TOPICS IN ALGEBRAIC LOGIC(ON NEAT REDUCTS OF ALGEBRAS OF LOGIC)

By

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Contents

Summary

This dissertation revolves around the notion of neat reducts and the related notion of neat embeddings. It has six chapters. Chapter one is a broad introduction to algebraic logic with emphasis on the significance of the notion of neat reducts. Every other chapter is preceded by an abstract and a more detailed technical preface, and is self-contained. In particular, every chapter can be read independently than the other chapters. In chapters two and three we relate results on neat embeddings to results on amalgamation. In chapter four we address amalgamation proper. In chapter five we relate results on neat embeddings to the algebraic notion of complete representations and the metalogical one of omitting types. We also solve a long-standing open problem of Tarski and his co-athours Andréka, Henkin, Monk, and Németi on neat reducts but only for the finite dimensional case. In chapter six, we extend this result to the infinite dimensional case.

In more detail, in chapter two we show that the class of α-dimensional neat reducts of β-dimensional algebras in several cylindric-like algebras of relations is not closed under forming subalgebras for any pair of ordinals $1 < \alpha < \beta$. As a corollary we settle a conjecture of Tarski for cylindric algebras in the affirmative. The algebras we deal with in chapter one are Tarski's cylindric algebras, Halmos' quasipolyadic algebras with and without equality and Pinter's substitution algebras. In particular, all algebras considered are reducts of Halmos' polyadic algebras. We give a constrasting result for Halmos' polyadic algebras of infinite dimension; for these we show that the class of neat reducts forms a variety. We also present a property of commutativity of two operators on algebras - that of taking neat reducts with that of forming subalgebras - as a definability condition related to the well-known definability properties of Beth and Craig in the corresponding logics. Finally, we present some positive results

on amalgamation answering questions posed by Pigozzi in his landmark paper on amalgamation published in Algebra Universalis in 1971.

In chapter three we show that several (other) reducts of Halmos' polyadic algebras, the so-called G polyadic algebras, have the strong amalgamation property. Here G is a semigroup of transformations on the set ω of non-negative integers, that specifies the signature of the ω-dimensional algebras in question. This will be done by abstracting away properties of G, the resulting (abstract) semigroup we call strongly rich. We present a concrete G that is strongly rich. We infer that the (well-known) definability Theorems of Beth, Craig and Robinson hold for certain extensions of first order logic without equality. The question remains as to whether the strongly rich semigroup G can be further chosen to be finitely presented. This result is relevant to the finitization problem in algebraic logic; which asks for a finitely axiomatizable quasivariety that is an adequate algebraization of first order logic.

In chapter four we review, and in the process unify, two techniques (due to Németi and Pigozzi) for proving results concerning strong amalgamation in algebraic logic. Presenting them in a functorial context as adjoint situations, we show that both techniques can be seen as instances of the use of the Shelah-Keisler ultrapower Theorem in proving Robinson's Joint consistency Theorem.

In chapter five we give a new characterization of the class of the so-called completely representable cylindric and relation algebras via special neat embeddings. We formulate and prove a statement about representing cylindric algebras that turns out to be equivalent (in ZF) (Zermelo-Frankeal set theory without choice) to the famous Martin's axiom; thus is also independent from $ZF + CH$ (CH abbreviates the continuum hypothesis.) Then we relate our results omitting types for finite variable fragments of first order logic. Using examples that abound in the literature of algebraic logic but oriented for other purposes, we show that the Henkin-Orey omitting types Theorem, fails for the finite variable fragments of first order logic. We discuss what implications our result has concerning omitting types for (usual) first order logic. We use a Fraisse-type construction to prove that the class of α neat reducts of β dimensional cylindric algebras for $1 < \alpha < \beta$, α finite is not closed under *elementary* subalgebras hence is not elementary. This solves (the finite dimensional part) of a long-standing open problem (the only one remaining) in the monograph on Cylindric algebras by Henkin, Monk and Tarski.

In the final chapter, chapter six, we extend our result in chapter five concerning neat reducts to the infinite dimensional case. We prove that for any pair of infinite ordinals $\alpha < \beta$, the class $Nr_\alpha CA_\beta$ is not elementary. Because an analogous Fraisse theorem is missing; we construct our desired model in a standard step-by-step argument, which can be seen as an infinite analogue of Fraisse's Theorem for certain infinitary logics, the so-called finitary logics of infinitary relations.

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Chapter 1

Introduction

When the idea thrown out by Boole shall have borne their full fruit, algebra though founded on the idea of number in the first instance will appear as a sectional model of the whole form of thought. Its forms considered apart from their matter will be seen to contain all the forms of thought in general. The anti-mathematical logician says that it makes thought a branch of algebra, instead of algebra a branch of thought. It makes nothing. It finds and it finds the laws of thought symbolized in the forms of algebra.

De Morgan in [37].

Pure mathematics was discovered by Boole, in a work which he called the Laws of Thought. His book was in fact concerned with formal logic and this is the same thing as mathematics.

Bertrand Russel as qouted in [25].

Algebraic logic has arisen as a subdiscipline of algebra mirrong constructions and theorems of mathematical logic. It is similar in this respect to such fields as algebraic goemetry and algebraic topology, where the main constructions and theorems are algebraic in nature, but the main intuitions underlying them are respectively goemetric and topological. The main intuitions underlying algebraic logic are, of course, those of formal logic.

Investigations in algebraic logic can proceed in two conceptually different, but often (and unexpectedly) closely related ways.

First one tries to investigate the algebraic essence of constructions and results in logic, at the hope of gaining more insight that could add to his understanding, thus his knowledge. Second, one can study certain "particular" algebraic structures (or simply algebras) that arise in the course of his first kind of investigations as objects of interest in their own right and go on to discuss questions which naturally arise independently of any connection with logic. But often such purely algebraic results have impact on the logic side.

In this dissertation, we will be concerned with both types of investigations and their unexpected and indeed intriguing interplay and interaction. Our results will mostly adress *algebraizable* (in the sense of [28]) modifications of first order logic, be it extensions or reducts. These are the finite-variable fragments of first order logic, and variants of the so-called finitary logics of infinitary relations studied in the monograph [51] sec 4.3. These logics were extensively studied lately, as multi-modal logics with applications in Computer Science. We will investigate definability and interpolation properties like those of Beth, Craig and Robinson as well as Henkin-Orey's omitting types theorem for these logics. Since the dessertation belongs to the field of algebraic logic, our investigations will of course be algebraic. Our theorems will be formulated (and proved) for the algebraic counterpart of such logics, i.e. for certain classes of algebras. The algebras we deal with are the particular boolean algebras with operators designed particularly to algebraize (such different modifications of) first logic. These are the cylindric algebras of Tarski and various reducts of Halmos' polyadic algebras.

History and background

In the middle of the nineteenth century Goerge Boole initiated the investigation of a class of algebraic structures which were subsequently called Boolean algebras. The theory of these algebras is directly related to the development of the most elementary part of mathematical logic, namely propositional logic. As is well known however, the theory of Boolean algebras can be developed in a purely algebraic fashon, it has at present numerous connections with several branches of mathematics- (independance results in) set theory, topology and analysis - and hence it can be understood and appreciated by mathematicians unfamiliar with the logical problems to which it owes its birth. The work of Boole was the starting point for a continous flow of inquiries into the algebraization of *quantifier* logics (like first order logic and infinitary reducts of Keisler's logic) which through various intermediate stages led to (but did not end with) the foundation of the theory of cylindric and polyadic algebras. The history of the subject is interesting and it is worth briefly reviewing. The story begins in the nineteenth century when the pre-modern tradition of algebraic logic began. We find De Morgan in a restive frame of mind trying to extend the Aristotle's syllogistic tradition that held sway for 2000 years to encompass more complex situations, his eye alighted on the slightly earliar work of Boole who had come forward with a highly succesful algebra of Propositions. In 1860 de Morgan published [37] thereby launching an investigation into the algebra of relations. This developed into the subject known now as algebraic logic though in the nineteenth century it was simply thought of as mathematical logic. This work, along with Frege's quantifier logic, became the foundation of modern logic and model theory.

So in the nineteenth century there were two approaches to the formalization of *quantification* in logic. The first [1] (algebraic) approach due to Boole and De Morgan and taken up by Pierce and Shroder led to what we now call (following Tarski) relation algebra. The other approach due to Frege, and boosted by Russel and Whitehead in their momenentoues work *Principia Mathematica*, became the standard formalism of first order logic with its explicit universal and existential quantifiers. Both can express quantification though in different ways. [2] Historians of mathematical logic frequently tell us that there are two traditions, the algebraic tradition of Boole, Schroder De Morgan and Pierce, arising from the algebraization of analysis as opposed to the quantification-theoretical (or logistic) tradition of Peano, Frege and Russel arising from the development of the theory of functions. It is said that these two traditions, together with the independent set-theoretic tradition of Cantor, Dedekind and Zermelo arising out of the search for a foundation for real analysis in the work of

[1]historically, too it was prior.

[2]For example, in first order logic we can say that there exists a person who is my parent and your sibling which could be expressed in relation algebra as you are either my uncle or my aunt. The fundamental operations on binary relations considered since De Morgan's time and highlighted by Tarski are the boolean operations of intersection and complementation, the unary converse operation and the binary operation of composition. It can be seen that composition gives a qualified (relativized) form of existential quantification, and indeed [25] argues that both can be considered foundations of modern logic.

Cauchy, Weistrass and others, were united by Whitehead and Russel in their Principia Mathematica to create Mathematical Logic. However, the dual *algebraic* and *quantification-theoretic* traditions, as a matter of of historical fact, simply did not exist for logicians in the turn of the century. It is a false retrospective duality which derives from the Principia and is post Principia phenomena. There was no such dichotomy in the nineteenth century, algebraic logic was simply *the* mathematical logic of its time.

In the twentieth century first order logic was given an algebraic setting by Tarski in the framework of *cylindric algebra*, which will be the central notion in this dissertation. This was a natural outcome of Tarski's formalization of the notion of truth in set theory, for indeed the prime examples of cylindric algebras are those algebras whose elements are sets of sequences (i.e. relations) satisfying first order formulas.

While relation algebras constitute an algebraization of binary relations, n-dimensional cylindric algebras are an algebraization of n-ary relations. (Here n is any (not necessarily finite) ordinal.) Ever since relation and cylindric algebras were defined, researchers have been investigating the connections between them. Tarski and Henkin investigated such connections, cf. [51] Thm 5.3.8. More recent references include the work of Maddux [83], [84] and the work of Hirsch and Hodkinson [55], [57]. [3] Making another leap into modern perspective, the motivation for studying the theory of relation algebra and cylindric algebra, or hereinafter algebras of relations for short, today comes from at least three areas:

(i) Logic (predicate logics, propositional multi-modal logics, dynamic logics.)

(ii) Algebra: the algebraic theory of these algebras is of interest in its own right. Indeed boolean and cylindric algebras served as a starting point for the (by now well-developed) theory of discriminator varieties.

(iii) The theory of relations: relations are of interest in themselves.

[3]Every cylindric algebra of dimension > 3, has a natural relation algebra reduct obtained by taking the 2-*neat reduct* i.e. the essentially 2-dimensional elements abstracting binary relations, and defining convese and composition using one spare dimension. Conversely, one can construct cylindric algebras of arbitrary dimension n, $3 < n \leq \omega$ from relation algebas that posses what Maddux calls an n-dimensional cylindric basis[83].

We could further motivate algebraic logic by surveying its numerous applications in temporal reasoning, planning data bases, modal logic and elsewhere, cf. the survey article [109] for an expose of such applications. Instead, we prefer to recommend the part of algebraic logic most relevant to our investigations here as a very well established mathematical procedure-*algebraization*- applied to a fundamental entity, viz. a *relation.*

For this purpose, let us focus for a while on cylindric algebras, which is an abstraction from algebras (of not necessarily binary) relations. The concept of cylindric algebras permits the use of algebraic methods in treating two related parts of mathematics. One of these is a very general kind of goemetry associated with basic set-theoretic notions, [4] and the other - as previously mentioned - is the theory of deductive systems in mathematical logic. The two parts are indeed interconnected, because models of deductive systems (like first order theories) give rise in a natural way to structures within the set-theoretic "spaces" (set algebras based on these models.)

To illustrate further this connection, let L be any first order relational language, i.e. L has no function symbols nor individual constants. Then L has an infinite countable sequence of individual variables $x_0,\ x_1 \ldots x_n : n < \omega$, logical constants $\neg$, $\implies$, $\forall$ $\exists$ $=$, and non-logical constants-say a system $\langle R_i : i \in I\rangle$ of relation symbols, where R_i is of rank $\rho_i < \omega$ for each $i \in I$. (ω is the set of all non-negative integers.) We assume as known the usual syntactical primitive notions defined in terms of L, e.g. the notions of a formula, a sentence (formula without free occurences of variables), the conjunction $\phi \wedge \psi$ of formulas of L, the notion of a formal proof of a formula from a set of sentences, etc. Now given a set Γ of sentences, sometimes referred to as a theory, we may call two formulas ϕ and ψ equivalent under (or modulu) Γ, in symbols

$$\phi \equiv_\Gamma \psi$$

provided that the biconditional $\phi \longleftrightarrow \psi$ is provable from Γ, in symbols $\Gamma \vdash \phi \longleftrightarrow \psi$. Here $\vdash$ denotes a standard (complete) proof system. [5] It is easy to check that the

[4]Indeed, Andréka, Madárasz and Németi have used their goemetric intuition originating from their investigations in the theory of cylindric algebra to formulate Einstein's general theory of relativity in first order logic, an indeed exotic and exciting novel application of algebraic logic [19].

[5]We defined the congruence relation via syntactical means, i.e. by fixing a standard complete

relation $\equiv_\Gamma$ is actually an equivalence relation on the set of formulas. In fact, it is a congruence relation compatible with the boolean operations of conjunction and negation on formulas. Even more, if we let A_Γ denote the set of equivalences classes under $\equiv_\Gamma$, we find that certain algebraic operations can be introduced on A_Γ which reflect the syntactical operations of building formulas: Besides the boolean operations

$$[\phi]_\Gamma + [\psi]_\Gamma = [\phi \vee \psi]_\Gamma,$$

$$[\phi]_\Gamma . [\psi]_\Gamma = [\phi \wedge \psi]_\Gamma,$$

$$-[\phi]_\Gamma = [\neg\phi]_\Gamma,$$

we have

$$c_i[\phi]_\Gamma = [\exists v_i \Gamma]_\Gamma,$$

$$d_{ij} = [v_i = v_j].$$

Here, of course, $[\phi]_\Gamma$ is the equivalence class of ϕ under Γ. Note that the congruence relation defined via provability is now compatible with the newly added operations c_i and d_{ij} for each $i, j < \omega$. The resulting quotient algebra $\mathcal{A}_\Gamma$ thus associated with L and Γ is one of the fundamental algebras studied in algebraic logic, and it is in fact the prime source of inspiration of cylindric algebras. It turns out that many constructions and theorems in logic can be algebraically reflected using these algebras. For example Γ is complete and consistent if $\mathcal{A}_\Gamma$ is a simple algebra, i.e. has no proper congruences; the theorem that any consistent theory can be extended to a complete and consistent

proof system and defining the relation in terms of provability. The congruence relation we defined is denoted by $\equiv_\Gamma^p$ in [51] sec 4.3. with the superscript p indicating that the relation is defined via some fixed proof system. This relation could have just as well been defined by semantical means i.e. by resorting to the notion of satisfiability in models. In more detail, define $\equiv_\Gamma^s$ - with the superscript s now indicating that we are adopting *symantical* means - as follows: $\phi \equiv_\Gamma^s \psi$ if for every model M of Γ and every assignment $s \in {}^\omega M$, s satisfies ϕ if and only if s satisfies ψ. Because of Godels's celebrated Completeness Theorem it follows these two congruences (on "ordinary" first order languages) coincide, i.e $\equiv_\Gamma^s = \equiv \Gamma^p$, for any Γ. However in other contexts, like for example the finite- variable fragments of first order logic and the finitary logics of infinitary relations studied in [51] sec 4.3. the semantical congruence defined via satisfiability in models, is not easily captured by "finitary" means like for example finite Hilbert- style axiomatizations. A lot of research in algebraic logic is devoted to investigating completeness for these logics. Such investigations are referred to in the literature as the *Representation Problem*; or rather the logical counterpart thereof. The Budapest group has called this problem, the *Finitization Problem*, the adjective "finitization" referring to the attempt to capture the infinity of validities by thorough finitary means, cf. problem 4.16 in [51], for one reincarnation of this problem, and Németi [109] for several other reincarnations. We will have occasion to discuss further the (algebraic aspect of the) Finitization Problem.

theory is mirrored by the theorem that any algebra $\mathcal{A}_\Gamma$ with $|A_\Gamma| > 1$ has a simple homomorphic image. For more on such connections between metalogical notions and algebraic ones, the reader is referred to [51] sec 4.3. The algebra $\mathcal{A}_\Gamma$ is referred to as the *Tarski-Lindenbaum cylindric algebra of formulas corresponding to* Γ, or simply an algebra of formulas. The notion of a cylindric algebra is obtained from algebras of formulas by a process of abstraction. Let α be an ordinal. A *cylindric algebra of dimension* α, for brevity a CA_α, is an algebra of the form

$$\mathcal{A} = \langle A, +, ., -, c_i, d_{ij} \rangle_{i,j<\alpha}$$

where $\langle A, +, ., - \rangle$ is a boolean algebra with $+$ denoting the *boolean join*, . denoting the *boolean meet* and $-$ denoting *boolean complementation.* The c_i's, $i \in \alpha$, are unary operations of *cylindrifications* on A (the domain of $\mathcal{A}$) and the d_{ij}'s, $i, j \in \alpha$ the *diagonal elements* are distinguished elements of A. The operation c_i is an abstract version of the unary operation on first order formulas of existential quantification with respect to the $i-th$ variable x_i. The diagonal element d_{ij} is an abstract version of the atomic identity formula $x_i = x_j$ in first order logic. The class CA_α of all cylindric algebras of dimenson α is axiomatized by finitely many equational schemata that aim at capturing the essential algebraic properties of existential quantification and atomic identity formulas (cf. [50] Def 1.1.1.) Intuitively, the axioms say the following for all $i, j, k \in \alpha$.

(i) c_i is an additive complemented closure operator.

(ii) $c_i c_j x = c_j c_i x$, i.e. the cylindrifications commute.

(iii) The diagonal elements satisfy the usual axioms one would expect of equality such as $d_{ij} \cap d_{jk} \leq d_{ik}$ (transitivity).

(iv) $d_{ij} \cdot c_i x \leq x$ for all $x \leq d_{ij}$.

Axiom (iv) for example is an algebraic version of Liebniz's law of equality saying that equal objects are indistinguishable by formulas, i.e.

$$\phi \implies v_i = v_j \vdash (v_i = v_j \wedge \exists v_i \phi) \implies \phi.$$

Now what are the advantages of working with the algebras rather than the concrete structures? There are a number of possible answers to this rather deep question. Firstly, the technique of focusing our whole attention on a small number of properties to the exclusion of all others is quite general in scientific analysis. Secondly and precisely because we are ignoring many concrete features of our structures, we may find that our approach is more general, and that the complexity of certain problems may be lower. Finally, algebra is a very thoroughly studied descipline and there is a range of methods and results that we can apply to the study of relations. Universal Algebra and Model Theory have techniques and theorems that can be fruitfully exploited here.

That the abstraction from the algebras of formulas to abstract algebras is indeed sound, is established by the following logical representation Theorem, which is not very difficult to prove [50]. Loosely speaking, the proof of Fact 0.1. consists of simply checking that the postulates characterizing cylindric algebras form an adequate algebraic transcription of the axioms of any standard complete system for first order logic, cf.[51] sec 4.3 Thm 4.2.28(iii).

Fact 0.1 . *For any algebra $\mathcal{A}$ having the same signature as CA_ω the following two conditions are equivalent:*

(i) $\mathcal{A} \cong \mathcal{A}_\Gamma$ *for some* Γ.

(ii) $\mathcal{A}$ *is a* CA_ω *such that the set* $\Delta x = \{i \in \omega : c_i x \neq x\}$ *is finite for every* $x \in A$.

Fact 0.1 justifies the choice of equations axiomatizing CA_α. In the infinite dimensional case; these equations force locally finite algebras satisfying them to be isomorphic to algebras of formulas. Δx is referred to as the *dimension set* of x, and sometimes the *support* of x. [6] An algebra for which Δx is finite for all x is called a locally finite algebra. The class of all such algebras of dimension α is denoted by Lf_α. Of course all finite dimensional algebras are locally finite, but for the ω - dimensional algebras, it is precisely the property of local finiteness that singles out algebras of formulas from arbitrary cylindric algebras (up to isomorphism.) In algebras of formulas,

[6] The support of x comes from terminology of polyadic algebras, due to Halmos.

that Δx is finite, refers to the fact that every first order formula is a finite string of symbols and hence contains only finitely many (free) variables.

The other more concrete source of cylindric algebras is that of *cylindric set algebras* which Tarski introduced as an algebraic counterpart of semantics of first order logic. [7] Such algebras, the cylindric set algebras, arise naturally from models of first order theories, and therefore they are closely related to the algebras of formulas. To explain further the connection of cylindric set algebras to algebras of formulas, we now turn to semantical notions.

Fix a first order relational language L. Let M be an L-structure. We take it as well-known what it means for a sequence $s \in {}^{\omega}M$ to *satisfy* a formula ϕ in M. We write $M \models \phi[s]$ if $s \in {}^{\omega}M$ satisfies ϕ in M. For $\phi \in L$, let ϕ^M be the set of all ω-ary sequences, or assignments, that satisfy ϕ in M, that is

$$\phi^M = \{s \in {}^{\omega}M : M \models \phi[s]\}.$$

Then ϕ^M is an ω-ary relation on M; it is a point-set in the ω-dimensional space ${}^{\omega}M$, and the set of all these, i.e the set

$$A_M = \{\phi^M : \phi \in L\}$$

is the universe of a cylindric algebra of dimension ω. It is easy to see that A_M is a boolean field of sets; it is closed under intersections and complementation (hence under unions), for indeed

$$\phi^M \cap \psi^M = (\phi \wedge \psi)^M,$$

and

$${}^{\omega}M \smallsetminus \phi^M = (\neg\phi)^M.$$

Certain set-theoretic operations similar to the classical operations of descriptive set theory can now be introduced correponding to the basic non-boolean operations. In harmony with the notation of Henkin, Monk and Tarski [50], we use capital letters

[7] The cylindric set algebras can be viewed as multi-dimensional boolean set algebras (their dimension could be finite or transfinite) and they play a role in the theory of cylindric algebras analogous to that played by boolean set algebras in boolean algebras and rings of matrices in ring theory. Such algebras are the building blocks, in the sense that the standard models of CA-theory the so-called *representable* algebras are defined to be *subdirect product* (in the universal algebraic sense) of set algebras.

for the interpretation of the cylindric operations in set algebras: The $i-th$ *cylindrification*, algebraising existential quantification with respect to the $i-th$ variable, is defined as follows:

$$C_i(\phi^M) = \{t \in {}^\omega M : \text{ there exists } s \in {}^\omega M \text{ such that } t(j) = s(j) \text{ for all } j \neq i\} = (\exists v_i \phi^M.)$$

This is simply the cylinder obtained by moving ϕ^M parallel to the i-axis, hence the terminology of $i-th$ cylindrification. On the other hand, the diagonal elements, denoted by D_{ij} for $i, j \in \omega$, are defined to be the following hyperplanes:

$$D_{ij} = \{s \in {}^\omega M : s_i = s_j\} = (x_i = x_j)^M.$$

Now let Γ be an L-theory. Suppose further that M is a model of Γ, i.e. that every sentence in Γ is true in M. Then this is reflected algebraically by the fact that $\mathcal{A}_M$ is a homomorphic image of A_Γ via the natural map

$$[\phi]/\Gamma \to \phi^M.$$

If ψ is satisfiable in M, then this can be expressed algebraically by the fact that the image of ψ under the above map is non-zero, for indeed ϕ^M contains a sequence that satisfies ψ in Γ, i.e. is non-empty.

Having at hand the concrete notion of a cylindric set algebra based on models, we now make another abstraction from this notion to obtain a more general (but still concrete) set-theoretic object. There is no reason why we should restrict our attention to ω-dimensional set algebras. Accordingly, relaxing the restriction of dimension, a cylindric set algebra of dimension α, α an *arbitrary* ordinal, and base U, for short a Cs_α^U or simply a Cs_α, when U is clear from context, is an algebra

$$\mathcal{A} = \langle A, \cup, \cap, \smallsetminus, C_i, D_{ij}\rangle_{i,j\in\alpha}$$

such that A is a boolean field of subsets of ${}^\alpha U$ closed under each C_i and containing the $D_{ij}'s$ for all $i, j \in \alpha$ defined as above. Note that set algebras can have arbitrary dimensions. It turns out that in the ω-dimensional case, the cylindric set algebras corresponding to models are precisely the locally finite *regular* ones. The property of local finiteness reflects the fact that such algebras consist of relations that are essentially "finitary", though of course the rank of such relations can grow without bound. On the other hand, $\mathcal{A} \in Cs_\omega$ is regular if for all $X \in A$ and $s, t \in {}^\omega U$, whenever s and t agree on ΔX, then $s \in X$ implies that $t \in X$. This, in turn, reflects the metalogical property that if two assignments agree on the indices of the free variables occuring in a formula, then they both satisfy the formula or none does. The class of regular cylindric set algebras of dimension ω is denoted by Cs_ω^{reg} in [51]. Because of its connection with first order models, the class $Cs_\omega^{reg} \cap Lf_\omega$ has been quite extensively studied in the literature. Therefore it is no coincidance that a lot of problems in algebraic logic frequently boils down to constructing certain models satisfying certain properties. [8]

In terms of the concept of a cylindric set algebra, a purely algebraic form of Godel's Completenesss Theorem can now be stated. For that we need to recall that a subdirect product of a family of algebras is a subalgebra of the product of this family such that the (natural) projections are onto.

Fact 0.2 . *(Tarski) If $\mathcal{A} \in \mathcal{CA}_\omega$ is locally finite, then $\mathcal{A}$ is isomorphic to a subdirect product of Cs_ω's, furthermore these Cs_ω's can be chosen to be locally finite and regular.*

The proof of Fact 0.2 is somewhat deeper than that of Fact 0.1. In fact, Fact 0.2 is equivalent in ZF (Zermelo-Fraenkel set theory without choice) to Godel's classical Completeness Theorem. To illustrate this connection we give a sketch of proof of one implication, namely that Godel's Completeness Theorem implies Fact 0.2. We use the (equivalent) form that every consistent set of first order formulas has a model.

Sketch of proof of Fact 0.2.

Let $\mathcal{A} \in CA_\omega$ be locally finite. It suffices to show that for every non-zero element $a \in A$, there is a regular locally finite Cs_ω $\mathcal{C}$, say and a homomorphism $h : \mathcal{A} \to \mathcal{C}$ such that $h(a) \neq 0$. We can assume without loss of generality that $|A| > 1$, for else there is nothing more to prove. By Fact 0.1 $\mathcal{A} \cong \mathcal{A}_\Gamma$, for some consistent theory Γ. Now let a be a non-zero element in A_Γ. Then a is the equivalence class of some ψ

[8]In this connection, we mention for example [?], [65] and [127], and our construction in chapter 4. Lately, Ian Hodkinson and Robin Hirsch [60] applied rather sophisticated machinery from Model Theory to solve hard problems in Algebraic Logic.

that is consistent with Γ i.e. $a = [\psi]/\Gamma$, such that $\Gamma \cup \psi$ is consistent meaning that no contradiction is derivable from it. By Godel's Completeness Theorem there is a model M of Γ in which ψ is satisfiable. Then it is not hard to check that the natural map $h : \mathcal{A}_\Gamma \to \mathcal{A}_M$ defined by

$$\phi/\Gamma \to \phi^M$$

is an onto homomorphism for which $h(a) = \psi^M$ is non-empty. We note that if Γ is a complete theory, then this map is further an isomorphism.

■

For the opposite implication, the reader is referred to [51] sec 4.3., where such and other deeper connections are further elaborated.

