimension t of $\mathfrak{B} \in \mathsf{Ca}_\alpha$. Then the term functions $\mathsf{c}_i\,\mathsf{c}_t$, $\mathsf{c}_t\,\mathsf{c}_i$, and $\mathsf{s}_i^t\,\mathsf{c}_j$ are the same in $\mathfrak{Cm}\,\mathfrak{B}'$ as in $\mathfrak{Cm}\,\mathfrak{B}$, provided $|\{t, i, j\}| = 3$.*

PROOF. We have already proved the first two statement of the lemma as (1) and (2) in the proof of 2.24. For the last part we have to show that for all $a, b \in B$,

$$\exists c \in E_{it}(a\ T_t\ c\ T_j\ b) \quad \text{iff} \quad \exists c \in E_{it}(a\ T_t'\ c\ T_j\ b).$$

($\Rightarrow$) Suppose $a\ T_t\ c\ T_j\ b$ and $c \in E_{it}$. If $c \notin \bigcup_{k \in I} \Xi_k$ or there is a $k \in I$ such that both c and a are in either Ξ_k' or Ξ_k'', then $a\ T_t'\ c$, so let's assume that

- $c \in \Xi_k'$ and $a \in \Xi_k''$, or
- $c \in \Xi_k''$ and $a \in \Xi_k'$

for some $k \in I$. In the first case, there is a $c' \in \Xi_{\pi^{-1}k}'$ that is T_j-related to c by (T3). By 2.10(ii) $c' \in E_{it}$, and we have $a\ T_t'\ c'\ T_j\ c\ T_j\ b$ and thus $a\ T_t'\ c'\ T_j\ b$, as desired. The proof of the second case is similar, the only difference is that we use (T3) to find a T_j-neighbour of c in $\Xi_{\pi k}''$ (instead of in $\Xi_{\pi^{-1}k}'$).
($\Leftarrow$) can be proved similarly, but it also follows from the $\Rightarrow$ direction by symmetry (inverse twisting). ☐

The following lemma describes the behaviour of $\breve{}$ in a twisted algebra.

LEMMA 2.26. *Let $\mathfrak{B} \in \mathsf{Ca}_\alpha$, $\alpha > 2$, and let $\mathfrak{B}'$ be the result of twisting T_1 in $\mathfrak{B}$, as in 2.23. Let K_{01}' denote the accessibility relation of $\breve{}$ in $\mathfrak{B}'$. Then for all and a, $b \in B$, $a\ K_{01}'\ b$ iff one of the following (mutually exclusive) conditions holds:*

(i) *$a\ K_{01}\ b$, and either $a \notin \bigcup_{i \in I} \Xi_i$ or $(\exists i \in I)(\{a, \mathsf{d}_{01} \cdot \mathsf{c}_1\, a\} \subseteq \Xi_i' \vee \{a, \mathsf{d}_{01} \cdot \mathsf{c}_1\, a\} \subseteq \Xi_i'')$;*
(ii) *$\mathsf{d}_{12} \cdot \mathsf{c}_1\, \xi_{\pi i}\ K_{01}\ b$, $a \in E_{12} \cap \Xi_i'$, and $\mathsf{d}_{01} \cdot \mathsf{c}_1\, a \in \Xi_i''$ for some $i \in I$;*
(iii) *$\mathsf{d}_{12} \cdot \mathsf{c}_1\, \xi_{\pi^{-1} i}\ K_{01}\ b$, $a \in E_{12} \cap \Xi_i''$, and $\mathsf{d}_{01} \cdot \mathsf{c}_1\, a \in \Xi_i'$ for some $i \in I$.*

PROOF. Let $a, b \in B$ and suppose first that $a\ K_{01}'\ b$. Then $a \in E_{12}$, and there are $a_{01} \in E_{01}$ and $a_{02} \in E_{02}$ such that

$$a\ T_1'\ a_{01}\ T_0\ a_{02}\ T_2\ b. \tag{1}$$

We consider the following possibilities.

Case 1. $a \notin \bigcup_{i \in I} \Xi_i$.

Then $a\ T_1\ a_{01}$ and hence $a\ K_{01}\ b$ by (1).

Case 2. $a \in \Xi_i$ for some $i \in I$.

We have three subcases to consider. First, if a and $\mathsf{d}_{01}\cdot\mathsf{c}_1\,a$ are both in Ξ_i' or Ξ_i'', then $a_{01}\ T_1'\ a\ T_1'\ \mathsf{d}_{01}\cdot\mathsf{c}_1\,a$, whence $a_{01}=\mathsf{d}_{01}\cdot\mathsf{c}_1\,a$ by Lemma 2.10(i) and 2.6(i),(v), so $a\ T_1\ a_{01}$ and thus $a\ K_{01}\ b$. Next, if $a\in\Xi_i'$ and $\mathsf{d}_{01}\cdot\mathsf{c}_1\,a\in\Xi_i''$, then $a_{01}\in\Xi_{\pi i}''$, for otherwise we would have $a_{01}\ T_1\ a\ T_1\ \mathsf{d}_{01}\cdot\mathsf{c}_1\,a$, and, as before, this would imply $\mathsf{d}_{01}\cdot\mathsf{c}_1\,a=a_{01}\in\Xi_i'$, a contradiction. But then $\mathsf{d}_{12}\cdot\mathsf{c}_1\,\xi_{\pi i}\ T_1\ a_{01}$ and hence $\mathsf{d}_{12}\cdot\mathsf{c}_1\,\xi_{\pi i}\ K_{01}\ b$ by (1). Finally, if $a\in\Xi_i''$ and $\mathsf{d}_{01}\cdot\mathsf{c}_1\,a\in\Xi_i'$, then $a_{01}\in\Xi_{\pi^{-1}i}''$, for otherwise we would have $a_{01}\ T_1\ a\ T_1\ \mathsf{d}_{01}\cdot\mathsf{c}_1\,a$, and, again, this would imply $\mathsf{d}_{01}\cdot\mathsf{c}_1\,a=a_{01}\in\Xi_i''$, a contradiction. Hence $\mathsf{d}_{12}\cdot\mathsf{c}_1\,\xi_{\pi^{-1}i}\ T_1\ a_{01}$ and we have $\mathsf{d}_{12}\cdot\mathsf{c}_1\,\xi_{\pi^{-1}i}\ K_{01}\ b$ by (1) as desired.

For the reverse implication, suppose first that $a\ K_{01}\ b$, i.e., that $a\in E_{12}$ and there are $a_{01}\in E_{01}$ and $a_{02}\in E_{02}$ such that

$$a\ T_1\ a_{01}\ T_0\ a_{02}\ T_2\ b. \tag{2}$$

Then, if $a\notin\bigcup_{i\in I}\Xi_i$, then $a\ T_1'\ a_{01}$ and hence $a\ K_{01}'\ b$; and if a and $\mathsf{d}_{01}\cdot\mathsf{c}_1\,a$ are both in Ξ_i' or Ξ_i'' for some $i\in I$, then $a_{01}\ T_1\ a\ T_1\ \mathsf{d}_{01}\cdot\mathsf{c}_1\,a$, whence $a_{01}=\mathsf{d}_{01}\cdot\mathsf{c}_1\,a$, and hence $a\ T_1'\ a_{01}$ and $a\ K_{01}'\ b$.

Suppose now that $\mathsf{d}_{12}\cdot\mathsf{c}_1\,\xi_{\pi i}\ K_{01}\ b$ (so there are $a_{01}\in E_{01}$ and $a_{02}\in E_{02}$ such that $\mathsf{d}_{12}\cdot\mathsf{c}_1\,\xi_{\pi i}\ T_1\ a_{01}\ T_0\ a_{02}\ T_2\ b$) and $a\in\Xi_i'$ but $\mathsf{d}_{01}\cdot\mathsf{c}_1\,a\in\Xi_i''$. By the last assumption and 2.6(v), $\Xi_i'\cap E_{01}$ is empty, whence so is $\Xi_{\pi i}'\cap E_{01}$ by (T3) and 2.10(ii). It follows that $a_{01}\in\Xi_{\pi i}''$, and thus $a\ T_1'\ a_{01}$. This shows that $a\ K_{01}'\ b$. A similar argument proves that $\mathsf{d}_{12}\cdot\mathsf{c}_1\,\xi_{\pi^{-1}i}\ K_{01}\ b$, $a\in\Xi_i''$ and $\mathsf{d}_{01}\cdot\mathsf{c}_1\,a\in\Xi_i'$ imply $a\ K_{01}'\ b$. □

While the preceeding lemma is fairly general, it is not very convenient to use. We will usually chose the parameters in twisting so that the following special version applies.

COROLLARY 2.27. *Let $\mathfrak{B}\in\mathsf{Ca}_\alpha$, $\alpha>2$, and let $\mathfrak{B}'$ be the result of twisting T_1 in $\mathfrak{B}$. Assume that $\xi_i\in E_{12}\cap\Xi_i'$ for all $i\in I$. Then*

(i)

$$a\ K_{01}'\ b \iff \begin{cases} a\ K_{01}\ b, \quad \textit{and } (\forall i\in I)(a=\xi_i\Rightarrow\Xi_i''\cap E_{01}=\emptyset), \textit{ or}\\ a=\xi_i \quad \textit{and} \quad \xi_{\pi i}\ K_{01}\ b \quad \textit{for some } i\in I \textit{ such that } \Xi_i''\cap E_{01}\neq\emptyset;\end{cases}$$

for all $a,b\in B$;

(ii) *for all $a\in B$ there is at most one $i\in I$ such that $a\ T_2\ \xi_i$, and*

$$(\mathsf{c}_2\,a)^{\smile\prime}=\begin{cases}(\mathsf{c}_2\,a)^{\smile} & \textit{if } (\forall i\in I)(a\ T_2\ \xi_i\Rightarrow\Xi_i''\cap E_{01}=\emptyset)\\ (\mathsf{c}_2\,\xi_{\pi i})^{\smile} & \textit{if } a\ T_2\ \xi_i \textit{ and } \Xi_i''\cap E_{01}\neq\emptyset\end{cases}$$

in the complex algebra $\mathfrak{Cm}\,\mathfrak{B}'$ of $\mathfrak{B}'$.

PROOF. (i) We show how the conditions of the lemma and the corollary correspond under the assumptions of the latter. So assume that $\xi_i\in E_{12}\cap\Xi_i'$ for all $i\in I$. Then condition (iii) of the lemma never holds, since $a\in E_{12}\cap\Xi_i$ implies $a=\xi_i\in\Xi_i'$ by 2.6(v).

We claim that the first condition in the corollary corresponds to (i) of the lemma, that is,

$$\begin{aligned}&a\ K_{01}\ b, \text{ and } (\forall i\in I)(a=\xi_i\Rightarrow\Xi_i''\cap E_{01}=\emptyset)\\ &\qquad\iff a\ K_{01}\ b, \text{ and either } a\notin\textstyle\bigcup_{i\in I}\Xi_i \text{ or } (\exists i\in I)(\{a,\mathsf{d}_{01}\cdot\mathsf{c}_1\,a\}\subseteq\Xi_i'\vee\{a,\mathsf{d}_{01}\cdot\mathsf{c}_1\,a\}\subseteq\Xi_i'').\end{aligned}$$

($\Rightarrow$) By $a\ K_{01}\ b$, $a\in E_{12}$; hence if $a\in\bigcup_{i\in I}\Xi_i$, then $a=\xi_i$ for some $i\in I$ by 2.6(v). Then $\Xi_i''\cap E_{01}=\emptyset$ and hence $\{a,\mathsf{d}_{01}\cdot\mathsf{c}_1\,a\}\subseteq\Xi_i'$.
($\Leftarrow$) If $a=\xi_i$ for some $i\in I$, then $a\in\Xi_i'$, whence $\mathsf{d}_{01}\cdot\mathsf{c}_1\,a\in\Xi_i'$, and from this, $\Xi_i''\cap E_{01}=\emptyset$ follows by 2.6(v).

Finally, to see that the second condition in the corollary corresponds to (ii) of the lemma, it suffices to show that

$$a = \xi_i,\ \xi_{\pi i}\, K_{01}\, b \text{ and } \Xi''_i \cap E_{01} \neq \emptyset$$
$$\iff \mathsf{d}_{12}\cdot \mathsf{c}_1\, \xi_{\pi i}\, K_{01}\, b,\ a \in E_{12} \cap \Xi'_i, \text{ and } \mathsf{d}_{01}\cdot \mathsf{c}_1\, a \in \Xi''_i$$

for all $i \in I$. Now $a = \xi_i$ iff $a \in E_{12} \cap \Xi'_i$ by 2.6(v), and $\xi_{\pi i}\, K_{01}\, b$ iff $\mathsf{d}_{12}\cdot \mathsf{c}_1\, \xi_{\pi i}\, K_{01}\, b$ since $\xi_{\pi i} = \mathsf{c}_1\, \xi_{\pi i}$, again by 2.6(v). Clearly, $\mathsf{d}_{01}\cdot \mathsf{c}_1\, a \in \Xi''_i$ implies $\Xi''_i \cap E_{01} \neq \emptyset$; conversely, since $\mathsf{d}_{01}\cdot \mathsf{c}_1\, a$ is the uniqe element of $E_{01} \cap \Xi_i$ (by 2.10(i)), if $\Xi''_i \cap E_{01} \neq \emptyset$ then $\mathsf{d}_{01}\cdot \mathsf{c}_1\, a \in \Xi''_i \cap E_{01}$. The proof of (i) of the corollary is now complete.
(ii) follows from (i) (cf. the proof of 2.16(ii)). □

2.4. Relativization

Of the constructions mentioned in the introduction to this chapter, relativization is the one that will be used least frequently in the rest of this dissertation. For this reason we will confine ourselves to giving only the basic definitions.

DEFINITION 2.28. *Let* $\mathfrak{A} = \langle A, +, \cdot, -, 0, 1, \mathsf{c}_i, \mathsf{d}_{ij} \rangle_{i,j<\alpha} \in \mathsf{CA}_\alpha$ *and let* $w \in A$. *The* ***relativization of*** $\mathfrak{A}$ ***to*** w *is the algebra*

$$\mathfrak{Rl}_w \mathfrak{A} \stackrel{\text{def}}{=} \langle \mathrm{Rl}_w \mathfrak{A}, +^w, \cdot^w, -^w, 0^w, 1^w, \mathsf{c}^w_i, \mathsf{d}^w_{ij} \rangle_{i,j<\alpha},$$

where

$$\mathrm{Rl}_w \mathfrak{A} \stackrel{\text{def}}{=} \{ x \in A : x \leq w \},$$

and $x +^w y = x + y$, $x \cdot^w y = x \cdot y$, $-^w x = w \cdot -x$, $0^w = 0$, $1^w = w$, $\mathsf{c}^w_i x = w \cdot \mathsf{c}_i x$ *and* $\mathsf{d}^w_{ij} = w \cdot \mathsf{d}_{ij}$ *for all* x, $y \in \mathrm{Rl}_w \mathfrak{A}$ *and* $i, j < \alpha$. *For a class* K *of* CA_α*'s,* $\mathbf{Rl}\, K \stackrel{\text{def}}{=} \{ \mathfrak{Rl}_w \mathfrak{A} : \mathfrak{A} \in K, w \in A \}$.

When $\alpha > 1$, $\mathsf{CA}_\alpha \subset \mathbf{Rl}\, \mathsf{CA}_\alpha$, that is, not every algebra obtained by relativization of a CA_α is a CA_α itself (see [HMT71, 2.2.4]). To distinguish relativizations that do not lead out of CA_α, we introduce the notation $\mathbf{Rl}^{ca} K \stackrel{\text{def}}{=} \mathsf{CA}_\alpha \cap \mathbf{Rl}\, K$, where $K \subseteq \mathsf{CA}_\alpha$. Finally, $\mathbf{Srl}^{ca} K \stackrel{\text{def}}{=} \mathsf{CA}_\alpha \cap \mathbf{S}\,\mathbf{Rl}\, K$.

The following theorem of D. Resek and R. Thompson will be an important building block in our proof of the "twisted-relativized" representation theorem for CA_3's (Theorem 3.11 in the next chapter).

THEOREM 2.29 (Resek-Thompson). $\mathsf{CA}_3 \cap \mathrm{Mod}(MGR) \subseteq \mathbf{Srl}^{ca}\, \mathsf{Cs}_3$.

PROOF. The most accessible proof of this theorem is in Andréka–Thompson [AT88]. □

Actually, the Resek-Thompson theorem is more general than the form quoted in Theorem 2.29, in two respects. First, it is about algebras of arbitrary dimensions ≥ 2; for dimensions greater than 3, MGR stands for the conjunction of ${}_3MGR$ and ${}_4MGR$ (cf. Remark 1.14). Second, it gives representation as concrete, set algebras (i.e., as subalgebras of relativized Cs_α's) of members of a slightly bigger class than $\mathsf{CA}_\alpha \cap \mathrm{Mod}(MGR)$.

CHAPTER 3

Twisting and untwisting cylindric algebras

In the first part of this chapter we show that *MGR* is not "preserved under subalgebra generation" (Theorem 3.3; this solves the first part of Problem 11 in the Problem Session paper of [AMN91]), and give the first example of an essentially infinite non-representable CA_3 (Theorem 3.6). We show that the class of RA-reducts of CA_3's is not closed under taking subalgebras (Theorem 3.7). Although we are not concerned here with RA-reducts of cylindric algebras of dimension > 3, for completeness we note that the class of RA-reducts of CA_4's is not closed under taking subalgebras either, by Maddux [Mad90].

The main result of this chapter (Theorem 3.11) solves a problem asked by Henkin [Hen67, p. 48]: roughly, our result says that every CA_3 can be obtained from a representable CA_3 (in fact, a Cs_3) by relativization and twisting. We will also show that our extension of the definition of twisting is essential, by exhibiting a CA_3 which cannot be obtained by relativization and finite twisting.

3.1. Constructing skewed cylindric algebras

The most important type of construction in the results of this chapter is twisting; so it is perhaps appropriate to start with a simple example of how it can be used.

EXAMPLE 3.1. Fix $1 < N \in \omega$ and let $U_N \overset{\text{def}}{=} 2\times N$, i.e., U_N consists of two disjoint copies of N. Let σ_N be the permutation on U_N that acts as successor modulo N on both copies of N, and let G_N be the group of permutations generated by σ_N. Let $\mathfrak{C}_N = \mathfrak{Fr}_{G_N}\mathfrak{A}_N$, where $\mathfrak{A}_N$ is the full Cs_3 on U_N. Recall from section 2.1 that the universe C_N of $\mathfrak{C}_N$ consists of those subsets of ${}^3U_N$ that are fixed by all elements of G_N (or, equivalently, by σ_N). We want to apply twisting to $\mathfrak{C}_N$ (or, rather, its atom structure $\mathfrak{B}_N$), so we set up things as in Definition 2.23. Let $t \overset{\text{def}}{=} 1$, $I \overset{\text{def}}{=} N$ and let π be the successor function modulo N. For $i < 2$, $k \in I$ and $n < N$ define $\xi_k^{\langle i,n\rangle} \overset{\text{def}}{=} \langle\langle 0,0\rangle,\langle i,n\rangle,\langle 1,k\rangle\rangle^-$, and let $\xi_k \overset{\text{def}}{=} \xi_k^{\langle 1,k\rangle}$. Condition (T2) then clearly holds.

Since the identity is the only element of G_N which fixes a point in U_N (G_N is *semiregular*), we have that $\xi_k^{\langle i,n\rangle} = \xi_l^{\langle j,m\rangle}$ implies $i = j$, $k = l$ and $n = m$. From this it follows that

$$\Xi_k \overset{\text{def}}{=} \xi_k/T_1 = \{\, \xi_k^{\langle i,n\rangle} : i < 2, n < N \,\}, \tag{1}$$

and the displayed elements are pairwise distinct. In particular, $\xi_k/T_1 \cap \xi_l/T_1 = \emptyset$ if $k \neq l$, so condition (T1) holds.

To complete the construction, we have to partition Ξ_k into two parts and check that condition (T3) holds. Using (1) and the semiregularity of G_N, we see that

$$\begin{aligned} T_0 \cap (\Xi_k \times \Xi_{\pi k}) &= \{\, \langle \xi_k^{\langle i,n\rangle}, \xi_{\pi k}^{\langle j,m\rangle}\rangle : i, j < 2,\ n, m < N, \text{ and } \langle\langle i,n\rangle,\langle 1,k\rangle\rangle \equiv_{G_N} \langle\langle j,m\rangle,\langle 1,\pi k\rangle\rangle \,\} \\ &= \{\, \langle \xi_k^{\langle i,n\rangle}, \xi_{\pi k}^{\langle i,n+1\rangle}\rangle : i < 2, n < N \,\} \end{aligned}$$

and

$$T_2 \cap (\Xi_k \times \Xi_{\pi k}) = \big\{ \langle \xi_k^{\langle i,n\rangle}, \xi_{\pi k}^{\langle j,m\rangle} \rangle : i, j < 2,\ n, m < N, \text{ and } \big\langle \langle 0,0\rangle, \langle i,n\rangle \big\rangle \equiv_{G_N} \big\langle \langle 0,0\rangle, \langle j,m\rangle \big\rangle \big\}$$
$$= \big\{ \langle \xi_k^{\langle i,n\rangle}, \xi_{\pi k}^{\langle i,n\rangle} \rangle : i < 2, n < N \big\},$$

where + is addition modulo N. Hence the partition

$$\Xi'_k \stackrel{\text{def}}{=} \{ \xi_k^{\langle 1,n\rangle} : n < N \} \quad \text{and} \quad \Xi''_k \stackrel{\text{def}}{=} \{ \xi_k^{\langle 0,n\rangle} : n < N \}$$

satisfies condition (T3).

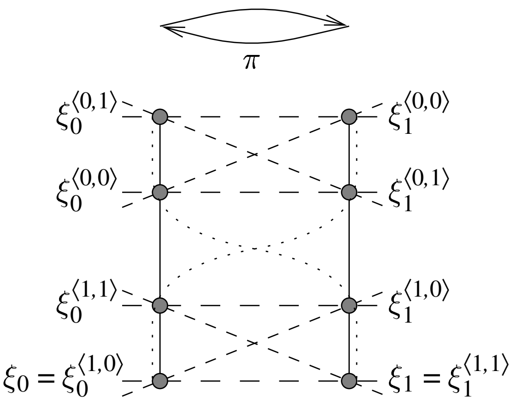


FIGURE 1. The twisted part of $\mathfrak{C}_2$

The solid vertical lines in Figure 1 show the two T_1-classes that are affected by twisting in $\mathfrak{B}_2$. The dotted curves are the T'_1 classes, the horizontal lines stand for T_0, and the sloped lines stand for T_2-connections.

Let $\mathfrak{B}'_N$ be the twisted version of $\mathfrak{B}_N$, and $\mathfrak{C}'_N$ its complex algebra. Since $\xi_k \in E_{12} \cap \Xi'_k$ and $\xi_k^{\langle 0,0\rangle} \in \Xi''_k \cap E_{01}$, Corollary 2.27 can be applied to conclude that in $\mathfrak{C}'_N$, $(\mathsf{c}_2\,\xi_k)^{\smile\prime} = (\mathsf{c}_2\,\xi_{\pi k})^{\smile}$ for all $k \in I$. Also, since $(\mathsf{c}_2\,\xi_{\pi k})^{\smile} = \mathsf{c}_2 \big\langle \langle 1, \pi k\rangle, \langle 0,0\rangle, \langle 1, \pi k\rangle \big\rangle^{-}$, and $\big\langle \langle 1, \pi k\rangle, \langle 0,0\rangle, \langle 1, \pi k\rangle \big\rangle^{-}$ is not T_2-related to any ξ_i (for $i \in I$), by 2.27 again we have $(\mathsf{c}_2\,\xi_{\pi k})^{\smile\smile\prime} = (\mathsf{c}_2\,\xi_{\pi k})^{\smile\smile} = \mathsf{c}_2\,\xi_{\pi k}$. Summing up, in $\mathfrak{C}'_N$ we have $(\mathsf{c}_2\,\xi_k)^{\smile\prime\smile\prime} = \mathsf{c}_2\,\xi_{\pi k}$. Hence the elements $\mathsf{c}_2\,\xi_k$ are of order $2N$ in $\mathfrak{C}'_N$ under ${}^{\smile\prime}$.

We note that every RA-axiom which is not valid in $\mathfrak{Ra}^*\,\mathsf{CA}_3$ fails in $\mathfrak{Ra}\,\mathfrak{C}'_N$ when $N > 1$. We have just seen that (R4) fails, and from this it follows that (R6) and (R7) fail, by Theorems 4.4 and 4.3 in Chapter 4 below. To see that the associative law (R1) fails, set $x = z = \big\langle \langle 1,1\rangle, \langle 0,0\rangle, \langle 0,0\rangle \big\rangle^{-}$ and $y = \xi_0$: it is not hard to see that e.g. in $\mathfrak{C}'_2$

$$(\mathsf{c}_2 x \,;\, \mathsf{c}_2 y) \,;\, \mathsf{c}_2 z = \mathsf{c}_2 \big\langle \langle 1,0\rangle, \langle 1,1\rangle, \langle 1,1\rangle \big\rangle^{-} \,;\, \mathsf{c}_2 z = \mathsf{c}_2 \big\langle \langle 1,0\rangle, \langle 0,0\rangle, \langle 0,0\rangle \big\rangle^{-},$$

and

$$\mathsf{c}_2 x \,;\, (\mathsf{c}_2 y \,;\, \mathsf{c}_2 z) = \mathsf{c}_2 x \,;\, \mathsf{c}_2 \big\langle \langle 0,0\rangle, \langle 0,0\rangle, \langle 0,0\rangle \big\rangle^{-} = \mathsf{c}_2 \big\langle \langle 1,1\rangle, \langle 0,0\rangle, \langle 0,0\rangle \big\rangle^{-}.$$

But these are indeed different elements because $\big\langle \langle 1,0\rangle, \langle 0,0\rangle \big\rangle \not\equiv_{G_2} \big\langle \langle 1,1\rangle, \langle 0,0\rangle \big\rangle$.

Theorem 3.3 below solves the first part of Problem 11 in the Problem Session paper of [AMN91]. First we need a lemma.

LEMMA 3.2 (Cf. [HMT71, 1.5.17]). $\mathsf{CA}_\alpha \models {}_k\mathsf{s}(l,m)\,\mathsf{c}_k\,\mathsf{c}_l\,x = \mathsf{s}_l^m\,\mathsf{c}_k\,\mathsf{c}_l\,x = {}_k\mathsf{s}(m,l)\,\mathsf{c}_k\,\mathsf{c}_l\,x$, *provided* $l \notin \{k, m\}$.

PROOF. By lemma 1.11(i), (ix) we have

$$ {}_k\mathsf{s}(m,l)\,\mathsf{c}_k\,\mathsf{c}_l\,x = {}_k\mathsf{s}(m,l)\,\mathsf{s}^l_m\,\mathsf{c}_k\,\mathsf{c}_l\,x = \mathsf{s}^m_l\,\mathsf{c}_k\,\mathsf{c}_l\,x $$

and

$$ \begin{aligned} {}_k\mathsf{s}(l,m)\,\mathsf{c}_k\,\mathsf{c}_l\,x &= \mathsf{s}^k_l\,\mathsf{s}^l_m\,\mathsf{s}^m_k\,\mathsf{c}_k\,\mathsf{c}_l\,x \\ &= \mathsf{s}^k_l\,\mathsf{s}^l_m\,\mathsf{c}_l\,\mathsf{s}^m_k\,\mathsf{c}_k\,x && \text{by 1.11(ii)} \\ &= \mathsf{s}^k_l\,\mathsf{c}_l\,\mathsf{s}^m_k\,\mathsf{c}_k\,x && \text{by 1.11(i)} \\ &= \mathsf{s}^k_l\,\mathsf{s}^m_k\,\mathsf{c}_k\,\mathsf{c}_l\,x && \text{by 1.11(ii)} \\ &= \mathsf{s}^m_l\,\mathsf{c}_k\,\mathsf{c}_l\,x && \text{by 1.11(vii)} \end{aligned} $$

as desired. □

THEOREM 3.3. *There are a $\mathfrak{C} \in \mathsf{CA}_3$ and an element x in C such that x satisfies the MGR-equations but MGR fails in the algebra generated by x.*

PROOF. Let $\mathfrak{C}$ be the algebra $\mathfrak{C}'_2$ defined in Example 3.1, and let $x = \xi_1 + \mathsf{c}_2\,\xi_0$. (Recall that we got $\mathfrak{C}'_2$ by twisting the T_1-classes of the atoms ξ_0 and ξ_1 of $\mathfrak{C}_2$. In this proof we use the notation introduced in 3.1, except that $'$ will be omitted from the twisted operations.) We are going to show that MGR_k holds for x (and all $k < 3$) but MGR_2 fails for $-x$.

First, we claim that

(1) $$ \mathsf{c}_2(-x) \geq \mathsf{c}_2\,\xi_1. $$

Indeed, we have

$$ \begin{aligned} & 0 = -\mathsf{c}_2\,\xi_1 \cdot \mathsf{c}_2\,\xi_1 \geq \mathsf{c}_2 - \mathsf{c}_2\,\xi_1 \cdot -\xi_1 \cdot \mathsf{c}_2\,\xi_1 && \text{by 1.6(iv)} \\ & \Rightarrow 0 = -\mathsf{c}_2\,\xi_1 \cdot \mathsf{c}_2(-\xi_1 \cdot \mathsf{c}_2\,\xi_1) && \text{by 1.6(ii)} \end{aligned} $$

and

$$ \begin{aligned} \neg(\xi_1\ T_2\ \xi_0) &\Rightarrow \xi_1 \leq -\mathsf{c}_2\,\xi_0 \\ &\Rightarrow \mathsf{c}_2\,\xi_1 \leq -\mathsf{c}_2\,\xi_0 && \text{by 1.6(iv)} \end{aligned} $$

and hence $\mathsf{c}_2\,\xi_1 \leq \mathsf{c}_2(-\xi_1 \cdot \mathsf{c}_2\,\xi_1) \leq \mathsf{c}_2(-\xi_1 \cdot -\mathsf{c}_2\,\xi_0) = \mathsf{c}_2 -x$.

From (1) we get

$$ {}_2\mathsf{s}(0,1)\,{}_2\mathsf{s}(0,1)\,\mathsf{c}_2 -x = (\mathsf{c}_2 -x)^{\smile\smile} \geq (\mathsf{c}_2\,\xi_1)^{\smile\smile} = \mathsf{c}_2\,\xi_0, $$

since $\pi 1 = 0$. But $-x \leq -\mathsf{c}_2\,\xi_0$, and thus $\mathsf{c}_2 -x \leq \mathsf{c}_2 - \mathsf{c}_2\,\xi_0 = -\mathsf{c}_2\,\xi_0$, that is, $\mathsf{c}_2\,\xi_0 \cdot \mathsf{c}_2 -x = 0$; this shows that MGR_2 fails for $-x$.

On the other hand,

$$ {}_2\mathsf{s}(0,1)\,{}_2\mathsf{s}(0,1)\,\mathsf{c}_2\,x = (\mathsf{c}_2\,\xi_0 + \mathsf{c}_2\,\xi_1)^{\smile\smile} = (\mathsf{c}_2\,\xi_0)^{\smile\smile} + (\mathsf{c}_2\,\xi_1)^{\smile\smile} = \mathsf{c}_2\,\xi_1 + \mathsf{c}_2\,\xi_0 = \mathsf{c}_2\,x, $$

so MGR_2 holds for x.

Now $\mathsf{c}_0\,\xi_0 = \mathsf{c}_0\,\xi_1$ because this is so in $\mathfrak{C}_2$, and the operation c_0 is the same in $\mathfrak{C}$ as in $\mathfrak{C}_2$. Hence

(2) $$ \mathsf{c}_0\,x = \mathsf{c}_0\,\xi_1 + \mathsf{c}_0\,\mathsf{c}_2\,\xi_0 = \mathsf{c}_0\,\mathsf{c}_2\,\xi_0. $$

Next, $\mathsf{c}_1\,\xi_1 \leq \mathsf{c}_2\,\mathsf{c}_1\,\xi_0$: this is because $\mathsf{c}_2\,\mathsf{c}_1\,\xi_1 \leq \mathsf{c}_2\,\mathsf{c}_1\,\xi_0$ in $\mathfrak{C}_2$ and hence in $\mathfrak{C}$, by Lemma 2.25. From this it follows that

(3) $$ \mathsf{c}_1\,x = \mathsf{c}_1\,\xi_1 + \mathsf{c}_1\,\mathsf{c}_2\,\xi_0 = \mathsf{c}_2\,\mathsf{c}_1\,\xi_0, $$

and from (2), (3) and lemma 3.2 we get that MGR_0 and MGR_1 hold for x. □

The following lemma is an obvious consequence of Lemma 1.11(xiv), but it has an important corollary.

LEMMA 3.4. *Let* $\mathfrak{C} \in \mathsf{CA}_\alpha$ *and suppose that there is an element* $x \in \mathfrak{Ra}\,\mathfrak{C}$ *such that* $x^{\smile\smile} < x$. *Then* $\langle x_i : i \in \omega\rangle$, *where* $x_0 = x$ *and* $x_{i+1} = x_i^{\smile\smile}$, *is a strictly decreasing chain in* $\mathfrak{Ra}\,\mathfrak{C}$.

COROLLARY 3.5. *Let* $\alpha > 2$. *The quasi-equation* $x^{\smile\smile} \leq x \to x^{\smile\smile} = x$ *is valid in the* RA*-reducts of finite* CA_α*'s. Equivalently, the quasi-equation*

(e) $\qquad {}_2\mathsf{s}(0,1){}_2\mathsf{s}(0,1)\,\mathsf{c}_2 x \leq \mathsf{c}_2 x \to {}_2\mathsf{s}(0,1){}_2\mathsf{s}(0,1)\,\mathsf{c}_2 x = \mathsf{c}_2 x$

is valid in every finite CA_α.

Clearly, (e) is valid in representable CA_3's, too. The next theorem shows that it does not hold in CA_3.

THEOREM 3.6. *There are a* CA_3 $\mathfrak{C}'$ *and an element* $x \in \mathfrak{Ra}\,\mathfrak{C}'$ *such that* $x^{\smile\smile} < x$ *in* $\mathfrak{Ra}\,\mathfrak{C}'$. *Hence* $\mathsf{CA}_3 \not\models e$, *where e is as in Corollary 3.5.*

PROOF. Let U be the set $2\times\mathbb{Z}$, suc the permutation $\langle i,z\rangle \mapsto \langle i,z+1\rangle$ and f the permutation $\langle i,z\rangle \mapsto \langle j,z\rangle$ of U, where $z \in \mathbb{Z}$ and $\{i,j\} = 2$. Let $\mathfrak{C}$ be the full Cs_3 with base U, G the group of permutations generated by $\{\mathrm{suc}, f\}$, and let $\mathfrak{C} = \mathfrak{Fr}_G\mathfrak{C}$. The proofs of some of the statements below make use of the simple fact that

(1) $\qquad G$ is semiregular.

Note that if $s = \langle\langle i_0,z_0\rangle,\langle i_1,z_1\rangle,\langle i_2,z_2\rangle\rangle$ and $\{i_0,j_0\} = \{i_1,j_1\} = \{i_2,j_2\} = 2$, then

$$s^- = \big\{\langle\langle i_0,z_0+z\rangle,\langle i_1,z_1+z\rangle,\langle i_2,z_2+z\rangle\rangle, \langle\langle j_0,z_0+z\rangle,\langle j_1,z_1+z\rangle,\langle j_2,z_2+z\rangle\rangle : z \in \mathbb{Z}, \{i_0,j_0\} = \{i_1,j_1\} = \{i_2,j_2\} = 2\big\}.$$

Let $\mathfrak{B}$ be the atom structure of $\mathfrak{C}$. Again, we want to set up things as in Definition 2.23. Let $t \overset{\text{def}}{=} 1$, $I \overset{\text{def}}{=} \mathbb{Z}$ and let π be the permutation $z \mapsto -(z+1)$ of I (hence π is of order 2). For $k \in I$ let $\xi_k = \langle\langle 0,0\rangle,\langle 1,k\rangle,\langle 1,k\rangle\rangle^-$. Conditions (T1) then follows from (1), and (T2) obviously holds. For future reference we note that

$$(\mathsf{c}_2\xi_k)^\smile = (\mathsf{c}_2\langle\langle 0,0\rangle,\langle 1,k\rangle,\langle 1,k\rangle\rangle^-)^\smile = \mathsf{c}_2\langle\langle 1,k\rangle,\langle 0,0\rangle,\langle 0,0\rangle\rangle^- = \mathsf{c}_2\langle\langle 0,k\rangle,\langle 1,0\rangle,\langle 1,0\rangle\rangle^- = \mathsf{c}_2\langle\langle 0,0\rangle,\langle 1,-k\rangle,\langle 1,-k\rangle\rangle^- = \mathsf{c}_2\xi_{-k}.$$

To complete the construction, we partition $\Xi_k = \xi_k/T_1$ into two parts and check that condition (T3) holds. Using (1), it is easy to see that

$$\Xi_k = \{\,\langle\langle 0,0\rangle,u,\langle 1,k\rangle\rangle^- : u \in U\,\}$$

and that the displayed atoms are pairwise distinct. Hence (again by (1)), $T_0 \cap (\Xi_k\times\Xi_{\pi k})$ is the bijection

$$\langle\langle 0,0\rangle,\langle i,z\rangle,\langle 1,k\rangle\rangle^- \mapsto \langle\langle 0,0\rangle,\langle i,z-2k-1\rangle,\langle 1,-(k+1)\rangle\rangle^-$$

from Ξ_k onto $\Xi_{\pi k}$ and $T_2 \cap (\Xi_k\times\Xi_{\pi k})$ is the bijection

$$\langle\langle 0,0\rangle,u,\langle 1,k\rangle\rangle^- \mapsto \langle\langle 0,0\rangle,u,\langle 1,-(k+1)\rangle\rangle^-$$

from Ξ_k onto $\Xi_{\pi k}$. From this it follows that the partition

$$\Xi'_k \stackrel{\text{def}}{=} \{ \langle\langle 0,0\rangle, \langle 1,z\rangle, \langle 1,k\rangle\rangle^- : z \in \mathbb{Z} \}, \qquad \Xi''_k \stackrel{\text{def}}{=} \{ \langle\langle 0,0\rangle, \langle 0,z\rangle, \langle 1,k\rangle\rangle^- : z \in \mathbb{Z} \}$$

of Ξ_k satisfies condition (T3). We let $\mathfrak{B}'$ be the twisted atom structure and $\mathfrak{C}'$ its complex algebra.

Since $\xi_k \in E_{12} \cap \Xi'_k$ and $\Xi''_k \cap E_{01} \neq \emptyset$ for all $k \in I$, Corollary 2.27 can be applied to conclude that in $\mathfrak{Ra}\,\mathfrak{C}'$,

$$(\mathsf{c}_2\,\xi_k)^{\smile\prime} = (\mathsf{c}_2\,\xi_{\pi k})^{\smile} = (\mathsf{c}_2\,\xi_{-(k+1)})^{\smile} = \mathsf{c}_2\,\xi_{k+1}$$

for all $k \in I$. Hence there is a $\mathbb{Z}$-type chain

$$\ldots \xrightarrow{\smile} \mathsf{c}_2\,\xi_{-1} \xrightarrow{\smile} \mathsf{c}_2\,\xi_0 \xrightarrow{\smile} \mathsf{c}_2\,\xi_1 \xrightarrow{\smile} \ldots$$

in $\mathfrak{Ra}\,\mathfrak{C}'$. Therefore, letting $x = \sum_{k\in\omega} \mathsf{c}_2\,\xi_k$, by 1.11(v) we get $x^{\smile\prime\smile\prime} = \sum_{1<k\in\omega} \mathsf{c}_2\,\xi_k < x$ in $\mathfrak{Ra}\,\mathfrak{C}'$, as desired. $\square$

The algebra in the previous theorem is the first example of an "essentially infinite" non-representable CA_3.

THEOREM 3.7. *$\mathfrak{Ra}^*\,\mathsf{CA}_3 \subset \mathbf{S}\mathfrak{Ra}^*\,\mathsf{CA}_3$, i.e., $\mathfrak{Ra}^*\,\mathsf{CA}_3$ is not closed under taking subalgebras.*

PROOF. We use the fact that in $\mathfrak{Ra}^*\,\mathsf{CA}_3$, $^\smile$, being a Boolean automorphism by Lemma 1.20(i), is onto. Our plan is to find a $\mathfrak{C}'' \in \mathsf{CA}_3$, an element $x \in A$, where A is the universe of $\mathfrak{A} = \mathfrak{Ra}\,\mathfrak{C}''$, and a subalgebra $\mathfrak{D}$ of $\mathfrak{A}$ such that $x^\smile \in D$ but $x \notin D$. Clearly, a necessary condition on $\mathfrak{C}''$ for this to be possible is that *MGR* fails in it (but see Problem 3.10 below).

We start out with the algebra constructed in 3.6 (and use the notation introduced in its proof). In particular, $\mathfrak{B} = \mathfrak{At}\,\mathfrak{C} = \langle B, T_i, E_{ij}\rangle_{i,j<3}$ is the atom structure with $B = \{ s^- : s \in {}^3U \}$, $T_i = \{ \langle s^-, z^-\rangle \in {}^2B : (\exists f \in s^-)(\forall j < 3, j \neq i) f_j = z_j \}$ and $E_{ij} = \{ s^- \in B : s_i = s_j \}$, and $\mathfrak{B}' = \mathfrak{At}\,\mathfrak{C}' = \langle B, T'_i, E_{ij}\rangle_{i,j<3}$ is the same as $\mathfrak{B}$ except that T'_1 is the twisted version of T_1. Since G is transitive (i.e., $U^2 = \cup G$), we have $T_i \,|\, T_j = {}^2B$ for all distinct $i, j < 3$, and hence

(1) $$T'_i \,|\, T'_j = {}^2B$$

by (1) and (2) in the proof of Theorem 2.24. Let $\Psi = (B \setminus E_{12}) \times (B \setminus E_{02}) \times (B \setminus E_{01})$. We want to dilate $\mathfrak{B}'$ with all elements of Ψ, so that composition in the complex algebra of the dilated $\mathfrak{B}'$ (denoted by $\mathfrak{B}''$ below) will get "as big as possible". (The point is to make composition as few valued as possible, and this is one way to achieve it.) We can dilate with each $\psi \in \Psi$ since it satisfies conditions (D1) and (D2) by (1). The only problem is that we did not define dilation for arbitrary sets of sequences of atoms. One way around this is to enumerate Ψ as $\{ \psi_n : n \in \omega \}$, dilate $\mathfrak{B}'$ with ψ_0, then dilate the resulting Ca_3 with ψ_1 (this is possible by Lemma 2.21) etc., and finally to take the union $\mathfrak{B}''$ of the resulting chain. (Remark 3.8 below describes how one can dilate with an infinite set in one step.) Since the Ca_3-axioms are all equivalent to universal-existential sentences, $\mathfrak{B}'' \in \mathsf{Ca}_3$. Let $\mathfrak{C}'' = \mathfrak{Cm}\,\mathfrak{B}''$.

We are going to define a subalgebra of $\mathfrak{A} = \mathfrak{Ra}\,\mathfrak{C}''$ and show that it is not in $\mathfrak{Ra}^*\,\mathsf{CA}_3$, but first we point out some properties of $\mathfrak{A}$.

(2) 1' is an atom in $\mathfrak{A}$.

By lemma 1.19, to see that (2) holds, it suffices to show that $1' = \mathsf{d}_{01} = \mathsf{c}''_2\,a$ for some $a \in B''$, where c''_2 denotes cylindrification in $\mathfrak{C}''$. Equivalently, we need to show that $E_{01} = a/T''_2$ for some $a \in B''$. Let a be the atom $\langle\langle 0,0\rangle, \langle 0,0\rangle, \langle 0,0\rangle\rangle^-$ of $\mathfrak{C}$ (and hence of $\mathfrak{C}'$ and $\mathfrak{C}''$). By transitivity of G, and since $T'_2 = T_2$,

$E_{01} = a/T_2 = a/T_2'$. Since $T_2'' \supseteq T_2'$, $E_{01} \subseteq a/T_2''$. Conversely, since $a \in E_{01}$, $a/T_2'' \subseteq T_2''^{*} E_{01} = E_{01}$ by 2.10(ii).

Next,

$$(3') \qquad (\forall a, b, c \in B'' \setminus E_{01})\big(a \in E_{12} \wedge b \in E_{02} \to C''(a, b, c)\big),$$

for if a, b, c are as in (3'), then, by 2.6(iv), $b \notin E_{12}$, $a \notin E_{02}$ and $c \notin E_{01}$ whence $\psi = \langle b, a, c\rangle \in \Psi$, and ν_ψ (the new atom we got by dilating with ψ) witnesses $C''(a, b, c)$ (cf. Definition 2.8). By Lemmas 2.9 and 2.10(i), (3') implies

$$(3) \qquad \text{if } x, y \notin \{0, 1'\}, \text{ then } x \,;\, y \geq -1'$$

in $\mathfrak{A}$.

Now we are ready to define the subalgebra $\mathfrak{D}$ of $\mathfrak{A}$ that is not the RA-reduct of any CA_3. Throughout the rest of the proof we drop the double prime superscripts from the operations of $\mathfrak{C}''$ and $\mathfrak{A}$. Let

$$X = \{\, \mathsf{c}_2\, \xi_k : k \geq 0 \,\} \cup \{1'\} \cup \{w\},$$

where $w = -\big(1' + \sum\{\, \mathsf{c}_2\, \xi_k : k \geq 0 \,\}\big)$ (and the operations are computed in $\mathfrak{C}''$) and let

$$D = \{\, \textstyle\sum Y : Y \subseteq X \,\}.$$

We claim that

$$(4) \qquad D \text{ is a subuniverse of } \mathfrak{A}.$$

D is clearly closed under the Boolean operations, and $1' \in D$. To see that it is closed under converse and composition, it is enough to check that $x^{\smile}$ and $x\,;y$ (computed, of course, in $\mathfrak{A}$) are in D whenever $x, y \in X$, because these operations are completely additive by 1.6(iii) and 1.11(v). Now $(\mathsf{c}_2\, \xi_k)^{\smile} = \mathsf{c}_2\, \xi_{k+1}$ because dilation does not touch converse (Lemma 2.16) and $1'^{\smile} = 1'$ (cf. Lemma 1.20(iii)); and since $^{\smile}$ is a Boolean automorphism in $\mathfrak{A}$ (Lemma 1.20(i)), these imply

$$\begin{aligned} w^{\smile} &= \big[-\big(1' + \textstyle\sum\{\, \mathsf{c}_2\, \xi_k : k \geq 0 \,\}\big)\big]^{\smile} = -\big(1' + \textstyle\sum\{\, \mathsf{c}_2\, \xi_k : k \geq 0 \,\}\big)^{\smile} \\ &= -\big(1'^{\smile} + \textstyle\sum\{\, \mathsf{c}_2\, \xi_k{}^{\smile} : k \geq 0 \,\}\big) = -\big(1' + \textstyle\sum\{\, \mathsf{c}_2\, \xi_k : k \geq 1 \,\}\big) = w + \mathsf{c}_2\, \xi_0 \in D. \end{aligned}$$

That composition of members of X is in D can be seen as follows. Let $x, y \in X$. If $\{x, y\} \cap \{0, 1'\} \neq 0$, then $x\,; y \in \{0, x, y\}$, and if $\{x, y\} \cap \{0, 1'\} \neq 0$, then $x\,; y \in \{1, -1'\}$ by (3) and (2). Since 0, 1 and $-1' = w + \sum\{\, \mathsf{c}_2\, \xi_k : k \geq 0 \,\}$ are in D, this shows that D is indeed closed under composition, and thus the proof of (4) is complete.

Let $\mathfrak{D}$ be the subalgebra of $\mathfrak{A}$ with universe D. It remains to check that $\mathfrak{D} \notin \mathfrak{Ra}^{*}\,\mathsf{CA}_3$. Converse is injective in members of $\mathfrak{Ra}^{*}\,\mathsf{CA}_3$ (by 1.20(i)), and $(\mathsf{c}_2\, \xi_{-1})^{\smile} = \mathsf{c}_2\, \xi_0$ in $\mathfrak{A}$, but $\mathsf{c}_2\, \xi_{-1} \notin D$, so converse is not onto in $\mathfrak{D}$. By 1.20(i) this implies that $\mathfrak{D} \notin \mathfrak{Ra}^{*}\,\mathsf{CA}_3$. □

REMARK 3.8. The "infinite" dilation used in the construction in Theorem 3.7 can be described more directly, as follows. Given a Ca_α $\mathfrak{B} = \langle B, T_i, E_{ij}\rangle_{i,j<\alpha}$ and a set $\Psi \subseteq {}^{\alpha}B$ such that each $\psi \in \Psi$ satisfies conditions (D1) and (D2), let the result of dilating with Ψ be the structure $\mathfrak{B}^{\Psi} = \langle B^{\Psi}, T_i^{\Psi}, E_{ij}^{\Psi}\rangle_{i,j<\alpha}$, where $B^{\Psi} \stackrel{\text{def}}{=} B \cup \{\, \nu_\psi : \psi \in \Psi \,\}$ (here the ν_ψ's are assumed to be distinct and not in B), $E_{ii}^{\Psi} \stackrel{\text{def}}{=} \mathfrak{B}^{\Psi}$ and $E_{ij}^{\Psi} \stackrel{\text{def}}{=} E_{ij}$ if $i \neq j$. To define T_i^{Ψ} it is convenient to introduce the following notation: for $a \in B^{\Psi}$ and $i < \alpha$,

$$|a|_i = \begin{cases} a & \text{if } a \in B \\ \psi_i & \text{if } a = \nu_\psi. \end{cases}$$

Then we let $a\ T_i^\Psi\ b \overset{\text{def}}{\iff} |a|_i\ T_i\ |b|_i$. It is not hard to show that $\mathfrak{B}^\Psi$ is then isomorphic to the structure we get by dilating with the elements of Ψ one by one, and then taking the union of the resulting chain (as in the proof of 3.7). □

REMARK 3.9. We describe the algebras in 3.6 and 3.7 more directly.

Let U be $2\times\mathbb{Z}$, G the group of permutations of U generated by suc and f (defined in 3.6) and let $\mathcal{R}$ be the set of equivalence relations on 3 with at most 2 equivalence classes. For $k,l\in\mathbb{Z}$ and $R\in\mathcal{R}$, let

$$a^R_{k,l} = \big\langle\langle 0,0\rangle,\langle i_1,k\rangle,\langle i_2,l\rangle\big\rangle^-,$$

where

$$i_j = \begin{cases} 0 & \text{if } 0Rj \\ 1 & \text{otherwise} \end{cases} \qquad \text{for } j\in\{1,2\}.$$

Note that $a^R_{k,l}\cap a^{R'}_{k',l'}\neq\emptyset$ implies $a^R_{k,l}=a^{R'}_{k',l'}$, which, in turn, implies $\langle R,k,l\rangle=\langle R',k',l'\rangle$. The first implication holds because $a^R_{k,l}$ and $a^{R'}_{k',l'}$, being orbits of the action of G on ${}^3U$, are either disjoint or coincide, and the second implication is true because the identity is the only element of G which fixes a point in U; hence if $g\big\langle\langle 0,0\rangle,\langle i_1,k\rangle,\langle i_2,l\rangle\big\rangle=\big\langle\langle 0,0\rangle,\langle i'_1,k'\rangle,\langle i'_2,l'\rangle\big\rangle$ for some $g\in G$, then $g=\mathrm{Id}_U$ and thus $k'=k$, $l'=l$ and $i'_j=i_j$ for $j=1,2$. From the latter $R=R'$ follows.

Next, $\cup\big\{a^R_{k,l} : R\in\mathcal{R}\ \text{ and }\ k,l\in\mathbb{Z}\big\}={}^3U$, because of the following. If $s\in{}^3U$, say $s=\big\langle\langle i_0,z_0\rangle,\langle i_1,z_1\rangle,\langle i_2,z_2\rangle\big\rangle$ with $z_j\in\mathbb{Z}$ and $i_j\in 2$ (for all $j\in 3$), then, setting $R=\big\{\langle j,j'\rangle : i_j=i_{j'}\big\}$ and chosing i'_1 and i'_2 such that $\{i_1,i'_1\}=\{i_2,i'_2\}=2$, we see that

$$s\equiv_G \begin{cases} \big\langle\langle 0,0\rangle,\langle i_1,z_1-z_0\rangle,\langle i_2,z_2-z_0\rangle\big\rangle & (\text{if } i_0=0) \\ \big\langle\langle 0,0\rangle,\langle i'_1,z_1-z_0\rangle,\langle i'_2,z_2-z_0\rangle\big\rangle & (\text{if } i_0=1) \end{cases} \in a^R_{z_1-z_0,z_2-z_0}.$$

From these and similarly simple observations it follows that the atom structure of $\mathfrak{C}=\mathfrak{Fr}_G\mathfrak{A}$ is $\mathfrak{B}=\langle B,T_i,E_{ij}\rangle_{i,j<3}$, where

$$B=\{\,a^R_{k,l} : R\in\mathcal{R}\quad\text{and}\quad k,l\in\mathbb{Z}\,\},$$

$$a^R_{k,l}\ T_i\ a^{R'}_{k',l'} \iff R\restriction 3\setminus\{i\}=R'\restriction 3\setminus\{i\}\quad\text{and}\quad \begin{cases} k-l=k'-l' & \text{if } i=0 \\ l=l' & \text{if } i=1 \\ k=k' & \text{if } i=2, \end{cases}$$

and if $i,j<3$ are distinct, then

$$a^R_{k,l}\in E_{ij}\iff iRj\quad\text{and}\quad \begin{cases} k=l & \text{if } \{i,j\}=\{1,2\} \\ l=0 & \text{if } \{i,j\}=\{0,2\} \\ k=0 & \text{if } \{i,j\}=\{0,1\}. \end{cases}$$

The atom ξ_k ($k\in I=\mathbb{Z}$) of 3.6 and 3.7 is $a^{\{\{0\},\{1,2\}\}}_{k,k}$ in this notation, and the partition of $\Xi_k=\xi_k/T_1=\big\{a^R_{l,k} : \langle 0,2\rangle\notin R\in\mathcal{R}\ \text{ and }\ l\in\mathbb{Z}\big\}$ is

$$\Xi'_k=\Big\{a^{\{\{0\},\{1,2\}\}}_{l,k} : l\in\mathbb{Z}\Big\},\qquad \Xi''_k=\Big\{a^{\{\{0,1\},\{2\}\}}_{l,k} : l\in\mathbb{Z}\Big\}.$$

□

The example in Theorem 3.7 motivates the following questions.

PROBLEM 3.10. Is $\mathfrak{Ra}^*\big(\mathsf{CA}_3 \cap \mathrm{Mod}(MGR)\big)$ closed under taking subalgebras? If the answer to this question is positive, then it would be interesting to see whether $\mathbf{S}\mathfrak{Ra}^*\big(\mathsf{CA}_3 \cap \mathrm{Mod}(FO)\big) = \mathfrak{Ra}^*\big(\mathsf{CA}_3 \cap \mathrm{Mod}(FO)\big)$, where FO (for "finite orbits") is the formula $\forall x(\exists n \in \omega)x^{\smile n} = x$, and, in particular, whether the class of RA-reducts of finite CA_3's is closed under taking subalgebras.

In Chapter 5 below we will introduce the schema MGR^+, a strengthening of MGR. We note that $\mathfrak{Ra}^*\big(\mathsf{CA}_3 \cap \mathrm{Mod}(MGR^+)\big)$ is closed under taking subalgebras by Theorem 5.9, but, as shown by Theorem 4.10, 5.9 does not hold if we replace MGR^+ with MGR in it.

3.2. All CA_3's by relativization and twisting

So far, twisting was used for destroying the MGR equation in CA's. In the next theorem it will be put to a more constructive use.

THEOREM 3.11. $\mathsf{CA}_3 \subseteq \mathbf{S}\,\mathbf{Rl}^{ca}\,\mathbf{Tw}\,\mathbf{Srl}^{ca}\,\mathsf{Cs}_3$. *That is, for every* $\mathfrak{A} \in \mathsf{CA}_3$ *there are* $\mathfrak{A}_1, \mathfrak{A}_2, \mathfrak{A}_3 \in \mathsf{CA}_3$ *and* $\mathfrak{A}_4 \in \mathsf{Cs}_3$ *such that* $\mathfrak{A}_3 \subseteq \mathfrak{Rl}\,\mathfrak{A}_4$*,* $\mathfrak{A}_2$ *is a twisted version of* $\mathfrak{A}_3$*,* $\mathfrak{A}_1 = \mathfrak{Rl}\,\mathfrak{A}_2$*, and* $\mathfrak{A} \subseteq \mathfrak{A}_1$.

PROOF. The plan of the proof is as follows. We start with a CA_3 $\mathfrak{A}$, and see if MGR holds in it. If yes, then $\mathfrak{A}$ is a subalgebra of a relativized Cs_3 by the Resek-Thompson 2.29. Otherwise we embed $\mathfrak{A}$ in its canonical embedding algebra $\mathfrak{A}^\sigma$ ($\mathfrak{A}_1$ in the formulation of the theorem) in order to be able to repair the failure of MGR by twisting its atomstructure. ($\mathfrak{A}_1$ is complete and atomic, and is in CA_3 by [HMT71] 2.7.5 and 2.7.15.) It will be clear how to choose the parameters in twisting but we have to apply dilation first to be able to satisfy the conditions of Definition 2.23. This is where the first "$\mathbf{Rl}^{ca}$" in the theorem comes in: $\mathfrak{A}^\sigma$ can be gotten back by relativizing the dilated algebra $\mathfrak{A}_2$ with the top element (i.e., the sum of the atoms) of $\mathfrak{A}^\sigma$. Next we apply twisting to the dilated algebra and get a CA_3 $\mathfrak{A}_3$ in which MGR holds and use the Resek-Thompson theorem to represent the latter as an $\mathbf{Srl}^{ca}\,\mathsf{Cs}_3$. Since the effect of twisting can always be undone by twisting the twisted algebra, the procedure we have described now will indeed show that $\mathfrak{A}$ can be gotten from a subalgebra of a relativized Cs_3 by applying twisting, relativization and the operation of taking subalgebras.

From this plan it is clear that the hard part of the proof is to dilate and twist a complete and atomic CA_3 in such a way that the resulting algebra satisfies MGR. So let $\mathfrak{A} \in \mathsf{CA}_3$ be complete and atomic, and let $\mathfrak{B}$ be its atom structure. By Lemma 1.13(ii) we can concentrate on MGR_2 (but for simplicity, we will write MGR nevertheless). By 2.10(iv), in every Ca_3, K_{01} is a permutation of E_{12} (we will use the more suggestive notation $^\smile$ for K_{01} in the rest of this proof). Hence the algebra $\langle E_{12}, ^\smile \rangle$ is the disjoint union of "cycles", that is, of its 1-generated subalgebras, each of which is isomorphic to a mono-unary algebra $\langle A, f \rangle$, where A is either the set of integers and f is the successor function, or $A = n = \{0, \dots, n-1\}$ for some $n \in \omega$ and f is the successor function modulo n. Clearly, MGR is valid in the complex algebra of a Ca_3 iff all the cycles are of length at most 2. With Corollary 2.27 in mind, we can try to bring this about by twisting T_1 the following way. Let us first concentrate on one cycle X of length at least 3, and enumerate it as $\langle \xi_i : i \in n \rangle$ or $\langle \xi_i : i \in \mathbb{Z} \rangle$ in such a way that $\xi_i^\smile = \xi_{i+1}$, where $i \in n$ and $+$ denotes addition $\mod n$ or $i \in \mathbb{Z}$, depending on whether $|X| = n \in \omega$ or infinite. So X looks either like

$$\xi_0 \xrightarrow{\smile} \xi_1 \xrightarrow{\smile} \dots \xrightarrow{\smile} \xi_{n-1} \xrightarrow{\smile} \xi_0$$

or

$$\dots \xrightarrow{\smile} \xi_{-1} \xrightarrow{\smile} \xi_0 \xrightarrow{\smile} \xi_1 \xrightarrow{\smile} \dots$$

Let $t=1$, let $I_X = I$ be $|X|$ if X is finite and $\mathbb{Z}$ otherwise, and let $\pi_X = \pi$ be the permutation $i \mapsto -i$ (computed $\mod n$ if $|X| = n$) of I. Note that condition (T1) of Definition 2.23 holds by (v) of 2.6, and condition (T2) holds, for if there were an $i \in I$ with $\xi_i \in E_{02}$, then $\xi_i \in E_{01}$ (by (v) of 2.6), whence $\xi_i{}^{\smile} = \xi_i$ (by the definition of K_{01} and 2.6(v)), and this would imply $i=0$ and $|X|=1$. So if we can partition each $\Xi_i = \xi_i/T_1$ into two parts Ξ_i' and Ξ_i'' in such a way that $\xi_i \in \Xi_i'$ and $\mathsf{d}_{01}\cdot\mathsf{c}_1\,\xi_i \in \Xi_i''$ for all $i \in I$, and the partition satisfies (T3), then, in the the twisted atomstructure, we have

$$\xi_i{}^{\smile\prime\smile\prime} = (\xi_{\pi i}{}^{\smile})^{\smile\prime} = \xi_{1+\pi i}{}^{\smile\prime} = \xi_{\pi(1+\pi i)}{}^{\smile} = \xi_{1+\pi(1+\pi i)} = \xi_i$$

by Corollary 2.27 and $1-(1-i)=i$. Hence *MGR* holds for all elements of X in the twisted structure.

Since the cycles (and hence the T_1-classes of their elements) are disjoint, it is clear that if for each cycle X of length at least 3 there are partitions of ξ/T_1 (for all $\xi \in X$) as above, then we can repair all cycles simultaneously, with one twisting. We just let I be the disjoint union of $\{I_X : X \text{ is a cycle of length at least } 3\}$ (of course we have to make the I_X's disjoint, but that's not a problem), and π the disjoint union of $\{\pi_X : X \text{ is a cycle of length at least } 3\}$. The problem is that there may not be such partitions in $\mathfrak{B}$ itself: so what we will do is dilate $\mathfrak{B}$ first to get a bigger Ca_3 $\mathfrak{B}^+ = \langle B^+, T_i^+, E_{ij}^+\rangle_{i,j<3}$ (with "enough atoms" in ξ/T_1^+ when ξ is in one of the cycles, so that condition (T3) can be satisfied) and apply the above procedure to $\mathfrak{B}^+$ instead. Note that by the definition of dilation and lemma 2.16, $\langle E_{12}^+, {}^{\smile +}\rangle = \langle E_{12}, {}^{\smile}\rangle$ (where ${}^{\smile +}$ denotes ${}^{\smile}$ in $\mathfrak{B}^+$).

Next, if for each cycle X, we can dilate $\mathfrak{B}$ in such a way that all new atoms are T_1^+-related to some $\xi \in X$, and X can be repaired in the dilated atom structure by twisting (i.e. if we can partition ξ/T_1^+ as above), then we can dilate $\mathfrak{B}$ in such a way that all cycles can be repaired by twisting. For, if X and Y are different cycles, then dilating in order to be able to repair X doesn't affect the twisting conditions on Y. This is clear for (T1) (since $Y \subseteq B$ and $T_1^+ \cap {}^2B = T_1$) and for (T2) (since dilation does not change E_{02}). That (T3) is unaffected can be seen as follows: all new atoms are T_1^+-related to some $\xi \in X$ and $\langle \zeta, \xi\rangle \in {}^2B \setminus T_1$ (so $\langle \zeta, \xi\rangle \notin T_1^+$), and thus $\zeta/T_1^+ = \zeta/T_1$ for all $\zeta \in Y$. In particular, $\zeta/T_1^+ \subseteq B$. But then

$$T_j^+ \cap (\zeta/T_1^+ \times \zeta'/T_1^+) = T_j^+ \cap (\zeta/T_1 \times \zeta'/T_1) = T_j \cap (\zeta/T_1 \times \zeta'/T_1)$$

for $j = 0, 2$ and all $\zeta, \zeta' \in Y$.

So from now on we only have to deal with one cycle X (but have to dilate carefully, i.e. make sure that all new atoms are T_1^+-related to some element of X). In the rest of the proof we will use the notation introduced in Remark 3.8

Claim 1. Suppose $\xi_0, \xi_1 \in E_{12} \setminus E_{01}$ are distinct and $\xi_0{}^{\smile\smile} = \xi_1$ in some $\mathfrak{B} \in \mathsf{Ca}_3$. Then there is a subset $\Psi \subseteq {}^3B$ such that

(1) $$\cup\{\operatorname{ran}\psi : \psi \in \Psi\} \subseteq \xi_0/T_1 \cup \xi_1/T_1,$$

(2) *for each $\psi \in \Psi$, if $\psi_0 \neq \psi_2$, then $\langle \psi_0, \psi_2\rangle \in E_{01} \times E_{12}$*

(3) *each $\psi \in \Psi$ satisfies conditions* (D1) *and* (D2) *of Definition 2.13*

and in the atom structure resulting from dilating $\mathfrak{B}$ by Ψ there are partitions $\xi_0/T_1 = \Xi_0 = \Xi_0' \cup \Xi_0''$ and $\xi_1/T_1 = \Xi_1 = \Xi_1' \cup \Xi_1''$ such that

(4) $$\xi_k \in \Xi_k' \qquad \textit{and} \qquad \mathsf{d}_{01}\cdot\mathsf{c}_1\,\xi_k \in \Xi_k'',$$

for $k \in \{0,1\}$, *and* (T3) *holds, that is,*

$$\begin{aligned} &\operatorname{dom} T_j \cap (\Xi_0' \times \Xi_1') \supseteq \Xi_0', &&\operatorname{ran} T_j \cap (\Xi_0' \times \Xi_1') \supseteq \Xi_1', \\ &\operatorname{dom} T_j \cap (\Xi_0'' \times \Xi_1'') \supseteq \Xi_0'', &&\operatorname{ran} T_j \cap (\Xi_0'' \times \Xi_1'') \supseteq \Xi_1'' \end{aligned} \tag{5}$$

for $j \in \{0,2\}$.

Before proving the claim we show that it implies the theorem. Let X be a cycle of length > 2. Recall that the permutation π we chose is a composition of disjoint transpositions $(i,-i)$; this means that inserting new atoms in T_1-classes of ξ_i and $\xi_{\pi i}$ in order to be able to satisfy condition (T3) for i will leave all the other T_1-classes, as well as the conditions on $\xi_j/T_1, \xi_{\pi j}/T_1$ $(j \notin \{i,-i\})$, alone. (This is the same argument we used before for showing that different cycles can be treated independently.) Hence, if

(†) for every $i > 0$ (or $0 < i < \lceil |X|/2 \rceil$ if X is finite) there is a $\Psi_i \subseteq {}^3B$ such that dilating $\mathfrak{B}$ with Ψ_i we get a Ca_3 in which every new atom is T_1-related to either ξ_i or $\xi_{\pi i}$, and the T_1-classes of ξ_i and $\xi_{\pi i}$ can be partitioned to $\Xi_i' \cup \Xi_i''$ and $\Xi_{\pi i}' \cup \Xi_{\pi i}''$ in such a way that (T3) holds and $\xi_i \in \Xi_i'$,

then, dilating $\mathfrak{B}$ with $\cup\{\Psi_i : 0 < i\}$ (or $\cup\{\Psi_i : 0 < i < \lceil |X|/2 \rceil\}$) we get a Ca_3 that can be twisted as above, i.e., so that in the twisted structure $\xi^{\smile\smile} = \xi$ for all $\xi \in X$. We prove (†) by induction on i. [1]

More precisely, we prove by induction that for every $0 < i$ or $0 < i < \lceil |X|/2 \rceil$ (depending on whether $|X|$ is infinite or not) there is a set $\Psi_i \subseteq {}^3B$ satisfying the conjunction of the following five statements:

$$\cup\{\operatorname{ran}\psi : \psi \in \Psi_i\} \subseteq \xi_i/T_1 \cup \xi_{\pi i}/T_1, \tag{1'}$$

$$\text{for each } \psi \in \Psi_i, \text{ if } \psi_0 \neq \psi_2, \text{ then } \langle \psi_0, \psi_2 \rangle \in E_{01} \times E_{12} \tag{2'}$$

$$\text{each } \psi \in \Psi_i \text{ satisfies conditions (D1) and (D2) of Definition 2.13} \tag{3'}$$

and in the atom structure resulting from dilating $\mathfrak{B}$ by Ψ_i there are partitions $\xi_i/T_1 = \Xi_i = \Xi_i' \cup \Xi_i''$ and $\xi_{\pi i}/T_1 = \Xi_{\pi i} = \Xi_{\pi i}' \cup \Xi_{\pi i}''$ such that

$$\xi_k \in \Xi_k' \qquad \text{and} \qquad \mathsf{d}_{01} \cdot \mathsf{c}_1\, \xi_k \in \Xi_k'', \tag{4'}$$

for $k \in \{i, \pi i\}$, and

$$\begin{aligned} &\operatorname{dom} T_j \cap (\Xi_i' \times \Xi_{\pi i}') \supseteq \Xi_i', &&\operatorname{ran} T_j \cap (\Xi_i' \times \Xi_{\pi i}') \supseteq \Xi_{\pi i}', \\ &\operatorname{dom} T_j \cap (\Xi_i'' \times \Xi_{\pi i}'') \supseteq \Xi_i'', &&\operatorname{ran} T_j \cap (\Xi_i'' \times \Xi_{\pi i}'') \supseteq \Xi_{\pi i}'' \end{aligned} \tag{5'}$$

for $j \in \{0,2\}$.

For $i = 1$ we have $\xi_{-1}{}^{\smile\smile} = \xi_1$, so the claim can be applied to get some $\Psi_1 \subseteq {}^3B$ satisfying (1')–(5'). Suppose now that the induction hypothesis holds for some $i \geq 1$. By (4') and (5'), in the dilated atom structure $\mathfrak{B}_i = \mathfrak{B}^{\Psi_i}$, the T_1-classes of ξ_i and $\xi_{\pi i}$ can be twisted is such a way that (denoting by ${}^{\smile\prime}$ the operation ${}^\smile$ in the twisted version $\mathfrak{B}_i'$)

$$\xi_{\pi(i+1)}{}^{\smile\prime\smile\prime} = \xi_{-(i+1)}{}^{\smile\prime\smile\prime} = \xi_{-(i+1)}{}^{\smile\smile\prime} = \xi_{-i}{}^{\smile\prime} = \xi_i{}^{\smile} = \xi_{i+1}$$

[1] In the proof of (†), we will dilate and twist $\mathfrak{B}$ in every step, but only because this is a simple way to satisfy the assumptions of the claim. The crucial point in the proof is transferring the conclusions of the claim back from the dilated and twisted structure to $\mathfrak{B}$. We point this out because it can be rather confusing that $\mathfrak{B}$ is dilated and twisted just in order to let us conclude that $\mathfrak{B}$ can be dilated in such a way that the dilated structure satisfies the twisting conditions — which is precisely what is happening here.

by 2.27 and 2.16. The claim (applied to $\xi_{-(i+1)}$ and ξ_{i+1} in $\mathfrak{B}'_i$) thus implies that there is a $\Psi \subseteq {}^3B'_i = {}^3B_i$ satisfying (1)–(5) in the claim. We complete the induction by showing that $\Psi_{i+1} \stackrel{\text{def}}{=} \Psi$ satisfies (1')–(5') in $\mathfrak{B}$. In the proof, T_k, T_k^+ and $T_k^{+\prime}$ denote the appropriate relations in $\mathfrak{B}$, $\mathfrak{B}_i$ and $\mathfrak{B}'_i$, respectively.

First we note that

$$(*) \qquad \xi_{i+1}/T_1^{+\prime} = \xi_{i+1}/T_1 \quad \text{and} \quad \xi_{-(i+1)}/T_1^{+\prime} = \xi_{-(i+1)}/T_1$$

because of the following. Let $a \in B'_i = B_i$. Then $a\ T_1^{+\prime}\ \xi_{i+1}$ iff $a\ T_1^+\ \xi_{i+1}$, because $\langle \xi_{i+1}, \xi_i\rangle, \langle \xi_{i+1}, \xi_{-i}\rangle \in {}^2B \setminus T_1 = {}^2B \setminus T_1^+$, whence the T_1^+-class of ξ_{i+1} was left alone when we twisted $\mathfrak{B}_i$. So we only have to show that $a\ T_1\ \xi_{i+1}$ holds iff $a\ T_1^+\ \xi_{i+1}$ does. Since $T_1 \subseteq T_1^+$, the "only if" direction is obvious. Conversely, if $a\ T_1^+\ \xi_{i+1}$, then $a \in B$ and hence $a\ T_1\ \xi_{i+1}$, for if $a = \nu_\varphi$ for some $\varphi \in \Psi_i$, then $\xi_{i+1}\ T_1\ \varphi_1 \in \xi_i/T_1 \cup \xi_{-i}/T_1$ (by (1') for i), contradicting $\langle \xi_{i+1}, \xi_i\rangle, \langle \xi_{i+1}, \xi_{-i}\rangle \notin T_1$ again. This proves the first part of (*), and the proof of the second part is the same.

From (*), (1') follows easily. For, if $\psi \in \Psi_{i+1}$ and $k < 3$ then

$$\psi_k \in \xi_{i+1}/T_1^{+\prime} \cup \xi_{-(i+1)}/T_1^{+\prime} = \xi_{i+1}/T_1 \cup \xi_{-(i+1)}/T_1$$

by (1) and (*). This proves (1') for $i+1$, and in particular, we got that $\Psi_{i+1} \subseteq {}^3B$.

Next, (2') obviously holds by (2), since E_{01}, E_{12} are the same in $\mathfrak{B}'_i$ as in $\mathfrak{B}$. For the same reason, only (D2) needs checking in (3'). Let $j, k < 3$ be different and let $\psi \in \Psi_{i+1}$. By (3), $\psi_j\ T_j^{+\prime} \,|\, T_k^{+\prime}\ \psi_k$ and we have to show that this holds already in $\mathfrak{B}$, that is, $\psi_j\ T_j \,|\, T_k\ \psi_k$. Now if $1 \in \{j, k\}$, say $j = 1$, then we are done because $\psi_1\ T_1^{+\prime}\ a\ T_k^{+\prime}\ \psi_k$ implies $\psi_1\ T_1\ a$ by (1') and (*); hence $a \in B$, and since $T_k^{+\prime} = T_k^+$ and $\psi_k \in B$, we get $\psi_1\ T_1\ a\ T_k\ \psi_k$. Suppose now that $\{j, k\} = \{0, 2\}$. If $\psi_0 = \psi_2$, then, since it is in B, obviously $\psi_0\ T_0 \,|\, T_2\ \psi_2$. Otherwise we have $\psi_0 \in E_{01}$ and $\psi_2 \in E_{12}$ by (2'). Since $\langle \psi_0, \psi_2\rangle \in T_2^{+\prime} \,|\, T_0^{+\prime} = T_2^+ \,|\, T_0^+$ (only T_1^+ was twisted), there is an $a \in B_i$ such that $\psi_0\ T_2^+\ a\ T_0^+\ \psi_2$. But then a must be in E_{01} (in fact $E_{01} \cap E_{12}$) by 2.10(ii), whence $a \in B$ and we get $\psi_0\ T_2\ a\ T_0\ \psi_2$ and hence $\psi_0\ T_0 \,|\, T_2\ \psi_2$ as desired.

It remains to check (4') and (5'). But this is easy: we know that in $\mathfrak{B}'_i$ there are partitions $\Xi'_{i+1} \cup \Xi''_{i+1}$ of $\xi_{i+1}/T_1^{+\prime}$ and $\Xi'_{-(i+1)} \cup \Xi''_{-(i+1)}$ of $\xi_{-(i+1)}/T_1^{+\prime}$ which satisfy (4) and (5), and by (*), these partitions exist already in $\mathfrak{B}$. They must satisfy (5'), because if $j \in \{0, 2\}$, then

$$T_j^{+\prime} \cap (\Xi'_{i+1} \times \Xi'_{-(i+1)}) = T_j^+ \cap (\Xi'_{i+1} \times \Xi'_{-(i+1)}) = T_j \cap (\Xi'_{i+1} \times \Xi'_{-(i+1)})$$

by $j \neq 1$ and $\Xi'_{i+1} \cup \Xi'_{-(i+1)} \subseteq B$, and similarly,

$$T_j^{+\prime} \cap (\Xi''_{i+1} \times \Xi''_{-(i+1)}) = T_j^+ \cap (\Xi''_{i+1} \times \Xi''_{-(i+1)}) = T_j \cap (\Xi''_{i+1} \times \Xi''_{-(i+1)})$$

by $j \neq 1$ and $\Xi''_{i+1} \cup \Xi''_{-(i+1)} \subseteq B$. Finally, (4') is direct consequence of (4) and (*).

Proof of Claim 1. Let $\xi_0, \xi_1 \in E_{12} \setminus E_{01}$ be distinct and suppose that $\xi_0^{\smile\smile} = \xi_1$. By the definition of K_{01} we have atoms $a_{ij}, b_{ij} \in E_{ij}$ such that

$$(6) \qquad \xi_0\ T_1\ a_{01}\ T_0\ a_{02}\ T_2\ a_{12} = b_{12}\ T_1\ b_{01}\ T_0\ b_{02}\ T_2\ \xi_1$$

in $\mathfrak{B}$. Let c_{01} denote the unique element of E_{01} which is T_1 related to ξ_1 (cf. Lemma 2.10(i)). Then we have

$$(7) \qquad \xi_0\ T_0\ \xi_1 \qquad \text{and} \qquad a_{01}\ T_2\ c_{01}.$$

To see that the first statement in (7) holds we argue as follows. Let a be the unique T_0 neighbour of ξ_0 in E_{01}. Then $a \in E_{01} \cap E_{12} = E_{01} \cap E_{12} \cap E_{02}$ by 2.10(ii) and 2.6(iv), and $a\ T_0 \,|\, T_1\ a_{02}$ because of 2.6(ii), (i). Hence there is an a' with $a\ T_0\ a'$ and $a'\ T_1\ a_{02}$; from the latter we get $a' \in E_{02}$ by 2.10(ii) and thus $a' = a$ by 2.6(v). We conclude that $a\ T_1\ a_{02}$. Proceeding in this fashion (using (6) with 2.6(ii), (i) and 2.10(ii), 2.6(v)) we get

$$\begin{array}{ll} a\,(T_1\,|\,T_2)\,b_{01} & a\,T_1\,a_{02}\,T_2\,b_{12}\,T_1\,b_{01} \\ a\,T_2\,b_{01} & a, b_{01} \in E_{01} \\ a\,(T_0\,|\,T_2)\,\xi_1 & a\,T_2\,b_{01}\,T_0\,b_{02}\,T_2\,\xi_1 \\ a\,T_0\,\xi_1 & a, \xi_1 \in E_{12} \end{array}$$

and hence $\xi_0\ T_0\ \xi_1$ by 2.6(i). This proves the first part of (7) and the proof of the second part is similar.