As regular locally finite set algebras of dimension ω are simple, thus Fact 0.2 shows that every locally finite algebra is actually *semisimple* i.e. isomorphic to a subdirect product of simple algebras. Since, on universal algebraic grounds, the kernel of a homomorphism is a maximal ideal if and only if its range is simple, this in turn, is equivalent to that the intersection of maximal ideals in any locally finite algebra is the trivial algebra. [9]

[9]We note that Halmos' proof that semisimplicity of (locally finite) polyadic algebras is equivalent to Godel's Completeness Theorem was to him a revelation. In fact, Halmos in [47] was concerned with the algebraisation of two of the deepest theorems in logic; the original formulations and proofs of these justly celebrated theorems are due to Godel; these are Godel's Completeness and Godel's Incompleteness Theorems. The algebraic counterpart of Godels Completeness Theorem is semisimplicity of locally finite polyadic algebras, as illustrated in Fact 0.2, bearing in mind that locally finite polyadic algebras are essentially the same as locally finite cylindric algebras. Now algebraizing Godel's Incompleteness results is more intricate, and is still (to the best of our knowledge) an unfinished task. However, Halmos [48] has work in this direction, which we now briefly review. There is no difficulty in constructing locally finite polyadic algebras, and for that matter locally finite cylindric algebras, with sufficiently rich structure to mirror the axiomatic system of complex systems like for example Peano arithmetic or even the stronger Zermelo-Fraenkel set theory in detail, so that the resulting algebra is what Halmos calls a Peano algebra. A Peano algebra is adequate to mirror elementary arithmetic; so for example we can "talk about" recursive functions in a Peano algebra. Now Godel's Incompleteness Theorem says (very roughly) that there are propositions that are neither refutable nor provable in any strong enough formal system to encode recursive functions. The algebraic version thus states that not every Peano algebra is simple, in the universal algebraic sense of the word. Halmos identifies a Peano algebra with Mathematics, and rephrases his result in the following pun: "Mathematics is not simple". We refer the interested reader to [47] for a very interesting survey of the history of Polyadic algebras. In passing, we note that Tarski and Givant [149] formalized set theory in the calculus of relations, i.e in relation algebras. A similar task was done by Németi, when he formalized an essentially undecidable fragment of Peano Arithmetic due to Tarski and Robinson in the so -called class of semi-associative relation algebras of Maddux. This interpretation of essentially undecidable fragments of set theory, into the calculas of relations, was

Soon however in the development of the subject, it was realized that the class Lf_ω, the algebraic counterpart of first order logic, has some serious defects when treated as the sole subject of research in an autonomous algebraic theory. In Universal Algebra one prefers to deal with *equational classes* of algebras i.e. classes of algebras characterized by a postulate systems in which every postulate has the form of an equation (an identity); such classes are also referred to as *varieties.* Classes of algebras which are not varieties are often introduced in discussions as specialized subclasses of varieties. One often treats fields as a special case of rings. Probably the reason for this preference is the fact that every variety is closed under certain general operations frequently used to construct new algebras from given ones; we mean here the operation of forming subalgebras, homomorphic images and direct products. By a well known theorem of Garrett Birkhoff, the varieties are precisely those classes of algebras that have all three of these closure properties. Local finiteness does not have the the form of an identity, nor can it be equivalently replaced by any identity or system of identities, nor indeed any set of first order axioms. This follows from the simple observation that the ultraproduct of infinitely many Lf_ω's is not, in general locally finite and a first order axiomatizable class is necessarily closed under ultraproducts. When Alfred Tarski introduced cylindric algebras, he introduced them in the form satisfying (ii) in Fact 0.1, and proved the representation Theorem expressed in Fact 0.2. However, Tarski's proof of Fact 0.2 was never published. A simple purely algebraic proof due to Andréka and Németi can be found in [13]. But some modifications in the

used to prove undecidability of the equational theory of several classes of relation algebras, cf.[8]. Recent results on undecidability of various classes of relation and cylindric algebras involve the more sophisticated methods of the interpretation of the (undecidable) tiling problem and the equational theory of semigroups [51] and [60]. Conversely, Godel's Incompleteness Theorem was used to solve problems in algebraic logic. In cylindric algebra, Godels' Incompleteness was used by Németi [**?**] to show that the (boolean reduct of the) free cylindric algebras of dimension at least 3 are not atomic. Recently, [144] the author proved that the free quasipolyadic algebras are not atomic either, by showing that their equational theory is still strong enough to encode Peano arithmetic. A long-standing open question here posed by Maddux and independently by Németi is whether the analagous result holds for the diagonal free reducts of cylindric algebras (Df). In fact, it is conjectured by Németi that set theory might be interpreted in Df_3, in the manner of Tarski-Givant [149]. The latter is also related to the "Finitization Problem" in algebraic logic to be discussed in a while. Another interesting application of Godel's Incompleteness results is the Németi-Sagi result in [110], which shows that the equational theory of Halmos' polyadic equality algebras is strong enough to "encode" second order Peano arithmetic; thus the representable polyadic *equality* algebras cannot be axiomatized by any reasonable schema, let alone a finite one, which is an interesting contrast to the classical Daigneualt-Monk representation theorem for polyadic algebras *without* equality [39].

definition of Tarski's cylindric algebras seemed desirable. The definition contains certain assumptions which considerably restrict the applicability of the definition and thus can be despensed with. One such is the fixed dimension ω. The other is local finiteness. The restrictive character of these two notions becomes obvious when we turn our attention to cylindric set algebras. We find there are algebras of all dimensions, and set algebras that are not locally finite easily constructed. For these reasons the original conception of a cylindric algebra has been extended: the restriction to dimension ω and local finiteness were removed, and the class CA_α was introduced.

A central and indeed still active part of research in algebraic logic is the vaguely posed lengthly discussed problem concerning improvements of Fact 0.2.

This problem is referred to as *The Finitization Problem* by the Budapest group specifically by Andréka and Németi while it is referred to as the *Representation Problem* by the London group specifically by Ian Hodkinson and Robin Hirsch. This problem has invoked extensive amount of research and is still creating employment till the present day.[10] To make the problem a little bit more precise, let RCA_α be the class of CA_α's isomorphic to subdirect products of Cs_α's. We now explain why the class RCA_α is indeed a plausible "natural" candidate for subsituting boolean set algebras in the quest of a Representation Theorem for cylindric algebras, analagous to that of Stone. It is easily seen that every cylindric set algebra of given dimension $\alpha < \omega$ is simple (has no proper congruences) and therefore subdirectly (and directly) indecomposable in the sense of the general theory of algebras, cf.[50]. Hence when discussing the problem as to which CA_α's are isomorphic to cylindric set algebras, it is natural to restrict ourselves to subdirectly indecomposable algebras. On the other hand, as a consequence of a classical theorem of Birkhoff, every CA_α is isomorphic to a subdirect product of subdirectly indecomposable CA_α's. Therefore we are naturally led to the problem of characterizing those CA_α's which are isomophic to subdirect products of set algebras. Henkin, Monk and Tarski declare that these are the *representable* algebras, thus the notation RCA_α. [11] The definition of representability ,

[10] Indeed in chapter three we will have occasion to deal with certain reducts of polyadic algebras that were introduced by Ildiko Sain as one possible solution to this problem.

[11] RCA_α plays a role in cylindric algebra analagous to the role played by boolean set algebras in boolean algebra and rings of matrices in ring theory. The Representation problem in boolean

without any change in its formulation, is extended to algebras of infinite dimension. In this case, however, an intuitive justification is less clear since cylindric set algebras of infinite dimension are not in general subdirectly indecomposable. In fact, for $\alpha \geq \omega$ no intrinsic property is known which singles out the algebras isomorphic to set algebras among all representable CA_α's, as opposed to the finite dimensional case where such algebras can be intrinsically characterized by the property of being simple. But in any case, members of RCA_α can be still represented as algebras consisting of genuine α-ary relations over a disjoint union of cartesian squares, the class consisting of all such algebras is denoted by Gs_α, with Gs standing for *generalized set algebras.* Generalized set algebras thus differ from the ordinary cylindric set algebras in one respect only: the unit of the algebra i.e. the α dimensional cartesian space ${}^\alpha U$ is replaced everywhere in their construction by any set which is a disjoint union of arbitrary many pairwise disjoint cartesian spaces of the same dimension. The class of generalized cylindric set algebras, just as that of ordinary cylindric set algebras, has many features which make it well qualified for representing CA_α. The construction of the algebras in this (bigger) class retains its concrete character, all the fundamental operations and distinguished elements are unambigiuosly defined in set-theoretic terms, and the definitions are uniform over the whole class; goemetric intuition underlying the construction gives us good insight into the structures of the algebras. Thus there is (goemetric) justification that RCA_α consists of the standard models of CA-theory. Its members consist of genuine α-ary relations. Now, in elementary terms the definition of RCA_α runs as follows: $\mathcal{A}$ is representable iff for every non-zero $x \in A$ there is a homomorphism h from $\mathcal{A}$ onto a Cs_α such that $h(x) \neq 0$. Thus Fact 0.2 is equivalent to the statement that every locally finite CA_ω is representable. But it soon transpired that the CA axioms (originating from the (complete) axiomatization of locally finite algebras, cf. Fact 0.2) do not exhaustively generate all valid principles governing α-ary relations, when $\alpha > 1$. More precisely, for $\alpha > 1$, RCA_α is properly contained in CA_α. CA_α, for $\alpha > 1$, is only an *approximation* of RCA_α. Tarski proved that RCA_α is a variety [109]. However for $\alpha > 2$ RCA_α cannot be

algebras is completey resolved by Stone's Theorem. In Ring Theory we see for example Wederbin Artin's Theorem and Goldie's Theorem giving nice intrinsic conditions on when an abstract ring is isomorphic to a subdirect product of rings of matrices. As opposed to Boolean algebras, the representation problem for the CA case proves to be much harder.

axiomatized by a finite schema of equations analagous to that axiomatizing CA_α, a classical result of Monk [95], cf. Fact 0.5. below. Furthermore, there is an unaviodable and inevitable degree of complexity to any (potential) axiomatization of RCA_α, as shown by Andréka [3] for any $\alpha > 2$. The Finitization Problem is thus the attempt to circumvent or sidestep such complexity. If we look at RCA_α as the standard models to which the CA_α's aspire, the Finitization Problem can thus be rephrased as the attempt to capture the essence of the standard models by thorough "finitary" means, or else find other broader comprehensible classes of "standard models" that are sufficiently concrete and tangible and most important of all would exhaust the class CA_α, or at worst possibly a slightly smaller class i.e. a variety that is finitely axiomatizable (by equations) over CA_α. [12]

A different slant on the same problem is: How can we abstract away from the subject matter of specific concrete α-ary relations to arrive at their essential forms? For unary relations the answer is given by Stone's Theorem: Every boolean algebra is isomorphic to a set algebra i.e. is representable. The significance of this is that the (finitely many) axioms of boolean algebras exactly capture the true properties of unary relations: all and only those properties that hold in every domain of individuals endowed with unary relations are derivable from the axioms. This is a great (perhaps the greatest) success for the algebraic viewpoint. The same situation holds for locally finite cylindric algebras. The axiomatization provided by Tarski and recalled herein is sufficient for representability as indicated in Fact 0.2 above. However, if we want to *algebraize first order logic*, then we have to remove the (non-first order)

[12] There has been work in this direction by representing CA's using quasigroups, cf. [98], [51] p.91-93, or sheaves [33] . A classical result of Resek that is relevant in this connection , shows that algebras satisfying the CA axioms plus the-so called *merry go round identities*, or MGR for short, can be represented as *relativized* set algebras, a primary advance in development of the theory of CA's, as indicated in the introduction of [51]. Resek's result was polished and "finitized" by Andréka and Thompson [24], where they replaced the infinite MGR by a finite schema, but, in the process, had to modify slightly the notion of representability. Recently, Simon [147] proved that any abstract 3-dimensional cylindric algebra satisfying the MGR can be obtained from a Cs_3 by the so-called methods of *twisting and dilation* studied in [51]p.86-91 which adds to our understanding as to the distance between the abstract notion of cylindric algebra and its concrete one, at least in the case of dimension 3. However, Simon had to broaden Henkin's notion of dilation to exhaust the class CA_3. The analogous problem for higher dimension is an intriguing open problem. This problem, due to Leon Henkin, asks whether every CA_n, $3 < n < \omega$ can be obtained from a Cs_n by a finite sequence of the operations of relativization, twisting and dilating - as defined in [51]- not necessarily in this order.

restriction of local finiteness, and keep only equations or at worst quasi-equations.[13] But then infinity seems unaviodable, it backlashes in a strong sense: Any first order axiomatization of the class RCA_α for $\alpha > 2$ is essentially infinite. [14]

Several different strategms to get round the obstacle of the non-finite axiomatizability of the class of representable algebras were evolved. One promulgated by Tarski especially was to find elegant intrinsic conditions for representability. For example, certain comprehensible subclasses of abstractly defined CA_α's turn out to be representable. In this connection, examples include locally finite, dimension complemented, semisimple and diagonal algebras of infinite dimension, cf. [51] Theorem 3.2.11. Another sample of such results in this direction is the classical result of Henkin and Tarski, formulated as Thm 3.2.14 in [51], that states that any atomic CA_α whose atoms are rectangular is representable. This was strengthened by Andréka et all by looking at dense subsets consisting of rectangular elements, that are not necessarily atoms [7] and Venema [151] extended this result to the diagonal free case. This approach of finding simple intrinsic sufficient conditions for representability has continued to the present, and now forms an extensive field, cf. [109]. Another strategy of attacking the Finitization Problem is to define variants of RCA_α, $\alpha > 2$ that are finitely axiomatizable and are still adequate to algebraize first order logic. Such an approach originates with William Craig [34]. The reasoning here is that maybe the negative results we already mentioned are merely a historical accident resulting from the particular (far from unique) choice of extra non-boolean operations, namely the cylindrifications and diagonal elements. This approach typically involves changing

[13] According to Blok and Pigozzi, *Algebraisable* logics are those logics whose algebraic counterparts are quasivarieties, i.e. classes of algebras axiomatized by quasiequations, cf.[28].

[14] We will have occasion to elaborate on the metalogical aspect of the Finitization Problem in chapter four, which in a nut shell asks for a *structural* extension of first order logic that admits a finite complete and sound Hilbert style axiomatization of the valid formula schemas involving only valid formula schemas. Here structural is understood in the sense of [28]. This is formulated as problem 4.16 in [51]. This form of the Finitization Problem, it seems, is not unrelated to Hilbert's progamme of finitizing metamathematics, and it adds to our knowledge of reasoning about reaoning. The Finitization Problem (in its algebraic form) is indeed non-trivial as the following quote from Henkin, Monk and Tarski in [50] might suggest "An outstanding open problem in cylindric algebra theory is that of exhibiting a class of cylindric algebras which contains an isomorphic image of every cylindric algebra and hence serves to represent the class of all these algebras, and which at the same time is sufficiently concrete and simply constructed to qualify for this purpose from an intuitive point of view. It is by no means certain or even higly plausible that a satisfactory solution of this problem will ever be found!" (our exclamation mark).

the signature of CA_α by either taking reducts or expansions or perhaps even changing the signature altogether but bearing in mind that cylindrifications and diagonal elements are term definable in the new-signature, cf. [34] [116], or / and broadening the notion of representability allowing representation on abitrary subsets of α-ary relation, rather than just (disjoint unions of) cartesian squares [90]. [15] Another more adventerous approach is to "stay inside" so to speak the "CA-RCA infinite discrepency" and to try to capture the essence of (the equations holding in) RCA_α in a simple as manner as possible. It is not hard to show that the set of equations holding in RCA_α for any α is recursively enumerable, and indeed, using a well-known trick of Craig, several (recursive) axiomatizations of RCA_α exist in the literature, the first such axiomatization originating with Monk, building on work of McKenzie [51]. Robinson's finite forcing in Model theory proves extremely useful here as shown by Hirsch and Hodkinson. The very powerful recent approach of synthesizing axioms by games due to Hirsch and Hodkinson [59], building on work of Lyndon [**?**], is a typical instance of giving an intrinsic characterization of the class of representable algebras by providing an *explicit* axiomatization of this class in a step-by step fashon. This approach is of a very wide scope; using Robinson's finite forcing in the form of games, Hirsch and Hodkinson [60] axiomatize, not only the variety of representable algebras, but almost all pseudo-elementary classes existing in the literature, an indeed remarkable achievement. [16] Another approach initiated by Van Banthem and Venema

[15]The approach of broadening the permissable units is referred to in the literature as *relativization* or the *non-square* approach. The former terminolgy comes from the Budapest group [90], while the second comes from the Amsterdam group, in fact it is due to Venema [150]. Relativization might involve adding new operations that become no longer term definable after relativization, like for example the difference operator [150]. This approach is related to *Dynamic Logic*, cf. [10]. Modalizing set algebras yield variants of the n-variable fragment of first order logic differing from the classical Tarskian view because the unit W of the set algebra in question may not be of the form of a "square" ${}^nU$, but merely a subset thereof. These mutant logics are under intensive study at the present time and we cite [10] as a source.

[16]It turns out, as discovered by Hirsch and Hodkinson, that being a representation of an algebra can be described in a first order 2-sorted language. The first sort of a model of this *defining theory* is the algebra itself, while the second sort is a representation of it. The defining theory specifies the relation between the two, and its axioms depend on what kind of representation we are considering. Thus the representable algebras are those models of the first sort of the defining theory with the second sort providing the representation. The class of all structures that arise as the first sort of a model of a two sorted first order theory is a venerable old notion in Model theory introduced by Maltsev in the forties and since studied by Makkai and others. It is known as a pseudo-elementary class. We mean here a PC_Δ class in the sense of [67] but expressed in a two sorted language.

consists of viewing the class of cylindric set algebras as complex algebras of Kripke frames that have the same signature as atom structures of cylindric algebras [150], thus opening an avenue to techniques and methods coming from modal logic. This typically involves introducing Gabbay-style rules on the logic side. These extremely liberal Gabbay-style inference systems coorespond to classes that are inductive i.e axiomatized by $\forall\exists$ -formulas. An example of such classes is the class of rectangularly dense cylindric algebras, cf.[7] [151]. We should mention that this approach is an instance, or rather, an application of the triple duality, in the sense of Goldblatt [45], existing between abstract modal logic, Kripke frames or relational structures, and boolean algebras with operators. We refer to the article by Venema in [93] for explaining and applying this duality to relation algebras and to [150] and [92], [8]

The term pseudo elementary class strictly means PC_{Δ} when the second sort is empty, but the two notions were proved to be equivalent by Makkai. Any psuedo elementary class is elementary, but the converse is not true; the class of neat reducts is an example as proved in chapters five and six herein, another is the class of strongly representable atom structures and the completely representable ones as proved by Hirsch and Hodkinson in [60]. Many classes in algebraic logic can be seen as psuedo elementary classes. The defining theory is usually finite and simple and essentially recursively enumerable, since we expect that a Turing machine can write down what we mean by a representation. A fairly but not completely general *definition* of the notion of representation is just the second sort of a model of a two sorted (perhaps recursively enumerable) first order theory, the first sort of the theory being the algebra. Now model theoretic forcing as seen in Hodges in [67] and indeed the classical completeness Theorem of Henkin, and his Neat Embedding Theorem to be recalled below, typically involves constructing a model of a first order theory by a game. The game builds the model step-by-step, elements of the model being produced by the second player called $\exists$, in his reponse the first player's critisism called $\forall$. (In forcing, during the game itself elements of the models are treated syntactically, e.g as Henkin's witnesses it is only at the end of the game that they become semantical objects of the model). The approach of Hirsch and Hodkinson is basically to combine the forcing games with the pseudo elementary approach mentioned above to representations to build the second sort of a model of the defining theory whose first sort is already fixed to be the algebra whose representability is at essue. Taking the defining theory of the pseudo elementary class to be given, this defines the notion of representation to be axiomatized. We refer the reader to [53] for applying this technique to axiomatize the classes of representable relation and cylindric algebras. The step-by-step technique in op.cit of building representations, especially when viewed as a game, is extremely potent. Not only does it allow the construction of axiomatization of relation algebras and cylindric algebras and other kinds of related algebras, but close examination of the way that games can be played on given algebras elicit very fine and detailed information about their structure. We will have occasion to use a disguised form of Robinson's finite forcing in model theory to prove interpolation results in chapter three and omitting types in chapter five for certain infinitary expansions of first order logic. This involves building models in an essentially step-by step argument , however, we will not resort to games. It is sometimes argued that the use of games is just a presentational matter, but we believe like Hirsch and Hodkinson that it is important, because games make life so much easier and so permit us to go further in the analysis. But in any case, this is no longer controversial: The step-by-step approach, whether by games or otherwise is widely accepted and has been used by - hardly exagerating - many workers, from Andréka to Zakharyaschev.

for similar applications in cylindric algebra. However, it is debatable whether any of the intrinsic characterizations of the class of the representable algebras is satisfactory or final. In particular, all hitherto existing axiomatizations of this class existing in the literature see e.g. [51], [3], [150], [60] ,it seems, are considered not-satisfactory from the algebraic point of view. This applies to the modal approach initiated by Venema [150],[151] [17] and the game approach initiated by Hirsch and Hodinson, [53] [56],[57] and [60]. The precise objections are difficult to pin down, but broadly it seems that these axiomatizations are regarded as unsatisfactory one way or another: they are either too complicated, or perhaps too trivial, just paraphrasing the original problem without providing any significant new "algebraic insight". A form of the Representation Problem is to describe properties of the class RCA_α and try to give a useful characterization of it in *abstract* terms. An old result of Henkin which gives an abstract sufficient and necessary condition for representability fits here. An algebra $\mathcal{A}$ is in RCA_α if and only if for every $\beta > \alpha$, it can be embedded as a *neat subreduct* in some cylindric algebra of dimension β- equivalently using ultraproducts- into an algebra in ω extra dimensions. This brings us to the central notion in this dissertation namely the notion of *neat reducts.* The notion of neat reducts, which we now recall, is also due to Leon Henkin [50].

Definition 0.3 .

Let $\alpha < \beta$ be ordinals, and let $\mathcal{A} \in CA_\beta$. Then the neat-α reduct of $\mathcal{A}$, in symbols $\mathcal{N}r_\alpha\mathcal{A}$, is the CA_α whose domain is the set of all α-dimensional elements of $\mathcal{A}$ defined by

$$Nr_\alpha A = \{a \in A : c_i^{\mathcal{A}} a = a \text{ for all } i \in \beta \smallsetminus \alpha\}.$$

The operations of $\mathcal{N}r_\alpha\mathcal{A}$ are those of the α-dimensional reduct

$$\mathcal{R}d_\alpha\mathcal{A} = \langle A, -^{\mathcal{A}}, \cdot^{\mathcal{A}}, c_i^{\mathcal{A}}, d_{ij}^{\mathcal{A}}\rangle_{i,j\in\alpha}$$

of $\mathcal{A}$, restricted to $Nr_\alpha A$.

When no confusion is likely, we omit the superscript $\mathcal{A}$. $Nr_\alpha A$, as easily checked, is closed under the indicated operations and indeed is a CA_α. $\mathcal{N}r_\alpha\mathcal{A}$ is thus a special

[17] A similar situation occurs in modal logic when poeple say that Gabbay's irreflexitivity rule involve introducing variables by the back door and are *inimical* to the true nature of modal logic.

subreduct of $\mathcal{A}$ i.e. a special subalgebra of a reduct in the universal algebraic sense.

For a class K, let SK denote the class of all algebras embeddable into members of K. Neat reducts play a key role in the representation theory of cylindric algebras as indeed illustrated in the next celebrated Theorem of Henkin, which is basically a Completeness result:

Fact 0.4 . *(Henkin) For any ordinal α and any $\mathcal{A} \in CA_\alpha$, the following two conditions are equivalent:*

(i) $\mathcal{A} \in RCA_\alpha$

(ii) $\mathcal{A} \in SNr_\alpha CA_{\alpha+\omega}$.

Infinity manifests itself in (ii) above, and it does so essentially in the case when $\alpha > 2$, in the sense that if $\mathcal{A}$ neatly embeds into an algebra in finitely many extra dimensions, then it might not be representable, as shown by Monk (cf. Fact 0.5 below.) All ω extra dimensions are needed for representability, one could not truncate ω to any finite ordinal. [18] However an interesting result of Ferenszi [42] shows that we can despense with some of the CA axioms when we get to ω-extra dimensions, more precisely: For every $\alpha > 2$ there exists a variety $K_{\alpha+\omega}$ having the same signature as $CA_{\alpha+\omega}$ such that $CA_{\alpha+\omega} \subset K_{\alpha+\omega}$ (here the inclusion is proper) and $RCA_\alpha = SNr_\alpha K_{\alpha+\omega}$. Here the class $Nr_\alpha K_{\alpha+\omega}$ is defined like in the CA case. Coming back to Henkin's Neat Embedding Theorem, we have the following. On the one hand, the characterization established by Henkin is an abstract characterization of the class RCA_α, and it does help occasionally to prove that certain subclasses consist exlusively of representable algebras. This occurs when it is easier to prove that an algebra embeds neatly into another algebra in ω extra dimensions, and applies for example in the cases of the classes of the so-called diagonal cylindric algebras and semisimple ones, [50] Theorem 2.6.50 and [51] Theorem 3.2.11. On the other hand, this characterization is not satisfactory from the set-theoretic representation point

[18] The ω extra dimensions plays the role of added constants or witnesses in Henkin's classical Completeness proof. Therefore it is no coincidance that variations on Fact 0.4 lead to metalogical results concerning interpolation and omitting types for the corresponding logic, for such results can be proved by Henkin's methods of constructing models out of constants. A central theme in chapter three, this analogy will be further elaborated upon.

of view, since it is not *intrinsic* referring to algebras outside the algebra considered namely algebras in ω-extra dimensions. [19] In [50] algebras in the class $SNr_\alpha CA_{\alpha+\omega}$ are said to have the *neat embedding property*. Therefore an algebra has the neat embedding property if it neatly embeds into an algebra in ω extra dimensions, and according to Henkin's Neat Embedding Theorem the class RCA_α coincides with the class of cylindric algebras that have the neat embedding property. [20] The notion of the neat embedding property is more suitable for an abstract algebraic treatment than that of representability. It seems that this is the main reason why the discussion of neat reducts and their subalgebras is comprehensive and detailed in [50]. It is known (cf. [128] or [145]) that the Neat Embedding Theorem, or the NET for short, proved by Henkin in the fifties is an algebraization of Henkin's celebrated proof of the completeness of first order logic, or rather an extension thereof. Indeed it can be viewed as a typical instance of Robinson's finite forcing in Model Theory [53]. Referred to by Hirsch and Hodkinson [54] as one of the earliest examples of step-by-step building representations in algebraic logic, variants of the NET have been succesfully applied to (algebraically) prove the completeness of several versions of quantifier logics, that are extensions, variants or reducts of first order logic. Examples include Keisler's logics investigated algebraically in [39], and various reducts thereof like the logics studied in [9] and much later in [116], under the name of *typless finitary logics of infinitary relations*, see also [51] sec 4.3. for a systimatic treatment of such logics. Conversely the NET (conjuncted with some rather complicated form of Ramsey's theorem) has been applied to show the essential Incompleteness of L_n, the first order logic restricted to the first n variables when $\omega > n > 2$ see e.g. [95], [62]. This follows from the following classical algebraic result of Monk that established the "infinite distance" between CA's and RCA's. Monk's result marked a turning point in the development of the subject, and is considered one of the most, if not *the* most,

[19]By an intrinsic property of an algebra $\mathcal{A}$ we understand, loosely speaking, a property which can be expressed entirely in terms of symbols denoting the operations of A, and variables ranging exclusively over elements of the universe of A, subsets of A, relations between elements of A, sets of such subsets and relations, etc., and not e.g. in case A is a family of sets, in terms of variables ranging over $\cup A$. (Note that the notion of an intrinsic property is of mathematical nature; to be made precise it must be relativized to a well defined formal language).

[20]Since locally finite algebras have the neat embedding property, Fact 0.4 can be seen as an indeed substantial generalization of Fact 0.3. pushing it to the limit.

important Model-theoretic result concerning cylindric algebras:

Fact 0.5 . *(Monk)*

Let $\omega > n > 2$ and $m \in \omega$. Then RCA_n is properly contained in Nr_nCA_{n+m}. In particular, RCA_n is properly contained in SNr_nCA_{n+m}.

Monk used Ramsey's Theorem to construct for each $m \in \omega$ and $2 < n < \omega$, an algebra $A_m \in Nr_nCA_{n+m}$ that is not representable. [21] The ultraproduct of these algebras (relative to any cofinite ultrafilter on ω) is in $SNr_nCA_{n+\omega}$, hence by the NET, is representable. Using elementary model theory, it follows thus that the class RCA_n, for $\omega > n > 2$, is not finitely axiomatizable. Maddux [84] proves that Monk's algebras can be generated by a single element. Had it been otherwise, i.e. if for $\omega > n > 2$, RCA_n had turned out finitely axiomatizable by a finite set of equations Σ say, then this Σ would have been probably taken as the standard axiomatization of CA_n. Unfortunetly this turned out not to be the case. As it seemed, the hopes of workers over a hundred years starting with De Morgan and culminating in Tarski's work to produce a (simple, elegant, or at least finite) set of algebraic properties - or in modern terminology - equations that captured exactly the true properties of n-ary relations for $\omega > n > 2$ were shattered by Monk's result. This impasse is still invoking extensive research till the present day, in essentially two conflicting (but complementary) forms. [22] One form, which we already discussed is to try to circumvent this negative

[21] These algebras are referred to in the literature as Maddux's or Monk's algebras. Both authors used them. The key idea of the construction of a Monk's algebra is not so hard. Such algebras are finite, hence atomic, more precisely their boolean reduct is atomic. The atoms are given colours, and cylindrifications and diagonals are defined by stating that monochromatic triangles are inconsistent. If a Monk's algebra has many more atoms than colours, it follows from Ramsey's Theorem that any representation of the algebra must contain a monochromatic traingle, so the algebra is not representable. Andréka's splitting as seen in [3] is a variation on the same theme. Here splitting refers to splitting an atom into more atoms that enforces non-representability. However, Andréka's splitting does not appeal to any form of Ramsey's theorem. We will have occasion to use Andréka's splitting to show that the class of neat reducts n reducts of β dimensional cylindric algebras for $2 < n < \beta \cap \omega$ is not elementary. We note that Monk's algebras established a very interesting connection between finite combinatorics and algebraic logic [98], a recurrent theme in algebraic logic. A recent use - establishing this link - of Monk's algebras with a powerful combinatorial result of Erdos has been used to show that the class of the so called strongly representable atom structures of relation algebras is not elementary [55]. We will use such counterexamples in chapter five to show that the omitting types Theorem fails for the finite variable fragments of first order logic, as long as the number of variables available are > 2.