(8) $$\xi_0\,(T_0\,|\,T_2)\,a_{01} \iff \xi_1\,(T_0\,|\,T_2)\,c_{01}$$

This follows from (7), for, assuming the left hand side of (8) we have $\xi_1\ T_0\ \xi_0\ (T_0\,|\,T_2)\ a_{01}\ T_2\ c_{01}$ by (7), and this gives the right hand side by 2.6(ii). The proof of the reverse implication is similar.

Now we consider two possibilities.

Case 1. $\neg\xi_0\,(T_0\,|\,T_2)\,a_{01}$

In this case we do not have to dilate, that is, we can chose $\Psi = \emptyset$. For $i = 0, 1$, let $\Xi_i = \xi_i/T_1$. Before partitioning Ξ_0 and Ξ_1, we show the following:

(9) $$\operatorname{dom} T_j \cap (\Xi_0 \times \Xi_1) \supseteq \Xi_0, \qquad \text{and} \qquad \operatorname{ran} T_j \cap (\Xi_0 \times \Xi_1) \supseteq \Xi_1,$$

for $j \in \{0, 2\}$. For, if $x \in \Xi_0$, then $x\ T_1\ \xi_0\ T_0\ \xi_1$ and $x\ T_1\ a_{01}\ T_2\ c_{01}\ T_1\ \xi_1$ by (7), and hence $x\,(T_0\,|\,T_1)\,\xi_1$ and $x\,(T_2\,|\,T_1)\,\xi_1$ by 2.6(ii). Hence there are $y, z \in \Xi_1$ with $x\ T_0\ y$ and $x\ T_2\ z$. The proof of the other statement in (9) is similar.

Now let $\Xi_i' = \{x \in \Xi_i : \xi_i\,(T_0\,|\,T_2)\,x\}$ and $\Xi_i'' = \Xi_i \setminus \Xi_i'$ for $i \in \{0, 1\}$. Since Ψ is empty, (1)–(3) are obviously satisfied. As for (4), note that from the assumption $\neg\xi_0\,(T_0\,|\,T_2)\,a_{01}$, $\neg\xi_1\,(T_0\,|\,T_2)\,c_{01}$ follows by (8). Hence, by 2.10(i) and the definition of the partitions, $\mathsf{d}_{01} \cdot \mathsf{c}_1\,\xi_0 = a_{01} \in \Xi_0''$ and $\mathsf{d}_{01} \cdot \mathsf{c}_1\,\xi_1 = c_{01} \in \Xi_1''$. Since $\xi_0 \in \Xi_0'$ and $\xi_1 \in \Xi_1'$ by definition, this shows that (4) holds.

Next, we check (5). By (9) and the definition of the partitions it is enough to show that for all $x \in \Xi_0$ and $y \in \Xi_1$,

(10) $$\text{if } x\,(T_0 \cup T_2)\,y, \text{ then } \xi_0\,(T_0\,|\,T_2)\,x \leftrightarrow \xi_1\,(T_0\,|\,T_2)\,y.$$

Suppose that $x\ T_j\ y$, with $j = 0$ or 2. If $\xi_0\,(T_0\,|\,T_2)\,x$, then $\xi_1\ T_0\ \xi_0\ (T_0\,|\,T_2)\ x\ T_j\ y$ by (7), whence $\xi_1\,(T_0\,|\,T_2)\,y$, and similarly, if $\xi_1\,(T_0\,|\,T_2)\,y$, then $\xi_0\ T_0\ \xi_1\ (T_0\,|\,T_2)\ y\ T_j\ x$, and hence $\xi_0\,(T_0\,|\,T_2)\,x$. This completes the proof of (10) and hence of Case 1.

Case 2. $\xi_0\,(T_0\,|\,T_2)\,a_{01}$

We will build Ψ and the partitions of the T_1-classes of ξ_0 and ξ_1 in the atom structure got by dilating $\mathfrak{B}$ with Ψ simultaneously. We start with $\Psi = \emptyset$ and some initial partitions $\Xi_i' \cup \Xi_i''$ of $\Xi_i = \xi_i/T_1$ for $i = 0, 1$. At each stage in the proof, as we put in more sequences in Ψ, T_k will always denote the accessibility relation in $\mathfrak{B}^\Psi$ (i.e., $\mathfrak{B}$ after dilating with all elements of Ψ) and Ξ_i will be $\xi_i/T_1 (= \xi_i/T_1^{\mathfrak{B}^\Psi})$. Whenever we add a new sequence s to Ψ, we will specify whether the new atom ν_s corresponding to s will be in Ξ_i' or Ξ_i''. This way, $\Xi_i' \cup \Xi_i''$ will always be a partition of Ξ_i.

Note first that because of the assumption $\xi_0\,(T_0\,|\,T_2)\,a_{01}$, we have

(11) $$\xi_1\,(T_0\,|\,T_2)\,c_{01}$$

by (8), and

$$a_{01} \notin E_{02} \cup E_{12} \text{ and } c_{01} \notin E_{02} \cup E_{12} \tag{12}$$

because $\xi_0\ T_1\ a_{01} \in E_{12}$ implies $\xi_0\ T_1\ a_{01} \in E_{02}$, which in turn implies $\xi_0 \in E_{02}$ and hence $\xi_0 \in E_{01}$ by 2.6(iv), 2.10(ii), and the same argument (with ξ_1 in place of ξ_0) proves the second statement of (12).

Since $a_{01}\ T_1\ \xi_0\ T_0\ \xi_1$, and $\neg(a_{01}\ T_0\ \xi_1)$ (for that would imply $a_{01} \in E_{12}$ by 2.10(ii), contradicting (12)), by 2.6(ii), a_{01} has a T_0-neighbour in Ξ_1 different from ξ_1. Similarly, c_{01} has a T_0-neighbour in Ξ_0 different from ξ_0. So we can split Ξ_i ($i = 0, 1$) into two parts Ξ_i' and Ξ_i'' in such a way that

$$\xi_i \in \Xi_i',\ a_{01} \in \Xi_0'',\ c_{01} \in \Xi_1'', \tag{13}$$

and

$$a_{01} \text{ has a } T_0\text{-neighbour in } \Xi_1'' \text{ and } c_{01} \text{ has a } T_0\text{-neighbour in } \Xi_0''. \tag{14}$$

(This is the initial partition alluded to earlier.) We will put new atoms in Ξ_i (and divide them judiciously between Ξ_i' and Ξ_i'') but this will of course not change the fact that, by (13), (4) is satisfied.

Our strategy for satisfying (5) is as follows: Suppose that there is an atom $a \in \Xi_0'$ with no T_2-neighbour in Ξ_1'. Then we simply dilate a new atom into Ξ_1' that is T_2-related to a. Of course, we have to make sure that eventually, the new atom will have a T_0-neighbour in Ξ_0' (it already has a T_2-neighbour, namely a). The simplest way to achieve this is to dilate with the sequence $\langle a, \xi_1, a\rangle$ and put the resulting new atom $\nu_{\langle a,\xi_1,a\rangle}$ in Ξ_1' (note that it is by definition in Ξ_1). This way, besides having provided a T_2-neighbour of a in Ξ_1', a will serve as a T_0-neighbour of $\nu_{\langle a,\xi_1,a\rangle}$ in Ξ_0'. Hence we don't have to use a step-by-step construction. There is, however, a small price to be paid for this convenience: the four atoms in $E_{12} \cup E_{01}$ need special treatment, since dilating with, say $\langle \xi_0, \xi_1, \xi_0\rangle$, would violate condition (D1).

Let's take care of the identity atoms first. Because of (7) and (14), a_{01} and c_{01} already have T_2 and T_0-neighbours in Ξ_1'' and Ξ_0'', respectively. Also, ξ_0 and ξ_1 are each other's T_0-neighbours in Ξ_1' and Ξ_0', so we just need to make sure there will be a T_2-neighbour of ξ_0 in Ξ_1' and a T_2-neighbour of ξ_1 in Ξ_0'. To achieve this, we will dilate with (i.e., put in Ψ) the sequences

$$\langle a_{01}, \xi_1, \xi_0\rangle, \qquad \langle a_{01}, \xi_0, \xi_1\rangle.$$

Before examining the resulting configuration, let's first check that conditions (D1) and (D2) in 2.13 hold for the two sequences. We have seen already that $\xi_0, \xi_1 \notin E_{01} \cup E_{02}$; hence, and by (12), (D1) holds. Next, $a_{01}\ (T_0 | T_1)\ \xi_1$ by (14) and $a_{01}\ (T_0 | T_2)\ \xi_0$ because we are in Case 2; from the latter $a_{01}\ (T_0 | T_2)\ \xi_1$ follows by (7), and $a_{01}\ (T_0 | T_1)\ \xi_0$ is obvious, since $a_{01}\ T_1\ \xi_0$. Finally, $\xi_1\ T_1\ c_{01}\ T_2\ a_{01}\ T_1\ \xi_0$ by (7), so $\xi_1\ (T_1 | T_2)\ \xi_0$. This completes the proof of (D2) (i.e., condition (3) of the Claim) for the two sequences. It is clear that Ψ also satisfies conditions (1) and (2).

Now we put $\nu_{\langle a_{01},\xi_1,\xi_0\rangle}$ in Ξ_1', and $\nu_{\langle a_{01},\xi_0,\xi_1\rangle}$ in Ξ_0'. By definition, $\nu_{\langle a_{01},\xi_1,\xi_0\rangle}$ is T_1-related to ξ_1, and $\nu_{\langle a_{01},\xi_0,\xi_1\rangle}$ is T_1-related to ξ_0, so for $i = 0$, 1, Ξ_i', Ξ_i'' is still a partition of (the now slightly bigger) Ξ_i.

With this we have made sure that the identity atoms have T_0 and T_2-neighbours in the proper places, as desired in (5), and so do the new atoms. This is because the initial partition took care of a_{01} and c_{01}, and

$$\xi_0\ T_0\ \xi_1\ T_2\ \nu_{\langle a_{01},\xi_0,\xi_1\rangle}\ T_0\ \nu_{\langle a_{01},\xi_1,\xi_0\rangle}\ T_2\ \xi_0,$$

with ξ_0, $\nu_{\langle a_{01},\xi_0,\xi_1\rangle} \in \Xi'_0$ and ξ_1, $\nu_{\langle a_{01},\xi_1,\xi_0\rangle} \in \Xi'_1$. The situation is depicted in Figure 2, where the T_1-classes of ξ_0 and ξ_1 are represented by vertical lines (broken, to indicate the partition into Ξ' and Ξ''), the horizontal lines stand for T_0, and the sloped lines for T_2-connections.

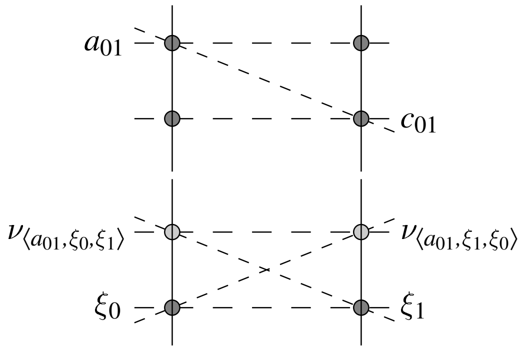


FIGURE 2. The relevant part of $\mathfrak{B}$ after the first dilation

Hence from now on we only have to deal with those atoms in $\Xi_0 \cup \Xi_1$ that are neither new, nor in $E_{01} \cup E_{12}$. Let Y be this set. As outlined above, to provide each element of Y with T_0 and T_2-neighbours as required in (5), we will dilate with all sequences in

$$\{\langle a,\xi_1,a\rangle : a \in \Xi_0 \cap Y\} \cup \{\langle a,\xi_0,a\rangle : a \in \Xi_1 \cap Y\}. \tag{15}$$

Again, we first check that putting these sequences in Ψ will not contradict conditions (1)–(3). For (1) and (2) this is obvious; for (3) we need to check the dilating conditions (D1) and (D2). Now (D1) is clear, because $Y \cap (E_{01} \cup E_{12}) = \emptyset$, and $\xi_0, \xi_1 \notin E_{02}$ by $\xi_0, \xi_1 \in E_{12} \setminus E_{01}$. To see that (D2) holds it is enough to check that

$$\langle a,\xi_1\rangle \in (T_0 \mid T_1) \cap (T_1 \mid T_2) \qquad \text{for all } a \in \Xi_0 \cap Y \tag{16}$$

and

$$\langle a,\xi_0\rangle \in (T_0 \mid T_1) \cap (T_1 \mid T_2) \qquad \text{for all } a \in \Xi_1 \cap Y. \tag{17}$$

Now (16) follows from 2.6(ii), for if $a \in \Xi_0$, then $a\ T_1\ \xi_0\ T_0\ \xi_1$ and $a\ T_1\ a_{01}\ T_2\ c_{01}\ T_1\ \xi_1$, both by (7). Similarly, if $a \in \Xi_1$, then $a\ T_1\ \xi_1\ T_0\ \xi_0$ and $a\ T_1\ c_{01}\ T_2\ a_{01}\ T_1\ \xi_0$ by (7). This finishes the proof of (D2).

Now if $a \in \Xi'_i$ then we put the resulting new atom $\nu_{\langle a,\xi_j,a\rangle}$ in Ξ'_j, where $\{i,j\} = 2$. Similarly, if $a \in \Xi''_i$, then we put the resulting new atom in Ξ''_j. This definition is unambiguous since $\Xi'_0, \dots, \Xi''_1$ are pairwise disjoint. It is also clear that the enlarged Ξ'_i, Ξ''_i is a partition of Ξ_i. We claim that (5) is now satisfied. The initial partition took care of the two E_{01} atoms a_{01} and c_{01}; the first dilation (the first two sequences we put in Ψ) took care of the two E_{12} atoms ξ_0 and ξ_1, and, as we have also seen, the corresponding two new atoms have T_0 and T_2-neighbours at the proper places. Dilating with the set of sequences in (15) took care of all other atoms and did not introduce new problems, because the new atoms we got in the second dilation already have proper neighbours, by their definition. For example, if a is in $\Xi'_0 \cap Y$, then we have $\nu_{\langle a,\xi_1,a\rangle}$ in Ξ'_1, and since $a\ (T_0 \cap T_2)\ \nu_{\langle a,\xi_1,a\rangle}$, a and $\nu_{\langle a,\xi_1,a\rangle}$ serve as each other's T_0 and T_2-neighbours in Ξ'_0 and Ξ'_1.

Because the initial partition already took care of condition (4), and we have checked (1)–(3) for all sequences we put in Ψ, the proof of Claim 1 is now complete. $\square$

□

Let $FB(\mathsf{Cs}_3)$ denote the class of 3-dimensional cylindric set algebras with finite base (this is a strictly smaller class than the class of finite Cs_3's), let $Fin(\mathsf{CA}_3)$ stand for the class of finite CA_3's, and if K is a class of cylindric algebras, then let $\mathbf{Tw}_f K$ denote the class of algebras that can be gotten from members of K by an application of *finite* twisting (i.e., by chosing a finite set I in 2.23). The following corollary of the proof of 3.11 was pointed out by Leon Henkin and Hajnal Andréka.

COROLLARY 3.12. $Fin(\mathsf{CA}_3) \subseteq \mathbf{Rl}^{ca}\,\mathbf{Tw}_f\,\mathbf{Srl}^{ca}\,FB(\mathsf{Cs}_3)$.

PROOF. If the algebra $\mathfrak{A}$ we start out with in the proof of 3.11 is finite, then there is no need to take its perfect extension since it is already complete and atomic. This is why the first "**S**" in the formulation of the theorem can be omitted. Next, what we get after dilating and twisting the atom structure of $\mathfrak{A}$ ($\mathfrak{A}_3$ in the formulation of 3.11) is also finite, because we put in only finitely many new atoms in the dilation phase, and twisting doesn't change the number of elements. By Theorem 1 in [AT88] it is isomorphic to a finite $\mathbf{Srl}^{ca}\,\mathsf{Cs}_3$. But then, by a recent result of Andréka-Hodkinson-Németi [AHN99] saying that every finite algebra in $\mathbf{S}\,\mathbf{Rl}\,\mathsf{Cs}_3$ is representable on a finite set, there is a finite set U such that $\mathfrak{A}_3$ is isomorphic to a subalgebra of a relativized Cs_3 with base U. □

One might ask whether the extension of twisting to infinite sets it necessary to obtain the "non-standard" representation theorem 3.11. Our next theorems indicate that it probably is.

THEOREM 3.13. *Let $\mathfrak{B} \in \mathsf{Ca}_3$, and let $\mathfrak{B}'$ be the result of twisting $\mathfrak{B}$ with a finite set I. If there is no $x \in \mathfrak{Ra}\,\mathfrak{Cm}\,\mathfrak{B}$ with $x^{\smile\smile} < x$, then the same is true for $\mathfrak{Ra}\,\mathfrak{Cm}\,\mathfrak{B}'$. In other words, Henkin's twisting does not lead out of $\mathsf{CA}_3 \cap \mathrm{Mod}(e)$, where e is the quasi-equation in the formulation of Corollary 3.5.*

PROOF. Suppose that there is an $x \in \mathfrak{Ra}\,\mathfrak{Cm}\,\mathfrak{B}'$ with $x^{\smile'\smile'} < x$, and let $\langle x_i : i \in \omega\rangle$ be the strictly decreasing sequence as in Lemma 3.4. Since $x \cdot -x^{\smile'\smile'}$ is non-zero and c_2-closed, by (vi)1.6 there is an atom $a \in E_{12}$ below it. Let f denote the permutation $K_{01} \cap^2 E_{12}$ of E_{12} (cf. Lemma 2.10(iv)), and let $g = f^2$. Define f' and g' similarly, using K'_{01} in place of K_{01}. Since g' is a permutation of E_{12}, either there is an $N > 0$ such that $g'^N a = a$, or the sequence $\langle g'^n a : n \in \omega\rangle$ is repetition-free. The first possibility is ruled out by the observation that it would imply $a = g'^N a \le x_N \le x_1$, a contradiction, since $a \le -x^{\smile'\smile'} \le -x_1$. From this it follows that $\langle f'^n a : n \in \omega\rangle$ is also repetition-free (if it were not, then there would be an $N > 0$ with $a = f'^N a$, and hence $a = f'^N a = f'^{2N} a = g'^N a$, a contradiction).

Now suppose that $\mathfrak{B}'$ was obtained by twisting T_t (for some $t < 3$) as in Definition 2.23 with I finite. Then

(1) $\quad f'$ and f differ on at most finitely many members of E_{12},

and in fact we have

$$\text{(2)} \qquad \text{for all } a \in E_{12}, \quad f'a \neq fa \Rightarrow a = \begin{cases} \mathsf{d}_{12} \cdot \mathsf{c}_1(\mathsf{d}_{01} \cdot \mathsf{c}_0 \xi_i) & \text{if } t = 0 \\ \mathsf{d}_{12} \cdot \mathsf{c}_1 \xi_i & \text{if } t = 1 \\ \mathsf{d}_{12} \cdot \mathsf{c}_1\big(\mathsf{d}_{01} \cdot \mathsf{c}_0(\mathsf{d}_{02} \cdot \mathsf{c}_2 \xi_i)\big) & \text{if } t = 2 \end{cases}$$

for some $i \in I$. We prove only the first part of (2), because the proof of the other parts are similar. By the definition of K_{01}, there are $b \in E_{01}$ and $c \in E_{02}$ with $a\;T_1\;b\;T_0\;c\;T_2\;fa$; hence if $t = 0$ and $f'a \neq fa$, then $\langle b, c\rangle \notin T'_0$. But then b must be in the T_0-class of ξ_i for some $i \in I$, and since $b \in E_{01}$, we must have $b = \mathsf{d}_{01} \cdot \mathsf{c}_0 \xi_i$ by 2.10(i). Similarly, since $b\;T_1\;a \in E_{12}$, $a = \mathsf{d}_{12} \cdot \mathsf{c}_1 b$ by 2.10(i). Hence $a = \mathsf{d}_{12} \cdot \mathsf{c}_1(\mathsf{d}_{01} \cdot \mathsf{c}_0 \xi_i)$ as desired.

Because of (1), there is a natural number N such that $g^n a = g'^n a$ for all $n > N$. Let $y = \mathsf{c}_2 \sum_{n>N} g^n a$. Then

$$\begin{aligned} y^{\smile\smile} &= \left(\sum\{\, \mathsf{c}_2\, g^n a : n > N\,\}\right)^{\smile\smile} \\ &= \sum\left\{\, (\mathsf{c}_2\, g^n a)^{\smile\smile} : n > N \,\right\} && \text{by 1.11(v)} \\ &= \sum\{\, \mathsf{c}_2\, g^{n+1} a : n > N \,\} && \text{by 2.10(iii)} \\ &= \sum\{\, \mathsf{c}_2\, g^n a : n > N+1 \,\} \end{aligned}$$

and since $\mathsf{c}_2\, g^{N+1} a \cdot \mathsf{c}_2\, g^n a = 0$ for all $n > N+1$ (for otherwise we would have $\langle g^{N+1} a, g^n a\rangle \in T_2$ and hence, by 2.6(v), $g^{N+1} a = g^n a$ for some $n > N+1$), we obtained an element y of $\mathfrak{Ra}\,\mathfrak{Cm}\,\mathfrak{B}$ with $y^{\smile\smile} < y$, contradicting our assumption that (e) holds in $\mathfrak{Cm}\,\mathfrak{B}$. ☐

THEOREM 3.14. *Let $\mathfrak{A}$, $\mathfrak{C} \in \mathsf{CA}_3$ with $\mathfrak{A} \subseteq \mathfrak{Rl}_w \mathfrak{C}$ for some $w \in C$. If $\mathfrak{C}$ satisfies the quasi-equation* (e) *in the formulation of Corollary 3.5, then so does $\mathfrak{A}$.*

PROOF. First, a simple observation:

$$\mathsf{CA}_3 \models \mathsf{c}_2 x < \mathsf{c}_2 y \leftrightarrow \mathsf{d}_{12} \cdot \mathsf{c}_2 x < \mathsf{d}_{12} \cdot \mathsf{c}_2 y \tag{1}$$

For, if $\mathsf{c}_2 x < \mathsf{c}_2 y$, then $\mathsf{d}_{12} \cdot \mathsf{c}_2 x \leq \mathsf{d}_{12} \cdot \mathsf{c}_2 y$, but equality would imply

$$\mathsf{c}_2 x = \mathsf{c}_2(\mathsf{d}_{12} \cdot \mathsf{c}_2 x) = \mathsf{c}_2(\mathsf{d}_{12} \cdot \mathsf{c}_2 y) = \mathsf{c}_2 y$$

by 1.6(viii). In the reverse direction, $\mathsf{d}_{12} \cdot \mathsf{c}_2 x < \mathsf{d}_{12} \cdot \mathsf{c}_2 y$ implies

$$\mathsf{c}_2 x = \mathsf{c}_2(\mathsf{d}_{12} \cdot \mathsf{c}_2 x) \leq \mathsf{c}_2(\mathsf{d}_{12} \cdot \mathsf{c}_2 y) = \mathsf{c}_2 y$$

again by 1.6(viii), but, of course, $\mathsf{c}_2 x = \mathsf{c}_2 y$ would give $\mathsf{d}_{12} \cdot \mathsf{c}_2 x = \mathsf{d}_{12} \cdot \mathsf{c}_2 y$.

Now suppose that $w \in C$ is as in the formulation of the theorem, i.e., such that $\mathfrak{Rl}_w \mathfrak{C}$ has a subalgebra $\mathfrak{A} \in \mathsf{CA}_3$. (Throughout this proof, unless otherwise noted, all terms are computed in $\mathfrak{C}$.) Then

$$w \leq \mathsf{s}^i_j w \quad \text{for every } i, j < 3 \tag{2}$$

$$_k\mathsf{s}(i,j)^w x = w \cdot {_k\mathsf{s}(i,j)} x \quad \text{for every } x \leq w, \tag{3}$$

where $_k\mathsf{s}(i,j)^w x \overset{\text{def}}{=} w \cdot \mathsf{s}^k_i(w \cdot \mathsf{s}^i_j(w \cdot \mathsf{s}^j_k x))$. Indeed, since $\mathfrak{A} \in \mathsf{CA}_3$, by Lemma 1.6(v) $\mathfrak{A} \models \mathsf{c}_i \mathsf{d}_{ij} = 1$, that is, $w \cdot \mathsf{c}_i(\mathsf{d}_{ij} \cdot w) = w$ holds in $\mathfrak{C}$. This proves (2), and (3) follows by repeated applications of (2) and Lemma 1.11(v):

$$\begin{aligned} w \cdot {_k\mathsf{s}(i,j)} x &= w \cdot \mathsf{s}^k_i\, \mathsf{s}^i_j\, \mathsf{s}^j_k x \\ &= w \cdot \mathsf{s}^k_i\, w \cdot \mathsf{s}^k_i\, \mathsf{s}^i_j\, \mathsf{s}^j_k x && \text{by (2)} \\ &= w \cdot \mathsf{s}^k_i(w \cdot \mathsf{s}^i_j\, \mathsf{s}^j_k x) && \text{by 1.11(v)} \\ &= w \cdot \mathsf{s}^k_i(w \cdot \mathsf{s}^i_j\, w \cdot \mathsf{s}^i_j\, \mathsf{s}^j_k x) && \text{by (2)} \\ &= w \cdot \mathsf{s}^k_i(w \cdot \mathsf{s}^i_j(w \cdot \mathsf{s}^j_k x)) && \text{by 1.11(v)} \\ &= {_k\mathsf{s}(i,j)^w} x. \end{aligned}$$

(Cf. the proofs of 2.2.3 and 2.2.7 in [HMT71].) These facts have some important consequences, such as

$$\mathsf{d}_{ij} \cdot \mathsf{c}_i w \leq w \quad \text{if } i, j < 3 \text{ are different} \tag{4}$$

$$w \leq {}_k\mathsf{s}(i,j)w \quad \text{if } \{i,j,k\} = 3 \tag{5}$$

$$\textit{MGR} \text{ holds for } w\text{, i.e.,} \quad \mathsf{c}_k w = {}_k\mathsf{s}(i,j){}_k\mathsf{s}(i,j)\,\mathsf{c}_k w \quad \text{if } \{i,j,k\} = 3 \tag{6}$$

and

$${}_k\mathsf{s}(i,j)^w{}_k\mathsf{s}(i,j)^w\,\mathsf{c}_k^w x = w \cdot {}_k\mathsf{s}(i,j){}_k\mathsf{s}(i,j)\,\mathsf{c}_k x \quad \text{if } \{i,j,k\} = 3 \text{ and } x \leq w. \tag{7}$$

To see that (4) holds, note that

$$\begin{aligned}
& \mathsf{c}_i(\mathsf{d}_{ij} \cdot w) \cdot \mathsf{c}_i(\mathsf{d}_{ij} \cdot -w) = 0 && \text{by (C7)}\\
\Rightarrow\quad & w \cdot \mathsf{c}_i(\mathsf{d}_{ij} \cdot -w) = 0 && \text{by (2)}\\
\Rightarrow\quad & \mathsf{c}_i w \cdot \mathsf{d}_{ij} \cdot -w = 0, && \text{by 1.6(ii)}
\end{aligned}$$

that is, $\mathsf{d}_{ij} \cdot \mathsf{c}_i w \leq w$, as desired. (5) is an obvious consequence of (2) and the fact that substitutions are order preserving (by 1.11(v)). Next, the $\leq$ part of (6) clearly follows from (5), whence

$$\begin{aligned}
& \mathsf{c}_k w \leq {}_k\mathsf{s}(j,i){}_k\mathsf{s}(j,i)\,\mathsf{c}_k w && \\
\Rightarrow\quad & {}_k\mathsf{s}(i,j)\,\mathsf{c}_k w \leq {}_k\mathsf{s}(j,i)\,\mathsf{c}_k w && \text{by 1.11(xiv)}\\
\Rightarrow\quad & {}_k\mathsf{s}(i,j){}_k\mathsf{s}(i,j)\,\mathsf{c}_k w \leq \mathsf{c}_k w, && \text{by 1.11(xiv)}
\end{aligned}$$

completing the proof of (6). For (7), we have

$$\begin{aligned}
{}_k\mathsf{s}(i,j)^w{}_k\mathsf{s}(i,j)^w\,\mathsf{c}_k^w x &= w \cdot {}_k\mathsf{s}(i,j){}_k\mathsf{s}(i,j)^w\,\mathsf{c}_k^w x && \text{by (3)}\\
&= w \cdot {}_k\mathsf{s}(i,j)(w \cdot {}_k\mathsf{s}(i,j)\,\mathsf{c}_k^w x) && \text{by (3)}\\
&= w \cdot {}_k\mathsf{s}(i,j)w \cdot {}_k\mathsf{s}(i,j){}_k\mathsf{s}(i,j)\,\mathsf{c}_k^w x && \text{by 1.11(v)}\\
&= w \cdot {}_k\mathsf{s}(i,j){}_k\mathsf{s}(i,j)\,\mathsf{c}_k^w x && \text{by (5)}\\
&= w \cdot {}_k\mathsf{s}(i,j){}_k\mathsf{s}(i,j)(w \cdot \mathsf{c}_k x) && \text{by def. of } \mathsf{c}^w\\
&= w \cdot ({}_k\mathsf{s}(i,j){}_k\mathsf{s}(i,j)w) \cdot ({}_k\mathsf{s}(i,j){}_k\mathsf{s}(i,j)\,\mathsf{c}_k x) && \text{by 1.11(v)}\\
&= w \cdot {}_k\mathsf{s}(i,j){}_k\mathsf{s}(i,j)\,\mathsf{c}_k x. && \text{by (5)}
\end{aligned}$$

Now suppose that (e) fails in $\mathfrak{A}$. Then there is an $x \in A$ such that

$$\mathfrak{A} \models {}_2\mathsf{s}(0,1){}_2\mathsf{s}(0,1)\,\mathsf{c}_2 x < \mathsf{c}_2 x,$$

and hence, by (1),

$$\mathfrak{A} \models \mathsf{d}_{12} \cdot {}_2\mathsf{s}(0,1){}_2\mathsf{s}(0,1)\,\mathsf{c}_2 x < \mathsf{d}_{12} \cdot \mathsf{c}_2 x.$$

In $\mathfrak{C}$, this means that $\mathsf{d}_{12}^w \cdot {}_2\mathsf{s}(0,1)^w{}_2\mathsf{s}(0,1)^w\,\mathsf{c}_2^w x < \mathsf{d}_{12}^w \cdot \mathsf{c}_2^w x$ for some $x \leq w$. Applying (7) we get $\mathsf{d}_{12} \cdot w \cdot {}_2\mathsf{s}(0,1){}_2\mathsf{s}(0,1)\,\mathsf{c}_2 x < \mathsf{d}_{12} \cdot w \cdot \mathsf{c}_2 x$, and thus

$$\mathsf{d}_{12} \cdot {}_2\mathsf{s}(0,1){}_2\mathsf{s}(0,1)\,\mathsf{c}_2 x < \mathsf{d}_{12} \cdot \mathsf{c}_2 x,$$

because $\mathsf{d}_{12} \cdot \mathsf{c}_2 x \leq \mathsf{d}_{12} \cdot \mathsf{c}_2 w \leq w$ by (4), and $\mathsf{d}_{12} \cdot {}_2\mathsf{s}(0,1){}_2\mathsf{s}(0,1)\,\mathsf{c}_2 x \leq \mathsf{d}_{12} \cdot {}_2\mathsf{s}(0,1){}_2\mathsf{s}(0,1)\,\mathsf{c}_2 w = \mathsf{d}_{12} \cdot \mathsf{c}_2 w \leq w$ by (6) and (4). But then ${}_2\mathsf{s}(0,1){}_2\mathsf{s}(0,1)\,\mathsf{c}_2 x < \mathsf{c}_2 x$ by (1), contradicting our assumption that (e) holds in $\mathfrak{C}$. $\square$

We note that is very easy to construct from an RCA_3 by finite twisting a CA_3 $\mathfrak{C}$ and an element $w \in C$ such that $\mathfrak{C} \models e$ but $(c_2 x)^{\smile} = 0$ in $\mathfrak{Rl}_w \mathfrak{C}$ for some nonzero $x \leq w$.

COROLLARY 3.15. *Let $\mathcal{S}$ be any finite string of the class operators $\{\mathbf{S}, \mathbf{Up}, \mathbf{P}, \mathbf{Tw}_f, \mathbf{Rl}^{ca}, \mathbf{Srl}^{ca}\}$. Then $\mathcal{S}\mathsf{RCA}_3 \subset \mathsf{CA}_3$.*

PROOF. By 3.6, the quasi-equation (e) in Corollary 3.5 fails in CA_3; but it is valid in RCA_3, and it is obviously preserved by $\mathbf{S}$, $\mathbf{Up}$ and $\mathbf{P}$. By 3.13 and 3.14, it is also preserved by $\mathbf{Tw}_f$ and $\mathbf{Srl}^{ca}$ (and hence by $\mathbf{Rl}^{ca}$, too). □

If we allow the operations of taking complex algebras and atom structures in $\mathcal{S}$, then our proof (and probably the corollary itself) breaks down. For, consider the sequence of algebras $\langle \mathfrak{C}'_N : 1 < N \in \omega \rangle$ constructed in Example 3.1; each of them is obtained from a Cs_3 by finite twisting, but if F is a non-principal ultrafilter over $\omega \setminus 2$, then $\mathfrak{Cm}\mathfrak{At}\prod_{1<N\in\omega}\mathfrak{C}_N/F$ is a CA_3 in which the quasi-equation (e) in Corollary 3.5 fails. This is because of the following: we have observed in 3.1 that, as N grows, the atom structure $\mathfrak{B}'_N$ of $\mathfrak{C}'_N$ contains bigger and bigger cycles of E_{12} atoms under the permutation $\breve{}$. Hence there is an infinite cycle (that is, a $\mathbb{Z}$-type chain) in the "limit", i.e., in $\prod_{1<N\in\omega}\mathfrak{B}'_N/F$. But then any non-trivial final segment of this chain is an element of $\mathfrak{Cm}\prod_{1<N\in\omega}\mathfrak{B}'_N/F$ that witnesses the failure of (e) (cf. the proof of 3.13). Thus (e) fails in $\mathfrak{Cm}\mathfrak{At}\prod_{1<N\in\omega}\mathfrak{C}'_N/F$, since $\mathfrak{At}\prod_{1<N\in\omega}\mathfrak{C}'_N/F \cong \prod_{1<N\in\omega}\mathfrak{B}'_N/F$. We note that $\prod_{1<N\in\omega}\mathfrak{C}'_N/F$, being the ultraproduct of atomic algebras in which (e) holds, is also atomic and validates (e), whence it cannot be complete by Theorem 2.4.

The proof of Theorem 3.11 and the preceding discussion motivates the following

PROBLEM 3.16. Is $\mathsf{CA}_3 \subseteq \mathbf{S}\,\mathbf{Tw}\,\mathbf{Srl}^{ca}\,\mathsf{Cs}_3$? Is $\mathsf{CA}_3 \subseteq \mathbf{S}\,\mathfrak{Cm}\,\mathfrak{At}\,\mathbf{Up}\,\mathbf{Tw}_f\,\mathbf{Srl}^{ca}\,\mathsf{Cs}_3$? More generally: what other "routes" *via* CA_3's are there to get to every CA_3 from representable ones?

In connection with the first question in 3.16 we note that the proof of Theorem 3.11 could probably be modified to give

$$\mathsf{CA}_3 \subseteq \mathbf{S}\,\mathbf{Tw}\,\mathbf{S}\,\mathbf{Rl}\,\mathsf{Cs}_3 \tag{*}$$

(after, of course, extending twisting to members of $\mathbf{S}\,\mathbf{Rl}\,\mathsf{Cs}_3$). But we believe that the presence of an algebra that is not in CA_3 in the chain leading from a CA_3 to a Cs_3 makes (*) less attractive.

The most important problem left open by the results in this section is the following.

PROBLEM 3.17 (Henkin). Does Theorem 3.11 extend in some form to dimensions > 3?

We note that if $\alpha > 3$, then, in order to be able to use the Resek-Thompson theorem, one has to cure the failures of *all* equations in 1.14.

CHAPTER 4

Independence of the RA-axioms within CA_3

We have already noted in Chapter 1 that the RA-reduct of a CA_3 is not necessarily an RA or even an SA. (By contrast, if $n \geq 4$, then $\mathfrak{Ra}^* \mathsf{CA}_n \subseteq \mathsf{RA}$.) By Monk [Mon61] and Maddux [Mad78] we have

LEMMA 4.1 (Monk, Maddux). $\mathfrak{Ra}^* \mathsf{CA}_3 \models$ (R0), ($\mathrm{R1}_0$), (R2), (R3), (R5).

It is known that each of the remaining axioms ((R1), (R4), (R6) and (R7)) may fail in $\mathfrak{Ra}^* \mathsf{CA}_3$, but one may still ask what subsets of these equations axiomatize RA or SA within $\mathfrak{Ra}^* \mathsf{CA}_3$ or $\mathbf{S}\mathfrak{Ra}^* \mathsf{CA}_3$ (note that these are separate questions by Theorem 3.7). For example, Monk [Mon61, Theorem 9.10] proves the following result (and credits the $\supseteq$ part to G. Fuhrken).

THEOREM 4.2 (Monk, Fuhrken). $\mathsf{RA} = \mathfrak{Ra}^* \mathsf{CA}_3 \cap \mathrm{Mod}\{$(R1), (R4), (R6)$\}$.

PROOF. This is Theorem 9.10 in [Mon61]. □

Problem 4 in [Mon61] asks whether both (R4) and (R6) are needed in the $\supseteq$ part of the above theorem. In Theorem 4.11 below we describe exactly what implications hold between the RA axioms within $\mathfrak{Ra}^* \mathsf{CA}_3$ and $\mathbf{S}\mathfrak{Ra}^* \mathsf{CA}_3$; as corollaries we obtain a characterization of those subsets of the RA-axioms which axiomatize RA or SA within $\mathfrak{Ra}^* \mathsf{CA}_3$ or $\mathbf{S}\mathfrak{Ra}^* \mathsf{CA}_3$. As an immediate corollary of these results we get the following solution to Monk's problem: in Theorem 4.2, (R6) is needed but (R4) is not.

We start by proving four implications in $\mathbf{S}\mathfrak{Ra}^* \mathsf{CA}_3$ (Theorems 4.3, 4.4(i),(ii) and 4.6 below) and then turn to the independence results. Throughout this chapter, a statement of the form $K \models \varphi \Rightarrow \psi$ means that $K \cap \mathrm{Mod}(\varphi) \models \psi$.

THEOREM 4.3. $\mathbf{S}\mathfrak{Ra}^* \mathsf{CA}_3 \models$ (R7) $\Rightarrow$ (R4).

PROOF. Let (R2') stand for the "mirror image" of the axiom (R2) for RA's, that is,

(R2') $x\,;(y+z) = (x\,;y)+(x\,;z)$.

It is easy to see that $\mathbf{S}\mathfrak{Ra}^* \mathsf{CA}_3 \models$ (R2'). Because of this, and Lemmas 4.1, 1.20(ii), we are done if we can show that for any RA-type algebra $\mathfrak{A}$,

(1) $$\mathfrak{A} \models \{\text{(R0), (R2'), (R3), (R7)}\} \quad \text{implies} \quad \mathfrak{A} \models x^{\smallsmile\smallsmile} \leq x.$$

So let $\mathfrak{A}$ be an RA-type algebra and assume that $\mathfrak{A} \models$ (R0), (R2'), (R3), (R7). By (R7) we have

(2) $$x^{\smallsmile\smallsmile}\,;-(x^{\smallsmile}\,;-x) \leq x.$$

Also by (R7), $x^{\smallsmile}\,;-(x\,;1') \leq -1'$, which by (R3), yields $x^{\smallsmile}\,;-x \leq -1'$, that is, $-(x^{\smallsmile}\,;-x) \geq 1'$. Putting these together, we get

$$\begin{aligned} x^{\smallsmile\smallsmile} &= x^{\smallsmile\smallsmile}\,;1' && \text{by (R3)}\\ &\leq x^{\smallsmile\smallsmile}\,;-(x^{\smallsmile}\,;-x) && \text{by (R2')}\\ &\leq x. && \text{by (2)} \end{aligned}$$

This completes the proof of (1) and the theorem. □

In passing we note that (1) in the proof shows that in axiomatizing RA, axiom (R4) can be replaced by the weaker postulate $x \le x^{\smile\smile}$ (if we use (R2') in place of (R2)).