[22] To understand the "essence" of representable algebras, one often deals with the non-representable ones, the "distorted images" so to speak. Simon's result in [147], of representing non-representable

non-finite axiomatizability result. The other form is to sharpen it. Indeed, Monk's negative result - as far as non-finite axiomatizability is concerned- stated in Fact 0.5 was refined and strengthened by many authors in many directions, to mention a few, Andréka, [3], Biro, [29], Maddux [84], Sagi [124], and Hirsch and Hodkinson [57]. The latest and strongest improvement, the author knows of, of Monk's negative result is due to Sagi [124]. [23] Historically, one motivation for defining the class SNr_nCA_m, of all subalgebras of n-dimensional neat reducts of m-dimensional cylindric algebras for $n < m < \omega$, was connected to proof theory of first order logic. Algebraic logic, via the notion of neat reducts, is a useful tool for analyzing the number of variables needed in proofs. The result of Monk, for example, implies that for any Hilbert style axiomatization and any $3 \leq n < m < \omega$, there are formulas built out of n variables that need more than m variables to be proved. Maddux's result implies that such incompleteness results hold in languages with finitely many variables and with only one binary relation symbol. Maddux's result was strengthened by Hirsch, Hodkinson and Maddux in [62], by showing that for a first order language with just one binary

algebras, seems to point out that this distortion is, after all, not completely choetic. This is similar to studying non-standard models of arithmetic.

[23]Biro [29] proves that RCA_n, $\omega > n > 2$, remains non-finitely axiomatizable if we add finitely many *first order definable* operations, a result that is already implicit in Monk's and Maddux's non-finite axiomatizability results in [95] and [84]. Andréka [3], building on work of Johnson for relation algebras, proves the same result in case we add other "kinds" of operations like for example *modalities*, i.e. operations distributing over the boolean join, as long as the added operations are finitely many. While Biro's results excludes axiomatizations by a finite set of equations, Andréka's , on the hand, exclude axiomatizations involving universal formulas in which only finitely many variables occur. Sagi, building on work of Lyndon, addresses the most general formulation of the problem showing that the Finitization Problem cannot be solved by adding finitely many *permutation invariant* operations in the sense of Tarski-Givant [149], as long as one hopes for particular (universal) axiomatizations. One general form of the Finitization Problem for both cylindric algebras and relation algebras is to the best of our knowledge still open. This more or less concrete form for cylindric algebras asks whether we can expand the language of RCA_n, $\omega > n > 2$ by finitely many permutation invariant operations so that the interpretation of these newly added operations in the resulting class of algebras is still of a concrete set-theoretic nature and the resulting class becomes a finitely axiomatizable variety or quasivariety, cf.[124]. The requirement of permutation invariance here is crucial for it correponds to the (meta-logical) fact that isomorphic models satisfy the same formulas, a basic requirement in abstract model theory. Without this requirement there are solutions to the Finitization Problem due to Biro and Maddux and Simon, cf. [29], [84], [147]. The current "belief" is that the answer to the more restrictive form of the problem is either negative, or perhaps even independent of ZF (Zermelo Frankeal set theory). Indeed, in [106] and [117] it is proved several versions of the Finitization problem are independent from ZF minus the axiom of foundation. The results of Hirsch and Hodkinson in [56] and [57] seems to be relevant to this form of the Finitization Problem, which in [60] is reduced to a problem about (colourings of) certain graphs. On the face of it, this seems to be bad news for graphs, rather than good news for the Finitization Problem.

relation, and for every $3 < n < m < \omega$, there is a formula built of n variables that can be proved using $m+1$ variables, but cannot be proved using m variables. [24] We note that from Monk's result, it follows that RCA_ω is also not finite schema axiomatizable, i.e cannot be axiomatized by a finite schema analagous to that axiomatizing CA_ω. (To aviod trivialities, here finite axiomatizability is replaced by finite schema axiomatizability because of the presence of infinitely many operations.) The logic corresponding to RCA_ω is L_ω the *structural algebraizable* formalism of first order logic, studied for example in [150], [50], [146] and [51]. Pigozzi, cf. [50] the remarks after problem 2.12, proved the analogous result of Hirch, Hodkinson and Maddux for L_ω. By this our general introduction concerning cylindric algebras is complete. However, we shall deal with other algebras as well. Their exact definition will be recalled in due time, i.e. at their first occurrence in the text. But now we can give at least the motivation for the introduction of such algebras, from algebras of formulas. Fix a set of sentences Γ in a first order language, and let $\mathcal{A}_\Gamma$ denote the Tarski-Lindenbuam algebra based on Γ.[25] From the cylindric algebra of formulas $\mathcal{A}_\Gamma$ one can add (other natural) operations, or remove operations to obtain new algebras. Unary functions s_i^j $(i, j < \omega)$ mimicking substitution. s_i^i is the identity function, and for $i \neq j$, and a formula ϕ, $s_j^i(\phi/\Gamma) = (\exists x_i(x_i = x_j \wedge \phi))/\Gamma)$. Unary functions s_{ij} $i, j < \omega$ where $s_{ij}(\phi/\Gamma) = \psi/\Gamma)$ and ψ is obtained from ϕ by swapping all x_i with x_j. The s_j^i's are called substitutions corresponding to replacements, while the s_{ij}'s are called substitutions corresponding to transpositions. These algebras satisfy various equations, corresponding to the

[24]The reader may already have wondered why the negative result of Monk was so terminal. After all we may choose to reason about n-ary relations using classical first order logic. There are Hilbert systems with finite schema axiomatizations in the textbooks and a Completeness Theorem. The properties expressible in the language of SNr_nCA_m for $n < m$ can be all expressed using m variables. This is correct but such an approach takes us outside the language of CA_n and so at least lacks elegance. We should mention that Lyndon's two papers [74], [75] on relation algebras were very influential. The one containing the negative results of constructing non-representable relation algebras from projective goemetries led Monk's Fact 0.5 proved in 1969. Monk though proved the analgous result for the class of representable relation algebras in 1964, so it took Monk quite some time to replace projective goemetries by graphs. While [75] is the base for Hirsch-Hodkinson's (step-by- step) approach to the representation problem.

[25]For what the boolean reduct of these algebras look like for common theories Γ, the reader is referred to the chapter by Myers in the Boolean algebra handbook [101]. For boolean algebras, the description consists of in describing a linear order $\leq$ such that the Lindenbuam-Tarski algebra is isomorphic to the interval algebra on $\leq$. It is somewhat surprising that analogous investigation for cylindric algebras of formulas has been almost neglected.

axioms of the (complete) proof system adopted in the definition of the congruence relation. Moving to an abstract point of view, in the process relaxing the condition of dimension like we did in the cylindric case, we consider the class of all algebras (of an arbitrary dimension) satisfying these equations. When we have all the above operations then we are dealing with Halmos' quasipolyadic equality algebras (QEA). However, we can choose to include only some (and not all of the operations in QEA) of the above operations, besides cylindrifications coresponding to quantification. If we keep diagonals and cylindrifications, but don't bother about substitutions (corresponding to replacements and transpositions) we get cylindric algebras. (However in cylindric algebras substitutions corresponding to replacements are term-definable. Substitutions corresponding to transpositions are not, so that QEA are proper expansions of CA.) Dropping diagonals together with substitutions corresponding to transpositions yields Pinter's substitution algebras (SC), while dropping diagonals alone and keeping all operations gives quasipolyadic equality algebras (QA). Keeping only cylindrifications, gives diagonal free reducts of cylindric algebras (Df). We could add infinititary substitutions and cylindrifications, corresponding to the operations of simulteneous substitution of infinitely many variables for variables and quantification on infinitely many variables, respectively. This shifts us to the realm of Halmos' polyadic algebras with and without equality (PEA, PA), which is an algebraisation of Keisler's infinitary logics. All these algebras have been studied in algebraic logic for a long time, and will be studied in the chapters to come.[26]

Notions like neat reducts and neat embeddings and results connecting representability to special neat embeddings generalize verbatim to such algebras. This completes our general introduction to algebraic logic. A comprehensive treatment of the algebraic theory of cylindric algebras can be found in Henkin, Monk, Tarski [50]. A review of results concerning Halmos' quasipolyadic algebras, and Pinter's substitution algebras can be found in [109]. The closely related theory of polyadic algebras can be found in Halmos[47]. Now let $K \in \{SC, QA, QEA, CA, PEA, PA\}$ and let $1 < \alpha < \beta$. In this dissertation we are interested in the class $Nr_\alpha K_\beta$ of neat α reducts

[26]In principle, one could add other operations on formulas like infinite conjunctions and disjunctions. This gives algebras in which certain supremums and infimums exist. However to the best of our knowledge not much work has been done in this direction.

of β-dimensional algebras in K, and the repercusions of the fact that this class is, more often than not, not closed under forming subalgebras.

What is this dessertation about?

In a paper written by J.D. Monk in 1991, but published in the logic Journal of IGPL in 2000, [102], Hajnal Andréka writes the final survey section on the subject, to update the article. It is clear from Andréka's section that among the important problems that are still open in algebraic logic are those in Pigozzi's Landmark paper on amalgamation [113] and two problems, both on neat reducts. [27] The two problems on neat reducts are the consecutive problems 2.11 and 2.12 posed by Henkin, Monk, Tarski in [50]. Hirsch and Hodkinson solve problem 2.12. They show that for $2 < n < \omega$ and $k \in \omega$, SNr_nCA_{n+k+1} is a proper subclass of SNr_nCA_{n+k} . Thus the decreasing sequence $CA_n \supseteq SNr_nCA_{n+1} \supseteq SNr_nCA_{n+2} \cdots$ converging to RCA_n is not only, not eventually constant, as proved by Monk, but is in fact *strictly* decreasing. Problem 2.11 in [50] asks whether the class $Nr_\alpha CA_\beta$ for $1 < \alpha < \beta$ is closed under forming subalgebras. Proved not to be closed under subalgebras by Németi [104], the question as to whether it is perhaps closed under elementary subalgebras appears as problem 4.4 in the monograph [51]. This is equivalent to asking whether this class is elementary, because it is closed under ultraproducts. In [104], Németi conjectures that for $1 < \alpha < \beta$, the class $Nr_\alpha CA_\beta$ is not closed under elementary subalgebras, hence is not elementary, i.e. cannot be axiomatized by any set of first order axioms. In this dessertation we solve some of Pigozzi's remaining open questions, confirm Németi's conjecture solving problem 4.4 in [51], and investigate variations on problem 2.11 in [50]. The closure of the class of neat reducts under forming subalgebras for CA-like algebras of relations turns out closely related to the algebraic property of amalgamation, which in turn is the algebraic equivalent of the metalogical property of interpolation. The elaboration on this connection is a central theme in chapters two, three and four herein. The closure of the class of neat reducts under *complete* subalgebras turns out closely related to the algebraic notion of complete representations, and the metalogical one of omitting types (for finite-variable fragments of first

[27] We believe that Maddux's conjecture formulated in [83] is important too. This conjecture was confirmed by Hirsch and Hodkinson [58].

order logics and certain multi-modal logics) as illustrated in chapter five. In fact, all the results in this dessertation can be seen as consequences of the (very innocent looking) fact that the class of neat reducts is not closed under forming subalgebras.

A brief outline and Organization

This dissertation consists of five other chapters. Every chapter, with the exception of chapter four, is based on at least one article that is accepted for publication. But chapter two contains more results than the two articles it originated from. Chapter two is based on the articles [145], and [129]. [145] entitled *On neat reducts of algebras of Logic* is a joint article with Istvan Németi that is published in Studia Logica [145]. [129] is accepted for publication in Fundemanta Mathematica. The results of Chapter two were presented by the author in Logic Colloquium 1996 and again in London Imperial College 1999. Some of results in chapter two are abstracted in the Bulletin of the Journal of Symbolic Logic [21]. The results in Chapter three are based on the article *On amalgamation of reducts of polyadic algebras* accepted for publication in Algebra Universalis, [130]. Some of the results in chapter three were presented by the Author in The Fried conference in Budapest 1999 [131]. The results in chapter four were presented in the universal algebra conference in Szeged 1999 and in Logic Colloquium in Paris 2000, and is abstracted in [138]. Most of the results in chapter five are based on the articles [136] and [142]. [136] entitled *Martin's axiom, omitting types and complete representations in algebraic logic* accepted for publication in a special eschoe of Studia Logica dedicated to many dimensional logical systems, [136]. The results in [136] were presented by the author in Logic Colloquium 1998, and it is abstracted in [23]. [142] entitled *A model-theoretic solution to a problem of Tarski* solves a long standing open problem in the monograph [51], and will appear in Mathematical Logic quaterly. The proof we present here is model-theoretic, based on ideas suggested by Ian Hodkinson. It addresses only the finite dimensional case. A different proof addressing also the infinite dimensional case appeared in logic Journal of IGPL, cf. [135]. This will be the content of chapter 6. The results of chapter 5 and 6 concerning neat reducts were presented by the author in the algebraic Logic Seminar in Imperial College London 1999, and in San Diego in 2000 and is abstracted in [132].

All the results in this dissertation were announced by the author in the algebraic Logic seminar in Budapest from 1998-2000.

As mentioned in the abstract, this dessertation has five other chapters. Every such chapter is completely self-contained and can be read independently. Accordingly, we decided to add an abstract before every chapter summarizing the results. The history and motivation of our results will be prefaced before every chapter. For the reader's conveniance we have unified the notation throughout. Like we did in the introduction, we have started every chapter with a quotation from a famous mathematician that we found appropriate to the contents of this chapter.

Chapter 2

On neat reducts and amalgamation

Abstract

SC_α, CA_α, QA_α and QEA_α stand for the classes of Pinter's substitution algebras, Tarski's cylindric algebras, Halmos' quasipolyadic algebras, and quasipolyadic equality algebras of dimension α, respectively. Generalizing a result of Németi on cylindric algebras, we show that for $K \in \{SC, CA, QA, QEA\}$ and any pair of ordinals $\alpha < \beta$, the class $Nr_\alpha K_\beta$ of α-dimensional neat reducts of β-dimensional K algebras, though closed under taking homomorphic images and products, is not closed under forming subalgebras (i.e. is not a variety) if and only if $\alpha > 1$. From this it easily follows that for $1 < \alpha < \beta$, the operation of forming α-neat reducts of algebras in K_β does not commute with that of forming subalgebras, a notion to be made precise. We show that $Nr_2 K_\beta$ is not closed under forming elementary subalgebras, hence is not first order axiomatizable. We give a contrasting result concerning Halmos' polyadic algebras (with and without equality). For such algebras, we show that the class of infinite dimensional neat reducts forms a variety. Following research initiated by Pigozzi, we also emphasize the strong tie that links the (apparently non-related) property of neat reducts commuting with forming subalgebras with proving amalgamation results in cylindric-like algebras of relations. We show that, like amalgamation, neat reducts commuting with forming subalgebras is another algebraic expression of definability and, accordingly, is also strongly related to the well-known metalogical properties of Craig, Beth and Robinson in the corresponding logics. We show that certain classes of

infinite dimensional algebras do not have the strong amalgamation property. We also present positive results concerning the strong amalgamation and embedding properties.

Key words: cylindric algebras, substitution algebras, quasipolyadic algebras, polyadic algebras neat reducts, neat embeddings, amalgamation.

History and Background

It is always exciting for a mathematician when close connections are discovered between seemingly distant notions and results from two different branches of mathematics, and this is especially true when the notions and results involved are important ones and the focus of considerable research in their respective areas. Thus for instance, some recent developments have brought to light close and unexpected connections between two groups of results - metalogical interpolation theorems of which the first and best known is Craig's interpolation theorem for first order logic and the algebraic theorems to the effect that certain classes of algebras have the amalgamation property. The main purpose of this paper is to study in detail the mutual implications between these two kinds of results and using these implications to obtain new results of either kind.

Pigozzi in [113].

In this chapter we deal with two seemingly unrelated (family of) problems. The first family concerns neat reducts of certain classes of algebras that abound in the literature of algebraic logic,[1] while the second family of problems concerns amalgamation properties in such classes of algebras. One main theme of this chapter is relating results on neat reducts to results on (the more familiar notion of) amalgamation. The connection is subtle and it will unfold as we go along. The first type of problems we address on neat reducts has a long history and it dates back to problem 2.11 in the monograph [50]. Problem 2.11 in [50] asks for which pair of ordinals $\alpha < \beta$ is the class of α-dimensional neat reducts of β-dimensional cylindric algebras closed under forming subalgebras.

[1]All algebras considered in this chapter and in fact throughout the whole dessertation are reducts of Halmos' polyadic equality algebras. More generally, they are boolean algebras with unary normal operators, i.e. unary operations that distribute over the boolean join.

While neat reducts is a central notion in the representation theory of several cylindric-like algebras of relations studied in algebraic logic (via the Neat Embedding Theorem of Henkin), the notion of amalgamation, on the other hand, is certainly of a wider scope and is a central notion in Model Theory. As Hodges puts it [67] "Everything that had happened in Model Theory has confirmed how important amalgamation is". In algebraic logic amalgamation for classes of algebras has acquired a special status too, because of its intimate connections to interpolation results (Like Craig interpolation) in the corresponding logics.

We first review the history of both family of problems that we are about to investigate. Then we highlight the connection. Finally we state our results. At first reading, the connections might not be clear, but we hope that after reading this chapter, the scattered results will fall into place and the picture will be complete.

Concerning the first kind of problems adressed in this chapter, the following two related problems on neat reducts appeared as problem 2.11 in [50] and as item (v) in the introduction of [51], respectively:

(i) For which pair of ordinals $\alpha < \beta$ is the class $Nr_\alpha CA_\beta$ of neat α reducts of β-dimensional cylindric algebras closed under forming subalgebras?

(ii) Now assume that α is infinite i.e. assume that $\omega \leq \alpha < \beta$. Is there a CA_α $\mathcal{A}$ and a CA_β $\mathcal{B}$, such that $\mathcal{A}$ is a generating subreduct of $\mathcal{B}$ distinct from $\mathcal{N}r_\alpha\mathcal{B}$?

Before illustrating the connection between (i) and (ii), we need to clarify some undefined notions in (ii). $\mathcal{A}$ is a generating subreduct of $\mathcal{B}$ means that $\mathcal{A}$ is a subalgebra of $\mathcal{N}r_\alpha\mathcal{B}$ and that A, as a set, generates the algebra $\mathcal{B}$. The process of generation here involves all the β operations of $\mathcal{B}$. Since A generates $\mathcal{B}$, then it is reasonable to expect that $\mathcal{B}$ would be "close" to $\mathcal{A}$. But what do we mean by close in this context? For this we need to recall from [50] that for $\mathcal{B} \in CA_\beta$, $x \in B$ and an ordinal α with $\alpha \leq \beta$, x is said to be *α-dimensional* if the set $\Delta x = \{i \in \beta : c_i x \neq x\}$ is contained in α. Δx is referred to as the *dimension set* of x in [50]. [2] Note that $Nr_\alpha B$ consists of precisely the α-dimensional elements of B. Now we explain what we mean

[2] Roughly, if x is viewed as an equivalence class of a formula ϕ, then Δx corresponds to the free variables occuring in ϕ.

by close. Since A generates $\mathcal{B}$, then one might expect that $\mathcal{A}$ - already consisting of α-dimensional elements - will exhaust the set of *all* α-dimensional elements of $\mathcal{B}$, namely the set $Nr_\alpha B$. Indeed, this is the case for example when $\mathcal{A}$ is locally finite or more generally *dimension complemented*, as proved in [50] Theorem 2.6.67. [3]

(ii) is a question based on a conjecture of Tarski, whose proof could not be reconstructed by Henkin and Monk. Tarski conjectured that if we choose our algebras carefully (avoiding of course the dimension complemented case) then the answer to (ii) would be yes, i.e. there exists $\mathcal{A} \in \mathcal{CA}_\alpha$ and $\mathcal{B} \in \mathcal{CA}_\beta$ with $\omega \leq \alpha < \beta$, such that $\mathcal{A} \subseteq \mathcal{N}r_\alpha B$, A generates $\mathcal{B}$, and $A \neq Nr_\alpha B$. Futhermore, Tarski conjectured that $\mathcal{A}$ and $\mathcal{B}$ can be chosen to be representable. Though (only) stated at the top of p.465 in part 1 of Henkin Monk Tarski [50], a *proof* of Tarski's conjecture was postponed to appear in part 2 of Henkin Monk Tarski [51]. But this proof never appeared. Instead (ii) appears as an open question in item (v) of the introduction of [51] among problems (supposedly) solved by Tarski, but their proof could not be supplied by his co-authors Henkin and Monk.

But how are (i) and (ii) related? One connection is the following: If Tarski's conjecture were false, then the class $Nr_\alpha CA_\beta$ for any pair of infinite ordinals $\alpha < \beta$ would be closed under forming subalgebras. To see this, let $\omega \leq \alpha < \beta$. Let $\mathcal{A} \in CA_\alpha$ and $\mathcal{B} \in CA_\beta$ be such that $\mathcal{A} \subseteq \mathcal{N}r_\alpha B$. Now if one takes $\mathcal{B}'$ to be the subalgebra of $\mathcal{B}$ generated by (the set) A, then the algebra $\mathcal{A}$ would be a generating subreduct of $\mathcal{B}'$, thus - remember we are assuming that Tarski's conjecture is wrong - A would exhaust the set of all α-dimensional elements of $\mathcal{B}$, i.e. $\mathcal{A} = \mathcal{N}r_\alpha \mathcal{B}'$. This shows that a subalgebra of a neat reduct of an algebra $\mathcal{B}$ is again a neat reduct of a possibly smaller algebra $\mathcal{B}'$, but a neat reduct all the same. It follows thus that for infinite $\alpha < \beta$, the class $Nr_\alpha CA_\beta$ is closed under forming subalgebras.

But Tarski was right. And indeed, in [104] Németi formulates the answer of (i) for every pair of ordinals $\alpha < \beta$. There he proves that the class $Nr_\alpha CA_\beta$, though closed under forming homomorphic images and products, is not a variety (i.e. not closed under forming subalgebras) if and only if $\alpha > 1$. Independently, Roger Maddux [82]

[3] $\mathcal{A} \in \mathcal{CA}_\alpha$ is dimension complemented if for all $x \in A$, there is an $i \in \alpha$ such that $c_i x = x$. In the infinite dimensional case dimension complemented algebras is a significant non-trivial generalization of the locally finite algebras. (cf.[50] Theorem 2.6.67. However, their elementary theories coincide [51].

proves that Nr_3CA_β is not closed under forming subalgebras for $\omega > \beta \geq 5$. The construction adopted by Németi in [104], though not explicitly stated therein, also settles the strongest from of (ii), namely that the algebras in question are representable. That is from Németi's construction in [104], it follows that for every pair of infinite ordinals $\alpha < \beta$, there exists representable algebras $\mathcal{A} \in \mathcal{CA}_\alpha$ and $\mathcal{B} \in \mathcal{CA}_\beta$ such that $\mathcal{A}$ is a generating subreduct of $\mathcal{B}$ that is not isomorphic to $\mathcal{N}r_\alpha\mathcal{B}$. This turns out to be surprisingly straightforward as will be illustrated below. It should be noted that the connection between (i) and (ii) was implicitly mentioned in the problem session paper [99] that appeared in [12]. In [99], Monk credits the confirmation of Tarski's conjecture to Andréka and Németi without explicitly citing the reference [104]. In [104], however, Tarski's conjecture is not mentioned at all. But the confirmation of Tarski's conjecture is not the aim of the present chapter since it is indeed implicit in [104]. This chapter will rather set out to investigate (among other things) (i) and (ii) for the most well-known variants of cylindric algebras that can be conceived as:

(a) Algebraizations of various versions of quantifier logics (like fragments of first order logic and Keisler's logics.)

(b) Expansions of boolean algebras from algebras of unary relations to relations of higher rank, at least as far as their concrete versions, the so called representable ones, are concerned.

These algebras are : Pinter's substitution algebras $SC's$, Halmos' quasipolyadic (equality) algebras $QA's$ ($QEA's$) and Halmos' infinite dimensional polyadic (equality) algebras $PA's$ (and $PEA's$). We note that $SC's$ are (generalized) reducts of all other algebras,[4] while $QEA's$ are expansions of $CA's$, $QA's$ and $QEA's$. Of course for all these classes of algebras the notion of neat reducts is meaningful and indeed the class of neat reducts can be defined for the above specified classes of algebras. In more detail, for any pair of ordinals $\alpha < \beta$, and $K \in \{SC, QA, QEA, PA, PEA\}$ the class $Nr_\alpha K_\beta$ of α-dimensional neat reducts of β-dimensional algebras in K is well-defined. Furthermore, $Nr_\alpha K_\beta \subseteq K_\alpha$. In particular, the questions (i) and (ii) formulated above for neat reducts of cylindric algebras make sense for such algebras but - unlike the CA case - are yet to be resolved. But why is it of importance to

[4]An algebra $\mathcal{A}$ is a *generalized reduct* of an algebra $\mathcal{B}$ if $\mathcal{A}$ is obtained from $\mathcal{B}$ by deleting some of the (term definable) operations of $\mathcal{B}$, in which case $\mathcal{B}$ is said to be an expansion of $\mathcal{A}$.

settle such questions on neat reducts. There are (at least) three possible answers to this question. First there are aesthetic reasons. Motivated by intellectual curiosity, the investigation of such questions is likely to lead to nice mathematics. The second reason concerns definability or classification. Now that we have the class of neat reducts infront of us, the most pressing need is to try to classify it. Classifying is a kind of defining. Most mathematical classification is by axioms (preferably first order) or, even better, equations (if the class in question is a variety.) It is known (and indeed not difficult to show) that the class of $Nr_\alpha CA_\beta$ is closed under products and homorphic images for all $\alpha < \beta$. By a famous Theorem of Birkhoff the class $Nr_\alpha CA_\beta$ would be a variety, i.e. definable by a set of equations if it is further closed under forming subalgebras. So question (i) asks whether the class of neat reducts forms a variety. Two of our main results in this chapter are that the class $Nr_\alpha K_\beta$ for $1 < \alpha < \beta$ and $K \in \{SC, CA, QA, QEA\}$ is not closed under forming subalgebras hence is not a variety, while, in contrast, the classes $Nr_\alpha PA_\beta$ and $Nr_\alpha PEA_\beta$ for $\omega \leq \alpha < \beta$ are varieties. Now we come to the third reason. Here we do not address (i) and (ii) as ends in themselves, but rather elaborate on (ii) as a *means* to prove results on the so-called strong amalgamation property. To illustrate the strong tie that links the property expressed in (ii) - inquiring when generating subreducts are actually neat reducts - to that of the strong amalgamation property, we start by reviewing the significance of the notion of amalgamation in (algebraic) logic.

We recall that a class K of models (in our case algebras) has the amalgamation property, if for all $\mathcal{A}, \mathcal{B}$ and $\mathcal{C} \in K$, and all monomorphisms f and g of $\mathcal{A}$ into $\mathcal{B}$, $\mathcal{C}$, respectively, there exists $\mathcal{D} \in K$, a monomorphism m from $\mathcal{B}$ into $\mathcal{D}$ and a monomorphism n from $\mathcal{C}$ into $\mathcal{D}$ such that $m \circ f = n \circ f$. $\mathcal{D}$ is reffered to an *amalgam* of $\mathcal{B}$ and $\mathcal{C}$ over the base algebra $\mathcal{A}$. Since $\mathcal{A}$ embeds in both $\mathcal{B}$ and $\mathcal{C}$ then it is sometimes referred to as the common subalgebra.

Discovered by Roland Fraisse, the idea of amalgamation proved very powerful. Subtly different kinds of amalgamation produce preservation theorems (like the Los-Tarski Theorem on the syntactic description of formulas preserved in substructures) and definability theorems (like the classical Beth-Definability Theorem.) In Model Theory we see Fraisse's method of amalgamation to construct universal models, cf.

[67] chapter 6. One is to build up a structure by taking smaller parts, extending them, and then amalgamating the extensions. Fraisse's method of constructing universal (and homogenoues) models, is a typical instance of amalgamating small parts of a model to get the desired one.

In algebraic logic, the notion of amalgamation occurs in essentially two forms. One of them is to construct representations in a step-by-step fashion in the spirit of Fraisse. See for example the mosaic method of Németi [**?**] and the work of Maddux, cf. [83] and [84], on relation algebras with n-dimensional basis and n-dimensional cylindric basis. [5]

We will not be concerned with this form of amalgamation until chapter 4, where we use a typical Fraisse construction to build a model that is ω-categorial and has quantifier elimination. The set algebra based on this model will be used to show that the class $Nr_\alpha CA_\beta$ for $1 < \alpha < \beta \cap \omega$ is not even closed under elementary subalgebras, so that it cannot be defined by any set of first order axioms. [6]

The other form of amalgamation occuring in algebraic logic is typically of the following form: Which classes frequently studied in algebraic logic - like for example the class of representable cylindric or the class polyadic algebras - have the amalgamation property.

It is this form of amalgamation that we will be addressing in this chapter and the

[5]Roughly a basis consists of (finite) approximations to a representation which Maddux call matrices. These matrices can be glued together or amalgamated in a step-by-step manner to give relativized representations, cf.[56] and [83]. Relativized semantics for algebras possesing n-dimensional relational or n-dimensional cylindric bases in the sense of Maddux [84] were provided by Ian Hodkinson and Robin Hirsch. Furthermore, Ian Hodkinson and Robin Hirsch generalized Maddux's bases by introducing the notions of hypernetworks and hyperbasis cf. [55], and [57], which are suitable for dealing with n-ary relations where $\omega > n > 2$, and not just binary relations. A basis (hyperbases) consists of matrices (tensors) whose entries are elements of the algebra, when they consist of atoms they are called atomic matrices (tensors). Every such matrix (tensor) can be viewed as a finite graph (hypergraph) giving finite information on the potential representation of an algebra. Under favarouble circumstances, specifically when these matrices form a cylindric basis or hyperbasis or relational basis, then they can be viewed as a saturated set of mosaics in the sense of Németi [**?**], and consequently can be amalgamated in a step-by-step fashon to give a limiting genuine relativized representation [60]. If A has an n-dimenional relation bases, the resulting representation is said to be n-square, while if it has an n- dimensional cylindric basis, the representation is said to be n-flat. The reader is referred to [60] for more details.

[6]Note the reverse in the order. In this chapter we use results on neat reducts to prove results on amalgamation, while in the chapter four we use a result on amalgamation to prove a result on neat reducts.

next one.