The second statement of the next theorem is proved in Monk [Mon61, 9.1]. For completeness, we give a proof here.

THEOREM 4.4. (i) $\mathbf{S}\mathfrak{Ra}^*\,\mathsf{CA}_3 \models (\mathrm{R6}) \Rightarrow (\mathrm{R4})$.
(ii) $\mathbf{S}\mathfrak{Ra}^*\,\mathsf{CA}_3 \models (\mathrm{R1}) \wedge (\mathrm{R6}) \Rightarrow (\mathrm{R7})$.

PROOF. Let $\mathfrak{A} \subseteq \mathfrak{Ra}\,\mathfrak{C}$ for some $\mathfrak{C} \in \mathsf{CA}_3$. We may assume that $\mathfrak{C}$ (and hence $\mathfrak{Ra}\,\mathfrak{C}$) is complete and atomic, because of the following: denoting the perfect extension (in the sense of Jónsson and Tarski [JT51]) of a BAO $\mathfrak{B}$ by $\mathfrak{B}^\sigma$, we have $\mathfrak{C}^\sigma \in \mathsf{CA}_3$ by [HMT71, 2.7.15], and $\mathfrak{A} \subseteq \mathfrak{Ra}\,\mathfrak{C} \subseteq (\mathfrak{Ra}\,\mathfrak{C})^\sigma \cong \mathfrak{Ra}\,\mathfrak{C}^\sigma$ because the operations $\mathfrak{Ra}$ and $^\sigma$ commute: this follows e.g. from Andréka-Givant-Németi [AGN95, Corollary 2.12].

Next we claim that in the RA-reduct of a complete and atomic CA_3,

(1) $$u\,;v \cdot 1' \ne 0 \quad \text{iff} \quad v = u^{\smile} \qquad \text{for all } u, v \in \mathrm{At}\,\mathfrak{Ra}\,\mathfrak{C},$$

or, equivalently (by 1.19, 2.4 and 2.7), if $\mathfrak{B} = \langle B, T_i, E_{ij}\rangle_{i,j<3} \in \mathsf{Ca}_3$, then in $\mathfrak{Cm}\,\mathfrak{B}$ we have

$$\mathsf{c}_2\,a\,;\mathsf{c}_2\,b \cdot 1' \ne 0 \quad \text{iff} \quad \mathsf{c}_2\,b = (\mathsf{c}_2\,a)^{\smile} \qquad \text{for all } a, b \in B.$$

Indeed, if $c \in \mathsf{c}_2\,a\,;\mathsf{c}_2\,b \cdot 1'$, then there are $a' \in E_{12}$, $b' \in E_{02}$ and $d \in B$ such that

$$a\; T_2\; a'\; T_1\; d\; T_0\; b'\; T_2\; b \quad \text{and} \quad d\; T_2\; c$$

by 2.9 and 2.8. But $d\,T_2\,c \in E_{01}$ implies $d \in E_{01}$ by 2.10(ii), whence $a'\,K_{01}\,b$ and hence $\mathsf{c}_2\,b = (\mathsf{c}_2\,a)^{\smile}$ by 2.9. Conversely, if $\mathsf{c}_2\,b = (\mathsf{c}_2\,a)^{\smile}$, then there are $a' \in E_{12}$, $b' \in E_{02}$ and $d \in E_{01}$ such that

$$a\; T_2\; a'\; T_1\; d\; T_0\; b'\; T_2\; b$$

and this shows $C(a', b', d)$, so $0 < \mathsf{c}_2\,d \le \mathsf{c}_2\,a\,;\mathsf{c}_2\,b \cdot 1'$. This completes the proof of (1).

Now for the proof of part (i) of the theorem. Suppose that (R6) holds but (R4) fails in $\mathfrak{A}$. By Lemma 1.20(ii) then there is an $x \in A$ with $x \not\le x^{\smile\smile}$.

Let $z = x \cdot -x^{\smile\smile}$ (non-zero since $x \not\le x^{\smile\smile}$). Then

$$\begin{aligned}
0 &\ne z\,;z^{\smile} \cdot 1' && \text{by (1)}\\
&= (z\,;z^{\smile} \cdot 1')^{\smile} && \text{by 1.20(iii)}\\
&= (z\,;z^{\smile})^{\smile} \cdot 1' && \text{by 1.20(i) and 1.20(iii)}\\
&= z^{\smile\smile}\,;z^{\smile} \cdot 1', && \text{by (R6)}
\end{aligned}$$

so there are atoms u, $v \le z$ of $\mathfrak{Ra}\,\mathfrak{C}$ with $0 \ne u^{\smile\smile}\,;v^{\smile} \cdot 1'$, and hence $u^{\smile\smile\smile} = v^{\smile}$ by (1). But this implies $v = u^{\smile\smile}$ (since $^{\smile}$ is injective), whence $v = u^{\smile\smile} \le z^{\smile\smile} \le x^{\smile\smile}$, contradicting $v \le z \le -x^{\smile\smile}$.

Suppose now that (R1) and (R6) hold in $\mathfrak{A}$. We may assume that besides $\mathfrak{C}$ (and hence $\mathfrak{Ra}\,\mathfrak{C}$), $\mathfrak{A}$ is complete and atomic as well. This is because from $\mathfrak{A} \subseteq \mathfrak{Ra}\,\mathfrak{C}$ we get $\mathfrak{A} \subseteq \mathfrak{A}^\sigma \subseteq (\mathfrak{Ra}\,\mathfrak{C})^\sigma \cong \mathfrak{Ra}\,\mathfrak{C}^\sigma$, as before; and (R1) and (R6), both being positive equations, are preserved under taking perfect extensions (cf. e.g. [HMT85, 2.7.13]). Because of (i), (R4) must also hold in $\mathfrak{A}$. We claim that

(2) $$\mathfrak{A} \models (x^{\smile}\,;-x) \cdot 1' = 0,$$

and

(3) $$\mathfrak{A} \models x \ne 0 \to (x^{\smile}\,;x) \cdot 1' \ne 0.$$

To see that (2) holds, suppose that $(x^{\smile} ; -x) \cdot 1' > 0$. Then there are atoms $u \leq x$ and $v \leq -x$ of $\mathfrak{Ra}\,\mathfrak{C}$ with $(u^{\smile} ; v) \cdot 1' > 0$ (in $\mathfrak{Ra}\,\mathfrak{C}$). By (1), $v = u^{\smile\smile}$, and hence $v \leq x^{\smile\smile} \cdot -x$, a contradiction, since $x \in A$, and (R4) holds in $\mathfrak{A}$. The proof of (3) is similar: if $x \neq 0$ then there is an atom $u \leq x$ of $\mathfrak{Ra}\,\mathfrak{C}$. By (1), and since (R4) holds in $\mathfrak{A}$ (and $x \in A$), $0 < u^{\smile} ; u^{\smile\smile} \cdot 1' \leq x^{\smile} ; x^{\smile\smile} \cdot 1' = x^{\smile} ; x \cdot 1'$.

Since $\mathfrak{A}$ is atomic, to show that it satisfies (R7) it suffices to show that

$$(x^{\smile} ; -(x ; y)) \cdot y_0 = 0$$

for every $x, y \in A$ and every atom $y_0 \leq y$ of A. Fix $x, y \in A$ and an atom $y_0 \leq y$ of A. Then

$$\begin{aligned}
[y_0{}^{\smile} ; ((x^{\smile} ; -(x ; y)) \cdot y_0)] \cdot 1' &\leq [y^{\smile} ; (x^{\smile} ; -(x ; y))] \cdot 1' \\
&= [(y^{\smile} ; x^{\smile}) ; -(x ; y)] \cdot 1' && \text{by (R1)} \\
&= [(x ; y)^{\smile} ; -(x ; y)] \cdot 1' && \text{by (R6)} \\
&= 0. && \text{by (2)}
\end{aligned}$$

On the other hand, $(y_0{}^{\smile} ; y_0) \cdot 1' \neq 0$ by (3); hence $(x^{\smile} ; -(x ; y)) \cdot y_0 < y_0$, and from this we get $(x^{\smile} ; -(x ; y)) \cdot y_0 = 0$, because y_0 is an atom. ☐

LEMMA 4.5. *In* $\mathbf{S}\mathfrak{Ra}^* \mathsf{CA}_3$ *we have*

(i) (R7) $\iff \big((x ; y) \cdot z = 0 \rightarrow (x^{\smile} ; z) \cdot y = 0\big)$;
(ii) (R7) $\iff \big((x ; y) \cdot z = 0 \leftrightarrow (x^{\smile} ; z) \cdot y = 0\big)$;
(iii) (R6) $\iff$ (R4)$\wedge$(R6) $\iff \big((x;y) \cdot z^{\smile} = 0 \leftrightarrow (y^{\smile};x^{\smile}) \cdot z = 0\big) \iff \big((x;y) \cdot z^{\smile} = 0 \rightarrow (y^{\smile};x^{\smile}) \cdot z = 0\big)$.

PROOF. (i) ($\Rightarrow$) If $(x ; y) \cdot z = 0$ then $z \leq -(x ; y)$ and hence

$$(x^{\smile} ; z) \cdot y \leq \big(x^{\smile} ; -(x ; y)\big) \cdot y \leq -y \cdot y = 0$$

by monotonicity of ; and (R7).

($\Leftarrow$) We have $(x ; y) \cdot -(x ; y) = 0$, so by assumption, $\big(x^{\smile} ; -(x ; y)\big) \cdot y = 0$, as desired.

Note that in the proof of (i) we only used the BA-axioms and monotonicity of ;.

(ii) follows easily from (i), by 4.3.

(iii) First we show that $(x ; y) \cdot z^{\smile} = 0 \rightarrow (y^{\smile} ; x^{\smile}) \cdot z = 0$ implies (R4). By 4.1 and 1.20(i) we have

$$(x^{\smile} ; 1') \cdot (-x)^{\smile} = x^{\smile} \cdot -x^{\smile} = 0,$$

so, by assumption, 4.1 and 1.20(iii),

$$x^{\smile\smile} \cdot -x = (1' ; x^{\smile\smile}) \cdot -x = (1'^{\smile} ; x^{\smile\smile}) \cdot -x = 0,$$

whence $x^{\smile\smile} \leq x$, and from this (R4) follows by 1.20(ii).

From this we easily get that $(x ; y) \cdot z^{\smile} = 0 \rightarrow (y^{\smile} ; x^{\smile}) \cdot z = 0$ implies $(y^{\smile} ; x^{\smile}) \cdot z = 0 \rightarrow (x ; y) \cdot z^{\smile} = 0$. Hence the last two formulas in (iii) are equivalent, and imply (R4). Because of 4.4(i), the same applies to the first two formulas. Next, if (R6) (and hence (R4)) holds, and $(x ; y) \cdot z^{\smile} = 0$ for some x, y and z, then $0 = ((x ; y) \cdot z^{\smile})^{\smile} = (x ; y)^{\smile} \cdot z^{\smile\smile} = (y^{\smile} ; x^{\smile}) \cdot z$ by 1.20(i), (R6) and (R4). This shows that (R6) implies the last formula in (iii).

Finally, we prove that (R6) follows from the assumption $(x ; y) \cdot z^{\smile} = 0 \leftrightarrow (y^{\smile} ; x^{\smile}) \cdot z = 0$. (Note that we can use (R4) in the proof). In fact, it suffices to show that $(x ; y)^{\smile} \cdot z = 0$ iff $(y^{\smile} ; x^{\smile}) \cdot z = 0$ for all x, y

and z. And this is clear, for

$$\begin{aligned}(x;y)^{\smile}\cdot z=0 &\iff \big((x;y)^{\smile}\cdot z\big)^{\smile}=0 && \text{by 1.20(i)}\\ &\iff (x;y)^{\smile\smile}\cdot z^{\smile}=0 && \text{by 1.20(i)}\\ &\iff (x;y)\cdot z^{\smile}=0 && \text{by (R4)}\\ &\iff (y^{\smile};x^{\smile})\cdot z=0 && \text{by assumption.}\end{aligned}$$

□

THEOREM 4.6. $\mathbf{S}\mathfrak{Ra}^*\mathsf{CA}_3\models(\mathrm{R1})\wedge(\mathrm{R7})\Rightarrow(\mathrm{R6})$

PROOF. Let $\mathfrak{A}\in\mathbf{S}\mathfrak{Ra}^*\mathsf{CA}_3$ satisfy (R1) and (R7). By Lemma 4.5(iii) and since (R4) holds in $\mathfrak{A}$ (by 4.3), to show that (R6) holds in $\mathfrak{A}$ it suffices to check that

(1) $$\mathfrak{A}\models(x;y)\cdot z^{\smile}\neq 0\rightarrow(y^{\smile};x^{\smile})\cdot z\neq 0.$$

Let x, y, $z\in A$ with $(x;y)\cdot z^{\smile}\neq 0$, and let $w=x;(y;z)\cdot 1'$. First we claim that w is non-zero. Indeed,

$$\begin{aligned}0\neq(x;y)\cdot z^{\smile}\Rightarrow\quad & 0\neq(x;y)^{\smile}\cdot z && \text{by 1.20(i) and (R4)}\\ \Rightarrow\quad & 0\neq(x;y)^{\smile};1'\cdot z && \text{by 4.1}\\ \Rightarrow\quad & 0\neq(x;y);z\cdot 1' && \text{by (R7) and 4.5(i)}\\ \Rightarrow\quad & 0\neq x;(y;z)\cdot 1'=w. && \text{by (R1)}\end{aligned}$$

Hence we have

$$\begin{aligned}0\neq w=x;(y;z)\cdot w\Rightarrow\quad & 0\neq(x^{\smile};w)\cdot(y;z) && \text{by (R7) and 4.5(ii)}\\ \Rightarrow\quad & 0\neq x^{\smile}\cdot(y;z) && \text{by 4.1 and } w\leq 1'\\ \Rightarrow\quad & 0\neq(y^{\smile};x^{\smile})\cdot z, && \text{by (R7) and 4.5(ii)}\end{aligned}$$

completing the proof of (1). □

The next two theorems show that neither (R1), nor (R7) in itself implies (R6) even in the strictly smaller class $\mathfrak{Ra}^*\mathsf{CA}_3$.

THEOREM 4.7. $\mathfrak{Ra}^*\mathsf{CA}_3\not\models(\mathrm{R7})\Rightarrow(\mathrm{R6})$

PROOF. Let $U=3=\{0,1,2\}$, and let G be the group of permutations on U generated by the successor function mod 3. Let $\mathfrak{C}$ be the algebra $\mathfrak{Fr}_G\mathfrak{D}$, where $\mathfrak{D}$ is the full Cs_3 on U. For $i<3$ let R_i be the atom $\langle 0,i,i\rangle^-$ of $\mathfrak{C}$ (so R_0 is just the main diagonal $\mathsf{D}_{01}\cdot\mathsf{D}_{12}$). Clearly, $\mathfrak{Ra}\,\mathfrak{C}$ has 3 atoms, $\mathsf{C}_2R_0=\mathsf{D}_{01}$, C_2R_1 and C_2R_2, with $^{\smile}$ interchanging C_2R_1 and C_2R_2 and leaving D_{01} fixed. Non-trivial compositions of the atoms are as in the first part of Table 1 (i.e., $\mathsf{C}_2R_i\circ\mathsf{C}_2R_j=\mathsf{C}_2R_{i+j \bmod 3}$). Let $\xi_0=\xi_2=\mathsf{D}_{02}\cdot\mathsf{C}_2R_1=\langle 0,1,0\rangle^-$ and $\xi_1=R_1$. It is clear that the sequence ξ satisfies conditions (D1) and (D2) in Definition 2.13. Let $\mathfrak{C}^\xi$ be the result of dilating $\mathfrak{C}$ with ξ. The atoms of $\mathfrak{Ra}\,\mathfrak{C}^\xi$ are D_{01}, $\mathsf{c}_2^\xi R_1$ and $\mathsf{c}_2^\xi R_2$. By lemma 2.16, converse in $\mathfrak{Ra}\,\mathfrak{C}^\xi$ is the same as in $\mathfrak{Ra}\,\mathfrak{C}$, and the composition of $\mathsf{c}_2^\xi R_i$ and $\mathsf{c}_2^\xi R_j$ changes only if $\xi_1\ T_2\ R_i$ and $\xi_0\ T_2\ R_j$, that is, if $i=j=1$, and

$$\mathsf{c}_2^\xi R_1\,;^\xi\,\mathsf{c}_2^\xi R_1=\mathsf{c}_2^\xi(\mathsf{C}_2R_1\circ\mathsf{C}_2R_1)+\mathsf{c}_2^\xi\xi_2=\mathsf{c}_2^\xi R_2+\mathsf{c}_2^\xi R_1,$$

as in the second part of Table 1. With this dilation we managed to kill (R6), but, unfortunately, (R7) also fails in $\mathfrak{Ra}\,\mathfrak{C}^\xi$, since $(\mathsf{c}_2^\xi R_1\,;^\xi\,\mathsf{c}_2^\xi R_1)\cdot\mathsf{c}_2^\xi R_1\neq 0$ while

$$\big((\mathsf{c}_2^\xi R_1)^{\smile}\,;^\xi\,\mathsf{c}_2^\xi R_1\big)\cdot\mathsf{c}_2^\xi R_1=(\mathsf{c}_2^\xi R_2\,;^\xi\,\mathsf{c}_2^\xi R_1)\cdot\mathsf{c}_2^\xi R_1=\mathsf{D}_{01}\cdot\mathsf{c}_2^\xi R_1=0$$

$\circ$	$\mathsf{C}_2 R_1$	$\mathsf{C}_2 R_2$
$\mathsf{C}_2 R_1$	$\mathsf{C}_2 R_2$	D_{01}
$\mathsf{C}_2 R_2$	D_{01}	$\mathsf{C}_2 R_1$

$;^{\xi}$	$\mathsf{c}_2^{\xi} R_1$	$\mathsf{c}_2^{\xi} R_2$
$\mathsf{c}_2^{\xi} R_1$	$\mathsf{c}_2^{\xi} R_2 + \mathsf{c}_2^{\xi} R_1$	D_{01}
$\mathsf{c}_2^{\xi} R_2$	D_{01}	$\mathsf{c}_2^{\xi} R_1$

$;^{\xi\zeta}$	$\mathsf{c}_2^{\xi\zeta} R_1$	$\mathsf{c}_2^{\xi\zeta} R_2$
$\mathsf{c}_2^{\xi\zeta} R_1$	$\mathsf{c}_2^{\xi\zeta} R_2 + \mathsf{c}_2^{\xi\zeta} R_1$	D_{01}
$\mathsf{c}_2^{\xi\zeta} R_2$	$\mathsf{D}_{01} + \mathsf{c}_2^{\xi\zeta} R_1$	$\mathsf{c}_2^{\xi\zeta} R_1$

TABLE 1. Composition in the RA-reducts in the proof of 4.7

(cf. Lemma 4.5(ii)). But we can rectify the situation by dilating $\mathfrak{C}^{\xi}$ with the sequence of atoms $\zeta_0 = \zeta_2 = \xi_0$, $\zeta_1 = R_2$. It is again easy to see that ζ satisfies conditions (D1) and (D2)(one only has to check this in $\mathfrak{C}$ by lemma 2.21(i)). The three atoms of $\mathfrak{Ra}\,\mathfrak{C}^{\xi\zeta}$ are D_{01}, $\mathsf{c}_2^{\xi\zeta} R_1$ and $\mathsf{c}_2^{\xi\zeta} R_2$, and by lemma 2.16, converse in $\mathfrak{Ra}\,\mathfrak{C}^{\xi\zeta}$ is the same as in $\mathfrak{Ra}\,\mathfrak{C}^{\xi}$. As for composition of the atoms, we have $\mathsf{c}_2^{\xi\zeta} R_i ;^{\xi\zeta} \mathsf{c}_2^{\xi\zeta} R_j = \mathsf{c}_2^{\xi\zeta}(\mathsf{c}_2^{\xi} R_i ;^{\xi} \mathsf{c}_2^{\xi} R_j)$ except when $\zeta_1\ T_2\ R_i$ and $\zeta_0\ T_2\ R_j$, that is, when $i=2$ and $j=1$. In this case we have

$$\mathsf{c}_2^{\xi\zeta} R_2 ;^{\xi\zeta} \mathsf{c}_2^{\xi\zeta} R_1 = \mathsf{c}_2^{\xi\zeta}(\mathsf{c}_2^{\xi} R_2 ;^{\xi} \mathsf{c}_2^{\xi} R_1) + \mathsf{c}_2^{\xi\zeta}\zeta_2 = \mathsf{D}_{01} + \mathsf{c}_2^{\xi\zeta} R_1.$$

The last part of Table 1 summarizes this.

We claim that $\mathfrak{C}^{\xi\zeta}$ is the algebra we are after. Let's first see that (R7) holds in $\mathfrak{Ra}\,\mathfrak{C}^{\xi\zeta}$. By Lemma 4.5(i) and since (R4) obviously holds in $\mathfrak{Ra}\,\mathfrak{C}^{\xi\zeta}$ we only have to check that $(x ;^{\xi\zeta} y)\cdot z \neq 0$ implies $(x^{\smile} ;^{\xi\zeta} z)\cdot y \neq 0$, and since $;^{\xi\zeta}$ and $^{\smile}$ are additive, it suffices to show this when x, y and z are atoms of $\mathfrak{Ra}\,\mathfrak{C}^{\xi\zeta}$. Also, since (R7) holds in $\mathfrak{Ra}\,\mathfrak{C}$ and compositions only got bigger in $\mathfrak{Ra}\,\mathfrak{C}^{\xi\zeta}$, we only have to consider atoms x, y and z for which $(x ; y)\cdot z = 0$ but $(x ;^{\xi\zeta} y)\cdot z \neq 0$. There are only two such triples of atoms, namely

$$x = y = z = \mathsf{c}_2^{\xi\zeta} R_1 \quad \text{and} \quad x = \mathsf{c}_2^{\xi\zeta} R_2,\ y = z = \mathsf{c}_2^{\xi\zeta} R_1$$

and we have

$$\big((\mathsf{c}_2^{\xi\zeta} R_1)^{\smile} ;^{\xi\zeta} \mathsf{c}_2^{\xi\zeta} R_1\big)\cdot \mathsf{c}_2^{\xi\zeta} R_1 = (\mathsf{c}_2^{\xi\zeta} R_2 ;^{\xi\zeta} \mathsf{c}_2^{\xi\zeta} R_1)\cdot \mathsf{c}_2^{\xi\zeta} R_1 = \mathsf{c}_2^{\xi\zeta} R_1 \neq 0$$

and

$$\big((\mathsf{c}_2^{\xi\zeta} R_2)^{\smile} ;^{\xi\zeta} \mathsf{c}_2^{\xi\zeta} R_1\big)\cdot \mathsf{c}_2^{\xi\zeta} R_1 = (\mathsf{c}_2^{\xi\zeta} R_1 ;^{\xi\zeta} \mathsf{c}_2^{\xi\zeta} R_1)\cdot \mathsf{c}_2^{\xi\zeta} R_1 = \mathsf{c}_2^{\xi\zeta} R_1 \neq 0,$$

whence $\mathfrak{Ra}\,\mathfrak{C}^{\xi\zeta} \models$ (R7). Finally, (R6) fails in $\mathfrak{Ra}\,\mathfrak{C}^{\xi\zeta}$, as shown by

$$\begin{aligned}(\mathsf{c}_2^{\xi\zeta} R_1 ;^{\xi\zeta} \mathsf{c}_2^{\xi\zeta} R_1)^{\smile} = (\mathsf{c}_2^{\xi\zeta} R_2 + \mathsf{c}_2^{\xi\zeta} R_1)^{\smile} = \mathsf{c}_2^{\xi\zeta} R_1 + \mathsf{c}_2^{\xi\zeta} R_2 \\ \neq \mathsf{c}_2^{\xi\zeta} R_1 = \mathsf{c}_2^{\xi\zeta} R_2 ;^{\xi\zeta} \mathsf{c}_2^{\xi\zeta} R_2 = \mathsf{c}_2^{\xi\zeta} R_1{}^{\smile} ;^{\xi\zeta} \mathsf{c}_2^{\xi\zeta} R_1{}^{\smile}.\end{aligned}$$

□

THEOREM 4.8. $\mathfrak{Ra}^{*}\,\mathsf{CA}_3 \not\models (\mathrm{R1}) \wedge (\mathrm{R4}) \Rightarrow (\mathrm{R6})$

PROOF. Let $\mathfrak{C}$ be the subalgebra of the full Cs_3 with base $\mathbb{Q}$ (the rationals) generated by the ordering $<$ (i.e., $\big\{\langle u,v,w\rangle : u,v,w \in \mathbb{Q}, u < v\big\}$). This algebra is described e.g. in Maddux [Mad91, p.

371]; it is atomic, and has thirteen atoms, including

$$\xi_0 = \{ \langle u, v, u \rangle : u, v \in \mathbb{Q}, u < v \}$$
$$\xi_1 = \{ \langle u, v, v \rangle : u, v \in \mathbb{Q}, u < v \}$$

and

$$\xi_2 = \{ \langle u, v, v \rangle : u, v \in \mathbb{Q}, u > v \},$$

and its RA-reduct has three atoms, $<$, $>$ (denoted by a and b in the rest of this proof) and the identity. It is easy to see that the sequence $\langle \xi_0, \xi_1, \xi_2 \rangle$ satisfies conditions (D1) and (D2) in 2.13. Let $\mathfrak{C}^\xi$ be the result of dilating $\mathfrak{C}$ with $\langle \xi_0, \xi_1, \xi_2 \rangle$. We claim that $\mathfrak{Ra}\,\mathfrak{C}^\xi \models$ (R1), (R4) but $\mathfrak{Ra}\,\mathfrak{C}^\xi \not\models$ (R6).

Let us observe first that $\mathfrak{Ra}\,\mathfrak{C}^\xi$ is also atomic, with atoms $\mathsf{c}_2^\xi a$, $\mathsf{c}_2^\xi b$ and $1' = \mathsf{D}_{01}$. By Lemma 2.16(ii), dilation does not change converse, that is, we have $(\mathsf{c}_2^\xi a)^\smile = \mathsf{c}_2^\xi b$, $(\mathsf{c}_2^\xi b)^\smile = \mathsf{c}_2^\xi a$ and of course $1'^\smile = 1'$ in $\mathfrak{Ra}\,\mathfrak{C}^\xi$. In particular, $\mathfrak{Ra}\,\mathfrak{C}^\xi \models$ (R4) (see also Theorem 2.17). By Lemma 2.16(iv), composition of the atoms in $\mathfrak{Ra}\,\mathfrak{C}^\xi$ is the same as in $\mathfrak{Ra}\,\mathfrak{C}$, except that $\mathsf{c}_2^\xi a \,;^\xi\, \mathsf{c}_2^\xi a = \mathsf{c}_2^\xi a + \mathsf{c}_2^\xi b$ (as opposed to $a\,;a = a$ in $\mathfrak{Ra}\,\mathfrak{C}$). Hence we have

$$(\mathsf{c}_2^\xi a \,;^\xi\, \mathsf{c}_2^\xi a)^\smile = (\mathsf{c}_2^\xi a + \mathsf{c}_2^\xi b)^\smile = \mathsf{c}_2^\xi b + \mathsf{c}_2^\xi a \neq \mathsf{c}_2^\xi b = \mathsf{c}_2^\xi b \,;^\xi\, \mathsf{c}_2^\xi b = (\mathsf{c}_2^\xi a)^\smile \,;^\xi\, (\mathsf{c}_2^\xi a)^\smile,$$

showing that (R6) fails in $\mathfrak{Ra}\,\mathfrak{C}^\xi$. It remains to show that $;^\xi$ is associative. For this it is enough to check that $(x \,;^\xi\, y)\,;^\xi\, z = x \,;^\xi\, (y \,;^\xi\, z)$ for $x, y, z \in \{\mathsf{c}_2^\xi a, \mathsf{c}_2^\xi b, 1'\}$. Now if $1' \in \{x, y, z\}$, or $|\{x, y, z\}| = 1$, then this is obvious. Otherwise, that is, if $\{x, y, z\} = \{\mathsf{c}_2^\xi a, \mathsf{c}_2^\xi b\}$, then

$$(x \,;^\xi\, y)\,;^\xi\, z = 1 = x \,;^\xi\, (y \,;^\xi\, z)$$

because this is clearly so in $\mathfrak{Ra}\,\mathfrak{C}$ (i.e., $(x\,;y)\,;z = 1 = x\,;(y\,;z)$ if $\{x, y, z\} = \{a, b\}$), and composition only got bigger in $\mathfrak{Ra}\,\mathfrak{C}^\xi$ by Lemma 2.16(iv). □

THEOREM 4.9. $\mathfrak{Ra}^*\,\mathsf{CA}_3 \not\models$ (R1) $\Rightarrow$ (R4)

PROOF. Consider the algebra $\mathfrak{A} = \mathfrak{Ra}\,\mathfrak{Cm}\,\mathfrak{B}''$ in the proof of Theorem 3.7 (we will denote composition and converse in $\mathfrak{A}$ by ; and $^\smile$). We have seen that (R4) fails in $\mathfrak{A}$ (e.g. we have $(\mathsf{c}_2\,\xi_0)^{\smile\smile} = \mathsf{c}_2\,\xi_2$), so we only have to check that composition in $\mathfrak{A}$ is associative. Let x, y, z be atoms in $\mathfrak{A}$. If $1' \in \{x, y, z\}$, then certainly $(x\,;y)\,;z = x\,;(y\,;z)$, so let's assume $1' \notin \{x, y, z\}$. Since $\mathfrak{A} \in \mathfrak{Ra}^*\,\mathsf{CA}_3$, $^\smile$ is onto by 1.20(i), so there is an element (in fact, an atom, by 1.20.(i)) z_0 in $\mathfrak{A}$ with $z_0{}^\smile = z$. Next, we have $x^\smile + z_0 \leq -1'$, because of the following. If $z_0 \cdot 1' \neq 0$, then we'd have $z_0 \geq 1'$ (since $1'$ is an atom by (2) in the proof of 3.7), whence $z = z_0{}^\smile \geq 1'$ by 1.20(iii) and since z is an atom, we get $z = 1'$, contradicting our assumption. To see that $x^\smile \leq -1'$, note that

$$x^\smile \cdot 1' = x^\smile \cdot 1'^\smile = (x \cdot 1')^\smile = 0$$

by 1.20(iii), 1.20(i) and the assumption x is an atom different from $1'$.

Hence, by (3) and (2) in the proof of 3.7 and (1) in the proof of 4.4 we get

$$(x\,;y)\,;z \geq -1'\,;z \geq z_0\,;z_0{}^\smile \geq -1' + 1' = 1$$

and

$$x\,;(y\,;z) \geq x\,;-1' \geq x\,;x^\smile \geq -1' + 1' = 1,$$

and since $\mathfrak{A}$ is atomic and ; is additive, this shows that ; is associative in $\mathfrak{A}$. □

The last independence result we need to prove in order to get a complete description of the dependencies between the RA-axioms within CA_3 is that (R6) does not imply (R7) in $\mathfrak{Ra}^*\mathsf{CA}_3$. The next theorem shows this, but it actually proves much more. First, it shows that (R6) does not even imply the following weakening of (R7):

(R7$^\smile$) $x = x^\smile \to x ; -(x ; y) \leq -y$

More importantly, Theorem 4.10 below is a specimen of a research direction which seeks answers to the following type of questions:

> What RA-type algebras can be generated by certain elements (or RA's) via a CA_3?

By a set X of elements of a CA_3 $\mathfrak{C}$ generating an RA-type algebra $\mathfrak{A}$ via $\mathfrak{C}$ we mean that $\mathfrak{A} = \mathfrak{Ra}\,\mathsf{Sg}^{(\mathfrak{C})}X$. 4.10 says that, via a CA_3, a representable RA can generate an RA-type algebra that is not even an RA. The specific RRA in the theorem below is generated by a single equivalence element, hence the result complements Jónsson's theorem in [Jón88], which states that in an RA, an equivalence element always generates a representable RA. By contrast, here we get that if generation goes via a CA_3, then the generated algebra need not even be an RA. In particular, Jónsson's theorem does not generalize to CA_3's.

THEOREM 4.10. *There are an* $\mathfrak{A} \in \mathsf{RRA}$ *and an* $\mathfrak{E} \in \mathsf{CA}_3$ *such that*

(i) $\mathfrak{A} \subseteq \mathfrak{Ra}\,\mathfrak{E}$, *A generates* $\mathfrak{E}$, *and* $\mathfrak{A}$ *is generated by a single equivalence element*;
(ii) (R6) *holds in* $\mathfrak{Ra}\,\mathfrak{E}$ *and* (R7$^\smile$) *fails in* $\mathfrak{Ra}\,\mathfrak{E}$.

We note that $\mathfrak{Ra}\,\mathfrak{E} \models$ (R6) implies $\mathfrak{Ra}\,\mathfrak{E} \models$ (R4), which, in turn, implies $\mathfrak{E} \models MGR$ by Theorem 4.4 and Lemma 1.13(ii).

PROOF. Let $U = \{0, 1, 0', 1'\}$ be the disjoint union of two copies of 2, and let G be the group of permutations of U generated by $\sigma = (01)(0'1')$ and $\rho = (00')(11')$. Let $\mathfrak{C} = \mathfrak{Fr}_G\langle \mathfrak{P}(^3U), \mathsf{C}_i, \mathsf{D}_{ij}\rangle_{i,j<3}$. Since G is transitive, $\mathsf{D}_{012} = \mathsf{D}_{01} \cap \mathsf{D}_{12}$ is an atom of $\mathfrak{C}$. Let $S = \langle 0, 1, 0\rangle^-$, $R = \langle 0, 1', 0\rangle^-$ and $Q = \langle 0, 0', 0\rangle^-$ (see Figure 1). First we claim that $\mathfrak{C}$ is generated by these four atoms, and in fact

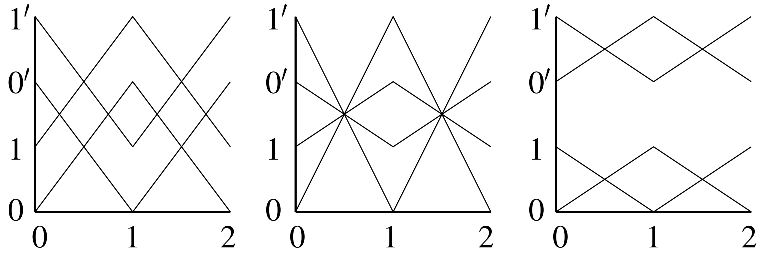
($\star$) each atom of $\mathfrak{C}$ is of the form $\mathsf{C}_0X \cap \mathsf{C}_2Y$ for some $X, Y \in \{\mathsf{D}_{012}, Q, R, S\}$.

For, if $s \in {}^3U$ then there are $X, Y \in \{\mathsf{D}_{012}, Q, R, S\}$ such that $s \in \mathsf{C}_0X \cap \mathsf{C}_2Y$, and of course this implies $s^- \subseteq \mathsf{C}_0X \cap \mathsf{C}_2Y$. But the reverse inclusion also holds, because of the following: Since X and Y are atoms (and hence orbits of the action of G on ${}^3U$), if $z \in \mathsf{C}_0X \cap \mathsf{C}_2Y$, then there are $\pi_1, \pi_2 \in G$ such that $\langle z_1, z_2\rangle = \pi_1\langle s_1, s_2\rangle$ and $\langle z_0, z_1\rangle = \pi_2\langle s_0, s_1\rangle$. G is semiregular (i.e., the identity is the only element of G which fixes a point), and π_1, π_2 agree on s_1, whence $\pi_1 = \pi_2$ and we get $z = \pi_1 s \in s^-$. This completes the proof of ($\star$), because all atoms of $\mathfrak{C}$ are of the form s^- for some $s \in {}^3U$.

We will obtain the algebra $\mathfrak{E}$ of the theorem by dilating $\mathfrak{C}$ twice, with the sequences of atoms ξ and ζ defined as

$$\xi_0 = \zeta_0 = \langle 0, 1', 0\rangle^- = R, \qquad \xi_1 = \zeta_1 = \langle 0, 1', 1'\rangle^- = \mathsf{D}_{12} \cap \mathsf{C}_2 R,$$
$$\xi_2 = \langle 0, 0', 0'\rangle^- = \mathsf{D}_{12} \cap \mathsf{C}_2 Q, \quad \text{and} \quad \zeta_2 = \langle 0, 1', 1'\rangle^- = \mathsf{D}_{12} \cap \mathsf{C}_2 R.$$

That both ξ and ζ satisfy condition (D1) in 2.13 is clear from the definition, and (D2) is a simple consequence of the fact that G is transitive. Hence we can dilate $\mathfrak{C}$ with ξ and ζ to get the CA_3's $\mathfrak{C}^\xi$ and $\mathfrak{C}^\zeta$. The CA_3 $\mathfrak{E}$ in the statement of the theorem will be $\mathfrak{C}^{\xi\zeta}$ (cf. Lemma 2.21).



Q R S

FIGURE 1. The generators of $\mathfrak{C}$

Notation. We denote cylindrifications in $\mathfrak{C}$, $\mathfrak{C}^\xi$, $\mathfrak{C}^\zeta$ and $\mathfrak{E} = \mathfrak{C}^{\xi\zeta}$ by C, c^ξ, c^ζ and c respectively. The corresponding accessibility relations (on the atoms of these algebras) will be denoted by T, T^ξ, T^ζ and $T^{\mathfrak{E}}$. Composition in $\mathfrak{Ra}\,\mathfrak{C}$ will be denoted by $\circ$ (to emphasize that this is concrete, set-theoretic composition of binary relations) and by $;^\xi$, $;^\zeta$ and $;$ in the RA-reducts of $\mathfrak{C}^\xi$, $\mathfrak{C}^\zeta$ and $\mathfrak{E}$. (Using the double superscript $^{\xi\zeta}$ on the operations of $\mathfrak{E}$ would be more in line with the formulation of 2.21 and 2.22 but we hope that its omission will not bee too confusing.) The diagonals D_{ij}, $i \neq j$ of these algebras are the same, so we need not give them different names (the D_{ii}'s are different, but we'll never mention them). We denote the new elements corresponding to ξ and ζ by ν and μ, respectively.

With the help of the results in section 2.2 it is easy to describe the RA-reduct of the algebras $\mathfrak{C}^\xi$, $\mathfrak{C}^\zeta$ and $\mathfrak{E}$, as follows. Since Q, R, S and D_{012} are atoms in $\mathfrak{C}$, $\mathsf{C}_2\,Q$, $\mathsf{C}_2\,R$, $\mathsf{C}_2\,S$ and $\mathsf{C}_2\,\mathsf{D}_{012} = \mathsf{D}_{01}$ are atoms in $\mathfrak{Ra}\,\mathfrak{C}$ (by 1.19), and since their sum is 1, they are the only atoms in $\mathfrak{Ra}\,\mathfrak{C}$.[1] Hence the RA-reducts of the other three algebras also have four atoms, the identity and the ones displayed in Table 2.