As previously mentioned, this is important (from the logical point of view) because it is proved in algebraic logic that the amalgamation property in classes of algebras corresponds to interpolation properties in the corresponding logics, see e.g [113] or [11]. Moreover, it is in the context of algebraic logic that the equivalence of the interpolation property on the logic side and the amalgamation property (for the corresponding class of algebras) becomes explicit. The connection is strongest in equational logic because here the algebras are built directly from the theories, and algebraic logic can be viewed as the process of formulating richer logical systems, or simply logics, as systems of equational logic, cf. [28]. The logics correponding to the algebras studied in this chapter are algebraizable extensions of first order logic, and have been studied quite intensely recently. Here algebraizable is understood in the sense of [28]. They are sometimes referred to in the literature as *typless logics* or more often as *finitary logics of infinitary relations*, cf.[109] [146], and [150]. Since the algebras considered are infinite dimensional, the corresponding logics have an "infinitary flavour". Such logics allow infinitely long atomic formulas, but like first order logic only the usual two place conjunctions and disjunctions and quantification over single variables are allowed. We refer the reader to [9], [113] and [51] sec 4.3. where such logics are studied for purely logical reasons. For the study of such logics as multi-modal logics, with applications in Computer Science, the reader is referred to [150][44] [45] and [6]. In algebraic logic, Daigneault [36] was the first to explicitly make the connection between the Craig interpolation theorem and amalgamation in the context of locally finite polyadic algebras, giving a new algebraic proof to the Beth-Definability Theorem for first order logic. The first systematic use of the link to obtain results about interpolation properties from results of amalgamation, or vice versa, can be found in Pigozzi's landmark paper [113]. The principal context of [113] is the class of infinite dimensional cylindric algebras, an equational formalism of first order logic. In [113] Pigozzi deals basically with the following question: Which subclasses of infinite dimensional cylindric algebras, different then the locally finite ones, still have the (strong) amalgamation property. The classes that Pigozzi deals with consist solely of algebras that are infinite dimensional and we assume, to simplify notation, that such classes of algebras are ω-dimensional, where ω is the least infinite ordinal. These classes include the class of ω-dimensional dimension complemented algebras (Dc_ω), the class of ω-dimensional diagonal algebras (Di_ω) (exact definitions will be recalled below), and the class of ω dimensional semisimple algebras (Sc_ω.) Here a semisimple algebra is a subdirect product of simple algebras. All of these classes consists exclusively of algebras that are representable, but unlike RCA_ω none of these classes is first order definable, least a variety. While the amalgamation property speaks about amalgamating algebras in such a way that the amalgam only agrees on the common subalgebra, the amalgamation is said to be *strong* if the common subalgebra is the *only* overlap between the two algebras in the amalgam. The positive results of section 2.2 in combination with the negative ones of section 2.3 of [113] answer most of the natural questions one could ask about amalgamation for cylindric algebras. In particular, Pigozzi proves that in the (strictly) increasing sequence

$$Lf_\omega \subset Dc_\omega \subset Di_\omega \subset RCA_\omega$$

the first and third classes have the amalgamation property while the second and fourth fail to have it. However, most questions concerning strong amalgamation, were posed as open questions and other closely related ones appeared after Pigozzi's paper was published. [7] Now we come to the connection between neat reducts or more precisely *special neat embeddings* and *strong amalgamation.* This connection was first observed by Pigozzi. While Henkin proves that an algebra $\mathcal{A}$ is representable it if neatly embeds into an algebra in ω-extra dimensions, Pigozzi observes that for a class $L \subseteq RCA_\alpha$, for L to have the strong amalgamation property, then every algebra in L should embed neatly into an algebra in ω-extra dimensions in a special way. [8] A crucial technical Lemma of Pigozzi, Lemma 2.2.12 in [113], relates neat embeddings to strong amalgamation. Describing the nature of such neat embeddings into algebras

[7] An article [80] by the present author and Judit Madarasz is devoted to answering all remaining questions.

[8] From a model-theoretic point of view, this is not too surprising. Let us explain why. Henkin's neat embedding Theorem is an algebraisation of his celebrated completeness proof, which is an instance of Robinson's finite forcing in model theory. As Hodges says [67]: if the forcing conditions are chosen cleverly, this yields interpolation results. Pigozzi did exactly that in his proof that the dimension-restricted free algebras that happen to be dimension complemented have the interpolation property. In the next chapter, we will adapt Pigozzi's proof to present an amalgamation proof, as an instance of Robinson's finite forcing for several reducts of Halmos' polyadic algebras.

in ω extra dimensions, Lemma 2.2.12 in [113] says the following: (Our formulation, though, is slightly different from Pigozzi's.) Let α be an arbitrary ordinal. Let $L \subseteq RCA_\alpha$. Assume that L has the amalgamation property. Then a sufficient condition for L to have the *strong* amalgamation property with respect to RCA_α, (meaning that the amalgam is not necessarily in L but is in the possibly bigger class RCA_α), is that L satisfies the following condition which we abbreviate by (*):

(*) For every $\mathcal{A} \in L$, there exists $\mathcal{B} \in CA_{\alpha+\omega}$ (here the Neat Embedding Theorem is used) such that $\mathcal{A} \subseteq \mathcal{N}r_\alpha\mathcal{B}$, and such that the following condition holds:

$$\text{For all } X \subseteq A \quad Sg^{\mathcal{A}}X = \mathcal{N}r_\alpha Sg^{\mathcal{B}}X.$$

Here $Sg^{\mathcal{C}}X$, for a given algebra $\mathcal{C}$, and an $X \subseteq C$ denotes the subalgebra of $\mathcal{C}$ generated by X. Let us dwell a little bit more on the property expressed in (*). Let $X \subseteq \mathcal{A} \subseteq \mathcal{N}r_\alpha\mathcal{B}$. Then we can form the subalgebra of $\mathcal{N}r_\alpha\mathcal{B}$ generated by X, which is the same as the subalgebra of $\mathcal{A}$ generated by X. Or alternatively, we can form the subalgebra of $\mathcal{B}$ generated by X (here we are using ω extra operations, so that in principle we can get new α-dimensional elements), and then form the neat-α reduct of the resulting algebra. Then (*) says that taking any of these two paths we will arrive at the same algebra (up to isomorphism), i.e. we *do not* get new α-dimensional elements. In short, *the operation of taking α-neat reducts commute with that of forming subalgebras.* For brevity, let us say that $L \subseteq RCA_\alpha$ satisfying (*) has NS (short for neat reducts commuting with forming subalgebras). Note that if L has NS, then $L \subseteq Nr_\alpha CA_{\alpha+\omega}$. [9] Having proved that the class Di_ω has the amalgamation property, as a means to prove strong amalgamation for Di_ω, Pigozzi asks the following instance of (ii) at the top of page 258 of [113]:

(ii)$'$ What if we take $\mathcal{A}$ to be a diagonal CA_ω and $\mathcal{B} \in \mathcal{CA}_{\omega+\omega}$ in (ii)? In particular, does Di_ω have the NS?

Our second main result concerning amalgamation is that any class containing the class of the so-called weak set algebras of any dimension > 1 and the class of ω dimensional diagonal algebras fail to have NS, and finally that this class fails to have the strong amalgamation property.

[9] It might be useful to mention at this early stage that NS is really a definablity condition. NS roughly says that terms definable in ω extra dimensions are already term definable.

The Main Results

Let $K \in \{SC, QA, CA, QEA\}$ and let $\alpha > 1$. SK denotes the class of all subalgebras of members of K. Recall that $SC's$ are (generalized) reducts of all other algebras, while $QEA's$ are expansions of $CA's$, $QA's$ and $SC's$. We first construct an α-dimensional algebra in $SNr_\alpha QEA_\beta$ for every $\beta > \alpha > 1$, such that its SC reduct is not in $Nr_\alpha SC_\beta$, and such that its CA reduct in the infinite dimensional case is a diagonal algebra (Theorem 1.) From this we readily infer that the class $Nr_\alpha K_\beta$ of α-dimensional neat reducts of β-algebras in K for $1 < \alpha < \beta$, is not closed under forming subalgebras (Corollary 1). As a byproduct, we confirm Tarski's conjecture and answer Pigozzi's question (ii)', together with their K- analogue (Corollary 2). From Theorem 1, it follows that the class for any pair of ordinals $1 < \alpha < \beta$, the class $Nr_\alpha K_\beta$ - though closed under homomorphic images and products, and hence under ultraproducts - [10] cannot be axiomatized by any set of equations. Motivated to give some (hopefully first order) characterization of such classes, it is thus natural to ask whether the class of neat reducts is perhaps closed under forming *elementary* subalgebras, so that it is an elementary (i.e. first order axiomatizable) class. We give a negative answer to this question but only for the lowest value of α for which this class is not a variety. We show that $Nr_2 K_\beta$ is not closed under elementary subalgebras (Theorem 2). [11] From our construction of Theorem 1, it easily follows that any class L containing the class of weak set algebras of dimension > 1 (and in the cylindric case ω-dimensional diagonal algebras) does not have NS (Corollary 3). Emphasizing on the connection between NS and the strong amalgamation property, we proceed to show that, in the class of ω dimensional diagonal algebras fails to have the strong amalgamation property (Theorem 3). In contrast to Theorem 1, we show that the class of neat reducts of polyadic algebras is a variety, in particular it is closed under forming subalgebras (Theorem 4). [12] We show that the classes of the so-called monadic generated $CA's$ and $CA's$ of positive characteristic have the strong amalgamation property (Theorem 6). In addition, to the (strong) amalgamation property, a weaker property, the so-called *(strong) embedding property* to be recalled

[10] Recall that an ultraproduct is a homomorphic image of a product

[11] We extend this result for SC's and CA's only to higher dimensions in chapters 4.

[12] That these classes have SAP was proved by Johnson in [71].

below, has been frequently investigated in the literature, see e.g. [113] and [87]. We show that the class of diagonal cylindric algebras and the class of representable algebras of infinite dimension have the strong embedding property (Theorem 7). The above results make us formulate a conjecture to the effect that the so called *strong amalgam base* of RCA_ω coincides with the class of algebras having the NS. If true, this answers a question of Monk in [100]. Finally, we show that the classes SC, QA together with their concrete versions have the strong embedding propery (Theorem 8). The work in this chapter is based on the articles [145], [129] and the talks [138] and [21].

Organization

This chapter has three sections. In section 0 we recall the basic notions and formulate our results. Proofs are given in section 1. In section 2, we discuss related results and pose some open questions.

0. Formulation of the main results

To formulate our results in this chapter, we review some basic notions and concepts from the monographs [50], [51] and the survey article [109]. In order to keep our presentation self-contained, we start off by defining the algebras we shall be dealing with throughout this chapter. Our notation throughout is standard; it is mostly in conformity with the monograph [50]. From now on, unless otherwise specified, $K \in \{SC, CA, QA, QEA\}$ where SC, CA, QA and QEA abbreviate the classes of substitution, cylindric, quasipolyadic, and quasipolyadic equality algebras, respectively. K_α, α an ordinal, stands for the class of all algebras in K of dimension α, sometimes referred to as α-dimensional algebras. According to a widespread custom, we distinguish between algebras and their universes or carriers. Algebras will be denoted by calligraphic letters, and when we write $\mathcal{A}$, then we will be tacitly assuming that A will denote the universe of $\mathcal{A}$.

Let $\mathcal{A} \in K_\alpha$. Then $\mathcal{A}$ is a boolean algebra with extra operations. In fact, $\mathcal{A}$ will be a boolean algebra with unary normal operators. In particular, the non-boolean extra operations are at most unary and are required to distribute over the boolean join. The basic boolean operations of $\mathcal{A}$ will be denoted by $+, \cdot, -, 0, 1$, standing for join, meet, complementation, least and greatest elements, respectively. All algebras considered are expansions of algebras in SC_α, the class of Pinter's substitution algebras of dimension α. An algebra in SC_α is of the form

$$\mathcal{A} = \langle A, +, \cdot, -, 0, 1, c_i, s_i^j\rangle_{i,j\in\alpha}$$

where

(1) $\langle A, +, \cdot, -, 0, 1\rangle$ is a boolean algebra (BA), hereinafter denoted by $Bl\mathcal{A}$.

(2) c_i and $s_i^j(i, j \in \alpha)$ are unary operations on A satisfying a finite schema of equations.

As we shall be dealing mostly with the concrete versions of algebras in SC_α, the so-called *representable* ones, we do not have to remember all those equations; we recall from [109] only the ones we need in the sequel.

Definition 0.1 . *Let $\mathcal{A} \in SC_\alpha$. Then the following equations hold for any $i, j, k \in \alpha$ and any $x, y \in A$:*

(E_1) $c_j0 = 0$, $x \leq c_ix$, $c_i(x \cdot c_iy) = c_ix \cdot c_iy$, *and* $c_ic_jx = c_jc_ix$.

In other words the c_is are complemented closure operators and c_i,c_j commute.

(E_2) $s_i^ix = x$.

(E_3) s_j^i *is a boolean endomorphism.*

(E_4) $s_j^ic_ix = c_ix$.

(E_5) $c_is_j^ix = s_j^ix$, *whenever* $i \neq j$.

(E_6) $s_j^ic_kx = c_ks_j^ix$, *whenever* $k \notin \{i, j\}$

(E_7) $c_is_i^jx = c_js_j^ix$.

(E_8) $s_j^is_i^kc_ix = s_j^kc_ix$.

We write $K_\alpha' s$ to abreviate algebras in K_α. In our treatment of $CA_\alpha' s$, we follow the standard terminolgy adopted in the monographs [50], [51], while in our treatment of $QA_\alpha' s$ and $QEA_\alpha' s$, we deviate from Halmos' terminology adopted in *op cit*, and follow rather the non-orthodox but more elegant axiomatization adopted in [118]. The latter formulation brings these algebras in a form (that of a system of varieties definable by a uniform schema of equations in the sense of [15]) that is easier to handle from the universal algebraic point of view.

In more detail, we think of $QA'_\alpha s$ as algebras of the form: $\langle \mathcal{A}, s_{ij}\rangle_{i,j\in\alpha}$, where $\mathcal{A}\in SC_\alpha$, and for all $i,j\in\alpha$, s_{ij} are unary operations on A, the universe of $\mathcal{A}$, which also happen to be boolean endomorphisms of $\mathcal{A}$. QEA_α's, on the other hand, are expansions of QA_α's by diagonal elements (d_{ij}'s ,$i,j\in\alpha$).

For an explicit finite schema of equations axiomatizing the variety K_α, we refer the reader to the appendix of [109] or [7]. For the purposes of the present chapter, however, it is enough to know that all algebras considered are expansions of SC_α's, that is, if $\mathcal{A}\in K_\alpha$, then its SC reduct obtained by deleting the s_{ij}'s and d_{ij}'s, if any, in symbols $Rd_{SC}\mathcal{A}$, is (term equivalent to) an SC_α. In particular, if $\mathcal{A}\in K_\alpha$, then $Rd_{SC}\mathcal{A}\models(E_1)-(E_8)$. Here by reduct we understand a *generalized* reduct. That is, the SC operations may not be basic ones in the algebras considered, but are term definable. For example in the standard formalism of CA's (cf. [50]) and that of QEA's (cf. [118], [109]), the substitution term s_i^j is not a basic operation but is term definable by $s_i^j=c_i(x\cdot d_{ij})$. However when dealing with QEA's and, for that matter, with CA's, we shall often treat the term definable operation s_i^j for all i,j as a basic operation.

We now recall a notion that prevails throughout this dissertation, namely that of neat reducts. The generalization of the notion of neat reducts from cylindric algebras to the other algebras considered in this chapter is absolutely straightforward. Roughly the neat α - reduct of an algebra in K_β is another algebra of lesser dimension $\alpha<\beta$ obtained by overlooking some the operations, namely the ones indexed by $\beta\smallsetminus\alpha$; and discarding some of its elements, more presicely:

Definition 0.2 .

(i) *Let $\alpha<\beta$ be ordinals. Let $\mathcal{B}\in K_\beta$. Then the neat α-reduct of $\mathcal{B}$, in symbols $\mathcal{N}r_\alpha\mathcal{B}$ is the K_α with universe $Nr_\alpha B=\{b\in B: c_ib=b \text{ for all } \alpha\le i<\beta\}$, and whose operations are those of the similarity type of K_α (evaluated in $\mathcal{B}$ and) restricted to $Nr_\alpha B$.*

(ii) *$\mathcal{A}$ is a subneat reduct of $\mathcal{B}$, or is neatly embeddable in $\mathcal{B}$, if $\mathcal{A}$ is a subalgebra of*

$\mathcal{N}r_\alpha\mathcal{B}$. When further A is a generating set of $\mathcal{B}$, then $\mathcal{A}$ is called a generating subneat reduct, or simply a generating subreduct of $\mathcal{B}$.

(iii) For a given class $L \subseteq K_\beta$, we let $Nr_\alpha L$ denote the class obtained by forming the neat α-reduct of algebras in L, that is

$$Nr_\alpha L = \{\mathcal{N}r_\alpha\mathcal{B} : \mathcal{B} \in L\}.$$

It is not hard to see that for any pair of ordinals $\alpha < \beta$ and $\mathcal{B} \in K_\beta$, $Nr_\alpha B$ is closed under the indicated operations so that the definition of $\mathcal{N}r_\alpha\mathcal{B}$ is sound. It is also easy to see that $\mathcal{N}r_\alpha\mathcal{B} \in \mathcal{K}_\alpha$. We should point out that neat reducts are subreducts (subalgebras of reducts) in the universal algebraic sense. Concepts related to neat reducts and neat embeddings of cylindric algebras, like for example the classical Neat Embedding Theorem of Henkin, generalize verbatim to SC, QA and QEA. In more detail, let SK denote the class of algebras that embed in members of K, here S is short for the operation of forming subalgebras. Borrowing terminology from [50] (cf. Remark 2.6.26), algebras in the class $SNr_\alpha K_{\alpha+\omega}$ are said to have the *neat embedding property*, or NEP for short. These are the algebras that embed in neat reducts of algebras in ω extra dimensions. $\mathcal{A} \in K_\alpha$ has the NEP iff $\mathcal{A}$ is neatly embeddable in a $K_{\alpha+n}$ for every $n \in \omega$. This equivalence is established by a typical compactness argument implemented via ultraproducts [50]. A central theorem in the representation theory of such algebras, the so-called Neat Embedding Theorem in [50] states that the class of algebras in K_α having the NEP is precisely the class of *representable* K_α's or RK_α, for short. In symbols (see [51] Theorem 3.2.10, [128] sec 5.2)

$$RK_\alpha = SNr_\alpha K_{\alpha+\omega} = \cap_{n\in\omega} SNr_\alpha K_{\alpha+n}.$$

According to the conventions of the monographs [50], [51], an RK_α is a K_α that is isomorphic to a subdirect product of (weak) set algebras of dimension α, and these, in turn, are algebras that can be represented as genuine fields of sets consisting of true α-ary relations. To unify notation, we write Ks (GKs) and Ks_α (GKs_α) to denote (generalized) K set algebras and (generalized) K set algebras of dimension α, respectively. $\mathcal{A} \in Ks_\alpha$ if its unit is a cartesian square of dimension α, i.e. a set of the form ${}^{\alpha}U$ for some set U. $\mathcal{A}$ is in GKs_{α}, if its unit is a disjoint union of cartesian squares each of dimension α, and the class of all isomorphic copies of GKs_{α} coincides with RK_{α} for $\alpha > 1$. By the same token, WKs_{α} stands for weak K set algebras of dimension α, that is $\mathcal{A} \in WKs_{\alpha}$ if A is a boolean field of sets with greatest element (unit) a weak cartesian space, which is a set of sequences agreeing cofinitely with a given sequence i.e. a typical unit of a weak set algebra is of the form

$$ {}^{\alpha}U^{(p)} = \{s \in {}^{\alpha}U : |\{i \in \alpha : s_i \neq p_i\}| < \omega\} $$

for some set U and some α-ary sequence $p \in {}^{\alpha}U$. U is called the base of the algebra.

In addition to this, A of course has to be closed under the extra non-boolean K_{α} operations. It is not hard to check that $WKs_{\alpha} \subseteq CA_{\alpha}$. For set algebras, i.e. algebras with universe $A \subseteq \wp(V)$ (the power set of V,) and greatest element V, for some $V \subseteq {}^{\alpha}U$, we write $\cup$, $\cap$, $\smallsetminus_V$, V, $\emptyset$ for the boolean operations, denoting the usual set-theoretic operations of intersection, union, etc. We write C_i^V, S_{ij}^V $(S_i^j)^V$ for the extra non-boolean operations, i.e. we use capital letters for the interpretation of the non-boolean operations in set algebras. So here we distinguish between the names of the operations (in abstract algebras) and the interpretation of these operations (in (the more concrete) set algebras). This is in harmony with the notation of the monograph [51], but admittedly is not very common among algebraists. When V is clear from context, we omit V from $\smallsetminus_V$, and write simply $\smallsetminus$. In a similar fashon, for the non-boolean operations, we may omit the superscript V and write for example C_i instead of C_i^V. For the sake of completeness, we recall from [109] the concrete interpretation of the non-boolean QEA_{α} operations on α-ary relations over a set $V \subseteq {}^{\alpha}U$ of α-ary sequences. Let $X \subseteq V$ and $i, j < \alpha$. Then

$$ C_i(X) = \{s \in V : \text{ there exists } t \in X, \text{ such that } s_j = t_j \text{ for all } j \neq i\}. $$

$$ S_i^j(X) = \{s \in V : s \circ [i|j] \in X\}. $$

Here, $[i|j]$, denotes the transformation on α, that sends i to j and is identity on $\alpha \smallsetminus \{i\}$. Next,

$$ S_{ij}(X) = \{s \in V : s \circ [i, j] \in X\} $$

where $[i, j]$ denotes the transposition on α that interhanges i and j. And finally,

$$ D_{ij} = \{s \in V : s_i = s_j\}. $$

For their geometric interpretation, C_i is called i-th *cylindrification* while D_{ij} is called a *diagonal element.* $S_i^j(S_{ij})$ is called the *substitution* operation corresponding to the transformation $[i|j]([i,j])$, or even simply a *substitution.* This stems rather from their metalogical interpretion, it being substitution of the variable x_i for x_j (interchanging the free occurences of x_i and x_j), cf. [109] or [128] chapter 2, for further elaboration on such connections.

As mentioned in the introduction, the representation problem for cylindric algebras is somewhat involved. The same applies to the other algebras considered herein with the sole exception of Halmos' polyadic algebras of infinite dimension. [13] In more detail, let $K \in \{SC, QA, CA, QEA\}$. Unlike boolean algebras, not every K_α, $\alpha > 2$ is representable. In fact, the following is known. Let $\alpha < \beta$. Then $SNr_\alpha K_\beta = K_\alpha = RK_\alpha$ if $\alpha \leq 1$. For $\alpha > 2$, the decreasing sequence $K_\alpha \supseteq SNr_\alpha K_{\alpha+1} \supseteq SNr_\alpha K_{\alpha+2} \cdots$ converging to RK_α is not eventually constant. In other words neat embeddability into algebras in finite extra dimensions is not sufficient for representability. This was proved and used by Monk for CAs in [95] to show that RCA_α is not finite schema axiomatizable when $\alpha > 2$, cf. Fact 0.5 in the introduction. The analogous result for SC, QA and QEA was proved by Johnson (the finite case) [70] and Sain and Thompson (the infinite case) [118]. Recently, Hirsch, Hodkinson and Maddux [61],[56] strengthened Monk's result for the finite dimensional CA case by showing that for $2 < \alpha < \omega$, and $k \in \omega$, $SNr_\alpha CA_{\alpha+k+1}$ is a proper subclass of $SNr_\alpha CA_{\alpha+k}$, i.e. that the above sequence (for finite $\alpha > 2$) and $K = CA$ is *strictly* decreasing. [14] Their proof generalizes verbatim to SCs QAs and QEA's [132]. When $\alpha = 2$, then all diagonal free algebras i.e. algebras in SC_2 and QA_2 are representable, see [51] Theorem 5.4.33. Though there are non-representable algebras in K_2 ($K \in \{CA, QEA\}$), RK_2 is finitely axiomatizable. Also $RK_2 = SNr_2K_\beta$ for all $\beta > 2$. The latter two results are due to Henkin. Finally we note that for arbitrary $\alpha < \beta$ the class $SNr_\alpha K_\beta$ is a variety that is finitely axiomatizable for $\alpha \leq 2$, but for $\omega > \alpha > 2$, $SNr_\alpha K_\beta$ is finitely axiomatizable if and only if $\beta \leq \alpha + 1$. The if part is due to Andréka [3], while the only if part is due to Hirsch and Hodkinson [56].

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[13] A classical result of Daigneualt and Monk [39] states that every PA_ω is representable. On the other hand, the representation problem for PEA_ω is much more complex as shown in [110].

[14] Hirsch and Hodkinson actually prove more. They prove that the gap between SNr_3CA_{m+1} and SNr_3CA_m for $m \geq 4$ cannot be finitely axiomatized [56].

We hasten to add that all algebras considered in this chapter are weak set algebras i.e. ones whose units are weak spaces, and hence representable.

Now we start formulating our results in this chapter.

Notation . *Let $\mathcal{A} \in QEA_\alpha$. Let $K \in \{SC, QA, CA, QEA\}$. Then $Rd_K\mathcal{A}$ is the K_α obtained from $\mathcal{A}$ by deleting the operations not in the similarity type of K_α.*

Our first main result in this chapter is:

Theorem 1 . *Let $\alpha > 1$. Let $K \in \{SC, CA, QA, QEA\}$. Then there exists $\mathcal{A} \in WQEAs_\alpha$, having unit a weak space of the form ${}^\alpha U^{(p)}$ where p is a constant sequence, such that $Rd_K\mathcal{A} \notin Nr_\alpha K_{\alpha+1}$.*

Unless otherwise specified, $K \in \{SC, CA, QEA, CA\}$. From Theorem 1, we readily obtain:

Corollary 1 . *Let $\alpha < \beta$. Then the class $Nr_\alpha K_\beta$ is not closed under forming subalgebras if $\alpha > 1$.*

Proof. Let $\alpha > 1$. Let $\mathcal{A}$ be as in Theorem 1. Then we have $\mathcal{A} \in SNr_\alpha QEA_\beta$ for all $1 < \alpha < \beta$. For $\beta \leq \alpha + \omega$, this follows from the Neat Embedding Theorem, while for bigger ordinals this follows by observing that $SNr_\alpha K_{\alpha+\omega} = SNr_\alpha K_{\alpha+\delta}$ for all infinite ordinals δ, cf. [50]. Thus by passing to reducts, we have $Rd_K\mathcal{A} \in SNr_\alpha K_\beta$ for all $1 < \alpha < \beta$, since the K_α reduct of $\mathcal{A}$ is obviously a weak set algebra, hence is representable. Also $Rd_K\mathcal{A}$ is not in $Nr_\alpha K_\beta$, for any such β, because for $\beta > \alpha$, as easily checked, we have $Nr_\alpha K_\beta \subseteq Nr_\alpha K_{\alpha+1}$ and $Rd_K\mathcal{A} \notin Nr_\alpha K_{\alpha+1}$. We conclude that for any pair of ordinal $1 < \alpha < \beta$ $Nr_\alpha K_\beta$ is not closed under the operation of forming subalgebras, and, in particular, it is a proper subclass of $SNr_\alpha K_\beta$. ■

In passing, we note that Corollary 1 admits a converse, i.e we can replace “If” by “if and only if” in Corollary 1. That $SNr_1 K_\beta \subseteq Nr_1 K_\beta$ is very close to Németi’s proof for the CA case (cf. [104], corollary 3), and therefore will be omitted. From this, together with (the K-analogue) of Theorem 3.1.128 in [51], it is in fact rather easy to show that $K_1 = Nr_1 K_\beta = Nr_1 Ks_\beta$, for all $\beta > 1$.

On the other hand, in [128] it is proved that for any pair of ordinals $\alpha < \beta$

the class $Nr_\alpha K_\beta$ is closed under the formation of homomorphic images and products for every $\alpha < \beta$. In symbols: $HNr_\alpha K_\beta = Nr_\alpha K_\beta$ and $PNr_\alpha K_\beta = Nr_\alpha K_\beta$. $HNr_\alpha CA_\beta = Nr_\alpha CA_\beta$ was settled by Németi in [104], while $PNr_\alpha CA_\beta = Nr_\alpha CA_\beta$ was proved in [50]. So with respect to these operations, $SC's$, $QA's$, and $QEA's$ also behave like $CA's$. In particular, $Nr_\alpha K_\beta$ is also closed under reduced products (and ultraproducts).

Closed under ultraproducts, an open question here is whether for $1 < \alpha < \beta$, the class $Nr_\alpha K_\beta$ is elementary, i.e. closed under ultraroots (or equivalently by the celebrated Keisler-Shelah ultrapower Theorem under elementary equivalence). We give a negative answer to this question for the lowest value of α, i.e. when $\alpha = 2$ herein. We shall prove:

Theorem 2 . *Let $K \in \{SC, QA, QEA, CA\}$. Let $\beta > 2$. Let L be any class such that $WKs_\beta \subseteq L \subseteq K_\beta$. Then Nr_2L is not closed under elementary subalgebras, hence is not elementary.*

In chapter four we generalize this result to higher dimensions, but only for Pinter's substitution algebras and cylindric algebras. A recent result of Hirsch and Hodkinson is that the class of neat reducts is psuedo-elementary, i.e a PC_Δ class in the sense of [67] (A pseudo-elementary class is a reduct of an elementary class.) We will return to such echoes in more detail in chapter 4.

The following Corollary of Theorem 1 confirms a conjecture of Tarski in the affirmative- cf. the introduction herein, and the introduction of [51] item (v)- concerning the existence (in the infinite dimensional case) of subreducts, that are not neat reducts; more presicely:

Corollary 2 . *Let $\omega \leq \alpha < \beta$. Then there exists $\mathcal{A} \in WKs_\alpha$ and $\mathcal{B} \in WKs_\beta$, such that $\mathcal{A}$ is a generating subreduct of $\mathcal{B}$ which is not isomorphic to $\mathcal{N}r_\alpha\mathcal{B}$.*

Proof. Let $\alpha \geq \omega$. Let $\mathcal{A}$ be the algebra constructed in Theorem 1. Then by [51] Theorem 3.1.122 we have that $Rd_K\mathcal{A} \in SNr_\alpha WKs_\beta$ for all $\beta > \alpha$. Now fix any such $\beta > \alpha$. Let $\mathcal{B} \in WKs_\beta$ be such that $\mathcal{A} \subseteq \mathcal{N}r_\alpha\mathcal{B}$. Let $\mathcal{B}'$ be the subalgebra of $\mathcal{B}$

generated by A. Then a moment's reflection reveals that $\mathcal{A}$ and $\mathcal{B}'$ are as required. ■

Though formulated only for CA, the proof of Theorem 3.1.122 in [51] easily generalizes to SC, QA and QEA. We refer the reader to [128] for a (systematic) generalization of this result (and other similar results) formulated for CA, to SC, QA and QEA.

The CA part of Corollary 2 contrasts Theorem 2.6.67 in [50]. The latter states that if $\mathcal{A}$ is a *dimension complemented* CA_α, in symbols, $\mathcal{A} \in Dc_\alpha$, α an infinite ordinal, and $\mathcal{A}$ is a generating subreduct $\mathcal{B} \in CA_\beta$, then $A = Nr_\alpha B$. We recall from [50] that $\mathcal{A} \in Dc_\alpha$, α an infinite ordinal, if for every $x \in A$ there exists $i \in \alpha$ such that $c_i x = x$. In [50] at the top of p.426, it was stated without proof that the class Dc_α cannot be replaced by arbitrary CA_α's, even when restricted to representable ones. In [51], this statement appears again as item (v) in the introduction, this time among results of Tarski, the proofs of which could not be reconstructed. Corollary 2 thus confirms what seems to have been a conjecture of Tarski's, together with its SC, QA and QEA analogues.

Our construction used in Theorem 1 to extend Nemeti's result to the other algebras considered herein is a non-trivial modification of Németi's construction, and it has the further benefit that it may (and will) be used to prove a (negative) result concerning amalgamation concerning the class of the so-called *Diagonal cylindric algebras.* We start by recalling from [113] the definition of such algebras:

Definition . *Let α be an infinite ordinal. $\mathcal{A}$ is a diagonal cylindric algebra of dimension α, a Di_α for short, if for every finite $\Gamma \subseteq \alpha$ and every non-zero $x \in A$, there exists distinct $k, l \in \alpha \smallsetminus \Gamma$ such that $x \cdot d_{kl} \neq 0$.*

Note that the definition of Diagonal algebras makes sense only in the infinite dimensional case. Diagonal algebras appear in [50] in Theorem 2.6.50 item (ii), where it is proved that such algebras have the neat embedding property, i.e. every diagonal algebra neatly embeds into an algebra in ω extra dimensions, thus is representable. Such algebras were introduced by Monk. However, the notation Di_α comes from Pigozzi [113]. Note that a weak set algebra of infinite dimension is a diagonal algebra if and only if its unit contains a constant sequence. [15] A different representability proof for Di_α's is given by Pigozzi, cf. [113], Theorem 2.2.19. We will adapt such a proof to show that the class Di_α does not the strong amalgamation property, answering a question of Pigozzi, witness the proof of Theorem 3 below.

Before addressing our results on amalgmation, we formulate a(n apparently) weaker result concerning NS. [16] To formulate the next Corollary we need to recall that a class of algebras $L \subseteq RK_\alpha$ has the NS, if for all $\mathcal{A} \in \mathcal{L}$, there exists $\mathcal{B} \in \mathcal{K}_{\alpha+\omega}$ such that

$$Sg^{\mathcal{A}}X = Nr_\alpha Sg^{\mathcal{B}}X.$$

Here $Sg^{\mathcal{C}}X$ for a given algebra $\mathcal{C}$ and $X \subseteq C$, denotes the subalgebra of C generated by X. NS is short for "**N**eat reducts commute with forming **S**ubalgebras."

Corollary 3 .

(i) Di_ω does not have the NS.

(ii) Let $\alpha > 1$. Then (any class L containing) WKs_α does not have NS.

Proof. Let $1 < \alpha < \beta$ be ordinals. Let $\mathcal{A}$ be the QEA_α constructed in Theorem 1. let $X = A$. Let $Rd_K\mathcal{A}$ denote the K_α which is the K-reduct of $\mathcal{A}$. Then

$$Sg^{Rd_K\mathcal{A}}A = A \neq Nr_\alpha B$$

for any $\mathcal{B} \in K_\beta$, since $\mathcal{A} \notin Nr_\alpha K_\beta$. This proves (ii). (i) follows from (ii) by noting that the CA reduct of the ω-dimensional $\mathcal{A}$ is a diagonal algebra, since its unit contains a constant sequence. ∎

Before formulating our results concerning strong amalgamation, we need to recall the related notion of "surjectiveness of epimorphisms" which we abbreviate by ES. Here epimorphisms is understood in the categorial sense. The equivalence of the

[15] Actually it is not hard to show that any $\mathcal{A} \in Di_\alpha$, α infinite, can be represented as a subdirect product of weak set algebras each having a unit of the form ${}^\alpha U^{(p)}$ where p agrees cofinitely with a *constant* sequence, i.e. p is eventually constant. This is equivalent to that $\mathcal{A}$ can be represented as a *generalized* weak set algebra whose unit has the form $\cup_{i\in I}^{\alpha} U_i^{(p_i)}$ where $p_i \in {}^\alpha U_i$ agrees cofinitely with a constant sequence and ${}^\alpha U_i^{(p_i)} \cap {}^\alpha U_j^{p_j} == \emptyset$ for distinct i and j.

[16] Our results in this chapter show that under reasonable conditions on $L \subseteq SNr_\omega K_{\omega+\omega}$, L has SAP iff L has AP and NS.

algebraic propery ES in a class of algebras viewed as a concrete category to that of the Beth Definability property in the corresponding logic was discovered by Németi. This equivalence is worked out in a rather general context in [51].

Definition .

Let L be any class of algebras.

(i) *Let $\mathcal{A}$ and $\mathcal{B} \in L$. An L epimorphism is a map $f : \mathcal{B} \to \mathcal{A}$, that is right cancellative. This means that for any $\mathcal{C} \in \mathcal{L}$ and any pair of homomorphisms $h, g : \mathcal{A} \to \mathcal{C}$ if $h \circ f = g \circ f$ then $h = g$.*

We note that surjective maps are epimorphisms, but the converse is not generally true. This prompts:

(ii) *ES holds in L if all L epimorphisms are surjective.*

In the next definition, we slightly relax the usual definition of amalgamation to permit that the (strong) amalgam can be found in a possibly bigger class, more precisely:

(iii) *Let V be a class of algebras (usually assumed to be a variety) and $L \subseteq V$. L is said to have the amalgamation property, or AP for short , with respect to V, if for all $\mathcal{A}, \mathcal{B}$ and $\mathcal{C} \in L$, and all monomorphisms f and g of $\mathcal{A}$ into $\mathcal{B}$, $\mathcal{C}$, respectively, there exists $\mathcal{D} \in V$, a monomorphism m from $\mathcal{B}$ into $\mathcal{D}$ and a monomorphism n from $\mathcal{C}$ into $\mathcal{D}$ such that $m \circ f = n \circ g$.*

$\mathcal{D}$ is called an amalgam *of $\mathcal{B}$ and $\mathcal{C}$ over $\mathcal{A}$ via m and n, or simply an amalgam; and we say that $\mathcal{B}$ and $\mathcal{C}$ can be amalgamated over $\mathcal{A}$ inside V.*

If in addition $m(B) \cap n(C) = m \circ f(A)$, then L is said to have the strong amalgamation property, or SAP for short, with respect to V.

In this case, we say that the amalgam $\mathcal{D}$ is a strong amalgam *of $\mathcal{B}$ and $\mathcal{C}$ over $\mathcal{A}$ via m and n, or simply a strong amalgam, and we say that $\mathcal{B}$ and $\mathcal{C}$ can be* strongly amalgamated *over $\mathcal{A}$ inside K.*

(iv) *L has $(S)AP$ if L has SAP with respect to L.*

It is straighforward to check that if ES fails in a class K, then K does not have SAP. Now, our next main result in this chapter is:

Theorem 3 . *Let Di_ω denote the class of diagonal cylindric algebras of dimension ω. Let $Rd_{CA}\mathcal{A}$ denote the (ω-dimensional) cylindric reduct of $\mathcal{A}$ provided by Theorem 1. Then there exists a proper superalgebra $\mathcal{A}_0$ of $Rd_{CA}\mathcal{A}$ such that the inclusion $Rd_{CA}\mathcal{A} \subseteq \mathcal{A}_0$ is a Di_ω epimorphism. In particular, ES fails in Di_ω and Di_ω does not have SAP.*

In fact, we shall prove something slightly stronger, but not as neat to formulate. Besides, Pigozzi's question adressed only the cylindric case, which made us explicitly formulate only the CA part of our result at this (earlier) stage. [17] During the writing of our dessertation we learnt that our Theorem 3 was overshadowed by the following (unpublished) *stronger* result of Judit Madárarz: ω-dimensional diagonal and semisimple cylindric algebras do not have SAP with respect to RCA_ω. In this connection, we do not know whether our algebras are semisimple.

Now let PA_α and PEA_α stand for the classes of polyadic and polyadic equality algebras of dimension α, respectively. (cf. [51] Def. 5.4.1). Then in contrast to the classes of algebras addressed in Theorem 1, we shall prove:

Theorem 4 . *Let $\omega \leq \alpha < \beta$. Then $Nr_\alpha PA_\beta$ and $Nr_\alpha PEA_\beta$ are varieties. In particular, these classes are closed under forming subalgebras.*

Establishing the tie between the property NS and that of the strong amalgamation property, Judit Madarasz proved the following:

Theorem 5 (Madarasz) . *Let α be an infinite ordinal. Let $L \subseteq K_\alpha$ be such that $WKs_\alpha \subseteq L \subseteq RK_\alpha$. Then L does not have SAP with respect to RK_α.*

The finite version of Theorem 5 can be found in [5]. Theorem 5 answers a question of Andréka and Németi in [17].

[17] What we actually show is the following: We first generalize the definition of diagonal cylindric algebras, to cover the cases when our algebras have no diagonals in their signature, like for example the case SC_ω. Call such a class $ReSC_\omega$. Here Re is short for replacement. This in turn points to algebras in which substitutions corresponding to replacements should behave in a certain way. This terminology will be further illustrated below. Then we prove that A_0, actually a $WQEAs_\omega$, satisfies that $Rd_{SC}\mathcal{A} \subseteq Rd_{SC}\mathcal{A}_0$ is an $ReSC_\omega$ epimorphism.

We end this chapter with several positive results concerning the strong amalgamation property, and the so called strong embedding property. Here we show that, conversely, when NS holds, then strong amalgamation and embedding properties follow. For that we need to recall a few distinguished classes of algebras. Recall that $\mathcal{A}$ is in GKs_α if it its unit is a disjoint union of cartesian squares. Each such cartesian square is called a subbase.

Definition .

(1) Let $\mathcal{A} \in K_\alpha$. Then $\mathcal{A}$ is of positive characteristic k, $0 < k < \alpha \cap \omega$, if $\mathcal{A}$ can be represented as a Gs_α each of whose subbases have cardinality $< k$. (Another syntactical definition of this class is given in [50] 2.6.43; the equivalence is due to Henkin, cf. [51] Thm 3.2.52.)

(2) $\mathcal{A}$ is monadic-generated (cf. [50] 2.2.24, if it is generated by a set X such that for every $x \in X$, $|\Delta x| \leq 1$. Monadic K_α are also representable. (cf. [50] Thm 2.6.56.)

Two important examples of monadic-generated algebras in the CA case are CA_1 and the *minimal* cylindric algebras. An algebra $\mathcal{A}$ is minimal, if it has no proper subalgebras. In particular, a minimal algebra coincides with the algebra generated by the (interpretation of the) constant symbols in its signature. For CA_α, these are the boolean constants $0, 1$ and the diagonals $(d_{ij}, i < j < \alpha)$. The class of minimal cylindric algebras of dimension α is denoted by Mn_α in [50] Def. 2.1.15.

In contrast to the negative results in Theorems 3 and 5, we shall show:

Theorem 6 . *Monadic generated algebras and algebras of positive characteristic have SAP.*

Theorem 6, answers two questions of Pigozzi posed in [113].

Before formulating our next positive results, we need to recall the notion of the embedding property. Roughly, the (strong) embedding property, is the (strong) amalgamation property when the common subalgebra is restricted to be the minimal subalgebra, more precisely:

Definition .

(1) *Let L be a class of algebras. Then L is said to have the embedding property, or EP for short, if any $\mathcal{A}, \mathcal{B}$ in L having isomorphic minimal subagebras can be amalgamated inside L.*

(2) *L is said to have the strong embedding property, or SEP for short, if any $\mathcal{A}$, $\mathcal{B} \in L$, having isomorphic minimal subalgebras can be strongly amalgamated inside L.*

Similarly to AP and SAP, one can define the EP and SEP for a class L with respect to a bigger class K.

While RCA_ω and Di_ω do not have SAP with respect to RCA_ω, as proved by Jusit Madarasz, concerning the strong embedding property, we shall prove the following interesting contrast:

Theorem 7 . *Let α be an infinite ordinal. Then RCA_α and Di_α have SEP with respect to RCA_α.*

Theorem 7 answers two other questions raised by Pigozzi in [113].

Finally we prove:

Theorem 8 . *Let α be any ordinal. Then QA_α,SC_α, RQA_α and RSC_α have SEP.*

In our proof of Theorem 8, we shall use a different technique (essentially due to Nemeti) inspired by investigations in modal logic. This technique will be compared to that initiated by Pigozzi concerning the NS in chapter four. Theorem 8 contrasts a remarkable result of Madarasz and Simon, namely that CA_ω does not have even the embedding property [147], chapter 6. Surprisingly, whether QEA_ω and QA_ω have SEP, is to the best of our knowledge still open. In connection to the connection of NS to SAP, we end this chapter by the following conjecture, but first a definition:

Definition . *Let K be a class of algebras.*

(i) *Let $\mathcal{A} \in \mathcal{K}$. $\mathcal{A}$ is in the (strong) amalgam base of K if any "amalgmation fork" $B \to A \to C$ in K can be (strongly) amalgamated inside K. I.e if B, C are in K then their joint embedding algebra is also in K.*

(ii) *$L \subseteq K$ lies in the (strong) amalgam base of K if every $\mathcal{A}$ in L is in the (strong) amalgam base of K.*

Now we formulate our conjecture that says that the strong amalgam base of RK_α coincides with the class of algebras that has AP and NS.

Conjecture . *Let α be any ordinal. Assume that $L \subseteq RK_\alpha$ has AP. Then if L lies in the strong amalgamation base of RCA_α then L has NS.*

We note that the converse of conjecture 1 is due to Pigozzi. If true, then Conjecture 1 answers a question formulated by Monk in [12], problem 45 p. 739. The problem of investigating the (strong) amalgamation base for important varieties like RCA_α *failing* to have $(S)AP$ (so here α should be > 1) seems to originate with Bjarni Johnson, [114]. In the next chapter we deal with a variant of RCA_ω for which all algebras have the NS. We use this to prove that this varient has SAP. [18]

[18] Bjarni Johnson though was concerned with relation algebras. A result of Sain [114] that is consistent with conjecture 1, is that a representable relation algebra is in the strong amalgamation base of RRA if it neatly embeds in a unique way in an Lf_ω. It would be nice to give an intrinsic charaterization of the (strong) amalgmation base of representable algebras; i.e one that does not refer to algebras outside the algebra considered, for conjecture 1, depends on how the given algebra (neatly) embeds into another algebra in ω extra dimensions.

1.Proofs

This section is devided to four subsections. In subsection 1 we prove Theorem 1. In subsection 2 we prove Theorem 2. In subsection 3 we prove Theorems 3 and 4. The final subsection, section 5, is devoted to proving our positive results on the strong amalgamation (and embedding) properties, i.e. in subsection 5, we prove Theorems 6,7 and 8.

1.1. Proof of Theorem 1.

In this subsection we prove Theorem 1 and formulate and prove sharper results as well. The work in this subsection is based on the article [145]. Unless otherwise specified, $K \in \{SC, CA, QA, QEA\}$ and α is an ordinal > 1. Our proof of Theorem 1 relies on statements 1 and 2 below. In statement 1 we formulate a property that algebras in $Nr_\alpha K_{\alpha+1}$ satisfy. In statement 2, we construct an algebra $\mathcal{A} \in WQEAs_\alpha$ for which this property fails for its K_α reduct.

To formulate statement 1, we need a few preparations. We start with a definition.

Definition 1.1.1 .

(i) Let $\mathcal{A} \in K_\beta$, with $1 < \beta$. $x \in A$ is said to be 2-rectangular [19] *if $x = c_0x \cdot c_1x$.*

(ii) Let $j \in \beta$. $x \in A$ is said to be j-closed if $c_jx = x$.

Note that if $\mathcal{A} \in Nr_\alpha K_\beta$ with $\alpha < \beta$ and $\alpha \leq j < \beta$, then all elements of $\mathcal{A}$ are j-closed. In particular, all such elements are α-closed.

Notation .

$$\tau(x, y) = c_1(c_0x \cdot s_1^0 c_1y) \cdot c_1x \cdot c_0y.$$

$$\tau_\alpha(x, y) = c_\alpha(s_\alpha^1 c_\alpha x \cdot s_\alpha^0 c_\alpha y).$$

[19] *2-rectangular elements, coincide with rectangular elements in CA_2's in the sense of [50], Def. 1.10.6. In a set algebra of dimension 2, an element is (2-) rectangular iff it is a " generalized rectangle" in the sense that whenever $\langle u_0, u_1\rangle$, $\langle v_0, v_1\rangle \in x$, we also have $\langle u_0, v_1\rangle$, $\langle v_0, u_0\rangle \in x$. (cf. [50] Def. 1.10.12).*

Remark 1.1.2

Note that $\tau(x,y)$ is a term in the language of SC_2. When $\alpha = 2$, the semicolon notation is more commonly used to abbreviate the term $\tau_\alpha(x,y)$. In this case, $x;y$ abstracts composition of binary relations. cf. [51] Def. 5.3.7.

The following Lemma, of computational nature, will be used in the proof of statement 1. It roughly states that τ "approximates" the binary operation of composition (of binary relations). Composition needs one extra dimension to be captured , cf. [51]. Lemma 1.1.3 says that the composition of certain special elements, namely, the 2-rectangular ones, can be defined (by τ) *without* the need to resort to extra dimensions. In our derivation, we use the axiomatization given in Definition 0.1.

Lemma 1.1.3 . *Let $1 < \alpha < \beta$ and let $\mathcal{A} \in K_\beta$. Then:*

(i) $\tau_\alpha(x,y) \leq \tau(c_\alpha x, c_\alpha y)$.

(ii) If x, y are α-closed and 2-rectangular, then $\tau_\alpha(x,y) = \tau(x,y)$.

Proof.

Computing we get

$$\tau_\alpha(x,y) = c_\alpha(s^1_\alpha c_\alpha x \cdot s^0_\alpha c_\alpha y)$$

$$\leq c_\alpha(s^1_\alpha(c_0 c_\alpha x \cdot c_1 c_\alpha x) \cdot s^0_\alpha(c_0 c_\alpha y \cdot c_1 c_\alpha y))$$

(Here, equality holds if x, y are α-closed and 2-rectangular)

$$= c_\alpha(s^1_\alpha c_0 c_\alpha x \cdot s^1_\alpha c_1 c_\alpha x \cdot s^0_\alpha c_0 c_\alpha y \cdot s^0_\alpha c_1 c_\alpha y).$$

$$= c_\alpha(s^1_\alpha c_0 c_\alpha x \cdot c_1 c_\alpha x \cdot c_0 c_\alpha y \cdot s^0_\alpha c_1 c_\alpha y)$$

$$= c_\alpha(s^1_\alpha c_0 c_\alpha x \cdot s^0_\alpha c_1 c_\alpha y) \cdot c_1 c_\alpha x \cdot c_0 c_\alpha y$$

$$= c_\alpha(s^1_\alpha c_0 c_\alpha x \cdot s^1_\alpha s^0_1 c_1 c_\alpha y) \cdot c_1 c_\alpha x \cdot c_0 c_\alpha y$$

$$= c_\alpha s^1_\alpha(c_0 c_\alpha x \cdot s^0_1 c_1 c_\alpha y) \cdot c_1 c_\alpha x \cdot c_0 c_\alpha y$$

$$= c_1 s^\alpha_1(c_0 c_\alpha x \cdot s^0_1 c_1 c_\alpha y) \cdot c_1 c_\alpha x \cdot c_0 c_\alpha y$$

$$= c_1 s^\alpha_1 c_\alpha(c_0 c_\alpha x \cdot s^0_1 c_1 c_\alpha y) \cdot c_1 c_\alpha x \cdot c_0 c_\alpha y.$$

$$= c_1c_\alpha(c_0c_\alpha x \cdot s_1^0 c_1c_\alpha y) \cdot c_1c_\alpha x \cdot c_0c_\alpha y$$

$$= c_1(c_0c_\alpha x \cdot s_1^0 c_1c_\alpha y) \cdot c_1c_\alpha x \cdot c_0c_\alpha y$$

$$= \tau(c_\alpha x, c_\alpha y).$$

■

Recall that an atom of an algebra $\mathcal{A}$ with a boolean reduct is a minimal non-zero element. The boolean reduct of $\mathcal{A}$, or simply $\mathcal{A}$, is atomic if every non-zero element in $\mathcal{A}$ contains an atom.

Notation . *Let $\mathcal{A} \in K_\alpha$ and $X \subseteq A$. Then $At\mathcal{A}$ denotes the set of all atoms of $Bl\mathcal{A}$, and $supX$ denotes the supremum of X, if it exists.*

Now we are ready to formulate and prove statement 1. Roughly statement 1 states that in neat reducts certain supremums must exist. Later we will show how the property expressed in statement 1 can be described by a first order formula (in the language of SC_α). This will enable us to sharpen Theorem 1, which we do in Corollary 1.1.5 below.

Statement 1 . *Let $1 < \alpha < \beta$ and let $\mathcal{C} \in Nr_\alpha K_\beta$. Suppose that $\mathcal{C}$ is atomic such that all its atoms are 2-rectangular. Let $X \subseteq At\mathcal{C}$ be such that $y = supX$ exists in $\mathcal{C}$. Suppose further that $c_i x \cdot y \in At\mathcal{C}$, for $i \in \{0,1\}$ and $x \in At\mathcal{C}$. Then $sup\{\tau(a,b) : a, b \in X\}$ also exists in $\mathcal{C}$.*

Proof. Let $X'' = \{\tau(a,b) :\ a, b \in X\}$ and $w = \tau_\alpha^{\mathcal{B}}(y,y)$, where $\mathcal{B}$ is any algebra in K_β such that $\mathcal{C} = \mathcal{N}r_\alpha\mathcal{B}$. Then $w \in C$. Let $a, b \in X$, and $i \in \{0,1\}$. Since s_α^i is a boolean endomorphism for $i < \alpha$ [20], we have

$$s_\alpha^0 a \leq s_\alpha^0 y \text{ and } s_\alpha^1 b \leq s_\alpha^1 y.$$

But a, b are 2-rectangular and together with y are α-closed, we obtain, by Lemma 1.1.3 that

$$\tau(a,b) = c_\alpha(s_\alpha^1 a \cdot s_\alpha^0 b) \leq c_\alpha(s_\alpha^1 y \cdot s_\alpha^0 y) = w.$$

[20] This term is evaluated in the big algebra $\mathcal{B}$, so strictly speaking we should write $s_\alpha^{i\,\mathcal{B}}$; but from now on we omit superscripts. It will be clear from context which terms are evaluated in the fixed (but arbitrary) $\mathcal{B}$.

This shows that $w \in C$, is an upper bound of X''. We claim that w is in fact the least such upper bound, i.e. that $w = supX''$. Towards this end, suppose that t is an upper bound of X'' in $\mathcal{C}$. We have to show that $w \leq t$. Assume, to the contrary, that this is not the case, i.e. that $w - t \neq 0$. Since $\mathcal{C}$ is atomic, there is an atom z say, below $w - t$ i.e.

$$(1)\quad z \leq w \text{ and } z \leq -t.$$

From $0 \neq z \leq w = c_\alpha(s_\alpha^1 y \cdot s_\alpha^0 y)$, and z is α-closed, it readily follows that

$$z \cdot (s_\alpha^1 y \cdot s_\alpha^0 y) \neq 0$$

thus

$$c_0 z \cdot c_1 z \cdot (s_\alpha^1 y \cdot s_\alpha^0 y) \neq 0.$$

From $s_\alpha^i c_i z = c_i z$, and the fact that the s_α^i's are boolean endomorphisms for each $i < \alpha$, we get

$$(2)\quad s_\alpha^1(c_1 z \cdot y) \cdot s_\alpha^0(c_0 z \cdot y) \neq 0.$$

Now let $a = c_1 z \cdot y$. Then by hypothesis, $a \in At\mathcal{C}$. Also $a \leq y$. But $y = supX$ and $a \cdot y = a \neq 0$, therefore $a \cdot x \neq 0$, for some atom $x \in X$, and so $a = x \in X$. Analogously, if $b = c_0 z \cdot y$, then $b \in X$. Being atoms, a and b are 2-rectangular, and therefore we have from Lemma 1.1.3 that

$$\tau(a, b) = c_\alpha(s_\alpha^1(c_1 z \cdot y) \cdot s_\alpha^0(c_0 z \cdot y)).$$

and from (2) we get that $\tau(a, b) > 0$. On the other hand, by invoking definitions and the fact that z is 2-rectangular a straightforward computation yields:

$$\tau(a, b) = \tau(c_1 z \cdot y, c_0 z \cdot y)$$

$$= c_1(c_0(c_1 z \cdot y) \cdot s_1^0 c_1(c_0 z \cdot y)) \cdot c_1(c_1 z \cdot y) \cdot c_0(c_0 z \cdot y)).$$

$$= c_1(c_0(c_1 z \cdot y) \cdot s_1^0 c_1(c_0 z \cdot y)) \cdot (c_1 z \cap c_1 y) \cdot (c_0 z \cdot c_0 y).$$

$$= c_1(c_0(c_1 z \cdot y) \cdot s_1^0 c_1(c_0 z \cdot y)) \cdot (c_0 z \cdot c_1 z) \cdot (c_0 y \cdot c_1 y).$$

$$= c_1(c_0(c_1 z \cdot y) \cdot s_1^0 c_1(c_0 z \cdot y)) \cdot z \cdot (c_0 y \cdot c_1 y).$$

That is, $0 < \tau(a, b) \leq z$. But z is an atom, and so $z = \tau(a, b)$. Since t is an upper bound for X", we get $z \leq t$. From (1) we have $z \leq -t$. So $z \leq t - t = 0$, i.e. $z = 0$

which is impossible, since z is an atom. This means that $w \leq t$, and so $w = supX$" as required. ■

We now turn to defining our weak set algebra providing the algebra specified in our Theorem 1. For that we need:

Notation . *Let X be a set. Then $\wp(X)$ denotes the set of all subsets (the powerset of) X. Let $\mathcal{A}$ be an algebra and $X \subseteq A$. Then $Sg^{\mathcal{A}}X$, or simply SgX, when $\mathcal{A}$ is clear from context, denotes the subalgebra of $\mathcal{A}$ generated by X.*

From now on, unless otherwise specified, α is a fixed ordinal with $\alpha > 1$, and Q denotes the set of rational numbers. We will construct a weak set algebra with base Q. For this purpose, we let $\mathbf{0}$ be the following constant α-ary sequence:

$$\mathbf{0} = \langle 0 : i < \alpha \rangle.$$

$\mathcal{C}$ denotes the *full* $WQEAs_\alpha$ with unit the weak space

$${}^{\alpha}Q^{(\mathbf{0})} = \{s \in {}^{\alpha}Q : |\{i \in \alpha : s_i \neq 0\}| < \omega\},$$

[21] i.e.

$$\mathcal{C} = \langle \wp({}^{\alpha}Q^{(\mathbf{0})}), \cup, \cap, \smallsetminus, \emptyset, {}^{\alpha}Q^{(\mathbf{0})}, C_i, D_{ij}, S_i^j, S_{ij} \rangle_{i,j \in \alpha}.$$

Let y denote the following α-ary relation over ${}^{\alpha}Q^{(\mathbf{0})}$:

$$y = \{s \in {}^{\alpha}Q^{(\mathbf{0})} : s_0 + 1 = \sum_{i>0} s_i\}.$$

Note that the sum on the right hand side is a finite one, since only finitely many of the s_i's are non-zero.

For each $s \in y$, we let y_s be the singleton containing s, i.e. $y_s = \{s\}$ and we define $\mathcal{A} \in WQEAs_\alpha$ as follows:

$$\mathcal{A} = Sg^{\mathcal{C}}\{y, y_s : s \in y\}.$$

Recall that for $\mathcal{A} \in QEA_\alpha$, we let $Rd_K\mathcal{A}$ denote the K_α reduct of $\mathcal{A}$. $Rd_K\mathcal{A}$ is of course in K_α.

[21] Note that ${}^{\alpha}Q^{(\mathbf{0})}$ is a vector space over the field Q; it is what algebraists call a direct summand. In passing, we note that our construction can be carried out over any weak set algebra with unit a direct summand of a field of characteristic 0. In particular, the countability of Q plays no essential role here.

Now we are ready to formulate (and prove) our statement 2.