It is clear that the four atoms of $\mathfrak{Ra}\,\mathfrak{C}$ are symmetric, and that they commute with each other: in fact the composition of any two distinct non-identity atoms is the third non-identity atom. From this it follows by Lemma 2.19 that $\mathfrak{Ra}\,\mathfrak{C}^\xi$, $\mathfrak{Ra}\,\mathfrak{C}^\zeta$ and $\mathfrak{Ra}\,\mathfrak{E}$ are also symmetric and commutative, since condition (1) in 2.19 is easily seen to hold for both ξ and ζ in $\mathfrak{C}$ and hence in $\mathfrak{C}^\xi$ (since $T^\xi \supseteq T$). To complete the picture we have to describe how composition behaves on the atoms of the RA-reducts of the dilated algebras (cf. Table 2). Let us first do this for $\mathfrak{Ra}\,\mathfrak{C}^\xi$. First of all, since the RA axiom (R3) holds in the RA-reduct of every CA_3 (Lemma 4.1), we only have to deal with compositions of non-identity atoms. From Lemma 2.16(iv) we see that we can almost copy the multiplication table of $\mathfrak{Ra}\,\mathfrak{C}$, i.e., that for all $a, b \in \{Q, R, S\}$, $\mathsf{c}_2^\xi a ;^\xi \mathsf{c}_2^\xi b = \mathsf{c}_2^\xi(\mathsf{c}_2\, a ; \mathsf{c}_2\, b)$, except when $\xi_1\ T_2\ a$ and $\xi_0\ T_2\ b$. But this holds iff $a = R = b$. Hence the only composition that changes is the composition of $\mathsf{c}_2^\xi R$ with itself, and we have

$$\mathsf{c}_2^\xi R ;^\xi \mathsf{c}_2^\xi R = \mathsf{c}_2^\xi \mathsf{D}_{01} + \mathsf{c}_2^\xi \xi_2 = \mathsf{D}_{01} + \mathsf{c}_2^\xi Q$$

by Lemma 2.16(iv). Now since $\zeta_0 = \xi_0$ and $\zeta_1 = \xi_1$, the situation is analogous in $\mathfrak{Ra}\,\mathfrak{C}^\zeta$ in that here again only $\mathsf{c}_2^\zeta R ;^\zeta \mathsf{c}_2^\zeta R$ changes, but in $\mathfrak{Ra}\,\mathfrak{C}^\zeta$ Lemma 2.16(iv) gives

$$\mathsf{c}_2^\zeta R ;^\zeta \mathsf{c}_2^\zeta R = \mathsf{c}_2^\zeta \mathsf{D}_{01} + \mathsf{c}_2^\zeta \zeta_2 = \mathsf{D}_{01} + \mathsf{c}_2^\zeta R.$$

[1] Incidentally, $\mathfrak{Ra}\,\mathfrak{C}$ is an example of what might be called a *perfectly symmetric* RRA: it is Galois closed, and its 4 atoms are exactly the members of its group of base automorphisms (D_{01} is the identity function on U, $\mathsf{C}_2\,Q = \rho$, $\mathsf{C}_2\,S = \sigma$, and $\mathsf{C}_2\,R = \rho\sigma = \sigma\rho$), the Klein group.

With this we have described composition in $\mathfrak{Ra}\,\mathfrak{C}^\xi$ and $\mathfrak{Ra}\,\mathfrak{C}^\zeta$. Since $\langle \xi_2, \zeta_2\rangle \notin T_2$, by Corollary 2.22(iii) an entry in the multiplication table of $\mathfrak{Ra}\,\mathfrak{C}$ is just the sum of the corresponding entries in the tables of $\mathfrak{Ra}\,\mathfrak{C}^\xi$ and $\mathfrak{Ra}\,\mathfrak{C}^\zeta$. We treat two cases in detail:

$$\begin{aligned} \mathsf{c}_2 S\,;\mathsf{c}_2 R &= \mathsf{c}_2^\xi S\,;^\xi \mathsf{c}_2^\xi R + \mathsf{c}_2^\zeta S\,;^\zeta \mathsf{c}_2^\zeta R && \text{by Corollary 2.22(iii)}\\ &= \mathsf{c}_2^\xi Q + \mathsf{c}_2^\zeta Q && \text{by the tables for } ;^\xi \text{ and } ;^\zeta\\ &= \mathsf{c}_2 Q && \text{by Corollary 2.22(i).} \end{aligned}$$

Note that all compositions can be computed in this simple manner except for

$$\begin{aligned} \mathsf{c}_2 R\,;\mathsf{c}_2 R &= \mathsf{c}_2^\xi R\,;^\xi \mathsf{c}_2^\xi R + \mathsf{c}_2^\zeta R\,;^\zeta \mathsf{c}_2^\zeta R && \text{by Corollary 2.22(iii)}\\ &= \mathsf{D}_{01} + \mathsf{c}_2^\xi Q + \mathsf{c}_2^\zeta R && \text{by the tables for } ;^\xi \text{ and } ;^\zeta\\ &= \mathsf{D}_{01} + \mathsf{c}_2^\xi Q + \mathsf{c}_2^\zeta Q + \mathsf{c}_2^\xi R + \mathsf{c}_2^\zeta R \end{aligned}$$

since $\mathsf{c}_2^\zeta Q = \mathsf{C}_2 Q \le \mathsf{c}_2^\xi Q$ and $\mathsf{c}_2^\xi R = \mathsf{C}_2 R \le \mathsf{c}_2^\zeta R$

$$= \mathsf{D}_{01} + \mathsf{c}_2 Q + \mathsf{c}_2 R \qquad \text{by Corollary 2.22(i).}$$

We are ready to establish (ii) of the theorem. Since $\mathfrak{Ra}\,\mathfrak{C}$ is symmetric and commutative, (R6) obviously holds in it. But (R7˘) fails because of the following: We have $(\mathsf{c}_2 R\,;\mathsf{c}_2 Q)\cdot \mathsf{c}_2 R = \mathsf{c}_2 S \cdot \mathsf{c}_2 R = 0$ by the table for ;, and hence

(1) $$\mathsf{c}_2 R \le -(\mathsf{c}_2 R\,;\mathsf{c}_2 Q).$$

So

$$\begin{aligned} 0 &\ne \mathsf{c}_2 Q\\ &= (\mathsf{c}_2 R\,;\mathsf{c}_2 R)\cdot \mathsf{c}_2 Q && \text{by the table for ;}\\ &\le \big(\mathsf{c}_2 R\,;-(\mathsf{c}_2 R\,;\mathsf{c}_2 Q)\big)\cdot \mathsf{c}_2 Q, && \text{by (1)} \end{aligned}$$

that is,

$$\mathsf{c}_2 R\,;-(\mathsf{c}_2 R\,;\mathsf{c}_2 Q) \nleq -(\mathsf{c}_2 Q)$$

although $(\mathsf{c}_2 R)^\smile = \mathsf{c}_2 R$. Hence the instance of (R7˘) when $x = \mathsf{c}_2 R$ and $y = \mathsf{c}_2 Q$ fails in $\mathfrak{Ra}\,\mathfrak{C}$.

Next we show that the equivalence element $\mathsf{c}_2 S$ generates a representable RA in $\mathfrak{Ra}\,\mathfrak{C}$. Let $\mathfrak{A}$ be the algebra generated by $\mathsf{c}_2 S$ in $\mathfrak{Ra}\,\mathfrak{C}$. Of course we cannot use Jónsson's theorem ([Jón88]) which says that the algebra generated by an equivalence element in an RA is always representable, because, as we have just seen, $\mathfrak{Ra}\,\mathfrak{C} \notin \mathsf{RA}$. But we can easily describe $\mathfrak{A}$ and show that it is representable, as follows.

It is clear that the BA generated by $\mathsf{c}_2 S$ and D_{01} is atomic and has three atoms, $\mathsf{c}_2 S$, D_{01} and $\mathsf{c}_2(R+Q)$. We claim that its universe A is closed under the RA-operations. Since $\mathsf{D}_{01} \in A$ and converse is the identity operation in $\mathfrak{Ra}\,\mathfrak{C}$, we only have to show that the composition of any two non-identity atoms is in A. But this can be read off the table of ; (Table 2) using distributivity of cylindrification and composition, as follows:

$$\begin{aligned} \mathsf{c}_2(R+Q)\,;\mathsf{c}_2(R+Q) &\ge \mathsf{c}_2 R\,;\mathsf{c}_2 R + \mathsf{c}_2 R\,;\mathsf{c}_2 Q\\ &= \mathsf{D}_{01} + \mathsf{c}_2 Q + \mathsf{c}_2 R + \mathsf{c}_2 S\\ &= 1, \end{aligned}$$

$\circ$	$\mathsf{C}_2 Q$	$\mathsf{C}_2 R$	$\mathsf{C}_2 S$
$\mathsf{C}_2 Q$	D_{01}	$\mathsf{C}_2 S$	$\mathsf{C}_2 R$
$\mathsf{C}_2 R$	$\mathsf{C}_2 S$	D_{01}	$\mathsf{C}_2 Q$
$\mathsf{C}_2 S$	$\mathsf{C}_2 R$	$\mathsf{C}_2 Q$	D_{01}

$;^\xi$	$\mathsf{c}_2^\xi Q$	$\mathsf{c}_2^\xi R$	$\mathsf{c}_2^\xi S$
$\mathsf{c}_2^\xi Q$	D_{01}	$\mathsf{c}_2^\xi S$	$\mathsf{c}_2^\xi R$
$\mathsf{c}_2^\xi R$	$\mathsf{c}_2^\xi S$	$\mathsf{D}_{01}+\mathsf{c}_2^\xi Q$	$\mathsf{c}_2^\xi Q$
$\mathsf{c}_2^\xi S$	$\mathsf{c}_2^\xi R$	$\mathsf{c}_2^\xi Q$	D_{01}

$;^\zeta$	$\mathsf{c}_2^\zeta Q$	$\mathsf{c}_2^\zeta R$	$\mathsf{c}_2^\zeta S$
$\mathsf{c}_2^\zeta Q$	D_{01}	$\mathsf{c}_2^\zeta S$	$\mathsf{c}_2^\zeta R$
$\mathsf{c}_2^\zeta R$	$\mathsf{c}_2^\zeta S$	$\mathsf{D}_{01}+\mathsf{c}_2^\zeta R$	$\mathsf{c}_2^\zeta Q$
$\mathsf{c}_2^\zeta S$	$\mathsf{c}_2^\zeta R$	$\mathsf{c}_2^\zeta Q$	D_{01}

$;$	$\mathsf{c}_2 Q$	$\mathsf{c}_2 R$	$\mathsf{c}_2 S$
$\mathsf{c}_2 Q$	D_{01}	$\mathsf{c}_2 S$	$\mathsf{c}_2 R$
$\mathsf{c}_2 R$	$\mathsf{c}_2 S$	$\mathsf{D}_{01}+\mathsf{c}_2 Q+\mathsf{c}_2 R$	$\mathsf{c}_2 Q$
$\mathsf{c}_2 S$	$\mathsf{c}_2 R$	$\mathsf{c}_2 Q$	D_{01}

TABLE 2. Composition in the RA-reducts in the proof of 4.10

$;$	$\mathsf{c}_2(Q+R)$	$\mathsf{c}_2 S$
$\mathsf{c}_2(Q+R)$	1	$\mathsf{c}_2(Q+R)$
$\mathsf{c}_2 S$	$\mathsf{c}_2(Q+R)$	D_{01}

TABLE 3. Composition in $\mathfrak{A}$

$$\begin{aligned}\mathsf{c}_2 S ; \mathsf{c}_2(R+Q) &= \mathsf{c}_2 S ; (\mathsf{c}_2 R + \mathsf{c}_2 Q)\\ &= \mathsf{c}_2 S ; \mathsf{c}_2 R + \mathsf{c}_2 S ; \mathsf{c}_2 Q\\ &= \mathsf{c}_2 Q + \mathsf{c}_2 R\\ &= \mathsf{c}_2(Q+R),\end{aligned}$$

and

$$\mathsf{c}_2 S ; \mathsf{c}_2 S = \mathsf{D}_{01}.$$

Since composition is commutative in $\mathfrak{Ra}\,\mathfrak{C}$, we have shown that A is indeed closed under the RA-operations, and that composition of the non-identity atoms in $\mathfrak{A}$ (the subalgebra of $\mathfrak{Ra}\,\mathfrak{C}$ with universe A) is as in Table 3. Hence $\mathfrak{A}$ is isomorphic to the RRA on a 6-element set generated by an equivalence relation with three two-element blocks (see Figure 2).

It remains to show that A generates $\mathfrak{C}$. Now since $\mathfrak{C}$ is finite, this boils down to showing that the atoms of $\mathfrak{C}$ are generated by A; but by (*), each atom of $\mathfrak{C}$ that is different from ν and μ is of the form $-(\nu+\mu)\cdot \mathsf{c}_0 X \cdot \mathsf{c}_2 Y$ for some X, $Y \in \{\mathsf{D}_{012}, Q, R, S\}$. Hence it suffices to show that $\{R, S, Q, \nu, \mu\} \subseteq \mathsf{Sg}^{(\mathfrak{C})} A$.

First, $S = \mathsf{D}_{02}\cdot \mathsf{c}_2 S$, whence $S \in \mathsf{Sg}^{(\mathfrak{C})} A$. It is easy to see that

(2) $$\mathfrak{C} \models \mathsf{D}_{01}+\mathsf{D}_{12}+\mathsf{D}_{02}+\mathsf{C}_0 S+\mathsf{C}_2 S+\mathsf{C}_1(\mathsf{D}_{12}\cdot \mathsf{C}_2 S) = 1,$$

since S, viewed as a binary relation on U is an equivalence relation with exactly two equivalence classes. As S is not T_i related to ξ_i or ζ_i for any $i < 3$, by Corollary 2.22(i) we have $\mathsf{c}_i S = \mathsf{C}_i S$, and

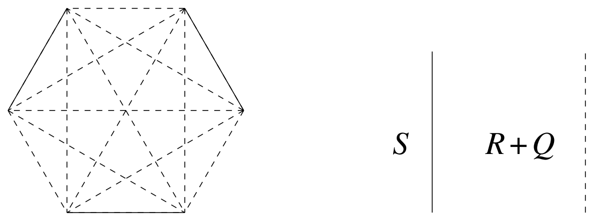


FIGURE 2. A representation of $\mathfrak{A}$

since $\mathsf{D}_{12} \cdot \mathsf{C}_2 S$ is not T_1 related to ξ_1 or ζ_1, we have $\mathsf{c}_1(\mathsf{D}_{12} \cdot \mathsf{c}_2 S) = \mathsf{C}_1(\mathsf{D}_{12} \cdot \mathsf{C}_2 S)$. Hence from (2) we get

$$\nu + \mu = 1^{\mathfrak{C}} - 1^{\mathfrak{C}} = -\big(\mathsf{D}_{01} + \mathsf{D}_{12} + \mathsf{D}_{02} + \mathsf{c}_0 S + \mathsf{c}_2 S + \mathsf{c}_1(\mathsf{D}_{12} \cdot \mathsf{c}_2 S)\big)$$

and thus $\nu + \mu \in \mathsf{Sg}^{(\mathfrak{C})} A$. Next we claim that

$$\mathsf{D}_{02} \cdot \mathsf{c}_0(\nu + \mu) = R. \tag{3}$$

This is because for every $b \in E_{02}$ we have

$$\begin{array}{lll} & b\, T_0^{\mathfrak{C}}\, \nu & \\ \text{iff} & b\, T_0^{\xi}\, \nu & \text{since } b, \nu \in B^{\xi} \\ \text{iff} & b\, T_0\, \xi_0 & \text{by def. of } T^{\xi} \\ \text{iff} & b = \xi_0 = R & \text{by 2.6(v), since } b, \xi_0 \in E_{02} \end{array}$$

and

$$\begin{array}{lll} & b\, T_0^{\mathfrak{C}}\, \mu & \\ \text{iff} & b\, T_0^{\xi}\, \zeta_0 & \text{by def. of } T^{\mathfrak{C}} \\ \text{iff} & b\, T_0\, \zeta_0 & \text{since } b, \zeta_0 \in B \\ \text{iff} & b = \zeta_0 = R & \text{by 2.6(v), since } b, \zeta_0 \in E_{02}. \end{array}$$

Hence R is the unique atom below $\mathsf{D}_{02} \cdot \mathsf{c}_0(\nu + \mu)$, so (3) holds and we have $R \in \mathsf{Sg}^{(\mathfrak{C})} A$.

Since $R\, T_2\, \zeta_2$, we have $R\, T_2^{\xi}\, \zeta_2$ and hence $R\, T_2^{\mathfrak{C}}\, \mu$; and since $\neg R\, T_2\, \xi_2$, we have $\neg R\, T_2^{\xi}\, \nu$, and hence $\neg R\, T_2^{\mathfrak{C}}\, \nu$ because $R, \nu \in B^{\xi}$. This shows that $\mu = (\nu + \mu) \cdot \mathsf{c}_2 R$, and hence both μ and $\nu = (\nu + \mu) \cdot -\mu$ are in $\mathsf{Sg}^{(\mathfrak{C})} A$. Finally, $Q\, T_2\, \xi_2$ implies $Q\, T_2^{\xi}\, \nu$, which, in turn, implies $Q\, T_2^{\mathfrak{C}}\, \nu$; and since $Q \in E_{02}$, we get $Q = \mathsf{D}_{02} \cdot \mathsf{c}_2 \nu \in \mathsf{Sg}^{(\mathfrak{C})} A$ by Lemma 2.10(i).

With this we have shown that $\{R, S, Q, \nu, \mu\} \subseteq \mathsf{Sg}^{(\mathfrak{C})} A$; hence A generates $\mathfrak{C}$, and the proof of the theorem is complete. $\square$

THEOREM 4.11. *Let* $E \cup \{\varphi\} \subseteq \{\text{(R1)}, \text{(R4)}, \text{(R6)}, \text{(R7)}\}$, $\varphi \notin E$. *Then* $\mathfrak{Ra}^* \mathsf{CA}_3 \cap \mathrm{Mod} E \models \varphi$ *iff* $\mathbf{S}\mathfrak{Ra}^* \mathsf{CA}_3 \cap \mathrm{Mod} E \models \varphi$ *iff one of the following conditions holds*:

(i) $\{\text{(R6)}, \text{(R7)}\} \cap E \neq \emptyset$ *and* φ *is* (R4);
(ii) $\{\text{(R1)}, \text{(R7)}\} \subseteq E$ *and* φ *is* (R6);
(iii) $\{\text{(R1)}, \text{(R6)}\} \subseteq E$ *and* φ *is* (R7).

PROOF. That the three conditions are sufficient (i.e. both (R6) and (R7) imply (R4) in $\mathbf{S}\mathfrak{Ra}^*\,\mathsf{CA}_3$ etc.) follows from Theorem 4.3, 4.4(i) (for condition (i)), Theorem 4.6 (for condition (ii)) and Theorem 4.4(ii) (for condition (iii)).

Now suppose that $\mathfrak{Ra}^*\,\mathsf{CA}_3 \cap \mathrm{Mod}\,E \models \varphi$, $\varphi \notin E$. Then φ cannot be (R1), since (R1) is not implied by {(R4), (R6), (R7)} in $\mathbf{S}\mathfrak{Ra}^*\,\mathsf{CA}_3$ (this is because $\mathsf{SA} \neq \mathsf{RA}$, proved in Maddux [Mad78], [Mad82]). If φ is (R4), then E must contain either (R6) or (R7) by Theorem 4.9. If φ is (R6), then E must contain (R7) by Theorem 4.8 and (R1) by Theorems 4.3, 4.7. Finally, if φ is (R7), then E must contain (R6) by Theorems 4.6, 4.8, and E must contain (R1) by Theorem 4.10. □

COROLLARY 4.12. *A subset of the* RA-*axioms defines* RA *within* $\mathfrak{Ra}^*\,\mathsf{CA}_3$ (*or within* $\mathbf{S}\mathfrak{Ra}^*\,\mathsf{CA}_3$) *iff it includes* {(R1), (R6)} *or* {(R1), (R7)}. *That is, for all* $E \subseteq \{(\mathrm{R0}), \ldots, (\mathrm{R7})\}$,

$$\begin{aligned}\mathsf{RA} = \mathfrak{Ra}^*\,\mathsf{CA}_3 \cap \mathrm{Mod}\,E &\iff \mathsf{RA} = \mathbf{S}\mathfrak{Ra}^*\,\mathsf{CA}_3 \cap \mathrm{Mod}\,E\\ &\iff E \supseteq \{(\mathrm{R1}), (\mathrm{R6})\} \quad \textit{or} \quad E \supseteq \{(\mathrm{R1}), (\mathrm{R7})\}.\end{aligned}$$

COROLLARY 4.13. *A subset of the* SA-*axioms defines* SA *within* $\mathfrak{Ra}^*\,\mathsf{CA}_3$ (*or within* $\mathbf{S}\mathfrak{Ra}^*\,\mathsf{CA}_3$) *iff it contains both* (R6) *and* (R7). *I.e., for all* $E \subseteq \{(\mathrm{R0}), (\mathrm{R1}_0), (\mathrm{R2}), \ldots, (\mathrm{R7})\}$,

$$\mathsf{SA} = \mathfrak{Ra}^*\,\mathsf{CA}_3 \cap \mathrm{Mod}\,E \iff \mathsf{SA} = \mathbf{S}\mathfrak{Ra}^*\,\mathsf{CA}_3 \cap \mathrm{Mod}\,E \iff \{(\mathrm{R6}), (\mathrm{R7})\} \subseteq E.$$

Corollary 4.13 shows an interesting property of the axiom sets {(R1), (R6)} and {(R1), (R7)}. All known axiomatizations of RA are such that replacing (R1) by $(\mathrm{R1}_0)$ in them yields a set defining SA. But by Corollaries 4.12 and 4.13 we have $\mathsf{RA} = \mathfrak{Ra}^*\,\mathsf{CA}_3 \cap \mathrm{Mod}\{(\mathrm{R1}), (\mathrm{R6})\}$ and $\mathsf{SA} \neq \mathfrak{Ra}^*\,\mathsf{CA}_3 \cap \mathrm{Mod}\{(\mathrm{R6})\} (= \mathfrak{Ra}^*\,\mathsf{CA}_3 \cap \mathrm{Mod}\{(\mathrm{R1}_0), (\mathrm{R6})\})$, so the above mentioned common phenomenon is by no means necessary.

PROBLEM 4.14. Is there a reason why the distinct classes $\mathfrak{Ra}^*\,\mathsf{CA}_3$ and $\mathbf{S}\mathfrak{Ra}^*\,\mathsf{CA}_3$ behave the same way wrt. the properties investigated in this chapter? More concretely, is there a simple sufficient condition for sets of equations $E \cup \{e\}$ such that $\mathfrak{Ra}^*\,\mathsf{CA}_3 \cap \mathrm{Mod}(E) \models e$ iff $\mathbf{S}\mathfrak{Ra}^*\,\mathsf{CA}_3 \cap \mathrm{Mod}(E) \models e$?

In connection with this problem cf. also Problem 3.10.

CHAPTER 5

Looking at the connections backwards

In the previous chapter we investigated the problem of axiomatizing RA's and SA's over the class of RA-reducts of cylindric algebras. The starting point of the present chapter is the equally natural problem of finding sets of natural CA-theoretic equations E and E' such that

$$\mathsf{SA} = \mathfrak{Ra}^*(\mathsf{CA}_3 \cap \mathrm{Mod}\, E) \tag{1}$$

$$\mathsf{RA} = \mathfrak{Ra}^*(\mathsf{CA}_3 \cap \mathrm{Mod}\, E'). \tag{2}$$

In connection with the second question we note the following: Maddux [Mad78] proved that $\mathsf{RA} = \mathfrak{Ra}^*(\mathbf{S}\,\mathbf{Nr}_3\,\mathsf{CA}_4)$, and Andréka [And] gave a finite set of equations axiomatizing $\mathbf{S}\,\mathbf{Nr}_3\,\mathsf{CA}_4$. This set of equations is recalled in Németi-Simon [NS97]. So Andréka's set of equations provides a set E' satisfying (2). For this reason, in this chapter we will concentrate on the first question.

It is important in this question that E be simple and natural in the language of CA's, for otherwise a mechanical translation of the SA axioms would do automatically the job. Such a translation might not be illuminating, it might look like a "Gödel number" from the CA-theoretic point of view. Actually, Maddux [Mad78, Theorem 19 (p. 150)] shows that indeed, the translation of the cycle law yields a set E that satisfies (1).

To obtain a natural set E of CA-equations satisfying the above desires, we start out with the *MGR*-equations (see Definition 1.12). The point here is that there does exist a natural, cylindric algebraic intuition behind *MGR*. In Definition 5.1 below we will introduce a strengthened version MGR^+ of *MGR* such that the intuition behind MGR^+ is the same as that behind *MGR*, but it is extended to more elements of the algebra. We will prove that MGR^+ is a good choice for E in (1) above, and in fact we have the following two stronger results (see 5.16)[1]: (i) For every $\mathfrak{C} \in \mathsf{CA}_3$, $\mathfrak{C}$ is a subreduct (i.e., subalgebra of a reduct) of a PEA_3 iff the RA-reduct of $\mathfrak{C}$ is an SA iff $\mathfrak{C} \models MGR^+$; and (ii) for every binary generated CA_3, $\mathfrak{C}$ is the reduct of a PEA_3 iff the RA-reduct of $\mathfrak{C}$ is an SA iff $\mathfrak{C} \models MGR^+$. As a corollary, we will find (see 5.18) that $\mathsf{SA} = \mathfrak{Ra}^*\,\mathsf{PEA}_3$.

There is an other respect in which the subvariety of CA_3 defined by MGR^+ seems better suited to the study of binary relations in 3-dimensional algebras than that defined by *MGR*. Consider the following statements about $K \subseteq \mathsf{CA}_3$:

(*) In members of K, the cylindric operations do not generate more c_2-closed elements from c_2-closed elements than the relation algebraic operations do. That is, $\mathsf{Sg}^{(\mathfrak{Ra}\,\mathfrak{C})}X = \mathfrak{Ra}\,\mathsf{Sg}^{(\mathfrak{C})}X$ for any $\mathfrak{C} \in K$ and $X \subseteq \mathfrak{Ra}\,\mathfrak{C}$.

(**) For any algebra $\mathfrak{A}$ there is at most one $\mathfrak{C} \in K$ (up to isomorphism) such that $\mathfrak{A} = \mathfrak{Ra}\,\mathfrak{C}$ and A generates $\mathfrak{C}$.

We note that (*) and (**) imply (***) below:

(***) For any algebra $\mathfrak{A}$ there is at most one $\mathfrak{C} \in K$ (up to isomorphism) such that $\mathfrak{A} \subseteq \mathfrak{Ra}\,\mathfrak{C}$ and A generates $\mathfrak{C}$.

[1] PEA_3 is the class of 3-dimensional polyadic equality algebras (defined in 5.10 below).

Theorems 5.3.12, 5.3.15 in [HMT85] state that (*) and (**) are true for $K = \mathbf{SNr}_3\,\mathsf{CA}_4$ (this was first proved by Monk [Mon61]), and Theorems 7,8 in Maddux [Mad78] state that (*) and (**) are true for $K = \mathsf{CA}_3'$, where

$$\mathsf{CA}_3' = \{\mathfrak{C} \in \mathsf{CA}_3 : \mathfrak{Ra}\,\mathfrak{C} \in \mathsf{SA}\}.$$

Theorem 4.10 shows that neither of these statements holds for $K = \mathsf{CA}_3 \cap \mathrm{Mod}\,MGR$. But, by 5.8 below, $\mathsf{CA}_3' = \mathsf{CA}_3 \cap \mathrm{Mod}\,MGR^+$, and therefore (*) and (**) hold for $\mathsf{CA}_3 \cap \mathrm{Mod}\,MGR^+$ (see 5.9).

Now we define MGR^+ and an equivalent variant of it. In Remark 5.2 we discuss the intuitive meaning of these formulas.

DEFINITION 5.1. *Let* $\{k,l,m\} = 3$. MGR^+_{klm} *is the equation*

$${}_k\mathsf{s}(l,m)\,\mathsf{c}_k(\mathsf{c}_l\,x \cdot \mathsf{c}_m\,y) = \mathsf{c}_k(\mathsf{s}^m_l\,\mathsf{c}_l\,x \cdot \mathsf{s}^l_m\,\mathsf{c}_m\,y),$$

MGR^*_{klm} *is the pair of quasi-equations*

$$\textstyle\prod_{i<3}\mathsf{c}_i x_i = 0 \;\leftrightarrow\; {}_k\mathsf{s}(l,m)\,\mathsf{c}_k x_k \cdot \mathsf{s}^m_l\,\mathsf{c}_l x_l \cdot \mathsf{s}^l_m\,\mathsf{c}_m x_m = 0$$

and MGR^+ *denotes the set of equations* $\{\, MGR^+_{klm} : \{k,l,m\} = 3\,\}$.

REMARK 5.2. In order to understand the meaning of MGR^+, it is useful to recall the notion of n-dimensional polyadic equality algebras (PEA_n for short). We will define polyadic equality algebras in 5.10; for the moment, it will suffice to say that these are cylindric algebras expanded with new unary operations s_τ (called *substitutions*) for every $\tau \in {}^n n$ (that is, for every transformation τ of n); and that representable PEA's are like representable cylindric algebras, with the new operations acting as follows: $\mathsf{s}_\tau X = \{\, s \in {}^n U : s \circ \tau \in X\,\}$, where U is the base of the algebra. So, for example, if τ is the permutation of n interchanging 0 and 1 (and leaving all $1 < j < n$ fixed), then $\mathsf{s}_\tau X$ is the relation we get from X by permuting its zeroth and first argument.

It is well known that these new operations are not term definable in CA_n; in fact a CA_n may not even be closed under all of them (see e.g. 5.17 below). However, if n is finite, then for those elements that are closed under at least one cylindrification (i.e., for "$n-1$-ary elements"), all substitutions can be simulated by cylindric operations. The reason is that

(i) the semigroup of transformations of n is generated by transpositions $\{\,(ij) : i,j < n\,\}$ and replacements $\{\,[i/j] : i,j < n\,\}$, where

$$[i/j](k) = \begin{cases} j & \text{if } k = i \\ k & \text{otherwise;} \end{cases}$$

(ii) for all transformation τ of n, s_τ is definable from transpositions and replacements, by (i) and since the equation schema $\mathsf{s}_\tau\,\mathsf{s}_\sigma X = \mathsf{s}_{\tau\sigma} X$ is valid;

(iii) the substitutions corresponding to transpositions and replacements are expressible for $n-1$-ary elements.

The substitution corresponding to the transposition (ij) is often denoted by p_{ij} (instead of $\mathsf{s}_{(ij)}$). Now, $\mathsf{s}_{[i/j]}X$ is just $\mathsf{s}^i_j X$, as defined in 1.10, and this definitition works for all X. We show how to define p_{ij} for c_k-closed elements; for simplicity we assume $n = 3$. If $\{i,j,k\} = 3$, then $\mathsf{p}_{ij}\,\mathsf{c}_k X = {}_k\mathsf{s}(i,j)\,\mathsf{c}_k X$, $\mathsf{p}_{ij}\,\mathsf{c}_i X = \mathsf{s}^j_i\,\mathsf{c}_i X$, and $\mathsf{p}_{ij}\,\mathsf{c}_j X = \mathsf{s}^i_j\,\mathsf{c}_j X$; it is easy to see that these definitions work in representable algebras. (For example, $\mathsf{s}^2_0 X$ is indeed what we get by interchanging the zeroth and second arguments of a c_0-closed relation X.)

Defining substitutions this way for $n-1$-ary elements in CA_n's we may ask how much of the polyadic laws are derivable from the CA-axioms. The answer is that without further assumptions, no interesting polyadic equations are derivable. Assuming the MGR equations makes the equation cited in (ii) derivable. In fact, it can be shown (cf. Thompson [Tho93]) that all polyadic equations that are "on the semigroup level" (i.e. those which reflect properties of the semigroup of transformations of n) follow from the CA_n axioms plus MGR. (If $n > 3$, then MGR stands for ${}_3MGR$ and ${}_4MGR$, as defined in 1.14.) But, even assuming MGR, there remain polyadic equations that are not derivable. To obtain an example, consider the polyadic equation

$$(*) \qquad \mathsf{p}_{02}\,\mathsf{c}_1(x\cdot y)=\mathsf{c}_1(\mathsf{p}_{02}\,x\cdot \mathsf{p}_{02}\,y).$$

This is true in representable PEA's (and in fact in PEA, because p_{02} is a Boolean endomorphism which commutes with c_1). Although there seems to be no way of expressing (*) in CA, the special case

$$(**) \qquad x=\mathsf{c}_0\,x\wedge y=\mathsf{c}_2\,y\longrightarrow \mathsf{p}_{02}\,\mathsf{c}_1(x\cdot y)=\mathsf{c}_1(\mathsf{p}_{02}\,x\cdot \mathsf{p}_{02}\,y)$$

of (*) *can* be expressed because of what we said above about the definability of p_{ij} for $n-1$-ary elements. That is, since $\mathsf{c}_1(x\cdot y)$ is c_1-closed, we may write ${}_1\mathsf{s}(0,2)$ in place of the first occurrence of p_{02} in (**), and, using the assumption that x is c_0-closed and y is c_2-closed, we can simulate p_{02} by s^2_0 on x, and by s^0_2 on y. What we get then is the cylindric algebraic quasi-equation

$$x=\mathsf{c}_0\,x\wedge y=\mathsf{c}_2\,y\longrightarrow {}_1\mathsf{s}(0,2)\,\mathsf{c}_1(x\cdot y)=\mathsf{c}_1(\mathsf{s}^2_0\,x\cdot \mathsf{s}^0_2\,y)$$

or, equivalently, the equation

$$(***) \qquad {}_1\mathsf{s}(0,2)\,\mathsf{c}_1(\mathsf{c}_0\,x\cdot \mathsf{c}_2\,y)=\mathsf{c}_1(\mathsf{s}^2_0\,\mathsf{c}_0\,x\cdot \mathsf{s}^0_2\,\mathsf{c}_2\,y)$$

which cannot be derived from the cylindric axioms and MGR. Note that (***) is MGR^+_{102}; hence MGR^+ is just the collection of all variants of (***) one gets by permuting the indices.

Theorem 5.16 near the end of this chapter shows that in the 3-dimensional case, adding MGR^+ to the cylindric axioms suffices to derive not just all polyadic equations, but all universal formulas that hold for binary elements in polyadic equality algebras.

The next two lemmas say that MGR^+ is indeed a strengthening of MGR, and that MGR^* is an equivalent form of MGR^+.

LEMMA 5.3. *Let* $\mathfrak{A}\in\mathsf{CA}_3$ *and* $\{k,l,m\}=3$. *Then*

(i) *if* $\mathfrak{A}\models MGR^+_{klm}$ *then* $\mathfrak{A}\models MGR_k$;
(ii) *if* $\mathfrak{A}\models MGR^+_{klm}$ *then* $\mathfrak{A}\models MGR^+_{kml}$.

PROOF. (i) Since $m\notin\{k,l\}$, we have

$$(1) \qquad \mathsf{c}_m\,\mathsf{d}_{kl}=\mathsf{d}_{kl}$$

by axiom (C6) and 1.6(i). Hence

$$\begin{aligned} {}_k\mathsf{s}(l,m)\,\mathsf{c}_k x &= {}_k\mathsf{s}(l,m)\,\mathsf{s}^k_l\,\mathsf{s}^l_k\,\mathsf{c}_k x && \text{by 1.11(viii)}\\ &= {}_k\mathsf{s}(l,m)\,\mathsf{c}_k(\mathsf{s}^l_k\,\mathsf{c}_k x\cdot \mathsf{d}_{kl}) && \text{def. of } \mathsf{s}^k_l\\ &= \mathsf{c}_k(\mathsf{s}^m_l\,\mathsf{s}^l_k\,\mathsf{c}_k x\cdot \mathsf{s}^l_m\,\mathsf{d}_{kl}) && \text{by } MGR^+_{klm} \text{ and (1)}\\ &= \mathsf{c}_k(\mathsf{s}^m_l\,\mathsf{s}^l_k\,\mathsf{c}_k x\cdot \mathsf{d}_{km}) && \text{1.11(vi)}\\ &= \mathsf{s}^k_m\,\mathsf{s}^m_l\,\mathsf{s}^l_k\,\mathsf{c}_k x\\ &= {}_k\mathsf{s}(m,l)\,\mathsf{c}_k x. \end{aligned}$$

(ii) By (i). □

LEMMA 5.4. *Let* $\mathfrak{A} \in \mathsf{CA}_3$ *and* $\{k,l,m\} = 3$. *Then*

$$\mathfrak{A} \models MGR^+_{klm} \qquad \textit{iff} \qquad \mathfrak{A} \models MGR^*_{klm}.$$

PROOF. ($\Rightarrow$)

$$\begin{aligned} & 0 = \textstyle\prod_{i<3} \mathsf{c}_i x_i\\ \iff & 0 = \mathsf{c}_k x_k \cdot \mathsf{c}_k(\mathsf{c}_l x_l \cdot \mathsf{c}_m x_m) && \text{by 1.6(i),(ii)}\\ \iff & 0 = {}_k\mathsf{s}(l,m)\,\mathsf{c}_k x_k \cdot {}_k\mathsf{s}(l,m)\,\mathsf{c}_k(\mathsf{c}_l x_l \cdot \mathsf{c}_m x_m) && \text{by 1.11(xiv)}\\ \iff & 0 = {}_k\mathsf{s}(l,m)\,\mathsf{c}_k x_k \cdot \mathsf{c}_k(\mathsf{s}^m_l\,\mathsf{c}_l x_l \cdot \mathsf{s}^l_m\,\mathsf{c}_m x_m) = 0 && \text{by } MGR^+_{klm}\\ \iff & 0 = {}_k\mathsf{s}(l,m)\,\mathsf{c}_k x_k \cdot \mathsf{s}^m_l\,\mathsf{c}_l x_l \cdot \mathsf{s}^l_m\,\mathsf{c}_m x_m = 0 && \text{by 1.11(iii), 1.6(ii)} \end{aligned}$$

($\Leftarrow$) Suppose that MGR^*_{klm} holds in $\mathfrak{A}$. We want to show that

$${}_k\mathsf{s}(l,m)\,\mathsf{c}_k(\mathsf{c}_l x_l \cdot \mathsf{c}_m x_m) = \mathsf{c}_k(\mathsf{s}^m_l\,\mathsf{c}_l x_l \cdot \mathsf{s}^l_m\,\mathsf{c}_m x_m),$$

or, equivalently by Lemma 1.11(xiv), that

$$\mathsf{c}_k(\mathsf{c}_l x_l \cdot \mathsf{c}_m x_m) = {}_k\mathsf{s}(m,l)\,\mathsf{c}_k(\mathsf{s}^m_l\,\mathsf{c}_l x_l \cdot \mathsf{s}^l_m\,\mathsf{c}_m x_m).$$

For this, it suffices to show that for every $x \in A$,

$$x \cdot \mathsf{c}_k(\mathsf{c}_l x_l \cdot \mathsf{c}_m x_m) = 0 \quad \text{iff} \quad x \cdot {}_k\mathsf{s}(m,l)\,\mathsf{c}_k(\mathsf{s}^m_l\,\mathsf{c}_l x_l \cdot \mathsf{s}^l_m\,\mathsf{c}_m x_m) = 0.$$

Now

$$\begin{aligned} & x \cdot \mathsf{c}_k(\mathsf{c}_l x_l \cdot \mathsf{c}_m x_m) = 0\\ \iff & \mathsf{c}_k x \cdot \mathsf{c}_l x_l \cdot \mathsf{c}_m x_m = 0 && \text{by 1.6(ii)}\\ \iff & {}_k\mathsf{s}(l,m)\,\mathsf{c}_k x \cdot \mathsf{s}^m_l\,\mathsf{c}_l x_l \cdot \mathsf{s}^l_m\,\mathsf{c}_m x_m = 0 && \text{by } MGR^*_{klm}\\ \iff & {}_k\mathsf{s}(l,m)\,\mathsf{c}_k x \cdot \mathsf{c}_k(\mathsf{s}^m_l\,\mathsf{c}_l x_l \cdot \mathsf{s}^l_m\,\mathsf{c}_m x_m) = 0 && \text{by 1.6(i),(ii)}\\ \iff & \mathsf{c}_k x \cdot {}_k\mathsf{s}(m,l)\,\mathsf{c}_k(\mathsf{s}^m_l\,\mathsf{c}_l x_l \cdot \mathsf{s}^l_m\,\mathsf{c}_m x_m) = 0 && \text{by 1.11(xiv)}\\ \iff & x \cdot {}_k\mathsf{s}(m,l)\,\mathsf{c}_k(\mathsf{s}^m_l\,\mathsf{c}_l x_l \cdot \mathsf{s}^l_m\,\mathsf{c}_m x_m) = 0, && \text{by 1.6(i),(ii)} \end{aligned}$$

as desired. □

By 5.3(ii) we may and will denote by MGR^+_k either of the equations MGR^+_{klm} or MGR^+_{kml} when $\{k,l,m\} = 3$, and by 5.4 the same applies to MGR^*. Corollary 5.7 (way below) shows that there are further connections between the MGR^+ equations.