Statement 2 . *$Rd_{SC}\mathcal{A} \notin Nr_\alpha SC_{\alpha+1}$. In particular, $Rd_K\mathcal{A} \notin Nr_\alpha K_{\alpha+1}$.*

Proof of statement 2.

For the sake of brevity, we let $\mathcal{A}$ denote $Rd_{SC}\mathcal{A}$. The proof goes by contradiction. We first show that had $\mathcal{A}$ been a neat reduct, then $\mathcal{A}$ would satisfy all the other hypotheses in statement 1, from which we infer that the supremum of the set in the conclusion, which we denote by w, must exist in A. Then we show, by the so-called method of *eliminating cylindrifications*, that $w \notin A$. [22] So assume, towards a contradiction, that $\mathcal{A} \in Nr_\alpha SC_{\alpha+1}$. We start by showing that $\mathcal{A}$ meets all the other hypotheses in statement 1, with $X = \{y_s : s \in y\}$. Of course every element of X, being a singleton, is an atom. Next we show that $\mathcal{A}$ is atomic. Towards this end, let $s \in {}^\alpha Q^{(\mathbf{0})}$ be an arbitrary sequence. Then

$$\langle s_0, s_0 + 1 - \sum_{i>1} s_i, s_i\rangle_{i>1}$$

and

$$\langle \sum_{0<i<\alpha} s_i - 1, s_i\rangle_{i\geq 1}$$

are elements in y. Since

$$\{s\} = C_1\{\langle s_0, s_0 + 1 - \sum_{i>1} s_i, s_i\rangle_{i>1}\} \cap C_0\{\langle \sum_{0\neq i<\alpha} s_i - 1, s_i\rangle_{i\geq 1}\},$$

it follows that $\{s\} \in A$. We have shown that A contains all singletons. Thus $\mathcal{A}$ is atomic. Moreover, as easily checked, every singleton is 2-rectangular. Also $y = \cup X$, and so $y = supX$ exists in $\mathcal{A}$. Finally, let $x = \{\langle s_i\rangle\}$ be an atom of $\mathcal{A}$. Then

$$C_0 x \cap y = \{\langle \sum_{i>0} s_i - 1, s_i\rangle_{i>0}\}$$

and

$$C_1 x \cap y = \{\langle s_0, s_0 + 1 - \sum_{i>1} s_i, s_i\rangle_{i>1}\};$$

[22] The method of eliminating cylindrifications is an algebraization of the metalogical method of "eliminating quantifiers", and indeed it can be and has been used to prove decidability results for certain classes of cylindric algebras, cf. [50] and [107].

which are singletons, hence atoms. For the sake of brevity let

$$Y = \{\tau(a, b) : a, b \in X\}.$$

Having shown that $\mathcal{A}$ satisfies the hypothesis of statement 1, it follows that $k = supY$ exists in A. We will determine k. We start by evaluating $\tau(a, b)$ for $a, b \in X$. Towards this end, let $a, b \in X$. Assume that $a = y_r$ and $b = y_t$ with

$$r = \langle r_i : i < \alpha\rangle \text{ and } t = \langle t_i : i < \alpha\rangle$$

are elements in y.

$$\text{If } r_1 \neq t_0 \text{ or } r_i \neq t_i \text{ for some } i > 1, \text{ then } \tau(y_r, y_s) = 0.$$

Else, $r_1 = t_0$ and $r_i = t_i$ for all $i > 1$. In this case,

$$\tau(y_r, y_t) = \{s\} \text{ where } s_0 = r_0, \text{ and } s_i = t_i \text{ for all } i > 0.$$

Moreover, an easy computation shows that

$$s_0 + 2 = s_1 + 2\sum_{i>1} s_i.$$

Now let w be the set of all such elements, i.e.

$$w = \{s \in {}^{\alpha}Q^{(\mathbf{0})} : s_0 + 2 = s_1 + 2\sum_{i>1} s_i\}.$$

Let $s \in w$. Then putting

$$r = \langle s_0, s_0 + 1 - \sum_{i>0} s_i, s_i\rangle_{i>1}$$

and

$$t = \langle s_0 + 1 - \sum_{i>0} s_i, s_1, s_i\rangle_{i>1},$$

we get $r, t \in y$ and $\tau(y_r, y_s) = \{s\}$. We have shown that

$$Y \smallsetminus \{0\} = \{\{s\} : s \in w\},$$

and so

$$w = \cup Y.$$

Now clearly $w \leq k$. We will show that $k \leq w$, i.e. that $w \in A$. But this easily follows from the fact that A contains all singletons. In more detail, let $z \in {}^{\alpha}Q^{(\mathbf{0})} \smallsetminus w$. Then $\{z\}$, and hence ${}^{\alpha}Q^{(\mathbf{0})} \smallsetminus \{z\} \in A$. Since $w \subseteq^{\alpha} Q^{(\mathbf{0})} \smallsetminus \{z\}$, we have $\tau(a, b) \subseteq {}^{\alpha}Q^{(\mathbf{0})} \smallsetminus \{z\}$, for every $a, b \in X$, i.e. ${}^{\alpha}Q^{(\mathbf{0})} \smallsetminus \{z\}$ is an upper bound of of Y. But this means that $k \subseteq {}^{\alpha}Q^{(\mathbf{0})} \smallsetminus \{z\}$, i.e. $z \notin k$. We have proved that $k \leq w$.

Next we proceed to show that $w \notin A$. In light of the above, this will prove statement 2. To this end, we modify the treatment carried out in [104] to handle the more general quasipolyadic equality case. The plan (in this order) is: We define the universe of a boolean algebra G^{**}, containing the set of generators of $\mathcal{A}$. We show that $w \notin G^{**}$. Finally we show that G^{**} is also closed under the quasipolyadic set operations, from which it follows that $w \notin A$. To execute our plan we need:

Definitions.

$$Pl = \{\{s \in {}^{\alpha}Q^{(\mathbf{0})} : t + \sum (r_i s_i) = 0\} : \{t, r_i : i < \alpha\} \subseteq Q\}.$$

Pl consists of all rational hyperplanes of dimension α. For $i < \alpha$, let

$$q_i = \{s \in {}^{\alpha}Q^{(\mathbf{0})} : s_i + 1 = \sum_{j \neq i} s_j\}.$$

Let $i \in \alpha$ and $k, l \in \alpha$ be distinct. Then if $i \in \{k, l\}$, $i = k$ say, then

$$S_{kl} q_i = q_l.$$

Else, we have $i \notin \{k, l\}$, so that i, k, l are pairwise distinct, in which case we have

$$S_{kl} q_i = q_i.$$

Now let

$$Pl^S = \{q_i : i < \alpha\}.$$

Then by the above Pl^S is closed under the operations S_{kl} for all $k, l \in \alpha$. Notice that $y = q_0 \in Pl^S$. Thus Pl^S consists of all hyperplanes obtained by interchanging the k, l co-ordinates of y, for all $k, l \in \alpha$, i.e. implementing the operation S_{kl}. Here the superscript S is short for substitutions (corresponding to replacements,) reflecting the fact that Pl^S consists of *substituted* versions of y. Let

$$Pl^{<} = \{p \in Pl : C_i p = p, \text{ for some } i < \alpha\}.$$

If $p \in Pl^<$ and $C_i p = p$, then p is a hyperplane parallel to the i-th axis.

Note that for $p \in Pl$, $p = \{s \in {}^\alpha Q^{(\mathbf{0})} : t + \sum_i r_i s_i = 0\}$ say, then $C_i p = p$ (i.e. p is parallel to the i-th axis) iff $r_i = 0$. Note too, that if $p \in Pl^<$ and $k, l \in \alpha$ then $S_{kl} p \in Pl^<$.

In the following we summarize the above, and state some other facts easy to check as well.

$$(1)\ \ y = q_0 \in Pl^S.$$

$$(2)\ \ Pl^S \cup \{w, D_{ij} : i, j \in \alpha\} \subseteq Pl.$$

$$(3)\ \ \text{If } q \in Pl^S(Pl^<) \text{ and } i, j \in \alpha, \text{ then } S_{ij} q \in Pl^S(Pl^<).$$

$$(4)\ \ Pl^S \cap Pl^< = \emptyset, w \notin Pl^< \text{ and } {}^\alpha Q^{\mathbf{0}} \in Pl^<.$$

$$(5)\ \{D_{ij} : i \neq j, i, j \in \alpha\} \subseteq Pl^< \text{ iff } \alpha \geq 3.$$

For further use, we shall need:

Notation. .

(i) Let X be a set. Then $Z \subseteq_\omega X$ abbreviates "Z is a finite subset X". $\wp_\omega X$ denotes all finite subsets of X, i.e.

$$\wp_\omega X = \{Z \in \wp(X) : Z \subseteq_\omega X\}.$$

(ii) Let $\mathcal{A} \in K_\alpha$, $a \in A$ and $\Delta \subseteq_\omega \alpha$. Then $c_{(\Delta)} a = c_{i_0} \cdots c_{i_{n-1}} a$, where $\Delta = \{i_0, \cdots i_{n-1}\}$. Because cylindrifications commute, the definition of $c_{(\Delta)}$ does not depend on any linear order defined on Δ. When $n = 0$, i.e. Δ is empty, then $c_{(\Delta)} a = c_\emptyset a := a$. As usual, for concrete algebras we use capital letters, i.e. we write $C_{(\Delta)}$ for the interpretation of $c_{(\Delta)}$ in set algebras. [23]

(iii) To simplify notation, from now on we may write $-a$ instead of ${}^\alpha Q^{\mathbf{0}} \smallsetminus a$. It will be clear from context whether $-$ refers to the operation of substraction in Q or the operation of complementation in $\mathcal{A}$. We also right $1^{\mathcal{A}}$, or just 1, instead of the more cumbersome ${}^\alpha Q^{\mathbf{0}}$.

[23] *The unary operation $c_{(\Delta)}$ is referred to as generalized cylindrification in [50]. Generalized cylindrifications, when Δ is finite, are term definable in cylindric algebras. In Halmos' formulation of polyadic algebras (which is the one recalled in [51]), these are basic operations. We note that in the case of infinite dimensional polyadic algebras, Δ is allowed to be infinite.*

Now we turn to defining G^{**}. For that we need:

Some more definitions.

$G = \{q, -q,\ p, -p,\ C_{(\Delta)}\{\mathbf{0}\}, -C_{(\Delta)}\{\mathbf{0}\},\ q \in Pl^S,\ p \in Pl^< \cup \{D_{01}\},\ \Delta \subseteq_\omega \alpha,\ 0 \in \Delta\}.$

Forming finite intersections of elements in G, we let

$$G^* = \{\cap_{i \in n} g_i : n \in \omega, g_i \in G\}.$$

and forming finite unions of elements in G^*, we let

$$G^{**} = \{\cup_{i \in n} g_i : n \in \omega, g_i \in G^*\}.$$

It is easy to see that $\{y, y_s : s \in y\} \subseteq G^{**}$, and G^{**} is a boolean field of sets. We start by proving that $w \notin G^{**}$. To this end, we set:

$$L = \{p \in Pl^< : C_0 p \neq p\} \text{ and } P(0) = L \cup \{D_{01}\}.$$

Notice that $L = P(0)$ iff $\alpha > 2$. L stands for the set of "lines" not parallel to the 0th axis. Being in $Pl^<$, any such line is parallel to some other axis. When $\alpha = 2$, i.e. in the rational plane $Q \times Q$, these are precisely the lines that are parallel to the "y axis". Next we define

$$G_1 = \{g \in G^* : g \subseteq q, \text{ for some } q \in Pl^S\}$$

and

$$G_2 = \{g \in G^* : g \not\subseteq q, \text{ for all } q \in Pl^S \text{ and } g \subseteq p, \text{ for some } p \in P(0)\}.$$

Note that $Pl^S \subseteq G_1$ and that $P(0) \subseteq G_2$. Note too, that $G_1 \cap G_2 = \emptyset$. Now let

$$G_3 = \{p_1 \cap p_2 \ldots \cap p_k :\ k \in \omega,\ \{p_1, p_2, \ldots, p_k\} \subseteq G \smallsetminus (Pl^S \cup P(0))\}.$$

It is easy to see that $G^* = G_1 \cup G_2 \cup G_3$. To prove that $w \notin G^{**}$ we need:

Claim . *If $g \in G_3$ and $0 \neq g$, then $g \not\subset w$.*

Proof of Claim

Assume that $g = p_1 \cap p_2 \ldots \cap p_k$ say, with $p_i \in G$ and $p_i \notin (Pl^S \cup P(0))$ for $1 \leq i \leq k$, and let $z \in g$. Let [] be the function from Pl into Q defined as follows:

$[p] = 1/r_0(-t - \sum r_i z_i)$ if $p = -\{s \in^\alpha Q^{(\mathbf{0})} : t + \sum r_i s_i = 0\}, r_0 \neq 0$ and else $[p] = 0$.

Let

$$r \in Q \smallsetminus ((\cup_{1 \leq i \leq k}[p_i]) \cup [-w])$$

be arbitrary, and let

$$z_r^0 = z \smallsetminus \{(0, z_0)\} \cup \{(0, r)\}.$$

Then

$$z_r^0 \in g \smallsetminus w \text{ i.e. } g \not\subseteq w.$$

(Here we are using that when $C_{(\Delta)}\{\mathbf{0}\} \in G$, then $0 \in \Delta$.)

We now proceed to show that $w \notin G^{**}$. Assume to the contrary, that

$$w = \cup\{g_i^1 : i < n_1\} \cup \cup\{g_i^2 : i < n_2\} \cup \cup\{g_i^3 : i < n_3\}$$

where

$$\{g_i^j : i < n_j\} \subseteq G_j \text{ for } j \in \{1, 2, 3\}.$$

By the above, we have $g_i^3 = 0$, for all $i < n_3$. From the definition of G_1 and G_2 , it follows that

$$w \subseteq \cup\{q_j : j < n_1\} \cup \cup\{p_j : j < n_2\}$$

where

$$\{q_j : j < n_1\} \subseteq Pl^S$$

and

$$\{p_j : j < n_2\} \subseteq P(0).$$

Since $w \subseteq -D_{01}$ for $\alpha = 2$, and $P(0) = L$ for $\alpha > 2$ we can, and will, assume that

$$\{p_j : j < n_2\} \subseteq L.$$

But it can be seen by implementing easy linear algebraic arguments that, for every $n \in \omega$, for every $m \in \omega \cap (\alpha + 1)$, and for every system

$$t_0 + \sum(r_{0i}x_i) = 0$$

$$\dots$$

$$\dots$$

$$t_n + \sum (r_{ni}x_i) = 0,$$

of equations, such that for all $j \leq n$, there exists $i < \alpha$, such that

$$r_{ji} = 0 \text{ and } r_{j0} \neq 0,$$

the equation

$$x_0 + 2 = x_1 + 2\sum x_j$$

has a solution s in the weak space ${}^{\alpha}Q^{(\mathbf{0})}$, such that $s \notin q_i$ for every $i < m$, and such that s is not a solution of

$$t_j + \sum (r_{ji}x_i) = 0,$$

for every $j \leq n$. Geometrically (and intuitively) a finite union of (rational) hyperplanes cannot cover a hyperplane, unless it is one of them. By taking $n = n_1$ and $m = n_2$ we get the desired conclusion. We have proved that $w \notin G^{**}$. To show that $w \notin A$, we will show that G^{**} is closed under the polyadic set operations. It only remains to show that G^{**} is closed under cylindrifications and substitutions, since by definition it is a boolean field of sets and contains the diagonal elements.

(1) G^{**} is closed under cylindrifications.

It is enough to show that (since the C_i's are additive), that for $j \in \alpha$ and $g \in G^*$ arbitrary, we have $C_j g \in G^{**}$. For this purpose, put for every $p \in Pl$

$$p(j|0) = C_j\{s \in p : s_j = 0\} \text{ and } (-p)(j|0) = -p(j|0).$$

Then it is not hard to see that

$$p(j|0) = \{s \in {}^{\alpha}Q^{(\mathbf{0})} : t + \sum_{i \neq j} (r_i s_i) = 0\},$$

if

$$p = \{s \in {}^{\alpha}Q^{\mathbf{0})} : t + \sum_{i < \alpha} (r_i s_i) = 0\},$$

and so

$$p(j|0) \in Pl^{<} \text{ for every } p \in Pl.$$

Let j and g be as indicated above. We can assume that

$$g = q_1 \cdots \cap q_l \cap p_1 \cap \cdots \cap p_n \cap -Q_1 \cdots \cap -Q_L$$

$$\cap - P_1 \cdots \cap -P_m \cap y \cap -C_{(\Delta_1)}\{\mathbf{0}\} \cdots \cap -C_{(\Delta_N)}\{\mathbf{0}\},$$

where

$$l, L, n, m, N \in \omega \smallsetminus \{0\}, q_i, Q_i \in Pl^S, p_i, P_i \in Pl^{<} \cup \{D_{01}\},$$

$$C_j p_i \neq p_i,\ C_j P_i \neq P_i,\ C_j q_i \neq q_i,\ C_j Q_i \neq Q_i,$$

$$y \in \{C_{(\Delta)}\{\mathbf{0}\}, 1 : \Delta \in \wp_\omega \alpha,\ 0 \in \Delta,\ j \notin \Delta\},$$

and

$$\{\Delta_1, \cdots, \Delta_n\} \subseteq \{x \in \wp_\omega \alpha :\ j \notin x, 0 \in x\}.$$

We distinguish between 2 cases:

Case 1.

$$y = C_{(\Delta)}\{\mathbf{0}\} \text{ and } j \notin \Delta.$$

Then a lenghly but routine computation gives

$$C_j(q_1 \cdots \cap q_l \cap -Q_1 \cdots \cap -Q_l \cap p_1 \cdots \cap p_n \cap -P_1 \cdots \cap -P_m$$

$$\cap C_{(\Delta)}\{\mathbf{0}\} \cap -C_{(\Delta_1)}\{\mathbf{0}\} \cdots \cap -C_{(\Delta_N)}\{\mathbf{0}\})$$

$$= q_1(j|0) \cap q_l(j|0) \cap -Q_1(j|0) \cdots \cap -Q_L(j|0) \cap p_1(j|0) \cdots$$

$$\cap p_n(j|0) \cap -P_1(j|0) \cdots \cap -P_m(j|0)$$

$$\cap C_j C_{(\Delta)}\{\mathbf{0}\} \cap -C_j C_{(\Delta_1)}\{\mathbf{0}\} \cdots \cap -C_j C_{(\Delta_N)}\{\mathbf{0}\}.$$

Case 2.

$$y = 1$$

In this case put

$$f(q_k) = \cap_{i \leq l}\{C_j(q_k \cap q_i)\} \cap \{\cap_{i \leq L} C_j(q_k \cap -Q_i)\} \cap \{\cap_{i \leq n} C_j(q_k \cap p_i)\}$$

$$\cap\{\cap_{i\leq m}C_j(q_k\cap -P_i)\}\cap\{\cap_{i\leq N}C_j(q_k\cap -C_{\Delta_i}\{\mathbf{0}\})\}$$

and similarly

$$f(p_k)=\cap_{i\leq l}\{C_j(p_k\cap q_i)\}\cap\{\cap_{i\leq L}C_j(p_k\cap -Q_i)\}\cap\{\cap_{i\leq n}C_j(p_k\cap p_i)\}$$

$$\cap\{\cap_{i\leq m}C_j(p_k\cap -P_i)\}\cap\{\cap_{i\leq N}C_j(p_k\cap -C_{\Delta_i}\{\mathbf{0}\})\}$$

Then it is not hard to check that

$$C_jg=\cap_{k\leq l}f(q_k)\cap\cap_{k\leq n}f(p_k).$$

Finally by noting that for every $p,q\in Pl$, there are p',q',p'' and $q''\in Pl^{<}$ such that

$$C_j(p\cap q)=p'\cap q',$$

$$C_j(p\smallsetminus q)=p''\smallsetminus q''$$

and if $j\in\Delta p\smallsetminus\Gamma$, then

$$C_j(p-C_{(\Gamma)}\{\mathbf{0}\})=^{\alpha}Q^{\mathbf{0}}\smallsetminus p(j|0)\cup(p(j|0)\smallsetminus C_jC_{(\Gamma)}\{\mathbf{0}\}),$$

it follows that $C_jg\in G^{**}$.

(2) G^{**} is closed under substitutions.

It is enough to consider the S_{ij}'s, since for all $i,j\in\alpha$ $S_i^j(x)=C_i(x\cap D_{ij})$, i.e. the S_i^j's are term definable, G^{**} is closed under the C_i's, and $D_{ij}\in G^{**}$. For this purpose, let

$$H=Pl^S\cup Pl^{<}\cup\{C_{(\Delta)}\{\mathbf{0}\}:\Delta\subseteq_\omega\alpha\}.$$

Here we are not requiring that $0\in\Delta$, so that H is bigger than G. However, since G^{**} is closed under cylindrifications we still have $G\subseteq H\subseteq G^{**}$. Now let $i,j\in\alpha$, and $x\in H$. Then it not hard to check that $S_{ij}x\in G^{**}$. By noting that the S_{ij}'s are boolean endomorphisms we get that G^{**}, generated as a boolean algebra by H (since $G\subseteq H$ is a set of generators), is closed under S_{ij}. By this, the proof is complete.

■

Remark

We note that our construction of the algebra $Rd_K\mathcal{A} \in SNr_\alpha K_\beta$ is uniform for each fixed dimension $\alpha > 1$. In particular, it does not vary by varying the ordinal β. Our Theorem 1 complements Monk's classical negative result stated as Fact 0.5 in the introduction herein. While Monk shows that for $\alpha > 2$, there is an algebra $\mathcal{A}_{\alpha,n} \in Nr_\alpha CA_{\alpha+n} \smallsetminus RCA_\alpha$ for all finite n, here we show, conversely, that for all $\alpha > 2$, there is an algebra $A_\alpha \in RCA_\alpha \smallsetminus Nr_\alpha CA_{\alpha+n}$, with $n \geq \omega$ allowed. (Note that for infinite n, we have $Nr_\alpha CA_{\alpha+n} \subseteq RCA_\alpha$.) [24]

Unlike Monk's algebras, the α-dimensional weak set algebra we construct is representable and does not depend on n. It depends only on α.

Before proving Theorem 2, we show in the next Corollary that the property formulated in statement 1 can be expressed by a first order formula. For this purpose, we let ElK denote the smallest elementary (i.e. first order axiomatizable) class containing K. Thus $ElK = UpUrK$, where Up is the operation of forming ultraproducts and Ur is that of forming ultraroots of members of K, respectively, cf. [51]. Recall that $RQEA_\alpha$ stands for the class of representable quasipolyadic equality algebras of dimension α.

Corollary 1.1.5 . *Let $\alpha > 1$. Then $Rd_{SC}RQEA_\alpha \not\subseteq ElNr_\alpha SC_{\alpha+1}$. In particular, for all $1 < \alpha < \beta$, we have $Rd_{SC}SNr_\alpha QEA_\beta \not\subseteq ElNr_\alpha SC_\beta$.*

Proof. Let $At(x)$ be the first order formula asserting that x is an atom. Let

$$Rc(x) := c_0x \cdot c_1x = x,$$

$$\phi := \forall x(x \neq 0 \to \exists y(At(y) \wedge y \leq x)) \wedge \forall x(At(x) \to Rc(x)),$$

$$A(x, y) := At(x) \wedge x \leq y,$$

and $\psi(y_0, y_1)$ be the following first order formula

$$\forall z(\forall x(A(x, y_0) \to x \leq z) \to y_0 \leq z) \wedge \forall x(At(x) \to At(c_0x \cap y_0) \wedge At(c_1x \cap y_0))$$

$$\to [\forall x_1 \forall x_2(A(x_1, y_0) \wedge A(x_2, y_0) \to \tau(x_1, x_2) \leq y_1)$$

[24] More succintly, the two results combined show that for $\alpha > 2$ and $n \in \omega$, neither of the classes $Nr_\alpha CA_{\alpha+n}$ and RCA_α contains the other.

$$\wedge \forall z(\forall x_1 \forall x_2(A(x_1, y_0) \wedge A(x_2, y_0) \to \tau(x_1, x_2) \leq z) \to y_1 \leq z)].$$

Then statement 1 can be rephrased more succintly as follows:

$$Nr_\alpha K_\beta \models \phi \to \forall y_0 \exists y_1 \psi(y_0, y_1).$$

Let $\mathcal{A}$ be the $WQEAs_\alpha$ constructed in statement 2. We have shown that $Rd_{SC}\mathcal{A}$ falsifies the above formula that holds in $Nr_\alpha K_\beta$, since as shown in statement 2, we have

$$Rd_{SC}\mathcal{A} \models \phi \text{ and not } Rd_{SC}\mathcal{A} \models \forall y_0 \exists y_1 \psi(y_0, y_1).$$

By this the proof is complete. ■

Then from Corollary 1.1.5 we have:

Corollary 1.1.6 . *Let $1 < \alpha < \beta$. Let $K \in \{SC, QA, QEA, CA\}$. Then $ElNr_\alpha K_\beta \subset SNr_\alpha K_\beta$. In particular, $ElNr_\alpha K_\beta$ cannot be axiomatized by universal formulas.*

Proof. $ElNr_\alpha K_\beta$ cannot be axiomatized by universal formulas since it is not closed under forming subalgebras. ■

Now that we have proved that the operation of taking neat reducts *does not commute* with that of forming subalgebras, a natural question here is which other (natural) operations like those of taking homomorphic images, products, ultraproducts, commute with the operation of taking neat reducts. It turns out that they all do as formulated in our next Theorem. Let HL and UpL stand for the classes of all homomorphic images, all ultraproducts of members in a class L of algebras, respectively. The following Theorem is proved in [128] and [140]:

Theorem 1.1.7 . *Let $K \in \{SC, CA, QA, QEA\}$. Let β be any ordinal. Let $L \subseteq K_\beta$. Then the following hold.*

(i) $HNr_\alpha SL \subseteq Nr_\alpha HSL$

(ii) $UpNr_\alpha SUpL = Nr_\alpha SUpL$.

(iii) In particular, for $L \in \{K_\beta, RK_\beta\}$, L is a variety, hence we have $HNr_\alpha L = Nr_\alpha L$, and $UpNr_\alpha L = Nr_\alpha L$.

(iv) $IGKs_\alpha \neq UfNr_\alpha GKs_\beta = ElNr_\alpha GKs_\beta$ and $ElCKs_\alpha \neq UfUpNr_\alpha CKs_\beta$.

Whether the inclusion $Nr_\alpha K_\beta \subseteq UfNr_\alpha K_\beta$ is proper, seems to be much harder. We now prove that this inclusion is strict for the lowest value of α for which $Nr_\alpha K_\beta$ with $\alpha < \beta$ is not closed under forming subalgebras, i.e. for $\alpha = 2$. In other words, we prove Theorem 2. For this purpose we use set algebras based on a fairly simple model, namely the real line expanded with constants. The extension of this result to higher dimensions, turns out rather involved, and will be done for SC and CA, in chapters 4 and 5. For this we shall need more sophisticated models (that has quantifier elimination) based on a Fraisse construction.

1.2 Proof of Theorem 2

We now proceed to prove Theorem 2. The work in this subsection is based on the article [129]. We construct an algebra $\mathcal{A} \in Ks_2 \cap Nr_2 WKs_\beta$ for all $\beta > 2$ and $\mathcal{B} \subseteq \mathcal{A}$, such that $\mathcal{B}$ is an elementary subalgebra of $\mathcal{A}$ and $\mathcal{B}$ is not (isomorphic to an algebra) in $Nr_2 K_\beta$ for all $\beta > 2$.

Construction of the algebras.

R denotes the set of real numbers. Let $U = R \times 3$. Note that U is simply 3 disjoint copies of R. $\mathcal{C}$ denotes the *full* polyadic set algebra with unit ${}^2U$, that is

$$\mathcal{C} = \langle \wp({}^2U), \cap, \cup, \smallsetminus, \emptyset, {}^2U, C_i, D_{ij}, P_{ij}, S_i^j \rangle_{i,j<2}.$$

Let $u = \langle u_0, u_1 \rangle \in {}^2 3$. For $r \in R$, we define $p(u, r)$ to be the following binary relation on U;

$$p(u, r) = \{ \langle \langle a_0, u_0 \rangle, \langle a_1, u_1 \rangle \rangle \in {}^2U : a_0 = a_1 + r \}$$

and we let

$$P(u) = \{ p(u, r) : r \in R \}.$$

Note that $P(u)$ is an uncountable set of binary relations on U. Let N be a fixed countably infinite subset of R[25] That is $N \subseteq R$ and $|N| = \omega$. We set

$$P_\omega(u) = \{ p(u, r) : r \in N \}.$$

[25]We point out that countability of N is not essential. In fact any infinite subset N of R with $|N| < |R|$ will do just as well.

The subscript ω indicates that $P_\omega(u)$ has countably many binary relations. Finally, we let

$$S = \{u \in {}^23 : u_0 \leq u_1\}.$$

Recall that for an algebra $\mathcal{A}$ and $X \subseteq A$, $Sg^{\mathcal{A}}X$ or simply SgX, denotes the subalgebra of $\mathcal{A}$ generated by X.

Now we define $\mathcal{A} \in QEAs_2$ and $\mathcal{B} \subseteq \mathcal{A}$ as follows:

$$\mathcal{A} = Sg^{\mathcal{C}}(\cup\{P(u) : u \in S\})$$

and

$$\mathcal{B} = Sg^{\mathcal{A}}(\cup\{P(u) : u \in S \smallsetminus \{\langle 0, 2\rangle\}\} \cup P_\omega(\langle 0, 2\rangle)).$$

Plan of the proof.

We will show that:

(1) $\mathcal{A} \in Nr_2WQEAs_\beta$, for all $\beta > 2$.

(2) $\mathcal{B}$ is an elementary subalgebra of $\mathcal{A}$

(3) $Rd_{SC}\mathcal{B} \notin Nr_2SC_\beta$, for all $\beta > 2$.

From (1), (2) and (3) Theorem 2 will immediately follow by passing to reducts. In more detail, let $K \in \{SC, QA, CA, QEA\}$ and let $\beta > 2$. Given $\mathcal{D} \in QEA_\alpha$, α an ordinal, let $Rd_K\mathcal{D} \in K_\alpha$ denote the K_α reduct of $\mathcal{D}$. Then it follows from (1) that $Rd_K\mathcal{A} \in Nr_2WKs_\beta$, from (3) that $Rd_K\mathcal{B} \notin Nr_2K_\beta$ and from (2) that $Rd_K\mathcal{B}$ is an elementary subalgebra of $Rd_K\mathcal{A}$ (in the first order language of K_2.) We devide the proof into three parts as indicated in the plan.

Part 1. $\mathcal{A}$ is a neat reduct.

We show that $\mathcal{A} \in Nr_2WQEAs_\beta$ for all $\beta > 2$. This will be done in two steps. First we show that $\mathcal{A} \in \cap_{2\leq k<\omega}Nr_2QEAs_k$, then we show that $\mathcal{A} \in Nr_2WQEAs_\beta$ for infinite β using a limiting construction. Throughout this subsection, unless otherwise specified, $n = \{0, \ldots, n-1\}$ denotes a finite ordinal > 1.

Notation . *$\mathbf{R} = \langle R, +, -, r\rangle_{r\in R}$ denotes the group of real numbers expanded with constants. L denotes the first order language of $\mathbf{R}$. For ϕ an L formula, we let $var(\phi)$ denote the set of variables occuring in ϕ, and we let $fr(\phi)$ denote the set of variables occuring free in ϕ.*

Let $u = \langle u_0, \dots, u_{n-1}\rangle \in {}^n3$. For ϕ an L formula with $fr(\phi) \subseteq \{x_0, \cdots, x_{n-1}\}$, we let $E(u, \phi)$ denote the following $n - ary$ relation on U:

$$E(u, \phi) = \{\langle\langle a_0, u_0\rangle, \cdots, \langle a_{n-1}, u_{n-1}\rangle\rangle \in {}^nU : \mathbf{R} \models \phi[a_0, \cdots, a_{n-1}]\}.$$

We let

$$P(n) = \{E(u, x_i = x_j + r) : u \in {}^n3,\ r \in R \text{ and } i, j \in n, i < j\}$$

and

$$\mathcal{A}(n) = Sg^{\mathcal{C}(n)}(P(n))$$

where $\mathcal{C}(n)$ is the full $QEAs_n$ with unit ${}^nU$.

It is easy to see that $\mathcal{A} = \mathcal{A}(2)$.

We will show that for $1 < l < m < \omega$, $\mathcal{A}(l) \cong \mathcal{N}r_l\mathcal{A}(m)$, so that in fact we are proving more than what we need. [26]

For that we need some more definitions specifying certain formulas in the first order language of $\mathbf{R}$. We set:

$$F(n) = \{x_i = x_j + r, x_i \neq x_j + r : i, j \in n \text{ and } r \in R\}.$$

Forming finite conjunctions of formulas in $F(n)$, we let

$$F(n)^* = \{\wedge J : J \subseteq_\omega F(n)\}.$$

Forming finite disjunctions of formulas in $F(n)^*$, we let

$$F(n)^{**} = \{\vee J : J \subseteq_\omega F(n)^*\}.$$

Now set

$$G(n) = \{E(u, \phi) : u \in {}^n3 \text{ and } \phi \in F(n)^{**}\};$$

[26] We strongly conjecture that for any $2 \leq n < \omega$, there is an elementary subalgebra of the neat reduct $A(n)$ that is not a neat reduct. We only succeeded to find such an algebra for the lowest value of n, i.e when $n = 2$.

and finally

$$G(n)^* = \{\cup J : J \subseteq_\omega G(n)\}.$$

Claim 1.2.1 . *Let $n > 1$. Then $A(n) = G(n)^*$.*

Proof. First we prove that $A(n) \subseteq G(n)^*$. Since $P(n) \subseteq G(n)^*$ and $\mathcal{A}(n)$ is generated by $P(n)$, it suffices to show that $G(n)^*$ is the universe of a QEA_n, i.e. is closed under the polyadic set operations.

(1) $G(n)^*$ is closed under the boolean operations.

Indeed let $u, v \in {}^n3$ and $\phi_1, \phi_2 \in F(n)^{**}$. Then

$$E(u, \phi_1) \cap E(v, \phi_2) = E(u, \phi_1 \wedge \phi_2) \text{ if } u = v$$

and is zero otherwise, while

$${}^nU \setminus E(u, \phi_1) = \cup\{E(v, \top) : v \in {}^n3 \setminus \{u\}\} \cup E(u, \neg\phi_1).$$

From the definition of $G(n)^*$, and by noting that for ϕ_1 and ϕ_2 in $F(n)^{**}$, there exists η and θ in $F(n)^{**}$ such that $\mathbf{R} \models \phi_1 \wedge \phi_2 \longleftrightarrow \eta$ and $\mathbf{R} \models \neg\phi_1 \longleftrightarrow \theta$, we get the desired conclusion.

(2) $G(n)^*$ contains the diagonal elements.

Let $i < j < n$. Then

$$D_{ij} = \cup\{E(u, x_i = x_j) : u \in {}^n3,\ u_i = u_j\}.$$

and so by the definition of $F(n)$ we get $D_{ij} \in G(n)^*$.

(3) $G(n)^*$ is closed under cylindrifications.

First a piece of useful notation. For $u, v \in {}^n3$ and $i < n$ we write $u \equiv_i v$ iff u and v agree except at i, i.e. $u(j) = v(j)$ for all $j \neq i$.

Now let $g \in G(n)^*$ and $i < n$.

Since the C_i's ($\exists$) distribute(s) over joins ($\vee$) we can and will assume that $g = E(u, \phi)$ with $u \in {}^n3$ and $\phi \in F(n)^{**}$. Computing we get

$$C_i g = C_i E(u, \phi) = \cup\{E(v, \exists x_i \phi) : v \equiv_i u\}.$$

Now we make use of the following consequence of the fact that $\mathbf{R}$ is in a sense "saturated" , which we abbreviate by $(*)$:

$(*)$ Given $\phi \in F(n)^{**}$ and $i < n$, there exists $\psi \in F(n)^{**}$ such that $x_i \notin var(\psi)$

and

$$\mathbf{R} \models \exists x_i \phi \longleftrightarrow \psi.$$

Note that (*) is a form of elimination of quantifiers. Now applying (*) we get that $C_i g \in G(n)^*$.

(4) $G(n)^*$ is closed under substitutions.

Since $S^i_j(x) = C_i(x \cap D_{ij})$ for $i \neq j$ and $S^i_i(x) = x$, so that the S^i_j's are term definable, it suffices to check only those substitutions of the form S_{ij}. So let $\phi \in F(n)^{**}$, $u \in {}^n3$ and $i, j < n$ be distinct. Abusing notation slightly, we let $S_{ij}\phi$, stand for the formula obtained from ϕ by interchanging the free occurences of x_i and x_j, and we let $S_{ij}u = u \circ [i, j]$. Since S_{ij} is a boolean endomorphism, it suffices to show that for all $u \in {}^n3$ and all $\phi \in F(n)$ we have $S_{ij}E(u, \phi) \in G(n)^{**}$. But this follows from the facts that

$$S_{ij}E(u, \phi) = E(S_{ij}u, S_{ij}\phi);$$

and that for $\phi \in F(n)$ we have $S_{ij}\phi \in F(n)$.

Next we show that $G(n)^* \subseteq A(n)$. It clearly suffices to show that for all $u \in {}^n3$ and $\phi \in F(n)$, we have $E(u, \phi) \in A(n)$.

Let u and ϕ be as indicated. Suppose that ϕ is $x_i = x_j + r$, for some $i, j \in n$ and some $r \in R$.

If $i < j$, then $E(u, \phi) \in P(n)$. If $i > j$ then $E(u, \phi) = E(u, x_j = x_i + (-r))$ which also belongs to $P(n)$. Thus we are left with the case that $i = j$. We have two subcases.

Case a. $r \neq 0$ and so $E(u, \phi) = 0 \in A(n)$.

Case b. $r = 0$, then $E(u, \phi) = E(u, \top)$. In order to show that the latter belongs to $A(n)$; we let $k \in n - \{i\}$. This is possible because $n > 1$. Then a straightforward computation gives

$$C_iE(u, x_i = x_k) \cap C_kE(u, x_i = x_k) = E(u, \top),$$

thus the latter is in $A(n)$. Finally by noting that $E(u, \neg\phi) = E(u, \top) \setminus E(u, \phi)$ we get (using the above reasoning) that $E(u, x_i \neq x_j + r)$ is also in $A(n)$, for all $u \in {}^n3$ and $i, j \in n$.

■

Having shown that $G(n)^* = A(n)$, and in particular that $G(n)^*$ is closed under the QEA_n operations we let $\mathcal{G}(n)^*$ denote the $QEAs_n$ with universe $G(n)^*$.

Before proceeding we fix some needed notation:

Notation .

(i) *For a function f, $Dom(f)$ denotes the domain of f and $Rng(f)$ denotes the range of f.*

(ii) *For a function f and $X \subseteq Dom(f)$, we write $f \upharpoonright X$ for the restriction of f to X.*

(iii) *For algebras $\mathcal{A}$, $\mathcal{B}$, we write $Ism(\mathcal{A}, \mathcal{B})$ for the set of all isomorphisms from $\mathcal{A}$ to $\mathcal{B}$.*

For further use we shall also need to highlight:

Definition 1.2.2 . *For $m \geq 1$, let $\mathcal{C}(m)$ denote the full $QEAs_m$ with unit ${}^mU$, i.e. the universe of $\mathcal{C}(m)$ is $\wp({}^mU)$. Now let $1 \leq l < k$. Let $i(l, k)$ be the following (neat embedding) function:*

$$i(l, k) : C(l) \to C(k)$$

$$a \to \{t \in {}^kU : t \upharpoonright l \in a\}.$$

Claim 1.2.3 . *Let $1 < l < k < \omega$. Then $i(l, k)\mathcal{G}(l)^* = \mathcal{N}r_l\mathcal{G}(k)^*$. In particular, $\mathcal{A} \in \cap_{2 \leq k < \omega} Nr_2 QEAs_k$.*

Proof. Let $1 \leq l < k$. Now assume that k, hence l is finite. Then it is easy to check that

$$i(l,k) \restriction G(l)^* \in Ism(\mathcal{G}(l)^*, \mathcal{N}r_l\mathcal{C}(k)).$$

We claim that, in fact, we have $i(l,k)G(l)^* \subseteq G(k)^*$. To this end let $u \in {}^l3$ and $\phi \in F(l)$. Then

$$i(l,k)E(u,\phi) = \{s \in {}^kU : s \restriction l \in E(u,\phi)\}.$$

$$= \cup\{E(v,\phi) : v \in {}^k3 \text{ and } v \restriction l \subseteq u\}.$$

The latter is in $G(k)^*$ because $F(l) \subseteq F(k)$, $\phi \in F(l)$ and $k < \omega$. Since $\mathcal{G}(k)^*$ is a $QEAs_k$, we get that $i(l,k)E(u,\phi) \in G(k)^*$, for any $u \in {}^l3$ and $\phi \in F(l)^{**}$. By $Nr_l(G(k)^*) = G(k)^* \cap Nr_lC(k)$, we get that

$$i(l,k) \restriction G(l)^* \in Ism(\mathcal{G}(l)^*, \mathcal{N}r_l(\mathcal{G}(k)^*)).$$

Next we show that $i(l,k)$ is actually onto $\mathcal{N}r_l\mathcal{G}(k)^*$, i.e that $i(l,k)G(l)^* = Nr_lG(k)^*$. This will be proved by using the fact that the structure $\mathbf{R}$ admits elimination of quantifiers as expressed in (*) above. In more detail, since the C_i's are additive, it suffices to show that for all $g \in G(k)$, there exists $a \in G(l)^*$ such that

$$i(l,k)a = C_lC_{l+1}\ldots C_{k-1}g.$$

So let $g = E(u,\phi)$ be as specified with $u \in {}^k3$ and $\phi \in F(k)^{**}$. Suppose further that $\phi = \phi_0 \vee \phi_1 \cdots \phi_m$, with $\phi_j \in F(k)^*$ for $0 \leq j \leq m$. For the sake of brevity set $v = u \restriction l$. Note that $v \in {}^l3$. Then an easy computation gives for every $j \leq m$

$$C_lC_{l+1}\cdots C_{k-1}E(u,\phi_j) = i(l,k)E(v, \exists x_l \exists x_{l+1} \cdots \exists x_{k-1}\phi_j).$$

Now (*) guarantees the existance of $\psi_j \in F(l)^{**}$ for every $j \leq m$ such that

$$\mathbf{R} \models \exists x_l \exists x_{l+1} \cdots \exists x_{k-1}\phi_j \longleftrightarrow \psi_j.$$

Now computing we get

$$C_lC_{l+1}\cdots C_{k-1}E(u,\phi) = C_lC_{l+1}\cdots C_l \cdots C_{k-1}E(u,\phi_0 \vee \cdots \phi_m)$$

$$= C_lC_{l+1}\cdots C_{k-1}E(u,\phi_0) \cup \ C_lC_{l+1}\ldots C_{k-1}E(u,\phi_m)$$

$$= i(l,k)E(v,\psi_0) \cup \cdots \cup i(l,k)E(v,\psi_m)$$

$$= i(l,k)E(v,\psi_0 \vee \cdots \psi_m).$$

Since by definition $\psi_0 \vee \cdots \psi_m \in F(l)^{**}$, we are done. ■

Claim 1.2.4 . *Let $2 \leq n < \omega$ and β be infinite. Then $\mathcal{G}(n)^* \in Nr_n QEA_\beta$. In particular, $\mathcal{A} \in Nr_2 QEA_\alpha$ for every $\alpha > 2$.*

Proof. Fix $\beta \geq \omega$. Let

$$\mathcal{G}(\beta) = Sg^{\mathcal{C}(\beta)} \cup \{i(k,\beta)G(k)^* : n \leq k < \omega\}.$$

Here as in Claim 1.2.3 $\mathcal{C}(\beta)$ denotes the full Cs_β with unit ${}^\beta U$; and for $a \in G(k)^*$ recall that $i(k,\beta)a = \{s \in {}^\beta U : s \upharpoonright k \in a\}$. Let $a \in G(\beta)$. We will show that $a \in Nr_n G(\beta)$ iff $a \in i(n,\beta)G(n)^*$, from which it readily follows that $i(n,\beta)G(n)^* = Nr_n G(\beta)$, and so $\mathcal{G}(n)^* \cong \mathcal{N}r_n \mathcal{G}(\beta)$. First note that $\cup\{i(k,\beta)G(k)^* : n \leq k < \omega\}$ is a subuniverse of $G(\beta)$. This is straightforward. Thus

$$G(\beta) = \cup\{i(k,\beta)G(k)^* : n \leq k < \omega\}.$$

It then follows that $a = i(l,\beta)g$ for some $n \leq l < \omega$ and $g \in G(l)^*$. Now suppose that $a \in Nr_n G(\beta)$. Then $g = C_j g$ for all $n \leq j < \omega$ and so $g \in Nr_n G(l)^*$. By Claim 1.2.3 we have $Nr_n G(l)^* = i(n,l)G(n)^*$; thus $g = i(n,l)g'$ say, for some $g' \in G(n)^*$. But

$$a = i(l,\beta)g = i(l,\beta) \circ i(n,l)g' = i(n,\beta)g'.$$

This shows that $a \in i(n,\beta)G(n)^*$. Conversely if $a \in i(n,\beta)G(n)^*$, then it is easy to see that $a \in Nr_n G(\beta)$. We have shown that $i(n,\beta)G(n)^* = Nr_n G(\beta)$. Since $\mathcal{A} \cong \mathcal{G}(2)^{**}$, and $\mathcal{G}(\beta) \in QEA_\beta$, we get that $\mathcal{A} \in Nr_2 QEA_\beta$, for all $\beta > 2$. ■

In Claims 1.2.3-4, we have proved that that $\mathcal{A} \in \cap_{2 \leq k < \omega} Nr_2 QEAs_k$ and that $\mathcal{A} \in Nr_2 QEA_\beta$ for infinite β, as well. In Claim 1.2.6 below we show that we can replace the class QEA_β by the smaller classes $WQEAs_\beta$ and $QEAs_\beta$ for infinite β. For this we need a definition.

Definition 1.2.5 . *Let β be an ordinal. We recall from [51] Def.3.1.1 (viii) that $\mathcal{A} \in Ks_\beta$ with unit ${}^\beta U$ is regular if for all $x \in A$, for all $f, g \in {}^\beta U$, whenever*

$f\restriction \Delta x\subseteq g$ and $f\in X$ then $g\in X$. $\mathcal{A}\in K_\beta$ is locally finite if Δx is finite for every element in A.

Notation . *Let β be an infinite ordinal. Then Ks_β^{reg} and LfK_β stand for the classes of regular K set algebras, and locally finite K algebras of dimension β, respectively.*

It is known (cf. [51] 3.1.70)[27] that for infinite β we have

$$Ks_\beta^{reg}\cap LfK_\beta\subseteq IWKs_\beta.$$

Claim 1.2.6 . *Let $n>1$. Then $\mathcal{G}(n)^*\in Nr_nWQEAs_\beta$ for infinite β. In particular, $\mathcal{A}\in Nr_2WQEAs_\beta$ for infinite β.*

Proof. Fix $\beta\geq\omega$. Let $\mathcal{F}(\beta)=Sg^{\mathcal{G}(\beta)}i(n,\beta)(\mathcal{G}(n)^*)$, where $\mathcal{G}(\beta)$, as usual, denotes the full quasipolyadic set algebra with unit ${}^\beta U$. Then, as easily checked, we have $\mathcal{F}(\beta)\in QEAs_\beta^{reg}\cap LfQEA_\beta$, and $\mathcal{G}(n)^*\cong\mathcal{N}r_n\mathcal{F}(\beta)$. From $IQEAs_\beta^{reg}\cap LfQEA_\beta\subseteq IWQEAs_\beta$, we are done. ∎

Remarks

(1) The class $LfK_\beta\cap Ks_\beta^{reg}$ is the algebraic counterpart of first order models. For more on such connections the reader is referred to [109], [51] sec 4.3., [9] and [128] Chapter 4. [28]

(2) In what follows, we give a simpler description of the algebra $\mathcal{A}$. Consider a first order language L consisting of binary relations symbols δ_u^r for all $u\in{}^23$ and all real numbers r. Let M be the L- structure with domain consisting of three disjoint copies R_0,R_1,R_2 of the real numbers, and with

$$M\models\delta_u^r(a,b)\text{ iff }a\in R_{u_0},b\in R_{u_1},\text{ and }b-a=r.$$

Now $\mathcal{A}$ is isomorphic to the algebra consisting of the first order definable subsets of ${}^2M$. In more detail, for $\beta>2$, any $v\in{}^\beta M$ can be regarded as an assignment

[27] We should point out that Theorem 3.1.70 in [51] is formulated only for cylindric algebras. However the proof easily adapts to SCs QAs and QEAs, as shown for example in [128] or [9].

[28] In Fact, one can define an isomorphism $h: Models\to LfK_\omega\cap Ks_\omega^{reg}$ definable in ZFC by an absolute formula without parameters, cf. [104] p.406.

of variables x_i, $(i < \beta)$ into M. Let $\mathcal{D}$ be the set of all subsets of ${}^{\beta}M$ of the form $\{v \in {}^{\beta}M : M \models \phi[v]\}$, where ϕ is a first order L-formula written with variables x_i $i < \beta$, and the operations defined as for set algebras. Here we are adopting the usual notion: $M \models \phi[v]$, if v satisfies ϕ in M. Then, of course, $\mathcal{D} \in QEAs_{\beta}$. In fact, $\mathcal{D} \in LfQEA_{\beta} \cap QEAs_{\beta}^{reg}$. On the other hand, $\mathcal{A}$ is isomorphic to the set algebra consisting of all subsets of ${}^{2}M$ of the form $\{v \in {}^{2}M : M \models \phi[v]\}$ where ϕ is an L-formula written with two variable x_0, x_1, and with operations defined by restricting those of $\mathcal{D}$ to the first two dimensions. By the proof of Fact 1.2.1, M has quantifier elimination, so $\mathcal{A} \cong \mathcal{N}r_2\mathcal{D}$. This proves Claim 1.2.6, too.

Henceforth we only deal with the 2-dimensional polyadic set algebras $\mathcal{A}$ and $\mathcal{B}$.

Part 2. $\mathcal{B}$ is an elementary subalgebra of $\mathcal{A}$.

We now show that the $QEAs_2$, $\mathcal{A}$ admit a particularly rich set of automorphisms; a property we use to show that $\mathcal{B}$ is an elementarily subalgebra of $\mathcal{A}$.

Notation . *For $u = \langle u_0, u_1\rangle \in {}^{2}3$ we let $1_{u_0u_1}$, and sometimes simply 1_u denote $E(u, \top)$. For $a \in A$ and $X \subseteq A$, we let $Rl_a(X)$ denote the relativization of X to a, that is $Rl_a(X) = \{x \in X : x \leq a\}$.*

Claim 1.2.7 . *Let $u \in {}^{2}3$. For brevity, set $V = E(u, \top)$. Then the following hold:*

(i) $Rl_V A = \{E(u, \phi) : \phi \in F(u)^{**}\}$

(ii) $Rl_V A$ is (the universe of) an atomic boolean algebra and the set of its atoms is equal to $P(u) = \{p(u, r) : r \in R\}$. In particular, the boolean algebra with universe $Rl_V A$ is generated by its atoms.

(iii) For all non-zero $a \in Rl_V A$ and for all $i < 2$ we have $C_i a = C_i V$.

(iv) For all $a \in A$ and for all $i < 2$ we have $C_i a \cap V \in \{0, V\}$.

Proof. (i) follows from the fact that $\mathcal{A} = \mathcal{G}(2)$. (ii) follows from the fact that for $u \in {}^{2}3$ and r_1, r_2 distinct elements of R we have

$$p(u, r_1) \cap p(u, r_2) = 0.$$

(iv) follows from (iii). Now we prove (iii).

Let $a \in Rl_V A$. Then $a \leq V$. By (i) $a = E(u, \phi)$ for some $\phi \in F(2)^{**}$. Now suppose that $0 \neq a$ and $i < 2$. Then by (ii) there exists an atom in $Rl_V A$ below a , i.e an $r \in R$ such that $E(u, x_0 = x_1 + r) = p(u, r) \leq a$. Now we have

$$C_i a \leq C_i V = \cup\{E(v, \top) : v \in {}^2 3 : v \equiv_i u\}$$

$$= C_i E(u, x_0 = x_1 + r) = C_i(p(u, r)) \leq C_i a.$$

We have proved (iii). ■

Remark

Notice that the set $\{1_u : u \in {}^2 3\}$ forms a partition of the unit $1 = \cup\{1_u : u \in {}^2 3\}$ of the algebra $\mathcal{A}$ so that the boolean structure of $\mathcal{A}$ is rather simple. Let $\mathcal{R}l_{1_u} A$ denote the boolean algebra with universe $Rl_{1_u} A$. Then $\mathcal{A}$ is isomorphic to the product of $\mathcal{R}l_{1_u} A$ indexed by ${}^2 3$; the isomorphism bieng:

$$h(a) = \langle a \cap 1_u : u \in {}^2 3 \rangle \text{ for } a \in A.$$

$\mathcal{R}l_{1_u} A$ has an even simpler boolean structure. It is isomorphic to the boolean algebra (with universe) $\wp(R)$ generated by the singletons, which is the same as the finite co-finite boolean algebra on R, i.e the algebra with universe

$$\{X \subseteq R : X \text{ or } R \smallsetminus X \text{ is finite }\}.$$

Note too that $P(u)$ constitutes further an uncountable partition of 1_u that is a *splitting* of 1_u in the sense of Andréka [3]. This means that every element in $P(u)$ is cylindrically equivalent to 1_u, i.e $C_i x = C_i 1_u$ for every $i < 2$ and every $x \in P(u)$. So in a sense (at the least in the above one) the elements in $P(u)$ are "big". Being atoms in $Rl_{1_u} A$ (and even more, as easily checked, in $\mathcal{A}$) they are also "small". Andréka in [3] refers to such elements as *big atoms.* It is precisely the "duality" in the nature of such elements (the big atoms) that will enable us, in Claim 1.2.8 below to extend given permutations on the set $P = \{p(u, r) : u \in {}^2 3 : u_0 \leq u_1, r \in R\}$ to automorphisms on $\mathcal{A}$. $\mathcal{A}$ in this respect resembles the free structure on P.

Claim 1.2.8 . *Let* $P = \{p(u,r) : u \in {}^23 : u_0 \leq u_1, r \in R\}$. *Let* f *be a permutation of* $P(\langle 0,2\rangle)$. *Then there exists an automorphism* h *of* $\mathcal{A}$ *such that* $f \subseteq h$ *and* $h \restriction (P \smallsetminus [P(\langle 0,2\rangle) \cup P(\langle 2,0\rangle)]) = Id$.

Proof. Let f be a given permutation of $P(\langle 0,2\rangle)$. Define

$$f^* = p_{01} \circ f \circ p_{01}.$$

Then, as easily checked, f^* is a permutation of $P(\langle 2,0\rangle)$. By Claim 1.2.7 (ii) $Rl_{1_{02}}A$ is generated as a boolean algebra by the set $P(\langle 0,2\rangle)$ of its atoms. It follows that there exists a boolean automorphism h_1 of $Rl_{1_{02}}A$ such that $f \subseteq h_1$. Similarly there exists a boolean automorphism h_2 of $Rl_{1_{20}}A$ such that $f^* \subseteq h_2$. For $x \in A$, let

$$h(x) = h_1(x \cap 1_{02}) \cup h_2(x \cap 1_{20}) \cup \bigcup\{x \cap 1_u : u \in {}^23 \text{ and } Rng(u) \neq \{0,2\}\}.$$

We show that h is the desired automorphism. Clearly h is a boolean automorphism that extends f and is the identity on $P \smallsetminus [P(\langle 0,2\rangle) \cup P(\langle 2,0\rangle)]$. So all we have to check is that h preserves the extra non- boolean operations, too.

(1) h preserves the diagonal element.

$$h(D_{01}) = h_1(D_{01} \cap 1_{02}) \cup h_2(D_{01} \cap 1_{20}) \cup \bigcup\{D_{01} \cap 1_u : u \in {}^23 \text{ and } Rgu \neq \{0,2\}\}.$$

Since $D_{01} \cap 1_{02} = D_{01} \cap 1_{20} = 0$, and h_1, h_2 are automorphisms of $Rl_{1_{02}}A$ and $Rl_{1_{20}}A$, respectively, so that $h_1(D_{01} \cap 1_{02}) = h_2(D_{01} \cap 1_{20}) = 0$, we have

$$h(D_{01}) = \bigcup\{D_{01} \cap 1_u : u \in {}^23 \text{ and } Rgu \neq \{0,2\}\} = D_{01}.$$

(2) h preserves S_{01}.