The next lemma should be compared to 4.5.

LEMMA 5.5. *Let* $\mathfrak{A} \in \mathsf{CA}_3$. *Then*

(i) $\mathfrak{A} \models MGR_0^*$ *iff* $\mathfrak{Ra}\,\mathfrak{A} \models z \cdot x \,;\, y = 0 \leftrightarrow x \cdot z \,;\, y^\smile = 0$;
(ii) $\mathfrak{A} \models MGR_1^*$ *iff* $\mathfrak{Ra}\,\mathfrak{A} \models z \cdot x \,;\, y = 0 \leftrightarrow y \cdot x^\smile \,;\, z = 0$;
(iii) $\mathfrak{A} \models MGR_2^*$ *iff* $\mathfrak{Ra}\,\mathfrak{A} \models z^\smile \cdot x \,;\, y = 0 \leftrightarrow z \cdot y^\smile \,;\, x^\smile = 0$.

PROOF. First we prove the ($\Rightarrow$) direction of the three statements. Let $x, y, z \in \mathrm{Nr}_2\,\mathfrak{A}$. By 1.6(ii) and the definition of ; in $\mathfrak{Ra}\,\mathfrak{A}$, we have that $z \cdot (x \,;\, y) = 0$ iff

$$z \cdot \mathsf{s}_2^1 x \cdot \mathsf{s}_2^0 y = 0. \tag{1}$$

Now (1) holds iff

$$\begin{aligned} & 0 = \mathsf{s}_1^2 \mathsf{s}_2^1 x \cdot \mathsf{s}_2^1 z \cdot {}_0\mathsf{s}(1,2)\, \mathsf{s}_2^0 y && \text{by } MGR_0^* \\ \Longleftrightarrow\; & 0 = x \cdot \mathsf{s}_2^1 z \cdot \mathsf{s}_2^0\, {}_2\mathsf{s}(0,1) y && \text{by 1.11(viii), (xi)} \\ \Longleftrightarrow\; & 0 = x \cdot \mathsf{c}_2(\mathsf{s}_2^1 z \cdot \mathsf{s}_2^0\, {}_2\mathsf{s}(0,1) y) && \text{by 1.6(ii)} \\ \Longleftrightarrow\; & 0 = x \cdot (z \,;\, y^\smile), \end{aligned}$$

and iff

$$\begin{aligned} & 0 = \mathsf{s}_0^2 \mathsf{s}_2^0 y \cdot \mathsf{s}_2^0 z \cdot {}_1\mathsf{s}(2,0)\, \mathsf{s}_2^1 x && \text{by } MGR_1^* \\ \Longleftrightarrow\; & 0 = y \cdot \mathsf{s}_2^0 z \cdot \mathsf{s}_2^1\, {}_2\mathsf{s}(0,1) x && \text{by 1.11(viii), (x)} \\ \Longleftrightarrow\; & 0 = y \cdot \mathsf{c}_2(\mathsf{s}_2^0 z \cdot \mathsf{s}_2^1\, {}_2\mathsf{s}(0,1) x) && \text{by 1.6(ii)} \\ \Longleftrightarrow\; & 0 = y \cdot (x^\smile \,;\, z). \end{aligned}$$

Now suppose that MGR_2^* holds. Then $\mathsf{s}_0^1 \mathsf{s}_2^0 y = \mathsf{s}_2^1\, {}_2\mathsf{s}(1,0) y = \mathsf{s}_2^1\, {}_2\mathsf{s}(0,1) y$ by Lemma 1.11(viii), Lemma 5.3(i) and Lemma 5.4; and $\mathsf{s}_1^0 \mathsf{s}_2^1 x = \mathsf{s}_2^0\, {}_2\mathsf{s}(0,1) x$ by Lemma 1.11(viii). Hence

$$\begin{aligned} & 0 = z^\smile \cdot x \,;\, y \\ \Longleftrightarrow\; & 0 = {}_2\mathsf{s}(0,1) z \cdot \mathsf{s}_2^1 x \cdot \mathsf{s}_2^0 y && \text{by 1.6(ii)} \\ \Longleftrightarrow\; & 0 = {}_2\mathsf{s}(0,1) z \cdot \mathsf{s}_0^1 \mathsf{s}_1^0 \mathsf{s}_2^1 x \cdot \mathsf{s}_1^0 \mathsf{s}_0^1 \mathsf{s}_2^0 y && \text{by 1.11(viii)} \\ \Longleftrightarrow\; & 0 = z \cdot \mathsf{s}_1^0 \mathsf{s}_2^1 x \cdot \mathsf{s}_0^1 \mathsf{s}_2^0 y && \text{by } MGR_2^* \\ \Longleftrightarrow\; & 0 = z \cdot \mathsf{s}_2^0\, {}_2\mathsf{s}(0,1) x \cdot \mathsf{s}_2^1\, {}_2\mathsf{s}(0,1) y \\ \Longleftrightarrow\; & 0 = z \cdot y^\smile \,;\, x^\smile. && \text{by 1.6(ii)} \end{aligned}$$

($\Leftarrow$) First we note that by 1.6(ii) and 1.11(viii), $\prod_{i<3} \mathsf{c}_i x_i = 0$ iff $\mathsf{c}_2 x_2 \cdot \mathsf{c}_2(\mathsf{c}_1 x_1 \cdot \mathsf{c}_0 x_0) = 0$ iff $\mathsf{c}_2 x_2 \cdot \mathsf{c}_2(\mathsf{s}_2^1 \mathsf{s}_1^2 \mathsf{c}_1 x_1 \cdot \mathsf{s}_2^0 \mathsf{s}_0^2 \mathsf{c}_0 x_0) = 0$, that is, iff

$$\mathsf{c}_2 x_2 \cdot (\mathsf{s}_1^2 \mathsf{c}_1 x_1 \,;\, \mathsf{s}_0^2 \mathsf{c}_0 x_0) = 0. \tag{2}$$

Assuming the right-hand side in (i), (2) holds iff

$$\begin{aligned} & \mathsf{s}_1^2 \mathsf{c}_1 x_1 \cdot (\mathsf{c}_2 x_2 \,;\, (\mathsf{s}_0^2 \mathsf{c}_0 x_0)^\smile) = 0 && \text{by assumption} \\ \Longleftrightarrow\; & \mathsf{s}_1^2 \mathsf{c}_1 x_1 \cdot \mathsf{c}_2(\mathsf{s}_2^1 \mathsf{c}_2 x_2 \cdot \mathsf{s}_2^0\, {}_2\mathsf{s}(0,1)\, \mathsf{s}_0^2 \mathsf{c}_0 x_0) = 0 && \text{def. of ; and } {}^\smile \\ \Longleftrightarrow\; & \mathsf{s}_1^2 \mathsf{c}_1 x_1 \cdot \mathsf{s}_2^1 \mathsf{c}_2 x_2 \cdot \mathsf{s}_2^0 \mathsf{s}_0^2\, {}_0\mathsf{s}(1,2)\, \mathsf{c}_0 x_0 = 0 && \text{by 1.11(x), 1.6(ii)} \\ \Longleftrightarrow\; & \mathsf{s}_1^2 \mathsf{c}_1 x_1 \cdot \mathsf{s}_2^1 \mathsf{c}_2 x_2 \cdot {}_0\mathsf{s}(1,2)\, \mathsf{c}_0 x_0 = 0, && \text{by 1.11(viii)} \end{aligned}$$

whence MGR_0^* holds. Assuming the right-hand side in (ii), (2) holds iff

$$\begin{aligned}
& \mathsf{s}_0^2\mathsf{c}_0 x_0 \cdot \big((\mathsf{s}_1^2\mathsf{c}_1 x_1)^{\smile} ; \mathsf{c}_2 x_2\big) = 0 && \text{by assumption}\\
\iff & \mathsf{s}_0^2\mathsf{c}_0 x_0 \cdot \mathsf{c}_2(\mathsf{s}_2^1 {}_2\mathsf{s}(0,1)\mathsf{s}_1^2\mathsf{c}_1 x_1 \cdot \mathsf{s}_2^0\mathsf{c}_2 x_2) = 0 && \text{def. of ; and } {}^{\smile}\\
\iff & \mathsf{s}_0^2\mathsf{c}_0 x_0 \cdot \mathsf{s}_2^1 {}_2\mathsf{s}(0,1)\mathsf{s}_1^2\mathsf{c}_1 x_1 \cdot \mathsf{s}_2^0\mathsf{c}_2 x_2 = 0 && \text{by 1.6(ii)}\\
\iff & \mathsf{s}_0^2\mathsf{c}_0 x_0 \cdot {}_1\mathsf{s}(2,0)\mathsf{c}_1 x_1 \cdot \mathsf{s}_2^0\mathsf{c}_2 x_2 = 0, && \text{by 1.11(xi), (viii)}
\end{aligned}$$

and this proves MGR_1^*. Now assume the right-hand side in (iii). Then $\mathfrak{A} \models MGR$, by 4.5(iii) and 1.13; hence

$$\begin{aligned}
& {}_2\mathsf{s}(0,1)\mathsf{c}_2 x_2 \cdot \mathsf{s}_0^1\mathsf{c}_0 x_0 \cdot \mathsf{s}_1^0\mathsf{c}_1 x_1 = 0 &&\\
\iff & {}_2\mathsf{s}(0,1)\mathsf{c}_2 x_2 \cdot \mathsf{s}_2^1\mathsf{s}_1^2\mathsf{s}_0^1\mathsf{c}_0 x_0 \cdot \mathsf{s}_2^0\mathsf{s}_0^2\mathsf{s}_1^0\mathsf{c}_1 x_1 = 0 && \text{by 1.11(viii)}\\
\iff & (\mathsf{c}_2 x_2)^{\smile} \cdot \mathsf{s}_1^2\mathsf{s}_0^1\mathsf{c}_0 x_0 ; \mathsf{s}_0^2\mathsf{s}_1^0\mathsf{c}_1 x_1 = 0 && \text{by 1.6(ii) and def. of ; and } {}^{\smile}\\
\iff & \mathsf{c}_2 x_2 \cdot (\mathsf{s}_0^2\mathsf{s}_1^0\mathsf{c}_1 x_1)^{\smile} ; (\mathsf{s}_1^2\mathsf{s}_0^1\mathsf{c}_0 x_0)^{\smile} = 0 && \text{by assumption}\\
\iff & \mathsf{c}_2 x_2 \cdot \mathsf{s}_2^1 {}_2\mathsf{s}(0,1)\mathsf{s}_0^2\mathsf{s}_1^0\mathsf{c}_1 x_1 \cdot \mathsf{s}_2^0 {}_2\mathsf{s}(0,1)\mathsf{s}_1^2\mathsf{s}_0^1\mathsf{c}_0 x_0 = 0 && \text{by 1.6(ii) and def. of ; and } {}^{\smile}\\
\iff & \mathsf{c}_2 x_2 \cdot {}_1\mathsf{s}(2,0) {}_1\mathsf{s}(2,0)\mathsf{c}_1 x_1 \cdot \mathsf{s}_2^0\mathsf{s}_0^2\mathsf{s}_1^0\mathsf{s}_2^1\mathsf{s}_1^2\mathsf{s}_0^1\mathsf{c}_0 x_0 = 0 && \text{by 1.11(x)}\\
\iff & \textstyle\prod_{i<3}\mathsf{c}_i x_i = 0, && \text{by } MGR\text{, 1.11(viii)}
\end{aligned}$$

that is, MGR_2^* holds in $\mathfrak{A}$. □

The following lemma says that any two instances of MGR^+ imply the third (see Corollary 5.7 below). By Lemmas 5.4 and 5.5 this can be viewed as a statement about RA-type algebras. We prove it in this more general setting because the assumption that the RA-type algebra in question is the RA-reduct of a CA_3 is unnecessarily strong.

LEMMA 5.6. *Let $\mathfrak{A}$ be an* RA*-type algebra such that the Boolean reduct of $\mathfrak{A}$ is a* BA *and*

$$\mathfrak{A} \models x ; 1' = x = 1' ; x. \tag{1}$$

Consider the formulas

$$z \cdot x ; y = 0 \leftrightarrow x \cdot z ; y^{\smile} = 0 \tag{e_0}$$

$$z \cdot x ; y = 0 \leftrightarrow y \cdot x^{\smile} ; z = 0 \tag{e_1}$$

$$z^{\smile} \cdot x ; y = 0 \leftrightarrow z \cdot y^{\smile} ; x^{\smile} = 0 \tag{e_2}$$

Then

(i) *if $\mathfrak{A} \models (e_0)$ or $\mathfrak{A} \models (e_1)$, then $\mathfrak{A} \models x^{\smile\smile} = x$;*
(ii) *if $\{i, j, k\} = 3$ and $\mathfrak{A} \models (e_i), (e_j)$, then $\mathfrak{A} \models (e_k)$.*

PROOF. To prove the first part of the lemma, it clearly suffices to show that if either of the equations (e_0), (e_1) holds in $\mathfrak{A}$, then for all $x \in A$,

$$\forall y \in A(y \cdot x = 0 \leftrightarrow y \cdot x^{\smile\smile} = 0).$$

But this is easy, for we have

$$\begin{array}{lll} & y\cdot x=0 & \\ \text{iff} & y\cdot 1';x=0 & \text{by (1)} \\ \text{iff} & 1'\cdot y;x^{\smile}=0 & \text{by } (e_0) \\ \text{iff} & y\cdot 1';x^{\smile\smile}=0 & \text{by } (e_0) \\ \text{iff} & y\cdot x^{\smile\smile}=0 & \text{by (1)} \end{array}$$

if (e_0) holds in $\mathfrak{A}$, and

$$\begin{array}{lll} & y\cdot x=0 & \\ \text{iff} & y\cdot x;1'=0 & \text{by (1)} \\ \text{iff} & 1'\cdot x^{\smile};y=0 & \text{by } (e_1) \\ \text{iff} & y\cdot x^{\smile\smile};1'=0 & \text{by } (e_1) \\ \text{iff} & y\cdot x^{\smile\smile}=0 & \text{by (1)} \end{array}$$

if (e_1) holds in $\mathfrak{A}$. This completes the proof of (i).

In proving (ii), we will use the fact that by (i), if any two of the formulas (e_i) hold in $\mathfrak{A}$, then $\mathfrak{A}\models$ (R4).

$(e_0)+(e_1)\Rightarrow(e_2)$:

$$\begin{array}{lll} & z^{\smile}\cdot x;y=0 & \\ \text{iff} & x\cdot z^{\smile};y^{\smile}=0 & \text{by } (e_0) \\ \text{iff} & y^{\smile}\cdot z;x=0 & \text{by } (e_1), \text{(R4)} \\ \text{iff} & z\cdot y^{\smile};x^{\smile}=0 & \text{by } (e_0) \end{array}$$

$(e_0)+(e_2)\Rightarrow(e_1)$:

$$\begin{array}{lll} & z\cdot x;y=0 & \\ \text{iff} & x\cdot z;y^{\smile}=0 & \text{by } (e_0) \\ \text{iff} & x^{\smile}\cdot y;z^{\smile}=0 & \text{by } (e_2), \text{(R4)} \\ \text{iff} & y\cdot x^{\smile};z=0 & \text{by } (e_0), \text{(R4)} \end{array}$$

$(e_1)+(e_2)\Rightarrow(e_0)$:

$$\begin{array}{lll} & z\cdot x;y=0 & \\ \text{iff} & y\cdot x^{\smile};z=0 & \text{by } (e_1) \\ \text{iff} & y^{\smile}\cdot z^{\smile};x=0 & \text{by } (e_2), \text{(R4)} \\ \text{iff} & x\cdot z;y^{\smile}=0 & \text{by } (e_1), \text{(R4).} \end{array}$$

□

In connection with the first part of this lemma we note that it is easy to construct an algebra $\mathfrak{A}$ satisfying the conditions of the lemma and such that the formula (e_2) holds in $\mathfrak{A}$ but $\mathfrak{A}\not\models$ (R4). We can take $\mathfrak{A}$ to be the $^{\smile}$-free reduct of any non-trivial RA and define $^{\smile}$ to be the function sending all elements of A to 0. On the other hand, if $\mathfrak{A}\in \mathbf{S}\mathfrak{Ra}^{*}\mathsf{CA}_3$ and (e_2) holds in $\mathfrak{A}$, then $\mathfrak{A}\models$ (R4) by 4.5(iii).

COROLLARY 5.7. *Let* $\mathfrak{C} \in \mathsf{CA}_3$, $\{k, l, m\} = 3$ *and suppose that* $\mathfrak{C} \models MGR^+_k, MGR^+_l$. *Then* $\mathfrak{C} \models MGR^+_m$.

PROOF. It is easy to see that the RA-reduct of any CA_3 satisfies the conditions of Lemma 5.6. Hence if MGR^*_k and MGR^*_l hold in $\mathfrak{C}$, then so does MGR^*_m by 5.5 and the lemma. This proves the corollary because of 5.4. □

Maddux [Mad78] (Definition 2 and 3 on pages 127 and 131) defines the classes

$$\mathsf{CA}'_3 \stackrel{\text{def}}{=} \{\, \mathfrak{C} \in \mathsf{CA}_3 : \mathfrak{Ra}\,\mathfrak{C} \models \text{(cycle law)}\,\} \quad \text{and}$$

$$\mathsf{CA}''_3 \stackrel{\text{def}}{=} \{\, \mathfrak{C} \in \mathsf{CA}'_3 : \mathfrak{C} \text{ is generated by } \mathrm{Nr}_2\,\mathfrak{C}\,\}.$$

COROLLARY 5.8. $\mathsf{CA}_3 \cap \mathrm{Mod}(MGR^+) = \mathsf{CA}'_3$.

PROOF. If $\mathfrak{C} \in \mathsf{CA}_3$ and the cycle law holds in $\mathfrak{Ra}\,\mathfrak{C}$, then MGR^+_0 and MGR^+_1 hold in $\mathfrak{C}$ by 5.5 and 5.4. Hence MGR^+ holds in $\mathfrak{C}$ by the previous corollary. Conversely, if MGR^+ holds in $\mathfrak{C}$, then the cycle law holds in $\mathfrak{Ra}\,\mathfrak{C}$ by 5.4 and 5.5. □

The next result says that (*) and (**) (on page 58 at the beginning of this chapter) hold for $K = \mathsf{CA}_3 \cap \mathrm{Mod}(MGR^+)$.

THEOREM 5.9. (i) *If* $\mathfrak{C} \in \mathsf{CA}_3 \cap \mathrm{Mod}(MGR^+)$ *and* $X \subseteq \mathrm{Nr}_2\,\mathfrak{C}$, *then* $\mathsf{Sg}^{(\mathfrak{Ra}\,\mathfrak{C})}X = \mathfrak{Ra}\,\mathsf{Sg}^{(\mathfrak{C})}X$.
(ii) *If* $\mathfrak{C}$, $\mathfrak{D} \in \mathsf{CA}_3 \cap \mathrm{Mod}(MGR^+)$ *are generated by their* RA*-reducts, and* f *is an isomorphism from* $\mathfrak{Ra}\,\mathfrak{C}$ *onto* $\mathfrak{Ra}\,\mathfrak{D}$, *then* f *extends to an isomorphism from* $\mathfrak{C}$ *onto* $\mathfrak{D}$.

PROOF. [Mad78, Theorem 7,8] proves this for CA'_3 (instead of for $\mathsf{CA}_3 \cap \mathrm{Mod}(MGR^+)$), so we are done by 5.8. We note however that in Maddux's proof, as well as in the proof of [HMT85, 5.3.12, 5.3.15] (which state the same result for $\mathbf{Nr}_3\,\mathsf{CA}_4$), the extra assumption that $\mathfrak{C}$ and $\mathfrak{D}$ are not only CA_3's but CA'_3's (or neat reducts of CA_4's) is used only for proving that the equations

$$\mathsf{c}_0(\mathsf{s}^1_2 x \cdot y) = \mathsf{s}^0_2(x^{\smile} ; y) \quad \text{and} \quad \mathsf{c}_1(\mathsf{s}^0_2 x \cdot y) = \mathsf{s}^1_2(y ; x^{\smile})$$

hold in $\mathfrak{C}$ and $\mathfrak{D}$ when $x, y \in \mathrm{Nr}_2 C$ or $\mathrm{Nr}_2 D$ (cf. [HMT85, 5.3.10]). We show how the first of these equations follow from MGR^+; the proof of the second is similar.

So let $\mathfrak{C} \in \mathsf{CA}_3 \cap \mathrm{Mod}(MGR^+)$, and let $x = \mathsf{c}_2 x$, $y = \mathsf{c}_2 y \in C$. Then, for all $z \in C$, we have

$$\begin{aligned}
z \cdot \mathsf{c}_0(\mathsf{s}^1_2 x \cdot y) = 0 &\iff \mathsf{c}_0 z \cdot \mathsf{s}^1_2 x \cdot y = 0 && \text{by 1.6(ii)}\\
&\iff \mathsf{s}^2_0 \mathsf{c}_0 z \cdot {}_1\mathsf{s}(2,0)\mathsf{s}^1_2 x \cdot \mathsf{s}^0_2 y = 0 && \text{by } MGR^*_1\\
&\iff \mathsf{s}^2_0 \mathsf{c}_0 z \cdot \mathsf{s}^1_2\, {}_2\mathsf{s}(0,1) x \cdot \mathsf{s}^0_2 y = 0 && \text{by 1.11(x)}\\
&\iff \mathsf{s}^2_0 \mathsf{c}_0 z \cdot x^{\smile} ; y = 0 && \text{by 1.6(ii) and def. of ; and } {}^{\smile}\\
&\iff \mathsf{c}_0 z \cdot \mathsf{s}^0_2(x^{\smile} ; y) = 0 && \text{by 1.11(viii),(xiv)}\\
&\iff z \cdot \mathsf{s}^0_2(x^{\smile} ; y) = 0 && \text{by 1.6(ii), 1.11(iii),}
\end{aligned}$$

showing that $\mathfrak{C} \models \mathsf{c}_0(\mathsf{s}^1_2 x \cdot y) = \mathsf{s}^0_2(x^{\smile} ; y)$. □

We note that because of 4.10, the assumption that $\mathfrak{C}$ satisfies MGR^+ cannot be replaced with the weaker requirement that $\mathfrak{C}$ satisfies MGR.

Now we recall the definition of finite-dimensional polyadic equality algebras. Instead of Halmos' axioms we use the simplified formulation in Sain-Thompson [ST91].

DEFINITION 5.10. *A **polyadic equality algebra of dimension** $2 < n < \omega$ is an algebra*

$$\mathfrak{C} = \langle \mathfrak{C}_0, \mathsf{c}_i, \mathsf{s}^i_j, \mathsf{p}_{ij}, \mathsf{d}_{ij} \rangle_{i,j<n}$$

such that $\mathfrak{C}_0$ is a Boolean algebra, c_i, s^i_j and p_{ij} are unary and d_{ij} are 0-ary operations, and the following postulates are satisfied for all $x, y \in C$ and all $i, j, k < n$.

(F0) $\mathsf{s}^i_i x = \mathsf{p}_{ii} x = x$ *and* $\mathsf{p}_{ij} x = \mathsf{p}_{ji} x$;
(F1) $x \le \mathsf{c}_i x$;
(F2) $\mathsf{c}_i(x+y) = \mathsf{c}_i x + \mathsf{c}_i y$;
(F3) $\mathsf{s}^i_j \mathsf{c}_i x = \mathsf{c}_i x$;
(F4) $\mathsf{c}_i \mathsf{s}^i_j x = \mathsf{s}^i_j x$ *whenever* $i \neq j$;
(F5) $\mathsf{s}^i_j \mathsf{c}_k x = \mathsf{c}_k \mathsf{s}^i_j x$ *whenever* $k \neq i, j$;
(F6) s^i_j *and* p_{ij} *are Boolean endomorphisms*;
(F7) $\mathsf{p}_{ij} \mathsf{p}_{ij} x = x$;
(F8) $\mathsf{p}_{ij} \mathsf{p}_{ik} x = \mathsf{p}_{jk} \mathsf{p}_{ij} x$ *whenever* i, j, k *are distinct*;
(F9) $\mathsf{p}_{ij} \mathsf{s}^i_j x = \mathsf{s}^j_i x$;
(F10) $\mathsf{s}^i_j \mathsf{d}_{ij} = 1$;
(F11) $x \cdot \mathsf{d}_{ij} \le \mathsf{s}^i_j x$.

PEA_n *denotes the class of n-dimensional polyadic equality algebras.*

REMARK 5.11. For completeness we note that for $n = 2$, the appropriate instances of (F0)–(F11) have to be supplemented with the axioms

$$\mathsf{c}_i(x \cdot \mathsf{c}_i y) = \mathsf{c}_i x \cdot \mathsf{c}_i y$$
$$\mathsf{c}_i \mathsf{c}_j x = \mathsf{c}_j \mathsf{c}_i x$$

but we will not be concerned with PEA_2's here. □

DEFINITION 5.12. *For $\mathfrak{C} \in \mathsf{PEA}_n$, $\mathfrak{Rd}_{ca}\mathfrak{C}$ denotes the **cylindric reduct** of $\mathfrak{C}$ and the* RA***-reduct*** *of $\mathfrak{C}$ is $\mathfrak{Ra}\,\mathfrak{C} \stackrel{\text{def}}{=} \mathfrak{Ra}\,\mathfrak{Rd}_{ca}\mathfrak{C}$. For $K \subseteq \mathsf{PEA}_n$, $\mathbf{Rd}_{ca} K \stackrel{\text{def}}{=} \mathbf{I}\{\mathfrak{Rd}_{ca}\mathfrak{C} : \mathfrak{C} \in K\}$ and $\mathfrak{Ra}^* K \stackrel{\text{def}}{=} \mathbf{I}\{\mathfrak{Ra}\,\mathfrak{C} : \mathfrak{C} \in K\}$.*

We collect all the facts that we will use about PEA's in the following Lemma.

LEMMA 5.13. *The following hold in every* PEA_n:

(i) *the cylindric reduct of a* PEA_n *is a* CA_n;
(ii) $\mathsf{s}^i_j \mathsf{c}_j x = \mathsf{p}_{ij} \mathsf{c}_j x$;
(iii) $\mathsf{c}_i \mathsf{p}_{ij} x = \mathsf{p}_{ij} \mathsf{c}_j x$;
(iv) *if* $k \notin \{i, j\}$ *then* $\mathsf{p}_{ij} \mathsf{c}_k x = \mathsf{c}_k \mathsf{p}_{ij} x$;
(v) *if* $i \neq j$ *then* $\mathsf{s}^i_j x = \mathsf{c}_i(\mathsf{d}_{ij} \cdot x)$;
(vi) *if* i, j, k *are distinct then* ${}_k\mathsf{s}(i,j) \mathsf{c}_k x = \mathsf{p}_{ij} \mathsf{c}_k x$;
(vii) *if* i, j, k *are distinct then* ${}_k\mathsf{s}(i,j) \mathsf{c}_k(\mathsf{c}_i x \cdot \mathsf{c}_j y) = \mathsf{c}_k(\mathsf{s}^j_i \mathsf{c}_i x \cdot \mathsf{s}^i_j \mathsf{c}_j y)$;
(viii) *if* $\sum_{k\in K} x_k$ *exists, then so does* $\sum_{k\in K} \mathsf{p}_{ij} x_k$, *and* $\mathsf{p}_{ij} \sum_{k\in K} x_k = \sum_{k\in K} \mathsf{p}_{ij} x_k$.

PROOF. (i) By [HMT85, 5.4.3]. From now on we will use CA-identities in PEA's without further comments, and suppress references to the PEA-axiom (F0).

In the proofs of (ii), (iii) and (iv) we may assume $i \neq j$.

(ii) $\mathsf{s}^i_j \mathsf{c}_j x = \mathsf{p}_{ij} \mathsf{s}^j_i \mathsf{c}_j x = \mathsf{p}_{ij} \mathsf{c}_j x$ by (F9) and (F3).

(iii) $\mathsf{p}_{ij}\,\mathsf{c}_j\,x = \mathsf{s}^i_j\,\mathsf{c}_j\,x = \mathsf{c}_i\,\mathsf{s}^i_j\,\mathsf{c}_j\,x = \mathsf{c}_i\,\mathsf{p}_{ij}\,\mathsf{c}_j\,x \geq \mathsf{c}_i\,\mathsf{p}_{ij}\,x$ by (ii), (F4), (F6) and (F1). The reverse inequality follows from this by (F6) and (F7).

(iv) We have

$$\mathsf{p}_{kj}\,\mathsf{c}_j\,\mathsf{p}_{kj}\,x = \mathsf{c}_k\,x$$

by (iii) and (F7). Using this we get

$$\begin{aligned}
\mathsf{p}_{ij}\,\mathsf{c}_k\,x &= \mathsf{p}_{ij}\,\mathsf{p}_{kj}\,\mathsf{c}_j\,\mathsf{p}_{kj}\,x \\
&= \mathsf{p}_{ik}\,\mathsf{p}_{ij}\,\mathsf{c}_j\,\mathsf{p}_{kj}\,x && \text{by (F8)} \\
&= \mathsf{c}_k\,\mathsf{p}_{ik}\,\mathsf{p}_{ij}\,\mathsf{p}_{kj}\,x && \text{by (iii)} \\
&= \mathsf{c}_k\,\mathsf{p}_{ik}\,\mathsf{p}_{ik}\,\mathsf{p}_{ij}\,x && \text{by (F8)} \\
&= \mathsf{c}_k\,\mathsf{p}_{ij}\,x. && \text{by (F7)}
\end{aligned}$$

(v) Applying c_i to both sides of (F11) and using (F4) we get

$$\mathsf{c}_i(\mathsf{d}_{ij}\cdot x) \leq \mathsf{c}_i\,\mathsf{s}^i_j\,x = \mathsf{s}^i_j\,x.$$

To show that the reverse inequality holds, note first that $\mathsf{c}_i\,\mathsf{d}_{ij} = 1$ by 1.6(v), so we have

$$\begin{aligned}
0 &= -\mathsf{c}_i\,\mathsf{d}_{ij} \\
&= -\mathsf{c}_i[(\mathsf{d}_{ij}\cdot x) + (\mathsf{d}_{ij}\cdot -x)] \\
&= -[\mathsf{c}_i(\mathsf{d}_{ij}\cdot x) + \mathsf{c}_i(\mathsf{d}_{ij}\cdot -x)] \\
&= -\mathsf{c}_i(\mathsf{d}_{ij}\cdot x)\cdot -\mathsf{c}_i(\mathsf{d}_{ij}\cdot -x),
\end{aligned}$$

and from this we get

$$-\mathsf{c}_i(\mathsf{d}_{ij}\cdot x) \leq \mathsf{c}_i(\mathsf{d}_{ij}\cdot -x) \leq \mathsf{s}^i_j -x = -\mathsf{s}^i_j\,x$$

by the inequality already established and (F6), and hence $\mathsf{s}^i_j\,x \leq \mathsf{c}_i(\mathsf{d}_{ij}\cdot x)$, as desired.

(vi) We have

$$\begin{aligned}
{}_k\mathsf{s}(i,j)\,\mathsf{c}_k\,x &= \mathsf{s}^k_i\,\mathsf{s}^i_j\,\mathsf{s}^j_k\,\mathsf{c}_k\,x \\
&= \mathsf{s}^k_i\,\mathsf{s}^i_j\,\mathsf{c}_j\,\mathsf{p}_{jk}\,x && \text{by (ii), (iii)} \\
&= \mathsf{s}^k_i\,\mathsf{c}_i\,\mathsf{p}_{ij}\,\mathsf{p}_{jk}\,x && \text{by (ii), (iii)} \\
&= \mathsf{p}_{ki}\,\mathsf{c}_i\,\mathsf{p}_{ij}\,\mathsf{p}_{jk}\,x && \text{by (ii)} \\
&= \mathsf{p}_{ki}\,\mathsf{p}_{ij}\,\mathsf{p}_{jk}\,\mathsf{c}_k\,x && \text{by (iii)} \\
&= \mathsf{p}_{ki}\,\mathsf{p}_{ik}\,\mathsf{p}_{ij}\,\mathsf{c}_k\,x && \text{by (F8)} \\
&= \mathsf{p}_{ij}\,\mathsf{c}_k\,x
\end{aligned}$$

by (F0) and (F7).

(vii) We have

$$\begin{aligned}
{}_k\mathsf{s}(i,j)\,\mathsf{c}_k(\mathsf{c}_i\,x\cdot\mathsf{c}_j\,y) &= \mathsf{p}_{ij}\,\mathsf{c}_k(\mathsf{c}_i\,x\cdot\mathsf{c}_j\,y) && \text{by (vi)} \\
&= \mathsf{c}_k(\mathsf{p}_{ij}\,\mathsf{c}_i\,x\cdot\mathsf{p}_{ij}\,\mathsf{c}_j\,y) && \text{by (iv) and (F6)} \\
&= \mathsf{c}_k(\mathsf{s}^j_i\,\mathsf{c}_i\,x\cdot\mathsf{s}^i_j\,\mathsf{c}_j\,y) && \text{by (ii)}
\end{aligned}$$

as desired.

(viii) By (F6) and (F7), $x \cdot \mathsf{p}_{ij} y = 0$ iff $y \cdot \mathsf{p}_{ij} x = 0$ for all x, y; that is, the operation p_{ij} is *self-conjugate*. From this, (viii) follows easily (cf. e.g. the remark preceding [HMT71, 1.2.6]). $\square$

DEFINITION 5.14. *A **triangle** in a* CA_3 $\mathfrak{C}$ *is an element of the form* $\prod_{i<3} \mathsf{c}_i x_i$*, with* $x_i \in C$ *for* $i < 3$.

We note that an element x of a CA_3 is a triangle iff $x = \prod_{i<3} \mathsf{c}_i x$. This is because if $x = \prod_{i<3} \mathsf{c}_i x_i$ for some x_0, x_1, x_2, then $\mathsf{c}_i x \le \mathsf{c}_i(\prod_{i<3} \mathsf{c}_i x_i) \le \mathsf{c}_i x_i$ for all $i < 3$, and hence $\prod_{i<3} \mathsf{c}_i x \le \prod_{i<3} \mathsf{c}_i x_i \le x$. We say that an element x of a CA_3 is 2-dimensional, if it is an i-cyilinder (i.e., $\mathsf{c}_i x = x$) for some $i < 3$.

LEMMA 5.15. *Let* $\mathfrak{C} \in \mathsf{CA}_3$ *and let* $X \subseteq C$ *consist of* 2*-dimensional elements. Then each element of* $\mathsf{Sg}^{(\mathfrak{C})} X$ *is a sum of finitely many triangles.*

PROOF. Let $F = \{\sum Y : Y$ is a finite set of triangles in $\mathfrak{C}\}$. We need to show that $X \subseteq F$ and F is closed under the cylindric operations. Two-dimensional elements are triangles (and hence $X \subseteq F$), for, if x is, say, c_0-closed, then $x = \mathsf{c}_0 x = \mathsf{c}_0 x \cdot \mathsf{c}_1 1 \cdot \mathsf{c}_2 1$. Since the diagonals are two-dimensional, we have $\mathsf{d}_{ij} \in F$ for all $i, j < 3$. For the same reason, F is closed under cylindrifications, and F is closed under + by definition. It remains to show that it is closed under complementation. But this follows from the following two simple facts. First, the complement of a triangle is in F, because $-(\mathsf{c}_0 x_0 \cdot \mathsf{c}_1 x_1 \cdot \mathsf{c}_2 x_2) = -\mathsf{c}_0 x_0 + -\mathsf{c}_1 x_1 + -\mathsf{c}_2 x_2$ is the sum of three two-dimensional elements by Lemma 1.6(iv). Second, F is closed under $\cdot$, because $\sum_{i\in I}(\mathsf{c}_0 x_i \cdot \mathsf{c}_1 x_i \cdot \mathsf{c}_2 x_i) \cdot \sum_{j\in J}(\mathsf{c}_0 y_j \cdot \mathsf{c}_1 y_j \cdot \mathsf{c}_2 y_j) = \sum_{i\in I, j\in J}(\mathsf{c}_0 x_i \cdot \mathsf{c}_0 y_j \cdot \mathsf{c}_1 x_i \cdot \mathsf{c}_1 y_j \cdot \mathsf{c}_2 x_i \cdot \mathsf{c}_2 y_j)$, a finite sum of triangles because $\mathsf{c}_k x_i \cdot \mathsf{c}_k y_j$ is c_k-closed by the cylindric axiom (C3). $\square$

THEOREM 5.16. *The following conditions are equivalent for every* $\mathfrak{C} \in \mathsf{CA}_3$:

(i) $\mathfrak{C}$ *is a subalgebra of the* CA_3*-reduct of a* PEA_3;
(ii) $\mathfrak{Ra}\,\mathfrak{C} \in \mathsf{SA}$;
(iii) $\mathfrak{C} \models MGR^+$.

If $\mathfrak{C}$ *is generated by* 2*-dimensional elements, then conditions (i)–(iii) are equivalent to*

(iv) $\mathfrak{C}$ *is the* CA_3*-reduct of a* PEA_3.

PROOF. The implication (iv)$\Rightarrow$(i) is clear, and (i) implies (ii), for, if $\mathfrak{C} \in \mathbf{SRd}_{ca}\mathsf{PEA}_3$, then $\mathfrak{C} \models MGR^+$ by 5.13(vii), and hence $\mathfrak{Ra}\,\mathfrak{C} \models$ (R6), (R7) by 5.4, 5.5 and 4.5; but then $\mathfrak{Ra}\,\mathfrak{C} \in \mathsf{SA}$ because of 4.13. To see that (ii) implies (iii), note that if $\mathfrak{Ra}\,\mathfrak{C} \in \mathsf{SA}$, then, by definition, the SA-axioms (R6) and (R7) hold in $\mathfrak{Ra}\,\mathfrak{C}$, and from this we get $\mathfrak{C} \models MGR^+$ by 4.5, 5.5, 5.4 and 5.7.

Assume now that $\mathfrak{C}$ is generated by 2-dimensional elements, $\mathfrak{C} \models MGR^+$ and hence $\mathfrak{C} \models MGR^*$ and MGR by 5.4 and 5.3(i). We show that (iv) holds by defining the operations p_{lm} for $l, m < 3$ on C and verifying that $\langle \mathfrak{C}, \mathsf{s}^l_m, \mathsf{p}_{lm}\rangle_{l,m<3} \in \mathsf{PEA}_3$. Throughout this proof, we assume $\{k, l, m\} = 3$.

Claim 1. Let $x, z \in {}^3C$*. Then*

$$\prod_{i<3} \mathsf{c}_i x_i \cdot {}_k\mathsf{s}(l,m)\mathsf{c}_k z_k \cdot \mathsf{s}^l_m \mathsf{c}_m z_m \cdot \mathsf{s}^m_l \mathsf{c}_l z_l = 0 \iff {}_k\mathsf{s}(l,m)\mathsf{c}_k x_k \cdot \mathsf{s}^l_m \mathsf{c}_m x_m \cdot \mathsf{s}^m_l \mathsf{c}_l x_l \cdot \prod_{i<3} \mathsf{c}_i z_i = 0$$

This follows from MGR^*_k because

$$\begin{aligned}\prod_{i<3} &\mathsf{c}_i x_i \cdot {}_k\mathsf{s}(l,m)\mathsf{c}_k z_k \cdot \mathsf{s}^l_m \mathsf{c}_m z_m \cdot \mathsf{s}^m_l \mathsf{c}_l z_l \\ &= {}_k\mathsf{s}(l,m)(\mathsf{c}_k z_k \cdot {}_k\mathsf{s}(m,l)\mathsf{c}_k x_k) \cdot \mathsf{s}^l_m(\mathsf{c}_m z_m \cdot \mathsf{s}^m_l \mathsf{c}_l x_l) \cdot \mathsf{s}^m_l(\mathsf{c}_l z_l \cdot \mathsf{s}^l_m \mathsf{c}_m x_m)\end{aligned}$$

by Lemma 1.11(viii), (xii) and (xiv).

Claim 2. If X, Y are finite subsets of ${}^3C$ and

$$\textstyle\sum_{x\in X}\prod_{i<3}\mathsf{c}_i x_i = \sum_{y\in Y}\prod_{i<3}\mathsf{c}_i y_i$$

then

$$\textstyle\sum_{x\in X}({}_k\mathsf{s}(l,m)\mathsf{c}_k x_k\cdot \mathsf{s}^l_m \mathsf{c}_m x_m \cdot \mathsf{s}^m_l \mathsf{c}_l x_l) = \sum_{y\in Y}({}_k\mathsf{s}(l,m)\mathsf{c}_k y_k\cdot \mathsf{s}^l_m \mathsf{c}_m y_m \cdot \mathsf{s}^m_l \mathsf{c}_l y_l).$$

By Lemma 5.15, to prove the Claim it is enough to show that for all triangles $z \in C$,

$$z\cdot\textstyle\sum_{x\in X}({}_k\mathsf{s}(l,m)\mathsf{c}_k x_k\cdot \mathsf{s}^l_m \mathsf{c}_m x_m \cdot \mathsf{s}^m_l \mathsf{c}_l x_l) = 0 \iff z\cdot\sum_{y\in Y}({}_k\mathsf{s}(l,m)\mathsf{c}_k y_k\cdot \mathsf{s}^l_m \mathsf{c}_m y_m \cdot \mathsf{s}^m_l \mathsf{c}_l y_l) = 0.$$

But this follows easily from Claim 1.

Now we are ready to define p_{lm} on C for $l, m < 3$. Let p_{ii} be the identity function on C for all $i < 3$, and if $l \neq m$ and X is a finite subset of ${}^3C$, then let

$$\mathsf{p}_{lm}(\textstyle\sum_{x\in X}\prod_{i<3}\mathsf{c}_i x_i) = \sum_{x\in X}({}_k\mathsf{s}(l,m)\mathsf{c}_k x_k\cdot \mathsf{s}^l_m \mathsf{c}_m x_m \cdot \mathsf{s}^m_l \mathsf{c}_l x_l).$$

This defines p_{lm} for all $x \in C$ by Lemma 5.15, and Claim 2 shows that p_{lm} is a function.

It remains to show that $\langle \mathfrak{C}, \mathsf{s}^l_m, \mathsf{p}_{lm}\rangle_{l,m<3}$ satisfies the PEA_3-axioms. We will check only those axioms in the list (F0)–(F11) in Definition 5.10 which mention the p_{lm}'s, since the other axioms are easily seen to be valid in every CA_3. (That those parts of (F0)–(F2) which do not mention the operations p_{lm} hold follow from (C2), 1.6(iii), and the fact that s^i_i is the identity function by definition; (F3)–(F5) appear as (i), (iii), (ii) of Lemma 1.11, the substitutions are Boolean endomorphisms by Lemma 1.11(v), (F10) holds because $s^i_i\mathsf{d}_{ii} = \mathsf{d}_{ii} = 1$ and if $i \neq j$ then $\mathsf{s}^i_j\mathsf{d}_{ij} = \mathsf{c}_i\mathsf{d}_{ij} = 1$ by the definition of s^i_j, (C5) and 1.6(v); finally, (F11) holds by the definition of s^i_j and (F1).)

In the rest of the proof we will use Lemma 5.15 without explicitly referring to it.

$\mathsf{p}_{lm}x = \mathsf{p}_{ml}x$: it is enough to check this for triangles; but then it is obvious by *MGR*.

p_{lm} is an endomorphism of $\mathfrak{Bl}\mathfrak{C}$: it clearly preserves $+$, 0 and it preserves 1 because of 1.11(v) and since $\mathsf{c}_i 1 = 1$ for all $i < 3$. So we only have to show that it preserves $\cdot$. Let X and Y be finite subsets of ${}^3C$. Then

$$(\textstyle\sum_{x\in X}\prod_{i<3}\mathsf{c}_i x_i)\cdot(\sum_{y\in Y}\prod_{i<3}\mathsf{c}_i y_i) = \sum_{x\in X, y\in Y}\prod_{i<3}(\mathsf{c}_i x_i\cdot \mathsf{c}_i y_i)$$

and hence

$$\begin{aligned}
&\mathsf{p}_{lm}\big((\textstyle\sum_{x\in X}\prod_{i<3}\mathsf{c}_i x_i)\cdot(\sum_{y\in Y}\prod_{i<3}\mathsf{c}_i y_i)\big)\\
&=\mathsf{p}_{lm}\big(\textstyle\sum_{x\in X, y\in Y}\prod_{i<3}(\mathsf{c}_i x_i\cdot \mathsf{c}_i y_i)\big)\\
&=\textstyle\sum_{x\in X, y\in Y}{}_k\mathsf{s}(l,m)(\mathsf{c}_k x_k\cdot\mathsf{c}_k y_k)\cdot\mathsf{s}^l_m(\mathsf{c}_m x_m\cdot \mathsf{c}_m y_m)\cdot \mathsf{s}^m_l(\mathsf{c}_l x_l\cdot\mathsf{c}_l y_l)
\end{aligned}$$

by the definition of p_{lm} and since $\prod_{i<3}(\mathsf{c}_i x_i\cdot\mathsf{c}_i y_i)$ is a triangle by (C3),

$$\begin{aligned}
&=\textstyle\sum_{x\in X, y\in Y}{}_k\mathsf{s}(l,m)\mathsf{c}_k x_k\cdot{}_k\mathsf{s}(l,m)\mathsf{c}_k y_k\\
&\qquad\cdot\mathsf{s}^l_m\mathsf{c}_m x_m\cdot\mathsf{s}^l_m\mathsf{c}_m y_m\cdot\mathsf{s}^m_l\mathsf{c}_l x_l\cdot\mathsf{s}^m_l\mathsf{c}_l y_l && \text{by 1.11(v)}\\
&=\textstyle\sum_{x\in X, y\in Y}\big(\mathsf{p}_{lm}(\prod_{i<3}\mathsf{c}_i x_i)\cdot\mathsf{p}_{lm}(\prod_{i<3}\mathsf{c}_i y_i)\big) && \text{by def. of } \mathsf{p}_{lm}\\
&=\big(\textstyle\sum_{x\in X}\mathsf{p}_{lm}(\prod_{i<3}\mathsf{c}_i x_i)\big)\cdot\big(\sum_{y\in Y}\mathsf{p}_{lm}(\prod_{i<3}\mathsf{c}_i y_i)\big)\\
&=\mathsf{p}_{lm}(\textstyle\sum_{x\in X}\prod_{i<3}\mathsf{c}_i x_i)\cdot\mathsf{p}_{lm}(\sum_{y\in Y}\prod_{i<3}\mathsf{c}_i y_i) && \text{since } \mathsf{p}_{lm} \text{ preserves } +.
\end{aligned}$$

$\mathsf{p}_{lm}\,\mathsf{p}_{lm}\,x = x$: again, it suffices to show that this holds for the triangles. (The same goes for the rest of the axioms.) So let $x \in {}^3C$. Then

$$\begin{aligned}\mathsf{p}_{lm}\,\mathsf{p}_{lm}\prod_{i<3}\mathsf{c}_i x_i &= \mathsf{p}_{lm}({}_k\mathsf{s}(l,m)\,\mathsf{c}_k x_k \cdot \mathsf{s}^l_m\,\mathsf{c}_m x_m \cdot \mathsf{s}^m_l\,\mathsf{c}_l x_l)\\ &= {}_k\mathsf{s}(l,m)\,{}_k\mathsf{s}(l,m)\,\mathsf{c}_k x_k \cdot \mathsf{s}^m_l\,\mathsf{s}^l_m\,\mathsf{c}_m x_m \cdot \mathsf{s}^l_m\,\mathsf{s}^m_l\,\mathsf{c}_l x_l\\ &= \prod_{i<3}\mathsf{c}_i x_i\end{aligned}$$

by 1.13(i), 1.11(viii).

$\mathsf{p}_{lm}\,\mathsf{p}_{lk}\,x = \mathsf{p}_{mk}\,\mathsf{p}_{lm}\,x$: let $x \in {}^3C$. Then

$$\begin{aligned}&\mathsf{p}_{lm}\,\mathsf{p}_{lk}\prod_{i<3}\mathsf{c}_i x_i\\ &= \mathsf{p}_{lm}({}_m\mathsf{s}(l,k)\,\mathsf{c}_m x_m \cdot \mathsf{s}^l_k\,\mathsf{c}_k x_k \cdot \mathsf{s}^k_l\,\mathsf{c}_l x_l)\\ &= \mathsf{s}^l_m\,{}_m\mathsf{s}(l,k)\,\mathsf{c}_m x_m \cdot \mathsf{s}^m_l\,\mathsf{s}^l_k\,\mathsf{c}_k x_k \cdot {}_k\mathsf{s}(l,m)\,\mathsf{s}^k_l\,\mathsf{c}_l x_l\\ &= \mathsf{s}^l_m\,{}_m\mathsf{s}(l,k)\,\mathsf{c}_m x_m \cdot \mathsf{s}^m_l\,\mathsf{s}^l_k\,\mathsf{c}_k x_k \cdot {}_k\mathsf{s}(m,l)\,\mathsf{s}^k_l\,\mathsf{c}_l x_l && \text{by } MGR\\ &= \mathsf{s}^m_k\,{}_k\mathsf{s}(m,l)\,\mathsf{c}_k x_k \cdot {}_l\mathsf{s}(k,m)\,\mathsf{s}^l_m\,\mathsf{c}_m x_m \cdot \mathsf{s}^k_m\,\mathsf{s}^m_l\,\mathsf{c}_l x_l && \text{by 1.11(xi),(viii)}\\ &= \mathsf{p}_{km}({}_k\mathsf{s}(l,m)\,\mathsf{c}_k x_k \cdot \mathsf{s}^l_m\,\mathsf{c}_m x_m \cdot \mathsf{s}^m_l\,\mathsf{c}_l x_l)\\ &= \mathsf{p}_{km}\,\mathsf{p}_{ml}\prod_{i<3}\mathsf{c}_i x_i\\ &= \mathsf{p}_{mk}\,\mathsf{p}_{lm}\prod_{i<3}\mathsf{c}_i x_i.\end{aligned}$$

$\mathsf{p}_{lm}\,\mathsf{s}^m_l\,x = \mathsf{s}^l_m\,x$: for $x \in {}^3C$ we have

$$\begin{aligned}&\mathsf{p}_{lm}\,\mathsf{s}^m_l\prod_{i<3}\mathsf{c}_i x_i\\ &= \mathsf{p}_{lm}(\mathsf{s}^m_l\,\mathsf{c}_m x_m \cdot \mathsf{s}^m_l\,\mathsf{c}_l x_l \cdot \mathsf{s}^m_l\,\mathsf{c}_k x_k) && \text{by 1.11(v)}\\ &= \mathsf{p}_{lm}(\mathsf{c}_m x_m \cdot \mathsf{s}^m_l\,\mathsf{c}_l x_l \cdot \mathsf{c}_k\,\mathsf{s}^m_l\,x_k) && \text{by 1.11(i), (ii)}\\ &= \mathsf{p}_{lm}(\mathsf{c}_m x_m \cdot \mathsf{s}^m_l\,\mathsf{c}_l x_l \cdot \mathsf{c}_l 1 \cdot \mathsf{c}_k\,\mathsf{s}^m_l\,x_k) && \text{by (C2)}\\ &= \mathsf{s}^l_m(\mathsf{c}_m x_m \cdot \mathsf{s}^m_l\,\mathsf{c}_l x_l) \cdot \mathsf{s}^m_l\,\mathsf{c}_l 1 \cdot {}_k\mathsf{s}(l,m)\,\mathsf{c}_k\,\mathsf{s}^m_l\,x_k\\ &= \mathsf{s}^l_m\,\mathsf{c}_m x_m \cdot \mathsf{c}_l x_l \cdot {}_k\mathsf{s}(l,m)\,\mathsf{s}^m_l\,\mathsf{c}_k x_k && \text{by 1.11(v), (viii), (ii)}\\ &= \mathsf{s}^l_m\,\mathsf{c}_m x_m \cdot \mathsf{s}^l_m\,\mathsf{c}_l x_l \cdot \mathsf{s}^l_m\,\mathsf{c}_k x_k && \text{by 1.11(i), (ix)}\\ &= \mathsf{s}^l_m\prod_{i<3}\mathsf{c}_i x_i. && \text{by 1.11(v)}\end{aligned}$$

This completes the proof of the implication (iii)⇒(iv) when $\mathfrak{C}$ is generated by 2-dimensional elements.

It remains to show that (iii) implies (i) without further assumption on $\mathfrak{C}$. The idea of the proof is that once we have defined the p_{ij}'s for triangles (as in the previous part of the proof), the definition can be extended almost freely to atoms that are below triangles. There are only some cardinality problems, but they can be eliminated by dilating new atoms into the algebra.

So let $\mathfrak{C} \in \mathsf{CA}_3$ satisfy MGR^+. We may assume that $\mathfrak{C}$ is complete and atomic, because CA_3 is closed under taking perfect extensions ([HMT71, 2.7.15]) and the equations in MGR^+ are all positive, and hence preserved under perfect extensions ([HMT71, 2.7.13]). Let $\mathfrak{B} = \langle B, T_i, E_{ij}\rangle_{i,j<3}$ be the atom structure of $\mathfrak{C}$.

First,

(1) if a is an atom below one of the diagonals, then a is a triangle.

For, if a is below, say d_{01}, then $a = \mathsf{d}_{01} \cdot \mathsf{c}_0 a = \mathsf{d}_{01} \cdot \mathsf{c}_1 a$ by 2.10(i), and hence $a = \mathsf{d}_{01} \cdot \mathsf{c}_0 a \cdot \mathsf{c}_1 a = \mathsf{c}_2 \mathsf{d}_{01} \cdot \mathsf{c}_0 a \cdot \mathsf{c}_1 a \geq \mathsf{c}_2 a \cdot \mathsf{c}_0 a \cdot \mathsf{c}_1 a \geq a$. It follows that if a is not a triangle, then we can dilate with the sequence $\langle a, a, a\rangle$. (This special case of dilation is also known as *splitting*.) Let $\mathfrak{B}^+ = \langle B^+, T_i^+, E_{ij}\rangle_{i,j<3}$ be the atom structure we get from $\mathfrak{B}$ by dilating κ many times by $\langle a, a, a\rangle$ for each non-tiangle $a \in B$, where κ is the cardinality of B. In other words, we get $\mathfrak{B}^+$ from $\mathfrak{B}$ by splitting each non-triangle into κ-many parts.

Next we show that

(2) $\mathfrak{C}$ embeds into $\mathfrak{C}^+ \overset{\text{def}}{=} \mathfrak{Cm}\,\mathfrak{B}^+$.

To see that (2) holds, it suffices to show that there is a function f mapping B^+ onto B with the following properties:

($*$) $b \in E_{ij}$ iff $f(b) \in E_{ij}$ for all $b \in B^+$ and $i, j < 3$;
($**$) if $b\, T_i^+\, b'$ then $f(b)\, T_i\, f(b')$ for all $b, b' \in B^+$;
($***$) if $f(b)\, T_i\, a'$ for some $b \in B^+$ and $a' \in B$ then there is a $b' \in B^+$ with $f(b') = a'$ and $b\, T_i^+\, b'$.

For, if f has these properties, then the map $f^+ : \mathfrak{Cm}\,\mathfrak{B} \longrightarrow \mathfrak{Cm}\,\mathfrak{B}^+$, defined as $f^+X = \{\, b \in B^+ : f(b) \in X\,\}$ for $X \subseteq B$, is a homomorphism by ($*$)–($***$), and is one-one because f is onto; this is a basic result in the duality theory of BAO's (see e.g. [Gol91, §2]), and it is quite easy to see directly. To define such a function f, let us introduce the following temporary notation: for $a \in B$, let

$$\Phi_a = \{a\} \cup \{\, b \in B^+ \setminus B : b \text{ is the result of dilating with } \langle a, a, a\rangle\,\}.$$

Clearly, $\{\,\Phi_a : a \in B\,\}$ is a partition of B^+; hence for $b \in B^+$, $f(b)$ can be defined as the unique $a \in B$ such that $b \in \Phi_a$. Condition ($*$) is then clear, since $E_{ij} \subseteq B$ if $i \neq j$, and by (1), Φ_a is a singleton if $a \in E_{ij}$ for some $j \neq i$. If $b\, T_i^+\, b'$ for $b \in \Phi_a$ and $b' \in \Phi_{a'}$, then $a\, T_i^+\, b\, T_i^+\, b'\, T_i^+\, a$ and hence $a\, T_i\, a'$, proving ($**$). In ($***$) we can choose $b' = a'$, because $f(a') = a'$ (since $a' \in B$) and $b\, T_i^+\, f(b)\, T_i\, a'$ and hence $b\, T_i^+\, a'$. Finally, f is onto, because $f \restriction B$ is the identity on B. This completes the proof of (2).

So from now on we can concentrate on $\mathfrak{B}^+$ and its complex algebra $\mathfrak{C}^+$. For $b \in B^+$, let $\psi_b = \{\, b' \in B^+ : (\forall i < 3) b\, T_i^+\, b'\,\}$. This is a partition of B^+, and for every $b \in B^+$, ψ_b (viewed as an element of $\mathfrak{C}^+$) is a triangle, and in fact the smallest one containing b. Hence the ψ_b's are the atoms of the subalgebra $\mathfrak{C}_0$ of $\mathfrak{C}^+$ generated by its 2-dimensional elements (in particular, $\psi_b \in C_0$, because $\psi_b = \prod_{i<3} \mathsf{c}_i b$). For suppose that $0 < x \leq \psi_b$ for some $x \in C_0$: by Lemma 5.15 we may assume that x is a triangle, and since it is non-0, there is an atom b' (of $\mathfrak{C}^+$) below x. But then $0 < b' \leq x \leq \psi_b = \psi_{b'}$ and since $\psi_{b'}$ is the smallest triangle containing b', we must have $x = \psi_{b'}$. This shows that the ψ_b's are atoms of $\mathfrak{C}_0$. Next we claim that $\mathfrak{C}_0$ is atomic. To see this, it is enough to show that for all triangles $x \in C_0$ there is an atom $b \in C^+$ such that $\psi_b \leq x$. But, if b is any atom of $\mathfrak{C}^+$ below x, then $b \leq x = \prod_{i<3} \mathsf{c}_i x$ and hence $\psi_b = \prod_{i<3} \mathsf{c}_i b \leq \prod_{i<3} \mathsf{c}_i x = x$.

Because of the implication (iii)⇒(iv), $\mathfrak{C}_0$ is the reduct of a PEA_3, and from the observations in the previous paragraph, it follows that the PEA_3-operations (denoted this time by P_{lm} ($l, m < 3$) to avoid confusion with the polyadic operations we are going to define on $\mathfrak{C}^+$) are permutations of the set $\{\,\psi_b : b \in B^+\,\}$. Our task is to extend these operations to all elements of $\mathfrak{B}^+$ and hence, by complete distributivity, to elements of $\mathfrak{C}^+$.

Claim 3. Suppose that p_{01} *and* p_{02} *are permutations of* B^+ *satisfying the following conditions:*

(A) $\mathsf{p}_{01} \restriction \psi_b$ *is a bijection from* ψ_b *to* $\mathsf{P}_{01}\,\psi_b$*, and* $\mathsf{p}_{02} \restriction \psi_b$ *is a bijection from* ψ_b *to* $\mathsf{P}_{02}\,\psi_b$;
(B) $\mathsf{p}_{01}\,\mathsf{p}_{01}\,b = b = \mathsf{p}_{02}\,\mathsf{p}_{02}\,b$ *for all* $b \in B^+$;
(C) $\mathsf{p}_{01}\,\mathsf{p}_{02}\,\mathsf{p}_{01}\,b = \mathsf{p}_{02}\,\mathsf{p}_{01}\,\mathsf{p}_{02}\,b$ *for all* $b \in B^+$.

Then $\langle \mathfrak{C}^+, \mathsf{p}^+_{ij} \rangle_{i,j<3} \in \mathsf{PEA}_3$*, where* $\mathsf{p}^+_{ii}x = x$*,* $\mathsf{p}^+_{01}\,x = \mathsf{p}^+_{10}\,x = \{\,\mathsf{p}_{01}\,b : b \in x\,\}$*,* $\mathsf{p}^+_{02}\,x = \mathsf{p}^+_{20}\,x = \{\,\mathsf{p}_{02}\,b : b \in x\,\}$*,* $\mathsf{p}_{12}\,b = \mathsf{p}_{21}\,b = \mathsf{p}_{01}\,\mathsf{p}_{02}\,\mathsf{p}_{01}\,b$ *and* $\mathsf{p}^+_{12}\,x = \mathsf{p}^+_{21}\,x = \{\,\mathsf{p}_{12}\,b : b \in x\,\}$ *for all* $x \in \mathfrak{C}^+$ *and* $b \in \mathfrak{B}^+$.

Proof. As in the proof of the implication (iii)⇒(iv), we only need to check those axioms that involve the permutations. Of these, (F6) is clear because p^+_{01}, p^+_{02} and p^+_{12} are all defined pointwise; (F7) holds for p^+_{01} and p^+_{02} by condition (B) and for p^+_{12} since $\mathsf{p}_{12}\,\mathsf{p}_{12}\,b = \mathsf{p}_{01}\,\mathsf{p}_{02}\,\mathsf{p}_{01}\,\mathsf{p}_{01}\,\mathsf{p}_{02}\,\mathsf{p}_{01}\,b = b$ for all $b \in B^+$. We only need to check (F8) (that is, $\mathsf{p}^+_{ij}\,\mathsf{p}^+_{ik}\,x = \mathsf{p}^+_{jk}\,\mathsf{p}^+_{ij}\,x$ for i, j, k distinct) for the atoms, and there are six cases to consider:[2]

$i = 0$**,** $j = 1$**:** $\mathsf{p}_{01}\,\mathsf{p}_{02}\,b = \mathsf{p}_{01}\,\mathsf{p}_{02}\,\mathsf{p}_{01}\,\mathsf{p}_{01}\,b = \mathsf{p}_{12}\,\mathsf{p}_{01}\,b$ by condition (B) and the definition of p_{12};
$i = 1$**,** $j = 0$**:** $\mathsf{p}_{01}\,\mathsf{p}_{12}\,b = \mathsf{p}_{01}\,\mathsf{p}_{01}\,\mathsf{p}_{02}\,\mathsf{p}_{01}\,b = \mathsf{p}_{02}\,\mathsf{p}_{01}\,b$ by the definition of p_{12} and condition (B);
$i = 0$**,** $j = 2$**:** $\mathsf{p}_{02}\,\mathsf{p}_{01}\,b = \mathsf{p}_{02}\,\mathsf{p}_{01}\,\mathsf{p}_{02}\,\mathsf{p}_{02}\,b = \mathsf{p}_{12}\,\mathsf{p}_{02}\,b$ by conditions (B), (C) and the definition of p_{12};
$i = 2$**,** $j = 0$**:** $\mathsf{p}_{02}\,\mathsf{p}_{12}\,b = \mathsf{p}_{02}\,\mathsf{p}_{02}\,\mathsf{p}_{01}\,\mathsf{p}_{02}\,b = \mathsf{p}_{01}\,\mathsf{p}_{02}\,b$ by the definition of p_{12} and conditions (C), (B);
$i = 1$**,** $j = 2$**:** $\mathsf{p}_{12}\,\mathsf{p}_{01}\,b = \mathsf{p}_{02}\,\mathsf{p}_{01}\,\mathsf{p}_{02}\,\mathsf{p}_{01}\,b = \mathsf{p}_{02}\,\mathsf{p}_{12}\,b$ by the definition of p_{12} and conditions (C), (B);
$i = 2$**,** $j = 1$**:** $\mathsf{p}_{12}\,\mathsf{p}_{02}\,b = \mathsf{p}_{01}\,\mathsf{p}_{02}\,\mathsf{p}_{01}\,\mathsf{p}_{02}\,b = \mathsf{p}_{01}\,\mathsf{p}_{12}\,b$ by the definition of p_{12} and conditions (C), (B).

In order to see that (F9), the last axiom we need to check, also holds, let us first observe that condition (A) is true for p_{12}, that is, p_{12} is a bijection from ψ_b to $\mathsf{P}_{01}\,\mathsf{P}_{02}\,\mathsf{P}_{01}\,\psi_b = \mathsf{P}_{12}\,\psi_b$; the last equality holds because $\psi_b \in \mathfrak{C}_0 \in \mathbf{Rd}_{ca}\,\mathsf{PEA}_3$ and $\mathsf{PEA}_3 \models \mathsf{p}_{01}\,\mathsf{p}_{02}\,\mathsf{p}_{01}\,x = \mathsf{p}_{12}x$. But, by Lemma 5.13(viii), condition (A), and by what we have said about the atoms of $\mathfrak{C}_0$, p^+_{01}, p^+_{02} and p^+_{12} agree with P_{01}, P_{02} and P_{12}, respectively, on $\mathfrak{C}_0$. But then, by 1.11(iv), (F9) in $\langle \mathfrak{C}_0, \mathsf{P}_{ij} \rangle_{i,j<3}$ and since $\mathsf{s}^j_i x \in C_0$ for all $x \in C^+$, we have $\mathsf{p}^+_{ij}\,\mathsf{s}^i_j x = \mathsf{p}^+_{ij}\,\mathsf{s}^i_j\,\mathsf{s}^j_i x = \mathsf{s}^j_i\,\mathsf{s}^j_i x = \mathsf{s}^j_i x$, that is, (F9) holds in $\langle \mathfrak{C}^+, \mathsf{p}^+_{ij} \rangle_{i,j<3}$. □

So we need to define permutations p_{01} and p_{02} on B^+ satisfying the conditions of the Claim. Condition (A) will be easy to satisfy because of the following:

(3) For all $b \in B^+$, ψ_b and $\mathsf{P}_{ij}\,\psi_b$ have the same cardinality.

To see that (3) holds, note first that for every $b \in B^+$, ψ_b has cardinality 1 if b is a triangle and κ otherwise. For, if b is a triangle, then $\psi_b = \{b\}$, and if b is not a triangle, then there is an $a \in B$ with $b \in \Phi_a$ (since Φ_a, $a \in B$, is a partition of B^+), and then $\Phi_a \subseteq \psi_b$, whence $\kappa \leq |\psi|$; since there are at most $\kappa \cdot \kappa = \kappa$ many elements of B^+, this shows that $|\psi_b| = \kappa$. This shows that (3) holds, because the P_{ij}'s are easily seen to preserve triangles.

We define p_{01} as follows: if $\mathsf{P}_{01}\,\psi_b = \psi_b$, then let p_{01} be the identity on ψ_b; otherwise, let p_{01} on ψ_b be any bijection from ψ_b to $\mathsf{P}_{01}\,\psi_b$ and let p_{01} on $\mathsf{P}_{01}\,\psi_b$ be its inverse. To define p_{02}, we consider the orbits of the action of the P_{ij}'s on the ψ_b's (i.e., the atoms of $\mathfrak{C}_0$). Each such orbit has cardinality 1, 2, 3 or 6, because the symmetric group on three elements have transitive representations on 1, 2, 3 and 6 element sets, as on Figure 1. If the orbit consists of one element ψ_b, then we let p_{02} be the identity function on ψ_b. If the orbit has two elements, then we let p_{02} be the same as p_{01} on both elements. In the third case we let p_{02} be the identity on the fixpoint of P_{02} and any bijection between the other two elements with the proviso that it be its own inverse. Finally, on a six-element orbit, we can define p_{02} on two pairs of ψ_b's connected by P_{02} as before to make p_{02} its own inverse, and on

[2]Of course, we could also deduce (F8) from the well-known fact that the symmetric group on three elements is defined by the relations $\{a^2 = \mathrm{Id} = b^2, aba = bab\}$.

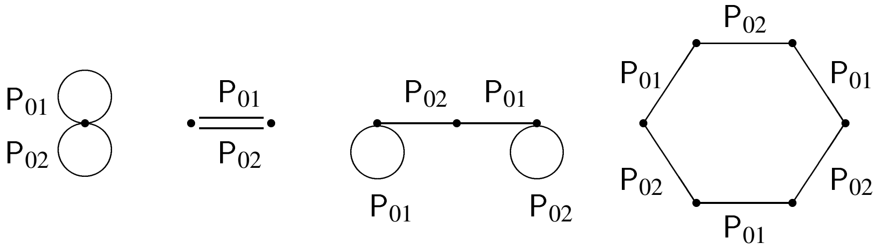


FIGURE 1

the remaining two ψ_b's we can take p_{02} to be $\mathsf{p}_{01}\,\mathsf{p}_{02}\,\mathsf{p}_{01}\,\mathsf{p}_{02}\,\mathsf{p}_{01}$. It is easy to see that in all cases the conditions of Claim 3 are satisfied. $\square$

REMARK 5.17. We give an example which shows that $\mathbf{Rd}_{ca}\,\mathsf{PEA}_3$ is not closed under taking subalgebras, and hence that in Theorem 5.16, the assumption that $\mathfrak{C}$ is generated by 2-dimensional elements is necessary for conditions (i)–(iii) to imply (iv).

Let $U = 6$, G the group of permutations of U generated by (01), (23) and (45), let

$$\mathfrak{C} = \mathfrak{Fr}_G\langle \mathfrak{P}(^3U), \mathsf{C}_i, \mathsf{D}_{ij}\rangle_{i,j<3},$$

and let $\mathfrak{B}$ be the atom structure of $\mathfrak{C}$. Now split the atom $R = \langle 0,2,4\rangle^- = \{0,1\}\times\{2,3\}\times\{4,5\}$ of $\mathfrak{C}$ into two parts, $R_0 = \{s \in R : s_0+s_1+s_2 \quad \text{is even}\}$ and $R_1 = \{s \in R : s_0+s_1+s_2 \quad \text{is odd}\}$. That is, define the atom structure $\mathfrak{B}' = \langle B', T_i', E_{ij}\rangle_{i,j<3}$ as follows: $B' = (B\setminus\{R\})\cup\{R_0,R_1\}$, $E_{ij}' = E_{ij}$ if $i \neq j$ and $E_{ii}' = B'$, and

$$T_i' = (T_i \cap {}^2B')\cup\{\langle R_0,R_0\rangle,\langle R_1,R_1\rangle\}\cup\{\langle R_j,a\rangle,\langle a,R_j\rangle : j<2, R\,T_i\,a\}.$$

It is not hard to show that for $\mathfrak{C}' = \mathfrak{Cm}\,\mathfrak{B}'$ we have

$$\mathfrak{C} \subseteq \mathfrak{C}' \subseteq \langle \mathfrak{P}(^3U), \mathsf{C}_i, \mathsf{D}_{ij}\rangle_{i,j<3}, \tag{1}$$

that is, the operations of $\mathfrak{C}'$ are still the concrete, set theoretic ones, and $\mathfrak{C}'$ differs from $\mathfrak{C}$ in one respect only, namely that R is the sum of the two new atoms R_0 and R_1 in $\mathfrak{C}'$. It follows easily from the definition of T_i' that $\mathsf{C}_i R_j = \mathsf{C}_i R$ for all $i<3$ and $j<2$.[3] Clearly, $\langle \mathfrak{P}(^3U), \mathsf{C}_i, \mathsf{D}_{ij}\rangle_{i,j<3} \in \mathbf{Rd}_{ca}\,\mathsf{PEA}_3$, and in fact it is the cylindric reduct of a *representable* PEA_3 (as defined e.g. in [HMT85, 5.4.22]): the operations p_{ij} can be defined by the formula

$$\mathsf{p}_{ij}X = \{s \in {}^3U : s\circ(ij) \in X\}, \tag{2}$$

for all $X \subseteq {}^3U$ and $i, j < 3$. Hence $\mathfrak{C}' \in \mathbf{S}\mathbf{Rd}_{ca}\,\mathsf{PEA}_3$. Now suppose for contradiction that $\mathfrak{C}'$ is isomorphic to the cylindric reduct of a PEA_3. Then $\mathfrak{C}'$ can be expanded with operations p_{ij} such that $\langle \mathfrak{C}', \mathsf{s}^i_j, \mathsf{p}_{ij}\rangle_{i,j<3} \in \mathsf{PEA}_3$. (We note that p_{ij} is not necessarily the concrete operation as defined in (2); all we can assume is that $\langle \mathfrak{C}', \mathsf{s}^i_j, \mathsf{p}_{ij}\rangle_{i,j<3}$ satisfies the axioms defining PEA_3.) Let R' be the atom $\langle 2,0,4\rangle^- = \{2,3\}\times\{0,1\}\times\{4,5\}$ of $\mathfrak{C}'$. We claim that

$$\mathsf{C}_2 R' = \mathsf{p}_{01}\,\mathsf{C}_2 R.$$

[3]The author has first seen the algebra $\mathfrak{C}'$ in a course given by H. Andréka, where it was used to show that the operation of taking the converse of an element is not definable in cylindric algebras. We note that (an isomorphic copy of) $\mathfrak{C}'$ could have been constructed by dilating $\mathfrak{C}$ with the sequence $\langle R,R,R\rangle$.

Indeed, by 5.13(vi) and (1), $\mathsf{p}_{01}\,\mathsf{C}_2 R = {}_2\mathsf{s}(0,1)\,\mathsf{C}_2 R = \mathsf{C}_2 R'$. Since p_{01} is a Boolean automorphism by axioms (F6) and (F7), it follows that p_{01} is a bijection between atoms below $\mathsf{C}_2 R$ and those below $\mathsf{C}_2 R'$. But this is a contradiction, because there are six atoms below $\mathsf{C}_2 R$,

$$R_0, \quad R_1, \quad \langle 0,2,2\rangle^- = \mathsf{D}_{12} \cap \mathsf{C}_2 R_0, \quad \langle 0,2,3\rangle^- = -\mathsf{D}_{12} \cap \mathsf{C}_2 R_0,$$
$$\langle 0,2,0\rangle^- = \mathsf{D}_{02} \cap \mathsf{C}_2 R_0, \quad \langle 0,2,1\rangle^- = -\mathsf{D}_{02} \cap \mathsf{C}_2 R_0,$$

and only five atoms

$$\langle 2,0,4\rangle^- = R', \quad \langle 2,0,0\rangle^- = \mathsf{D}_{12} \cap \mathsf{C}_2 R', \quad \langle 2,0,1\rangle^- = -\mathsf{D}_{12} \cap \mathsf{C}_2 R',$$
$$\langle 2,0,2\rangle^- = \mathsf{D}_{02} \cap \mathsf{C}_2 R', \quad \langle 2,0,3\rangle^- = -\mathsf{D}_{02} \cap \mathsf{C}_2 R',$$

below $\mathsf{C}_2 R'$. (A similar construction shows $\mathbf{Rd}_{ca}\,\mathsf{PEA}_n \subset \mathbf{S}\,\mathbf{Rd}_{ca}\,\mathsf{PEA}_n$ for all finite $n > 3$.)

To sum up,

$$\mathbf{Rd}_{ca}\,\mathsf{PEA}_3 \subset \mathbf{S}\,\mathbf{Rd}_{ca}\,\mathsf{PEA}_3 = \{\,\mathfrak{C} \in \mathsf{CA}_3 : \mathfrak{Ra}\,\mathfrak{C} \in \mathsf{SA}\,\} = \mathsf{CA}_3 \cap \mathrm{Mod}(MGR^+),$$

by this example and Theorem 5.16.

THEOREM 5.18. $\mathsf{SA} = \mathfrak{Ra}^*\big(\mathsf{CA}_3 \cap \mathrm{Mod}(MGR^+)\big) = \mathfrak{Ra}^*\,\mathsf{PEA}_3$.

PROOF. The first equality follows from 5.8 and Maddux [Mad78, Theorem 19 (p. 150)]: the latter states that $\mathsf{SA} = \mathfrak{Ra}^*\,\mathsf{CA}_3'$. Now suppose that $\mathfrak{A} = \mathfrak{Ra}\,\mathfrak{C}$ for some $\mathfrak{C} \in \mathsf{CA}_3 \cap \mathrm{Mod}(MGR^+)$, and let $\mathfrak{C}^- = \mathsf{Sg}^{(\mathfrak{C})}\,\mathrm{Nr}_2\,\mathfrak{C}$. Then $\mathfrak{C}^- \models MGR^+$, so $\mathfrak{C}^- = \mathfrak{Rd}_{ca}\,\mathfrak{B}$ for some $\mathfrak{B} \in \mathsf{PEA}_3$ by Theorem 5.16. But then $\mathfrak{Ra}\,\mathfrak{B} = \mathfrak{Ra}\,\mathfrak{C}^- = \mathfrak{Ra}\,\mathfrak{C} = \mathfrak{A}$, so we have shown $\mathfrak{Ra}^*\big(\mathsf{CA}_3 \cap \mathrm{Mod}(MGR^+)\big) \subseteq \mathfrak{Ra}^*\,\mathsf{PEA}_3$. Finally, the inclusion $\mathfrak{Ra}^*\,\mathsf{PEA}_3 \subseteq \mathsf{SA}$ follows from 5.13(vii) and 5.16. $\square$

CHAPTER 6

The embedding property for cylindric algebras

As yet another application of the method of twisting, in this chapter[1] we exhibit two CA_ω's which cannot be jointly embedded in any CA_ω. This solves a problem of Don Pigozzi [Pig69], [Pig71, p. 347].

Before formulating and proving this result, we recall the following definitition.

DEFINITION 6.1. *A class K of algebras is said to have the* ***embedding property*** *if any two algebras in K with a common minimal subalgebra can be embedded in an algebra in K.*

We note that the classes Cs_ω and RCA_ω both have the embedding property (see Pigozzi [Pig71]).

THEOREM 6.2. CA_ω *does not have the embedding property.*

PROOF. The plan of the proof is as follows. Since our aim is to show the failure of the embedding property, we need to find two CA_ω's that cannot be jointly embedded in a third. The first algebra will be a CA_ω $\mathfrak{C}$ contructed, of course, by twisting, in which *MGR* fails on a "small" element $\mathsf{c}_2\,\xi_0$. By "small" we mean that the 0th projection of $\mathsf{c}_2\,\xi_0$ has only finitely many elements, or, since $\mathfrak{C}$ is not representable, it would be more accurate to say that "$\mathfrak{C}$ thinks that $\mathsf{c}_2\,\xi_0$ is small". The other algebra we will consider is a CA_ω $\mathfrak{D}$ in which there is a total ordering r. Since the equations expressing the latter property do not involve free variables, if there is a CA_ω $\mathfrak{E}$ containg both $\mathfrak{C}$ and $\mathfrak{D}$ as subalgebras, r will still be a total ordering in $\mathfrak{E}$. For the same reason (i.e, because smallness can be expressed by an equation not involving free variables), $\mathsf{c}_2\,\xi_0$ will continue to be "small" in $\mathfrak{E}$. Hence r can be used to pick a maximal point (denoted by μ in the proof) in the 0-th projection of $\mathsf{c}_2\,\xi_0$, and this in turn can be used to define a non-zero element y below $\mathsf{c}_2\,\xi_0$ for which *MGR* holds, i.e., $y = y^{\smile\smile}$. But this will lead to a contradiction, because *MGR* fails in $\mathfrak{C}$ (and hence in $\mathfrak{E}$) in such a way that $\mathsf{c}_2\,\xi_0 \cdot (\mathsf{c}_2\,\xi_0)^{\smile\smile} = 0$.

We start the proof by constructing $\mathfrak{C}$. For $0 < k < \omega$, let π_k be the permutation $(01)(2k, 2k+1)$ of $U = \omega$, G the group generated by $\{\,\pi_k : 0 < k < \omega\,\}$, and let $\mathfrak{A} = \mathfrak{Fr}_G\mathfrak{E}$, where $\mathfrak{E}$ is the powerset algebra of dimension ω on U.

Claim 1. Let $H = \{\,X \subseteq \omega : |X|$ *is even*$\,\}$. *Then* H *is closed under* $\oplus$ *(symmetric difference), and* $\langle H, \oplus, {}^{-1}, \emptyset\rangle$ *(where* $X^{-1} = X$ *for all* $X \in H$*) is a group isomorphic to* G.

Proof. If $X, Y \subseteq \omega$ and $X \setminus Y$ are even, then $X \cap Y$ is even, because $X = (X \setminus Y) \mathbin{\dot\cup} (X \cap Y)$; hence $Y \setminus X$ must be even, since $Y = (Y \setminus X) \mathbin{\dot\cup} (X \cap Y)$. As $X \oplus Y = (X \setminus Y) \mathbin{\dot\cup} (Y \setminus X)$, this proves the first part of the claim.

Let f be the map $X \mapsto \{\,(2k, 2k+1) : k \in X\,\}$ from H into $\mathrm{Sym}U$. It is not hard to check that f is a one-one group homomorphism; and its range contains G, because $\pi_k = f(\{0,k\})$ for all $0 < k < \omega$. We show by induction on the size of $X \in H$ that f maps into G. $f(\emptyset) = 1^G$, and if $|X| = 2$, then either $X = \{0, k\}$ for some $k > 0$, and then $f(X) = \pi_k$, or $X = \{k, l\}$ $k, l > 0$, $k \neq l$, and then $f(X) = \pi_k\pi_l$. Let $n > 0$, suppose we know that $f(X) \in G$ for all $X \in H$ of size $2n$, and let $X \in H$ be of size $2(n+1)$.

[1]The results in this chapter are joint with Judit Madarász.

Then $X = X' \dot\cup \{k,l\}$ for some $X' \in H$ and $k \neq l$ such that $0 \notin \{k,l\}$. Hence $f(X) = f(X' \oplus \{k,l\}) = f(X')f(\{k,l\}) = f(X')\pi_k\pi_l \in G$.

Thus f is an isomorphism from H to G. $\square$

From the Claim it follows that G is commutative and it contains exactly those permutations of ω which "invert" an even number of blocks (where a block is a set $\{2k, 2k+1\}$) and leaves the rest of ω alone.

For $n \in \omega$, let $\xi_0^n = \langle 0, n, 2, 4, 6, \dots\rangle^-$ and $\xi_1^n = \langle 0, n, 3, 4, 6, \dots\rangle^-$, $\xi_0 = \xi_0^2$ and $\xi_1 = \xi_1^3$. Note that $\xi_i^n = \xi_j^m$ iff $j = i$ and $m = n$: this is because there is no $\pi \in G$ which fixes all even numbers except 2. It is clear that the twisting conditions (T1) and (T2) hold for $t = 1$ and $I = \{0,1\}$, and that $\Xi_0 \overset{\text{def}}{=} \xi_0/T_1 = \{\, \xi_0^n : n \in \omega \,\}$ and $\Xi_1 \overset{\text{def}}{=} \xi_1/T_1 = \{\, \xi_1^n : n \in \omega \,\}$. Next we claim that

$$T_0 \cap (\Xi_0 \times \Xi_1) = \Big\{ \langle \xi_0^m, \xi_1^n \rangle : m = n > 3 \text{ or } \{m,n\} \in \big\{\{0,1\},\{2,3\}\big\} \Big\}, \tag{1}$$

$$T_2 \cap (\Xi_0 \times \Xi_1) = \big\{ \langle \xi_0^m, \xi_1^n \rangle : m = n \big\} \tag{2}$$

and if $i \geq 2$, then

$$T_{i+1} \cap (\Xi_0 \times \Xi_1) = \Big\{ \langle \xi_0^m, \xi_1^n \rangle : \quad m = n \notin \{2, 3, 2i, 2i+1\} \text{ or } \{m,n\} \in \big\{\{2,3\},\{2i,2i+1\}\big\} \Big\}. \tag{3}$$

The first equality holds because $\xi_0^m\, T_0\, \xi_1^n$ iff $\langle m, 2, 4, 6, \dots\rangle \equiv_G \langle n, 3, 4, 6, \dots\rangle$, and the only element of G that fixes $2k$ for all $k > 1$ and sends 2 to 3 is π_1. Hence from $\pi_1(m) = n$ it follows that either $m = n > 3$ or $\{m,n\} \in \{\{0,1\},\{2,3\}\}$.

The second equality holds because $\xi_0^m\, T_2\, \xi_1^n$ iff $\langle 0, m, 4, 6, \dots\rangle \equiv_G \langle 0, n, 4, 6, \dots\rangle$, and the identity is the only $\pi \in G$ for which $\pi(0) = 0$ and $\pi(2k) = 2k$ for all $k > 1$.

The last equality holds, because if $i \geq 2$, then $\xi_0^m\, T_{i+1}\, \xi_1^n$ iff $\langle 0, m, 2, 6, \dots\rangle \equiv_G \langle 0, n, 3, 6, \dots\rangle$ (if $i = 2$) or $\langle 0, m, 2, \dots, 2(i-1), 2(i+1), 2(i+2), \dots\rangle \equiv_G \langle 0, n, 3, \dots, 2(i-1), 2(i+1), 2(i+2), \dots\rangle$ (if $i > 2$), and $\{(23)(2i\, 2i+1)\} = \pi_1\pi_i$ is the only $\pi \in G$ with $\pi(2) = 3$ and $\pi(2k) = 2k$ for all $k \in \omega \setminus \{1, i\}$. Hence either $m = n \notin \{2, 3, 2i, 2i+1\}$ or $\{m,n\} \in \{\{2,3\},\{2i,2i+1\}\}$.

From these equalities it is clear that if Ξ_i $(i = 0,1)$ are partitioned into

$$\Xi_i' = \{\xi_i^2, \xi_i^3\}, \qquad \Xi_i'' = \{\, \xi_i^n : n \in \omega \setminus \{2,3\} \,\},$$

then condition (T3) holds. Let $\mathfrak{C}$ denote the CA_ω we get by twisting $\mathfrak{A}$ according to this partition and the permutation (01). Then, by 2.27, in $\mathfrak{C}$ we have $(\mathsf{c}_2\,\xi_i)^{\smile} = \mathsf{c}_2\,\xi_j$ if $\{i,j\} = \{0,1\}$.

Now we prove some properties of the elements ξ_0, ξ_1 in $\mathfrak{C}$.

$$\mathsf{c}_2\,\xi_0 \cdot \mathsf{c}_2\,\xi_1 = 0 \tag{4}$$

Otherwise we'd have $\xi_0\, T_2\, \xi_1$, i.e. $\langle 0, 2, 4, 6, \dots\rangle \equiv_G \langle 0, 3, 4, 6, \dots\rangle$, contradicting the fact that there is no $\pi \in G$ which inverts exactly one block.

$$\mathsf{c}_2(\xi_0 + \xi_1) = \mathsf{c}_0\,\mathsf{c}_2\,\xi_0 \cdot \mathsf{c}_1\,\mathsf{c}_2\,\xi_0 \tag{5}$$

($\leq$) $\mathsf{c}_2\,\xi_0 \leq$ RHS is obvious. $\xi_1 \leq \mathsf{c}_0\,\xi_0$ by (1), and hence $\mathsf{c}_2\,\xi_1 \leq \mathsf{c}_0\,\mathsf{c}_2\,\xi_0$. Finally, $\xi_1 \leq \mathsf{c}_2\,\xi_0^3$ by (2), and $\xi_0^3 \leq \mathsf{c}_1\,\xi_0$, since $\xi_0^3, \xi_0 = \xi_0^2 \in \Xi_0'$; hence $\mathsf{c}_2\,\xi_1 \leq \mathsf{c}_2\,\xi_0^3 \leq \mathsf{c}_2\,\mathsf{c}_1\,\xi_0$. Thus $\mathsf{c}_2\,\xi_1 \leq$ RHS.

($\geq$) First note that the term function $\mathsf{c}_1\,\mathsf{c}_2$ is the same in $\mathfrak{C}$ as in $\mathfrak{A}$ by 2.25, so it is enough to check $\geq$ in $\mathfrak{A}$. Let $s \in {}^{\omega}\omega$ and assume that $s^{-} \leq \mathsf{c}_0\,\mathsf{c}_2\,\xi_0 \cdot \mathsf{c}_1\,\mathsf{c}_2\,\xi_0$. By $s^{-} \leq \mathsf{c}_0\,\mathsf{c}_2\,\xi_0$, $s \upharpoonright \omega \setminus \{0,2\} \equiv_G \langle 2,4,6,\dots\rangle$. We may as well assume that $s \upharpoonright \omega \setminus \{0,2\} = \langle 2,4,6,\dots\rangle$ (since s^{-} contains such a sequence). So $s = \langle s_0, 2, s_2, 4, 6, \dots\rangle$. From this, and $s^{-} \in \mathsf{c}_1\,\mathsf{c}_2\,\xi_0$ we get

$$\langle s_0, 4, 6, \dots\rangle = s \upharpoonright \omega \setminus \{1,2\} \equiv_G \langle 0,4,6,\dots\rangle,$$

so there is a $\sigma \in G$ with $\sigma(s_0) = 0$ and $\sigma(2k) = 2k$ for all $k \geq 2$. Hence σ is either the identity or π_1. In the first case we have $s_0 = 0$, and hence $s^{-} = \langle 0,2,s_2,4,6,\dots\rangle^{-}\; T_2\; \xi_0$ and in the second $s_0 = 1$ and $s^{-} = \langle 1,2,s_2,4,6,\dots\rangle^{-} = \langle 0,3,\pi_1 s_2,4,6,\dots\rangle^{-}\; T_2\; \xi_1$. Hence $s^{-} \leq \mathsf{c}_2\,\xi_0 + \mathsf{c}_2\,\xi_1$, completing the proof of (5).

$$\mathsf{c}_1\,\mathsf{c}_2\,\xi_0 = \mathsf{c}_1\,\mathsf{c}_2\,\xi_1 \quad \text{and} \quad \mathsf{c}_0\,\mathsf{c}_2\,\xi_0 = \mathsf{c}_0\,\mathsf{c}_2\,\xi_1 \tag{6}$$

These equations clearly hold in $\mathfrak{A}$ (the first is shown by the identity permutation, and the second by π_1), and since c_0, c_2 and $\mathsf{c}_1\,\mathsf{c}_2$ in $\mathfrak{C}$ are the same as in $\mathfrak{A}$ by 2.25, they continue to hold in $\mathfrak{C}$.

Intuitively, (7) below says that "$\mathfrak{C}$ thinks that the 0th projection (or domain) of ξ_0 is of cardinality ≤ 2".

$$\mathsf{c}_1\,\mathsf{c}_2\,\xi_0 \cdot \mathsf{s}^0_1\,\mathsf{c}_1\,\mathsf{c}_2\,\xi_0 \cdot \mathsf{s}^0_2\,\mathsf{c}_1\,\mathsf{c}_2\,\xi_0 \leq \mathsf{d}_{01} + \mathsf{d}_{02} + \mathsf{d}_{12} \tag{7}$$

Since c_0 and $\mathsf{c}_1\,\mathsf{c}_2$ in $\mathfrak{C}$ are the same as in $\mathfrak{A}$, it is enough to check that this holds in $\mathfrak{A}$. (And then it is clear, because $|\mathrm{dom}\,\mathsf{c}_2\,\xi_0| \leq 2$, but let's check it anyway.) Let $s \in {}^{\omega}\omega$ and suppose that $s^{-} \leq \mathsf{c}_1\,\mathsf{c}_2\,\xi_0 \cdot \mathsf{s}^0_1\,\mathsf{c}_1\,\mathsf{c}_2\,\xi_0 \cdot \mathsf{s}^0_2\,\mathsf{c}_1\,\mathsf{c}_2\,\xi_0$ and $s^{-} \nleq \mathsf{d}_{01} + \mathsf{d}_{02}$. Then $s \upharpoonright \omega \setminus \{1,2\} \equiv_G \langle 0,4,6,\dots\rangle$, and there are $z, w \in {}^{\omega}\omega$ such that $z_0 = z_1$, $z \upharpoonright \omega \setminus \{0\} \equiv_G s \upharpoonright \omega \setminus \{0\}$, $z \upharpoonright \omega \setminus \{1,2\} \equiv_G \langle 0,4,6,\dots\rangle$, and $w_0 = w_2$, $w \upharpoonright \omega \setminus \{0\} \equiv_G s \upharpoonright \omega \setminus \{0\}$, $w \upharpoonright \omega \setminus \{1,2\} \equiv_G \langle 0,4,6,\dots\rangle$. We may assume $s \upharpoonright \omega \setminus \{1,2\} = \langle 0,4,6,\dots\rangle$, $z \upharpoonright \omega \setminus \{1,2\} = \langle 0,4,6,\dots\rangle$ and $w \upharpoonright \omega \setminus \{1,2\} = \langle 0,4,6,\dots\rangle$. So

$$s = \langle 0, s_1, s_2, 4, 6, \dots\rangle, \quad z = \langle 0,0,z_2,4,6,\dots\rangle, \quad \text{and} \quad w = \langle 0, w_1, 0, 4, 6, \dots\rangle,$$

and there are $\sigma, \pi \in G$ such that $\sigma(s_1) = 0 = \pi(s_2)$ and $\sigma(2k) = 2k = \pi(2k)$ for all $k \geq 2$. From this last equality it follows that $\{\sigma, \pi\} \subseteq \{\mathrm{Id}, \pi_1\}$. But by the assumption $s^{-} \nleq \mathsf{d}_{01} + \mathsf{d}_{02}$, $0 \notin \{s_1, s_2\}$, whence $\mathrm{Id} \notin \{\sigma, \pi\}$. Hence $\sigma = \pi_1 = \pi$ and thus $s_1 = 1 = s_2$, that is, $s^{-} \leq \mathsf{d}_{12}$.

A similar argument shows that

$$\mathsf{c}_0\,\mathsf{c}_2\,\xi_0 \cdot \mathsf{s}^1_0\,\mathsf{c}_0\,\mathsf{c}_2\,\xi_0 \cdot \mathsf{s}^1_2\,\mathsf{c}_0\,\mathsf{c}_2\,\xi_0 \leq \mathsf{d}_{01} + \mathsf{d}_{02} + \mathsf{d}_{12},$$

i.e., that "$\mathfrak{C}$ thinks that the 1st projection of ξ_0 is of cardinality ≤ 2", but we will not need this in the sequel.

Now let $\mathfrak{D}$ be the full Cs_ω of the natural numbers, and let r be the ordering $<$ of ω, that is, $r \overset{\text{def}}{=} \{ s \in {}^{\omega}\omega : s_0 < s_1 \}$. We will show that there is no $\mathfrak{E} \in \mathsf{CA}_\omega$ which embeds both $\mathfrak{C}$ and $\mathfrak{D}$. But in order for this to witness tha failure of the embedding property in CA_ω, we need to prove that

(8) the minimal subalgebras of $\mathfrak{C}$ and $\mathfrak{D}$ are isomorphic.

By [HMT71, 2.5.30], simple minimal CA's of the same characteristic are isomorphic; hence (8) follows from Claim 2 below.

Claim 2. The minimal subalgebras $\mathfrak{C}_0$ *of* $\mathfrak{C}$ *and* $\mathfrak{D}_0$ *of* $\mathfrak{D}$ *are both simple, and of characteristic* 0.

Proof. In any CA_ω let $a_0 = 1$, and for every $1 \le k < \omega$, let $a_k = \mathsf{c}_0 \cdots \mathsf{c}_{k-1} \prod_{i<j<k} -\mathsf{d}_{ij}$. By [HMT71, 2.1.17], $\mathfrak{Nr}_0\,\mathfrak{C}$, the BA of zero-dimensional elements of a minimal CA_ω $\mathfrak{C}$ is $\mathsf{Sg}^{(\mathfrak{Bl}\,\mathfrak{C})}\{\, a_k : k < \omega \,\}$. It is easy to see that for all $k < \omega$, $a_k^{\mathfrak{A}} = 1^{\mathfrak{A}}$ and $a_k^{\mathfrak{D}} = 1^{\mathfrak{D}}$. From this it follows by 2.25 and $\mathfrak{C}_0 \subseteq \mathfrak{C}$, $\mathfrak{D}_0 \subseteq \mathfrak{D}$, that

(*) $$a_k^{\mathfrak{C}_0} = 1^{\mathfrak{C}_0} \text{ and } a_k^{\mathfrak{D}_0} = 1^{\mathfrak{D}_0} \text{ for all } k < \omega.$$

Hence $\mathfrak{Nr}_0\,\mathfrak{C}_0$ and $\mathfrak{Nr}_0\,\mathfrak{D}_0$ are isomorphic to the two-element BA. Because of this, and since $\mathfrak{C}_0$ and $\mathfrak{D}_0$ are minimal and hence locally finite (by [HMT71, 2.1.16]), it follows from [HMT71, 2.3.14] that they are simple. Finally, from (*) and [HMT71, 2.4.63(i)] we get that $\mathfrak{C}_0$ and $\mathfrak{D}_0$ are of characteristic 0. □

Now suppose that $\mathfrak{C}$ and $\mathfrak{D}$ can be jointly embedded in a CA_ω $\mathfrak{E}$. We can of course assume that $\mathfrak{C}$ and $\mathfrak{D}$ are subalgebras of $\mathfrak{E}$. Unless otherwise stated, the definitions and statements in the rest of the proof all refer to the algebra $\mathfrak{E}$. Let

$$\mu \overset{\text{def}}{=} \mathsf{c}_1\,\mathsf{c}_2\,\xi_0 \cdot -\mathsf{c}_1(\mathsf{s}_1^0\,\mathsf{c}_1\,\mathsf{c}_2\,\xi_0 \cdot r).$$

Our first goal is to show that

(9) $$\mu \text{ is non-zero.}$$

(10) $$0 < \mathsf{c}_1\,\mathsf{c}_2\,\xi_0 \cdot \mathsf{s}_2^0\,\mathsf{c}_1\,\mathsf{c}_2\,\xi_0 \cdot -\mathsf{d}_{02}$$

Of course we only need to see that this holds in $\mathfrak{C}$ and hence in $\mathfrak{A}$ by 2.25; and there we have

$$\langle 0,2,1,4,6,\dots\rangle^-\ T_2\ \langle 0,2,2,4,6,\dots\rangle^- = \xi_0\ T_1\ \xi_0 \quad \text{and}$$

$$\begin{aligned}\langle 0,2,1,4,6,\dots\rangle^-\ T_0\ \langle 1,2,1,4,6,\dots\rangle^-\ T_2\ \langle 1,2,3,4,6,\dots\rangle^-&\\ = \langle 0,3,2,4,6,\dots\rangle^-\ T_1\ \xi_0&\end{aligned}$$

and since $\langle 0,2,1,4,6,\dots\rangle^- \le -\mathsf{d}_{02}$ and $\langle 1,2,1,4,6,\dots\rangle^- \le \mathsf{d}_{02}$, this shows that

$$0 < \langle 0,2,1,4,6,\dots\rangle^- \le \mathsf{c}_2\,\mathsf{c}_1\,\xi_0 \cdot \mathsf{s}_2^0\,\mathsf{c}_2\,\mathsf{c}_1\,\xi_0 \cdot -\mathsf{d}_{02} = \mathsf{c}_1\,\mathsf{c}_2\,\xi_0 \cdot \mathsf{s}_2^0\,\mathsf{c}_1\,\mathsf{c}_2\,\xi_0 \cdot -\mathsf{d}_{02}.$$

Let $r_{02} = \mathsf{s}_2^1\, r$ and $r_{20} = \mathsf{s}_0^1\,\mathsf{s}_2^0\, r$.

(11) $$-\mathsf{d}_{02} \le r_{02} + r_{20} \quad \text{and} \quad r_{20} \cdot r \le -(\mathsf{d}_{01} + \mathsf{d}_{02} + \mathsf{d}_{12})$$

It suffices to check these statements in $\mathfrak{D}$; and there, since $r_{02} = \{\, s \in {}^\omega\omega : s_0 < s_2 \,\}$ and $r_{20} = \{\, s \in {}^\omega\omega : s_0 > s_2 \,\}$, the first is true by the trichotomy of $<$, and the second by the irreflexivity and transitivity of $<$ on the natural numbers.

(12) $$0 < \mathsf{c}_1\,\mathsf{c}_2\,\xi_0 \cdot \mathsf{s}_2^0\,\mathsf{c}_1\,\mathsf{c}_2\,\xi_0 \cdot r_{20}$$

For suppose that $\mathsf{c}_1\,\mathsf{c}_2\,\xi_0 \cdot \mathsf{s}_2^0\,\mathsf{c}_1\,\mathsf{c}_2\,\xi_0 \cdot r_{20} = 0$. Then

$$\begin{aligned} 0 &= {}_1\mathsf{s}(2,0)\,\mathsf{c}_1\,\mathsf{c}_2\,\xi_0 \cdot {}_1\mathsf{s}(2,0)\,\mathsf{s}_2^0\,\mathsf{c}_1\,\mathsf{c}_2\,\xi_0 \cdot {}_1\mathsf{s}(2,0) r_{20} && \text{by 1.11(v)}\\ &= \mathsf{s}_2^0\,\mathsf{c}_1\,\mathsf{c}_2\,\xi_0 \cdot \mathsf{c}_1\,\mathsf{c}_2\,\xi_0 \cdot r_{02} \end{aligned}$$

because ${}_1\mathsf{s}(2,0)\,\mathsf{c}_1\,\mathsf{c}_2\,\xi_0 = \mathsf{s}_2^0\,\mathsf{c}_1\,\mathsf{c}_2\,\xi_0$ by 3.2,

$$\begin{aligned} {}_1\mathsf{s}(2,0)\,\mathsf{s}_2^0\,\mathsf{c}_1\,\mathsf{c}_2\,\xi_0 &= \mathsf{s}_0^2\,\mathsf{s}_2^0\,\mathsf{c}_1\,\mathsf{c}_2\,\xi_0 && \text{by 1.11(ii), 3.2}\\ &= \mathsf{c}_1\,\mathsf{c}_2\,\xi_0 && \text{by 1.11(viii)} \end{aligned}$$

and

$$_{1}\mathsf{s}(2,0)r_{20} = \mathsf{s}^1_2\,\mathsf{s}^2_0\,\mathsf{s}^0_1\,\mathsf{s}^1_0\,\mathsf{s}^0_2\,r = \mathsf{s}^1_2\,\mathsf{s}^2_0\,\mathsf{s}^0_2\,r = \mathsf{s}^1_2\,r = r_{02}$$

by repeated applications of 1.11(viii). But this is a contradiction, since $\mathsf{c}_1\,\mathsf{c}_2\,\xi_0\cdot\mathsf{s}^0_2\,\mathsf{c}_1\,\mathsf{c}_2\,\xi_0\cdot r_{20}=0$ implies $\mathsf{c}_1\,\mathsf{c}_2\,\xi_0\cdot\mathsf{s}^0_2\,\mathsf{c}_1\,\mathsf{c}_2\,\xi_0\cdot r_{02} > 0$ by (10) and (11).

Hence to prove (9) it is enough to show that

$$\mathsf{c}_1\,\mathsf{c}_2\,\xi_0\cdot\mathsf{s}^0_2\,\mathsf{c}_1\,\mathsf{c}_2\,\xi_0\cdot r_{20} \le \mathsf{c}_1\,\mathsf{c}_2\,\xi_0\cdot -\mathsf{c}_1(\mathsf{s}^0_1\,\mathsf{c}_1\,\mathsf{c}_2\,\xi_0\cdot r),$$

or, equivalently, that

$$\mathsf{c}_1\,\mathsf{c}_2\,\xi_0\cdot\mathsf{s}^0_2\,\mathsf{c}_1\,\mathsf{c}_2\,\xi_0\cdot r_{20}\cdot\mathsf{c}_1(\mathsf{s}^0_1\,\mathsf{c}_1\,\mathsf{c}_2\,\xi_0\cdot r)=0.$$

By 1.6(ii) and since $\mathsf{c}_1\,\mathsf{c}_2\,\xi_0\cdot\mathsf{s}^0_2\,\mathsf{c}_1\,\mathsf{c}_2\,\xi_0\cdot r_{20}$ is c_1-closed (by 1.11(ii), 1.11(iii) and (C3)), the latter statement is equivalent to

$$\mathsf{c}_1\,\mathsf{c}_2\,\xi_0\cdot\mathsf{s}^0_2\,\mathsf{c}_1\,\mathsf{c}_2\,\xi_0\cdot r_{20}\cdot\mathsf{s}^0_1\,\mathsf{c}_1\,\mathsf{c}_2\,\xi_0\cdot r=0,$$

and this is an obvious consequence of (7) and (11).

Our next goal is to show that

$$\mu\cdot\mathsf{s}^0_2\,\mu \le \mathsf{d}_{02}. \tag{13}$$

This follows from (14) below, because applying s^1_2 to both sides of (14) we get $\mathsf{s}^1_2\,\mu\cdot\mathsf{s}^1_2\,\mathsf{s}^0_1\,\mu \le \mathsf{s}^1_2\,\mathsf{d}_{01}$ by 1.11(v), and $\mathsf{s}^1_2\,\mu=\mu$ by 1.11(i) and 1.6(iv),

$$\begin{aligned}
\mathsf{s}^1_2\,\mathsf{s}^0_1\,\mu &= \mathsf{s}^1_2\,\mathsf{s}^0_1\,\mathsf{c}_1\,\mathsf{c}_2\,\mu && \text{1.6(iv), 1.11(ii), (C3)}\\
&= \mathsf{s}^1_2\,\mathsf{c}_2\,\mathsf{s}^0_1\,\mathsf{c}_1\,\mu && \text{by 1.11(ii)}\\
&= \mathsf{s}^1_2\,\mathsf{s}^2_0\,\mathsf{c}_2\,\mathsf{s}^0_1\,\mathsf{c}_1\,\mu && \text{by 1.11(i)}\\
&= \mathsf{s}^1_2\,\mathsf{s}^2_0\,\mathsf{s}^0_1\,\mathsf{c}_1\,\mathsf{c}_2\,\mu && \text{by 1.11(ii)}\\
&= {}_{1}\mathsf{s}(2,0)\,\mathsf{c}_1\,\mathsf{c}_2\,\mu && \\
&= \mathsf{s}^0_2\,\mathsf{c}_1\,\mathsf{c}_2\,\mu && \text{by 3.2}\\
&= \mathsf{s}^0_2\,\mu &&
\end{aligned}$$

and $\mathsf{s}^1_2\,\mathsf{d}_{01}=\mathsf{d}_{02}$ by 1.11(vi). Now we prove

$$\mu\cdot\mathsf{s}^0_1\,\mu \le \mathsf{d}_{01}. \tag{14}$$

Since $-\mathsf{d}_{01} \le r + r^{\smile}$ (in $\mathfrak{D}$ and hence in $\mathfrak{C}$), to see that (14) holds it suffices to check that

$$\mu\cdot\mathsf{s}^0_1\,\mu\cdot r = 0 \quad\text{and}\quad \mu\cdot\mathsf{s}^0_1\,\mu\cdot r^{\smile}=0.$$

The first of these statements is true because

$$\begin{aligned}
\mu\cdot\mathsf{s}^0_1\,\mu\cdot r &\le -\mathsf{c}_1(\mathsf{s}^0_1\,\mathsf{c}_1\,\mathsf{c}_2\,\xi_0\cdot r)\cdot\mathsf{s}^0_1\,\mathsf{c}_1\,\mathsf{c}_2\,\xi_0\cdot r && \text{by def. of } \mu\\
&\le -\mathsf{c}_1(\mathsf{s}^0_1\,\mathsf{c}_1\,\mathsf{c}_2\,\xi_0\cdot r)\cdot\mathsf{c}_1(\mathsf{s}^0_1\,\mathsf{c}_1\,\mathsf{c}_2\,\xi_0\cdot r)\\
&= 0,
\end{aligned}$$

and the second statement follows from the first, because ${}_{2}\mathsf{s}(0,1)\mu=\mathsf{s}^0_1\,\mu$ by 3.2, and ${}_{2}\mathsf{s}(0,1)\,\mathsf{s}^0_1\,\mu = \mathsf{s}^1_0\,\mathsf{s}^0_1\,\mu=\mu$ by 3.2 and 1.11(viii).

The last statement about $\mathfrak{C}$ we need is

$$\mu \cdot \mathsf{c}_0(\mathsf{c}_2\,\xi_0 \cdot \mu) \le \mathsf{c}_2\,\xi_0. \tag{15}$$

To see that this holds, assume for contradiction that $\mu \cdot -\mathsf{c}_2\,\xi_0 \cdot \mathsf{c}_0(\mathsf{c}_2\,\xi_0 \cdot \mu) \ne 0$; then

$$0 \ne \mathsf{d}_{02} \cdot \mu \cdot -\mathsf{c}_2\,\xi_0 \cdot \mathsf{c}_0(\mathsf{c}_2\,\xi_0 \cdot \mu) = \\ \big[\mathsf{d}_{02} \cdot \mu \cdot -\mathsf{c}_2\,\xi_0 \cdot \mathsf{c}_0(\mathsf{c}_2\,\xi_0 \cdot \mu \cdot \mathsf{d}_{02})\big] + \big[\mathsf{d}_{02} \cdot \mu \cdot -\mathsf{c}_2\,\xi_0 \cdot \mathsf{c}_0(\mathsf{c}_2\,\xi_0 \cdot \mu \cdot -\mathsf{d}_{02})\big]$$

by 1.6(vi) and since

$$\begin{aligned} \mathsf{c}_0(\mathsf{c}_2\,\xi_0 \cdot \mu) &= \mathsf{c}_0(\mathsf{c}_2\,\xi_0 \cdot \mu \cdot (\mathsf{d}_{02} + -\mathsf{d}_{02})) \\ &= \mathsf{c}_0((\mathsf{c}_2\,\xi_0 \cdot \mu \cdot \mathsf{d}_{02}) + (\mathsf{c}_2\,\xi_0 \cdot \mu \cdot -\mathsf{d}_{02})) \\ &= \mathsf{c}_0(\mathsf{c}_2\,\xi_0 \cdot \mu \cdot \mathsf{d}_{02}) + \mathsf{c}_0(\mathsf{c}_2\,\xi_0 \cdot \mu \cdot -\mathsf{d}_{02}). \end{aligned}$$

But this is a contradiction, since

$$\mathsf{d}_{02} \cdot \mu \cdot -\mathsf{c}_2\,\xi_0 \cdot \mathsf{c}_0(\mathsf{c}_2\,\xi_0 \cdot \mu \cdot \mathsf{d}_{02}) \le \mathsf{c}_0(\mathsf{d}_{02} \cdot -\mathsf{c}_2\,\xi_0) \cdot \mathsf{c}_0(\mathsf{c}_2\,\xi_0 \cdot \mathsf{d}_{02}) = 0$$

by (C7) and $\mathsf{d}_{02} \cdot \mu \cdot -\mathsf{c}_2\,\xi_0 \cdot \mathsf{c}_0(\mathsf{c}_2\,\xi_0 \cdot \mu \cdot -\mathsf{d}_{02}) = 0$, for otherwise we would have

$$\begin{aligned} 0 &< \mathsf{c}_0(\mathsf{d}_{02} \cdot \mu \cdot -\mathsf{c}_2\,\xi_0) \cdot \mathsf{c}_2\,\xi_0 \cdot \mu \cdot -\mathsf{d}_{02} && \text{by 1.6(ii)} \\ &\le \mathsf{c}_0(\mathsf{d}_{02} \cdot \mu) \cdot \mu \cdot -\mathsf{d}_{02} \\ &= \mathsf{s}^0_2\,\mu \cdot \mu \cdot -\mathsf{d}_{02} \\ &= 0. && \text{by (13)} \end{aligned}$$

Now let $y = \mu \cdot \mathsf{c}_2\,\xi_0$. Then $y \ne 0$, since that would imply $0 = \mu \cdot \mathsf{c}_1\,\mathsf{c}_2\,\xi_0 = \mu$ by 1.6(ii) and the definition of μ (in particular, that it is c_1-closed), contradicting (9). If we succeed in showing that $y^{\smile\smile} = y$, then we have proved the theorem, for then

$$0 < y \cdot y^{\smile\smile} \le \mathsf{c}_2\,\xi_0 \cdot (\mathsf{c}_2\,\xi_0)^{\smile\smile} = \mathsf{c}_2\,\xi_0 \cdot \mathsf{c}_2\,\xi_1 = 0$$

by (4), whence our assumption that there is a CA_ω containing both $\mathfrak{C}$ and $\mathfrak{D}$ as subalgebras led to a contradiction.

We start showing $y^{\smile\smile} = y$ by proving first that y is a "0–1 rectangle", i.e., that $y = \mathsf{c}_0\,y \cdot \mathsf{c}_1\,y$. Of course, only the $\ge$ part needs proof; here is how it goes:

$$\begin{aligned} \mathsf{c}_0\,y \cdot \mathsf{c}_1\,y &= \mathsf{c}_0(\mu \cdot \mathsf{c}_2\,\xi_0) \cdot \mathsf{c}_1(\mu \cdot \mathsf{c}_2\,\xi_0) \\ &= \mathsf{c}_0(\mu \cdot \mathsf{c}_2\,\xi_0) \cdot \mu \cdot \mathsf{c}_1\,\mathsf{c}_2\,\xi_0 && \text{by } \mu = \mathsf{c}_1\,\mu \text{ and (C3)} \\ &= \mathsf{c}_0(\mu \cdot \mathsf{c}_2\,\xi_0) \cdot \mu && \text{by def. of } \mu \\ &\le \mathsf{c}_2\,\xi_0 \cdot \mu && \text{by (15)} \\ &= y \end{aligned}$$

Hence, by 1.11(xiv), to see that $y = y^{\smile\smile}$, it suffices to show that ${}_2\mathsf{s}(0,1)\,\mathsf{c}_i\,y = {}_2\mathsf{s}(1,0)\,\mathsf{c}_i\,y$ for $i = 0$, 1. But, since $y = \mathsf{c}_2\,y$, this follows from 3.2. ☐

We close this chapter with a discussion of possible extensions of Theorem 6.2. Throughout this discussion, the classes of algebras mentioned are all assumed to be subclasses of CA_ω. We have already mentioned that the representable algebras do have the embedding property. From this, it follows that the simple algebras also have the embedding property, because of the following. Simple algebras are representable by [HMT85, 3.2.11], hence for each pair $\mathfrak{A}_0$, $\mathfrak{A}_1$ of simple algebras with a common minimal subalgebra there is a representable algebra $\mathfrak{C}$ which embeds both. By [HMT71, 2.3.17], every proper ideal of CA can be extended to a maximal proper ideal. Hence the following lemma ensures that $\mathfrak{C}$ can be chosen to be simple.

LEMMA 6.3. *Let V be a variety of* BAO*'s, and let $\mathfrak{A} \subseteq \mathfrak{C} \in V$. Suppose that $\mathfrak{A}$ is simple, and let I be an ideal of $\mathfrak{C}$. Then $\mathfrak{A}$ can be embedded in $\mathfrak{C}/I$.*

PROOF. $I \cap A$ is an ideal of $\mathfrak{A}$, and is proper since $1^{\mathfrak{A}} = 1^{\mathfrak{C}} \notin I$. Hence $I \cap A = \{0\}$, because $\mathfrak{A}$ is simple. But then $\mathfrak{A} \cong \mathfrak{A}/(I \cap A)$ and the latter embeds in $\mathfrak{C}/I$. □

A further question one may ask is whether the subdirectly irreducible algebras also have the embedding property, perhaps in the weaker form that each pair of subdirectly irreducible algebras with a common minimal subalgebra embed in a (not necessarily subdirectly irreducible) CA_ω. It turns out that the answer even to this less ambitious question is no. For let $\mathfrak{C}$ and $\mathfrak{D}$ be the algebras constructed in the proof of 6.2, and fix subdirect decompositions of $\mathfrak{C}$ and $\mathfrak{D}$ with subdirectly irreducible factors $\{\mathfrak{C}_i : i \in I\}$ and $\{\mathfrak{D}_i : i \in J\}$, respectively. Since the common minimal subalgebra of $\mathfrak{D}$ and $\mathfrak{C}$ is simple (by Claim 2 in the proof of 6.2), it embeds in $\mathfrak{C}_i$ and $\mathfrak{D}_j$ for all $i \in I$, $j \in J$ by Lemma 6.3. Hence, if each pair of subdirectly irreducible algebras with a common minimal subalgebra embed in a CA_ω, then for all $i \in I$ and $j \in J$ there is an $\mathfrak{E}_{ij} \in \mathsf{CA}_\omega$ containing isomorphic copies of $\mathfrak{C}_i$ and $\mathfrak{D}_i$. But then $\mathfrak{C} \rightarrowtail \prod_{i\in I} \mathfrak{C}_i \rightarrowtail \prod_{i\in I, j\in J} \mathfrak{E}_{ij}$, and similarly for $\mathfrak{D}$ and $\prod_{j\in J} \mathfrak{D}_j$, that is, $\prod_{i\in I, j\in J} \mathfrak{E}_{ij}$ is a CA_ω which embeds both $\mathfrak{C}$ and $\mathfrak{D}$, a contradiction.

In our proof of 6.2 the *MGR* equation played a crucial role. A positive answer to the following problem would mean that this is not a coincidence.

PROBLEM 6.4. Does the class $\mathsf{CA}_\omega \cap \mathrm{Mod}(MGR)$ have the embedding property? Here *MGR* stands for the schema defined in Remark 1.14; hence, by the Resek-Thompson theorem [AT88], this question can be rephrased as follows: Does the class $\mathbf{Srl}^{ca}\,\mathsf{CA}_\omega$ have the embedding property?

The class QPEA_ω of ω-dimensional quasi-polyadic equality algebras is defined in Halmos [Hal62] (but see Sain–Thompson [ST91] for an equivalent definition that is more in line with the cylindric algebraic tradition; the latter definition can be obtained by replacing n with ω in Definition 5.10 in Chapter 5 above). The following problem is closely related to 6.4.

PROBLEM 6.5. Does QPEA_ω have the embedding property?

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