Let $Y = \{p(u,r) : u \in {}^23,\ r \in R\}$. Let $X = \{C_ia : a \in A,\ i < 2\} \cup \{D_{01}\}$, and let $\mathcal{D}$ be the boolean algebra generated by $X \cup Y$. We claim that D is a subuniverse of $\mathcal{A}$. Since D contains D_{01} and is closed under the boolean set operations, it suffices to show that D is closed under C_0,C_1 and S_{01}. So let $a \in D$ and $i < 2$. By $D \subseteq A$ we have $a \in A$ and so, by definition, we have $C_ia \in D$. Since S_{01} is a boolean endomorphism, it suffices to check that for $x \in Y \cup X$ we have $S_{01}x \in D$. But this is true because $x \in Y$ iff $S_{01}x \in Y$, $x \in X$ iff $x = 0$ or there is a subset J of ^{23} such that $x=\{1_u : u\in J\}$ and $S_{01}1_{u_0u_1}=1_{u_1u_0}$. By this we have shown that D is a subuniverse of $\mathcal{A}$. Since $Y\subseteq D$, Y generates $\mathcal{A}$, and $D\subseteq A$, we get $D=A$. Now let $X'=\{1_u : u\in {}^23\}\cup\{D_{01}\}$ and let $\mathcal{D}'$ be the boolean algebra generated by $X'\cup Y$. Then $D=D'$ because every non zero element of X is a finite union of elements in X'. Thus $D'=A$. Since S_{01} is a boolean endomorphism, to prove that h preserves S_{01}, it is enough to show that

$$h(S_{01}x)=S_{01}h(x) \text{ for all } x\in Y\cup X'.$$

To this end, let $x\in Y\cup X'$. If $u\in {}^23$, $Rng(u)\neq\{0,2\}$ and $x\in\{p(u,r),1_u\}$, then

$$S_{01}x\cap 1_v=0 \text{ for } v\in\{\langle 0,2\rangle,\langle 2,0\rangle\},$$

thus

$$h(S_{01}x)=\cup\{S_{01}x\cap 1_u : u\in {}^23 \text{ and } Rng(u)\neq\{0,2\}\}$$
$$=S_{01}x \text{ (by } h(x)=x) \ =S_{01}h(x).$$

Else

$$x=p(u,r) \text{ or } x=1_u \text{ with } Rng(u)=\{0,2\}.$$

Suppose that $u=\langle 0,2\rangle$ and that $x=p(\langle 0,2\rangle,r)$. (The subcases $x=1_{02}$ or $x=1_{20}$ or $x=p(\langle 2,0\rangle,r)$ can be treated analogously and are left to the reader.) By noting that $S_{01}x=p(\langle 2,0\rangle,-r)$; and computing we get

$$h(S_{01}x)=h_2(p(\langle 2,0\rangle,-r))=f^*(p(\langle 2,0\rangle,-r))=p_{01}\circ f\circ p_{01}[p(\langle 2,0\rangle,-r)]$$

$$=S_{01}\circ f\circ S_{01}\circ S_{01}(x)=S_{01}\circ f(x)=S_{01}(h(x)).$$

(3) Finally we check cylindrifications. Let $k<2$ and $x\in A$. Then computing we get (1)

$$C_kh(x)=C_k(h_1(x\cap 1_{02})\cup h_2(x\cap 1_{20})\cup\cup_{u\in {}^23}\{x\cap 1_u : Rng(u)\neq\{0,2\}\})$$

$$=C_kh_1(x\cap 1_{02})\cup C_kh_2(x\cap 1_{20})\cup\cup_{u\in {}^23}\{C_k(x\cap 1_u) : Rng(u)\neq\{0,2\}\}.$$

Since h_1 is an automorphism of $Rl_{1_{02}}A$ and $x\cap 1_{02}\leq 1_{02}$, we get

$$h_1(x\cap 1_{02})\leq h_1(1_{02})=1_{02}.$$

Since $x\cap 1_{02}=0$ iff $h_1(x\cap 1_{02})=0$, we then have by Claim 1.2.7 (iii) that

$$C_k(h_1(x\cap 1_{02}))=C_k(x\cap 1_{02}).$$

By exactly the same reasoning it follows that

$$C_k(h_2(x\cap 1_{20}))=C_k(x\cap 1_{20}).$$

Substituting in (1) we obtain

$$C_kh(x)=C_k(x\cap 1_{02})\cup C_k(x\cap 1_{20})\cup\cup_{u\in{}^23}\{C_k(x\cap 1_u):Rng(u)\neq\{0,2\}\}$$

$$=C_k(x\cap\cup\{1_u:u\in{}^23\})=C_kx.$$

Now computing $h(C_kx)$ we get (2)

$$h(C_kx)=h_1(C_kx\cap 1_{02})\cup h_2(C_kx\cap 1_{20})\cup\cup_{u\in{}^23}\{C_kx\cap 1_u:Rng(u)\neq\{0,2\}\}$$

That $h_1(C_kx\cap 1_{02})=C_kx\cap 1_{02}$ and $h_2(C_kx\cap 1_{20})=C_kx\cap 1_{20}$ follows from Claim 1.2.7 (iv) upon noting that h_1 and h_2 are boolean automorphisms ; thus we have

$$h(C_kx)=C_kx\cap\cup\{1_u:u\in{}^23\}=C_kx.$$

We have shown that $C_kh(x)=hC_k(x)=C_kx$ [29] , thus h preseves cylindrifications.

■

Now we are ready to prove:

Claim 1.2.9 . *$\mathcal{B}$ is an elementary subalgebra of $\mathcal{A}$*

Proof. We shall use the Tarski-Vaught criterion to show that $\mathcal{B}$ is an elementary subalgebra of $\mathcal{A}$. So let $\exists x_n\phi(x_0,\cdots,x_n)$ be any first order formula in the language of QEA_2. Suppose that $b_0,\cdots,b_{n-1}\in B$, and that $\mathcal{A}\models\phi(b_0,\cdots,b_{n-1},a)$, for some $a\in A$. We want to find $b_n\in B$ such that $\mathcal{A}\models\phi(b_0,\cdots,b_{n-1},b_n)$. Since $\mathcal{A}$ is

[29] We used the "bigness" condition to show that h preserves cylindrifications. Loosely speaking when a cylindric or polyadic algebra $\mathcal{A}$ is generated by a set of big atoms, then we can forget about cylindrifications when constructing automorphisms of $\mathcal{A}$. This method is also referred to as the method of eliminating cylindrifications.

generated by $P = \{p(u,r) : u \in {}^23 : u_0 \leq u_1, r \in R\}$, then there exists a finite subset H_0, say, of P such that $a \in Sg^{\mathcal{A}}H_0$. Let

$$P' = \{p(u,r) : u \in {}^23 : u_0 \leq u_1, u \neq \langle 0,2\rangle, r \in R\} \cup P_\omega(\langle 0,2\rangle).$$

Since $b_0, \cdots b_{n-1} \in B$, and $\mathcal{B}$ is generated by P', then there exists a finite subset H_1 say of P' such that $(H_0 \cap B) \cup \{b_0, \cdots, b_{n-1}\} \subseteq Sg^{\mathcal{B}}H_1$. Let f be a permutation of $P(\langle 0,2\rangle)$ such that

$$f \upharpoonright P(\langle 0,2\rangle) \cap H_1 \subseteq Id \text{ and } f(H_0 \smallsetminus H_1) \subseteq P_\omega(\langle 0,2\rangle).$$

Such a permutation exists, since $|H_0| < \omega$, $|H_1| < \omega$ and $|P_\omega(\langle 0,2\rangle)| = \omega$. By Claim 1.2.8 f extends to an automorphism h of $\mathcal{A}$, such that

$$h \upharpoonright (P \smallsetminus P(\langle 0,2\rangle) \subseteq Id.$$

But

$$\mathcal{A} \models \phi(b_0, \cdots, b_{n-1}, a).$$

and so

$$\mathcal{A} \models \phi(h(b_0), \cdots, h(b_{n-1}), h(a)).$$

Since $h \upharpoonright H_1 \subseteq Id$ and $\{b_0, \cdots, b_n\} \subseteq Sg^{\mathcal{B}}H_1$, it follows that $h(b_i) = b_i$ for all $i < n$. Since $h(H_0) \subseteq B$ and $a \in Sg^{\mathcal{A}}H_0$, we get $h(a) \in B$. This completes the proof. ■

Part 3. ***B*** **is not a neat reduct**

Here we show that $\mathcal{B}$ is not a neat reduct. Recall that $\mathcal{B}$ was obtained from $\mathcal{A}$ by an infinite "cardinality twist", namely by keeping only countably many atoms below 1_{02} and deleting the rest. In this section we show that first order logic cannot "see" this infinite cardinality twist. Though there are only countably many atoms below 1_{02}, in principle there could be uncountably many elements below 1_{02} coming from other components. We start off by excluding such a possibility. We show that the set of *all* elements in B below 1_{02} remains countable.

Claim 1.2.10 . *The set* $\{b \in B : b \leq 1_{02}\}$ *is countable.*

Proof. Let

$$Y = \{p(u,r) : u \in {}^{2}3,\ \text{and } Rng(u) \neq \{0,2\}\} \cup P_\omega(\langle 0,2\rangle) \cup P_\omega(\langle 2,0\rangle).$$

(Recall that $P_\omega(\langle u_0,u_1\rangle) = \{p(\langle u_0,u_1\rangle, r) : r \in N\}$, where N is a fixed countable subset of R.) Let

$$X = \{C_i a : i < 2, a \in A\} \cup \{D_{01}\}.$$

Let $\mathcal{D}$ be the boolean subalgebra of $\mathcal{A}$ generated by $Y \cup X$. Then D is a subuniverse of $\mathcal{A}$. Since $Y \subseteq D$, we have $B \subseteq D$. (In fact, $B = D$, but we do not need that much), and so $Rl_{1_{02}}B \subseteq Rl_{1_{02}}D$. It clearly suffices to show that $|Rl_{1_{02}}D| \leq \omega$. To this end, let $rl_{1_{02}}$ denote the following endomorphism of the structure $\langle A, \cup, \cap, -\rangle$:

$$rl_{1_{02}}a = a \cap 1_{02}, a \in A$$

Then $Rl_{1_{02}}(D) = Sg^{Bl(\mathcal{A})} rl_{1_{02}}(X \cup Y)$. But $rl_{1_{02}}(X \cup Y)$ is countable because $|P_\omega(\langle 0,2\rangle)| = \omega$ and $|X| < \omega$. Therefore $Rl_{1_{02}}D$, generated as a boolean algebra by a countable set, is also countable. ■

Next we show that had $\mathcal{B}$ been a neat reduct, then $Rl_{1_{02}}B$ would be an uncountable set. This contradicts Claim 1.2.10. From this we infer that $\mathcal{B}$ is not a neat reduct. For this we need the following terms[30]:

Claim 1.2.11 . *Let $\tau(x,y)$ denotes the SC_2 term $c_1(c_0 x \cdot s_1^0 c_1 y) \cdot c_1 x \cdot c_0 y$. Let $\tau_3(x,y)$ denote the SC_3 term $c_2(s_2^1 c_2 x \cdot s_2^0 c_2 y)$. Then (i) and (ii) below hold.*

$$(i) \tau^{\mathcal{B}}(1_{01}, 1_{12}) = 1_{02}.$$

$$(ii) SC_3 \models \tau_3(x,y) \leq \tau(c_2 x, c_2 y).$$

Proof of (i).

$$\tau^{\mathcal{B}}(1_{01}, 1_{12}) = C_1(C_0 1_{01} \cap S_1^0 C_1 1_{12}) \cap C_1 1_{01} \cap C_0 1_{12}.$$

We first compute $S_1^0 C_1 1_{12}$.

$$S_1^0 C_1 1_{12} = C_0(D_{01} \cap (1_{12} \cup 1_{11} \cup 1_{10})) = C_0(D_{01} \cap 1_{11})$$

[30]These were used in the proof of Theorem 1

$$= (\text{ by Claim 1.2.7 (iv)}) \ = C_0 1_{11} = C_0 1_{01}.$$

Thus

$$\tau^{\mathcal{B}}(1_{01}, 1_{12}) = C_1(C_0 1_{01} \cap C_0 1_{01}) \cap C_1 1_{01} \cap C_0 1_{12}.$$

$$= C_1 1_{01} \cap C_0 1_{12} = 1_{02}.$$

Proof of (ii). Identical to the proof of Lemma 1.1.1.

■

Claim 1.2.12 . *Let $Rd_{SC}\mathcal{B}$ denote the SC reduct of $\mathcal{B}$. Then the following hold:*

(i) Suppose that $Rd_{SC}\mathcal{B} \subseteq \mathcal{N}r_2(\mathcal{C})$ with $\mathcal{C} \in SC_3$. Let

$$D = \{\tau_3^{\mathcal{C}}(x, y) : x, y \in Nr_2(C) : x \leq 1_{01}, y \leq 1_{12}\}.$$

Then $|D| > \omega$. In fact, $P(\langle 0, 2 \rangle)) \subseteq D$.

(ii) $Rd_{SC}\mathcal{B} \notin Nr_2 SC_3$

Proof of (i). Assume first that $\mathcal{C}$ is a set algebra. Then the result follows by noting that

$$\tau_3^{\mathcal{C}}(p(\langle 0, 1 \rangle, r_1), p(\langle 1, 2 \rangle, r_2) = p(\langle 0, 2 \rangle, r_1 + r_2).$$

Now if C is not a set algebra, then replacing C by $\mathcal{E} = Sg^{\mathcal{C}}B$, we get that $\mathcal{E}$ is isomorphic to a set algebra by [86]. [31] This is so because $\mathcal{E}$ is an atomic algebra that is generated by a set of atoms, namely the set of atoms of B, that are functions. Indeed, every $p(u, r)$ for $u \in {}^2 3$ and $r \in R$, is a binary relation that is a function. The same argument as above, but now applied to $\mathcal{E}$, works.

Proof of (ii) Assume, seeking a contradiction, that $Rd_{SC}\mathcal{B} = \mathcal{N}r_2\mathcal{C}$, say, with $\mathcal{C} \in SC_3$. Let $x, y \in B$ be such that $x \leq 1_{01}$ and $y \leq 1_{12}$. Then x and y are $2-$*closed*, i.e $C_2 x = x$ and $C_2 y = y$. Also $\tau_3^{\mathcal{C}}(x, y) \in B$. Now

$$\tau_3^{\mathcal{C}}(x, y) \leq \tau_3^{\mathcal{C}}(1_{01}, 1_{12})$$

[31] Strictly speaking the result of Maddux and Tarski says that if $\mathcal{A}$ is an atomic relation algebra, whose atoms are functions, then A is representable. Unpublished work of Andréka and Givant generalizes this result to CA-like algebras of relations. For results of this kind - giving sufficient conditions for representability of atomic algebras of relations by imposing conditions on the atoms like *density* - we refer the reader to [7].

$$\leq \tau^{\mathcal{C}}(C_2 1_{01}, C_2 1_{12}) = \tau^{\mathcal{B}}(1_{01}, 1_{12}) = \text{ (by Claim (1.2.12) (i))} 1_{02}.$$

In other words we have

$$D = \{\tau_3^{\mathcal{C}}(x, y) : x \leq 1_{01}, y \leq 1_{12}\} \subseteq \{b \in B : b \leq 1_{02}\}.$$

By Claim 1.2.10 D must be countable. But this contradicts Claim 1.2.12(i). By this, the proof of Theorem 2 is complete. ■

1.3. Proofs of Theorems 3 and 4

In this subsection we prove Theorems 3 and 4. Unless otherwise specified, let α be a fixed infinite ordinal, and for notational conveniance, we let it be the least infinite ordinal, i.e. we assume throughout this subsection that $\alpha = \omega$. As usual, we let $K \in \{SC, CA, QA, QEA\}$. We first introduce a certain class of K algebras that coincides with the class Di_ω of diagonal cylindric algebras when $K = CA$. In our definition we simply replace diagonal elements by substitutions corresponding to replacement; for such operations are available in all algebras considered, more precisely:

Definition 1.3.1 . *$\mathcal{A} \in K_\omega$ is called a replacement algebra, an ReK_ω for short, if for all finite subset Γ of ω and every non-zero $a \in A$, there exists distinct $k, l \in \omega \smallsetminus \Gamma$ such that $s^l_k a \neq 0$.*

It is absolutely straightforward to check that $\mathcal{A} \in ReCA_\omega$ iff $\mathcal{A}$ is a diagonal algebra (i.e a Di_ω) in the sense of [113] p.327. See also [50] 2.6.50 and the immediately following remark.

To prove Theorem 3, we let $\mathcal{A}$ be the $WQEAs_\omega$ constructed in statement 2. Let w be the element proved (in statement 2) not to belong to A, i.e.

$$w = \{s \in {}^\alpha Q^{(0)} : s_0 + 2 = s_1 + 2\sum_{i>1} s_i\}$$

Now we define $\mathcal{A}_0$ to be the $WQEAs_\alpha$ obtained by adjoining the element w to $\mathcal{A}$. More precisely, we let

$$\mathcal{A}_0 = Sg^C\{y, y_s, w : s \in y\}.$$

In the our next statement we identify ReK_α with the concrete category whose objects (cf. [1] for undefined categorial notions) are algebras in ReK_α and whose morphisms are homomorphisms between algebras. Recall that an L-epimorphism $f : \mathcal{A} \to \mathcal{B}$, for $\mathcal{A}, \mathcal{B} \in L$ is a homomorphism such that whenever m, n are homomorphisms from $\mathcal{B}$ into $\mathcal{C} \in L$, such that $m \circ f = n \circ f$, then $m = n$.

Statement 3 . *The inclusion map from $Rd_{SC}\mathcal{A}$ into $Rd_{SC}\mathcal{A}_0$ is an epimophism in the category $ReSC_\omega$ that is not surjective.*

We leave it to the reader to check that Theorem 3 follows imediately from statement 3.

Now we prove statement 3. For that we need some preparations. We start with a definition.

Definition 1.3.2 .

(i) *Let $\mathcal{A} \in K_\omega$ be representable. Let $\mathcal{B} \in K_{\omega+\omega}$ for which there exists an embedding*

$$i : \mathcal{A} \to \mathcal{N}r_\omega \mathcal{B}.$$

Recall that the existance of such a $\mathcal{B}$ is guaranteed by the Neat Embedding Theorem. Then we refer to $\mathcal{B}$ as an ω dilation of $\mathcal{A}$ via i, or simply an ω dilation, if i is immaterial in the context.

(ii) *Let I be a set and $\langle \mathcal{A}_i : i \in I\rangle$ be a system of algebras having the same similarity type. Let F be filter on I. Then $\prod_{i\in I} \mathcal{A}_i$ denotes the product of the $\mathcal{A}_i$'s and $\prod_{i\in I} \mathcal{A}_i/F$ denotes the reduced product of the $\mathcal{A}_i$'s relative to F.*

We shall also need the following notion of generalized reducts, obtained by renaming some of the operations. For conveniance, we recall from [50] Def 2.6.1 the definition only for cylindric algebras. The definition for the other cases is entirely analagous:

(iii) *Let $\mathcal{B} = \langle B, +, ., 0, 1, c_k, d_{kl}\rangle_{k,l\in\beta}$ be a CA_β, α an ordinal, and ρ a sequence of length α with $Rng(\rho) \subseteq \beta$. Then the algebra*

$$\mathcal{A} = \langle B, +, ., -, 0, 1, c_{\rho_k}, d_{\rho_k\rho_l}\rangle_{k.l<\alpha}.$$

is called the ρ-reduct of $\mathcal{B}$, and is denoted by $Rd^\rho_\alpha\mathcal{B}$ or simply $Rd^\rho\mathcal{B}$, when the dimension α is clear from context.

We leave it to the reader to check that $Rd^\rho_\alpha\mathcal{B}$, so defined, is indeed a CA_α.

In our proof of statement 3 we shall also need the following Lemma, which is of some interest in its own right. [32]

[32]Indeed Lemma 1.3.3 implies the representability of several distinguished classes of abstractly

Lemma 1.3.3 . *Let $\mathcal{A}_0$, and $\mathcal{A}_1 \in ReK_\omega$, Then there exist ω dilations $\mathcal{B}_0$ and $\mathcal{B}_1$ of $\mathcal{A}_0$ and $\mathcal{A}_1$ via i_0 and i_1 respectively, such that for every homomorphism $f : \mathcal{A}_0 \to \mathcal{A}_1$ there exists a homomorphism $g : \mathcal{B}_0 \to \mathcal{B}_1$ such that $g \circ i_0 = i_1 \circ f$.*

Proof. Let $\mathcal{A}_0, \mathcal{A}_1 \in ReK_\omega$. We let $X \subseteq_\omega Y$ stand for X is a finite subset of Y. We construct the desired dilations using ultraproducts. Let R be the set of all ordered quadruples

$$\langle \Gamma, n, k, l \rangle$$

such that:

$$\Gamma \subseteq_\omega \omega \text{ and } n \in \omega,$$

k, l are one to one (finite) sequences with

$$(1) \quad k, l \in {}^n(\omega \smallsetminus \Gamma),$$

and

$$Rng(k) \cap Rng(l) = \emptyset.$$

For $\Gamma \subseteq_\omega \omega$, and $n \in \omega$ put

$$X_{\Gamma,n} = \{\langle \Delta, m, k, l \rangle \in R : \ \Gamma \subseteq \Delta \text{ and } n \leq m\}.$$

Roughly $X_{\Gamma,n}$ consists of all quadruples in R extending Γ. It is straighforward to check that the set consisting of all $X_{\Gamma,n}$'s is closed under finite intersections. Accordingly, we let M be the proper filter of $\wp(R)$ generated by the $X_{\Gamma,n}$'s. so that

$$M = \{Y \subseteq R : X_{\Gamma,n} \subseteq Y \text{ for some } \Gamma \subseteq_\omega \omega \text{ and } n \in \omega\}.$$

For each $\langle \Gamma, n, k, l \rangle \in R$, choose a bijection $\rho(\langle \Gamma, n, k, l \rangle)$ from $\omega + \omega$ onto ω such that

$$(2) \quad \rho(\langle \Gamma, n, k, l \rangle) \upharpoonright \Gamma \subseteq Id$$

and

$$(3) \quad \rho(\langle \Gamma, n, k, l \rangle)(\alpha + j) = k_j, \text{ for each } j < n.$$

defined algebras like the locally finite, dimension complemented, semisimple and replacement algebras of infinite dimension. This follows from the observation that these classes consist exclusively of replacement algebras, and replacement algebras have the neat embedding property, a result that is implicit in our Lemma.

Now fix $i \in \{0,1\}$. Let

$$\mathbf{F}\mathcal{A}_i = \prod_{\phi \in R} Rd^{\rho(\phi)}\mathcal{A}_i.$$

Here $Rd^{\rho(\phi)}\mathcal{A}_i$ - the $\rho(\phi)$ reduct of $\mathcal{A}_i$ - is a $K_{\omega+\omega}$, and so $\mathbf{F}\mathcal{A}_i$ - a direct product of these - is also a $K_{\omega+\omega}$. Note too, that for each $\phi \in R$, the algebra $Rd^{\rho\phi}\mathcal{A}_i$ has universe A_i.

Now let j_i be the function from A_i into $\prod_{\phi \in R} A_i/M$ defined as follows

$$j_i x = \langle (s_{l_0}^{k_0})^{\mathcal{A}_i} \circ \ldots (s_{l_{n-1}}^{k_{n-1}})^{\mathcal{A}_i} x : \langle \Gamma, n, k, l \rangle \in R \rangle / M.$$

We now proceed to show that $\mathcal{B}_i = \mathbf{F}\mathcal{A}_i/M$ is an ω dilation of $\mathcal{A}_i$ via j_i. For simplicity of notation, we omit the subscript i everywhere, i.e we write simply j, $\mathcal{A}$ and $\mathcal{B}$ instead of j_i $\mathcal{A}_i$ and $\mathcal{B}_i$. Now for the proof. First of all $(s_{l_0}^{k_0})^{\mathcal{A}} \circ \ldots (s_{l_{n-1}}^{k_{n-1}})^{\mathcal{A}}$ is a composition of boolean homomorphisms, so it is a boolean homomorphism from $\mathcal{A}$ into $Rd^{\rho(\phi)}\mathcal{A}$, for each $\phi \in R$. It readily follows, thus, that j is a boolean homomorphism from the boolean reduct of $\mathcal{A}$ to that of $\mathbf{F}\mathcal{A}/M$. Now consider any $n < \omega$. Then for each $(\Gamma, \mu, k, l) \in R$ such that $n \in \Gamma$ we have

$$s_{l_0}^{k_0} \cdots \circ s_{l_{n-1}}^{k_{n-1}} c_n x = c_n s_{l_0}^{k_0} \cdots s_{l_{n-1}}^{k_{n-1}} x$$

for every $x \in A$. This equality follows from [50] 1.5.8 (ii) since $n \notin Rng(k) \cup Rng(l)$ by (1). From (2), properties of ultraproducts and the obvious fact that

$$\{(\Gamma, \mu, k, l) : (\Gamma, \mu, k, l) \in R, n \in \Gamma\} \in M$$

we readily conclude that

$$j(c_n x) = c_n(jx) \text{ for all } x \in A \text{ and } n < \omega.$$

We have proved j preserves cylindrifications. Using the same reasoning one can show that j preseves substitutions corresponding to replacements, transpositions (in the case of QA and QEA) and diagonal elements in the case of CAs and QEAs. Next, we prove that $j(A)$ consists only of ω-dimensional elements, i.e. that

$$j(A) \subseteq Nr_\omega B.$$

This part involves only cylindrifications. Let $\eta \in (\omega+\omega) \smallsetminus \omega$. . Then for each $\phi = (\Gamma, \mu, k, l) \in R$, such that $\eta < \omega + \mu$, we have

$$c_{\rho(\phi)} s^{k_0}_{l_0} \cdots \circ s^{k_{n-1}}_{l_{n-1}} s^{k_0}_{l_0} \cdots s^{k_{n-1}}_{l_{n-1}} x.$$

for every $x \in A$. This equality follows from [50] 1.5.8 (ii) and [50] 1.5.9(ii) since $\rho(\phi) \in Rng(k) \smallsetminus Rng(\Lambda)$. by (1) and (3). But

$$I_\eta = \{(\Gamma, \mu, k, l) : (\Gamma, \mu, k, l) \in R, \eta < \omega + \mu\} \in M$$

we see that $j(A) \subseteq Nr_\omega B$.

It remains to show that j is an embedding, i.e. one to one. Now here we use the fact that $\mathcal{A}$ is a replacement algebra. More precisely, let $x \in A$ and $x \neq 0$. We shall show that $j(x)$ is non-zero. For any finite $\Gamma \subseteq \alpha$ and any $n < \omega$, there exists one to one sequences $k, l \in {}^n(\omega \smallsetminus \Gamma)$ with the property that

$$Rng(k) \cap Rng(l) = \emptyset$$

and

$$s^{k_0}_{l_0} \cdots \circ s^{k_{n-1}}_{l_{n-1}} x \neq 0;$$

that such k, l must always exist follow directly from the definition of replacement algebras by a simple inductive argument. Now putting back subscripts we have shown that $\mathcal{B}_i$ is an ω-dilation of A_i via j_i. Now assume that $f : \mathcal{A}_0 \to \mathcal{A}_1$ is given. Then let g be the function from $\mathbf{F}A_0/M$ into $\mathbf{F}A_1/M$ defined by:

$$g(\langle x_\phi : \phi \in R\rangle/M) = \langle f x_\phi : \phi \in R\rangle/M.$$

Then it is not hard to check that g is the desired "lifting" function.

■

Now we have all the machinery to prove Statement 3.

Proof of Statement 3

Proof..

We will show that the inclusion $Rd_{SC}\mathcal{A} \subseteq Rd_{SC}\mathcal{A}_0$ is an epimorphism in the category $ReSC_\omega$ that is not surjective. We have already proved that $w \notin A_0$, so that the inclusion map is indeed not surjective. For the sake of brevity, we write $\mathcal{C}$ instead of $Rd_{SC}\mathcal{C}$. To prove that $\mathcal{A} \subseteq \mathcal{A}_0$ is an epimorphism in $ReSC_\omega$, we have to show that for any $\mathcal{C} \in ReSC_\omega$ and any $h : \mathcal{A}_0 \to \mathcal{C}$ and $k : \mathcal{A}_0 \to \mathcal{C}$, if h and k agree on $\mathcal{A}$ then they agree on $\mathcal{A}_0$. So let $\mathcal{A}_1 \in ReSC_\omega$, and assume that h and k be homomorphisms from $\mathcal{A}_0$ to $\mathcal{A}_1$ that agree on $\mathcal{A}$. It clearly suffices to show that that $k(w) = h(w)$. We start off by lifting k and h to ω extra dimensions. More precisely, by appeal to Lemma 1.3.3, let $\mathcal{B}_0$,$\mathcal{B}_1$ be ω-dilations of $\mathcal{A}_0$ and $\mathcal{A}_1$ via i_0 and i_1 repectively, let $k^* : \mathcal{B}_0 \to \mathcal{B}_1$ be a homomorphism such that

$$k^* \circ i_0 = i_1 \circ k,$$

and let $h^* : \mathcal{B}_0 \to \mathcal{B}_1$ be a homomorphism such that

$$h^* \circ i_0 = i_1 \circ h.$$

Define the composition-like term

$$\tau_\omega(x_1, x_2) = c_\omega(s^1_\omega c_\omega x_1 \cdot s^0_\omega c_\omega x_2).$$

Note that this term is in the language of $CA_{\omega+1}$. We will show that (*)

$$\tau_\omega^{B_0}(i_0 y, i_0 y) = i_0 w$$

which means that when we go up one extra dimension, then w can be “recaptured” by applying this “composition like” term to y. By (*) we will be done because of the following:

$$k^* \circ i_0(w) = k^*(\tau_\omega^{B_0}(i_0 y, i_0 y)) = \tau_\omega(k^* \circ i_0(y), k^* \circ i_0(y)).$$

But

$$k^* \circ i_0(y) = h^* \circ i_0(y).$$

Indeed, we have

$$k^* \circ i_0(y) = i_1 \circ k(y) = i_1 \circ h(y) = h^* \circ i_0(y).$$

From this we get that

$$\tau_\omega(k^* \circ i_0(y), k^* \circ i_0(y)) = \tau_\omega(h^* \circ i_0(y), h^* \circ i_0(y))$$

$$= h^*(\tau_\omega(i_0(y), i_0(y)) = h^*(i_0 w) = h^* \circ i_0(w).$$

We have shown that

$$k^* \circ i_0(w) = h^* \circ i_0(w).$$

Thus

$$i_1 \circ k(w) = i_1 \circ h(w).$$

But since i_1 is one to one, it follows thus that

$$k(w) = h(w)$$

We are done modulu (*). We now prove (*).

For the sake of brevity and better readability, we let $i = i_0$ and $\mathcal{B} = \mathcal{B}_0$. Now let $\tau(x_1, x_2)$ be the following SC_2 term:

$$\tau(x_1, x_2) = c_1(c_0 x_1 \cdot s_1^0 c_1 x_2) \cdot c_1 x_1 \cdot c_0 x_2.$$

Recall that

$$y = \{s \in {}^\alpha Q^{(\mathbf{0})} : s_0 + 1 = \sum_{i>0} s_i\},$$

and that for $s \in y$, $y_s = \{s\}$. Let

$$X = \{y_s : s \in y\} \text{ and let } a, b \in X.$$

Then repeating the proof of Lemma 1. 1.3 we have (1)

$$i(\tau(a, b)) = i(\tau_\omega^{\mathcal{B}}(a, b)) = \tau_\omega^{\mathcal{B}}(i(a), i(b)) \leq \tau_\omega(i(y), i(y).$$

Now let

$$X" = \{\tau(a, b) : a, b \in X\}.$$

Then by the proof of statement 2 we have (2)

$$w = \cup X"$$

Replacing $\mathcal{B}$ by $\mathcal{D} = Sg^{\mathcal{B}} i(A_0)$, and by noting that $\mathcal{A}$ contains all singletons, we obtain the following density condition

(3) For all non-zero $d \in Nr_\omega D$ there exists an atom in $x \in A$ such that $i(x) \leq d$.

From (1) and (2) and (3) we get the desired because any atom in $\mathcal{A}$ below w is of the form $\tau(a,b)$ for singletons a,b below y. In more detail, we have

$$i(w) \leq \tau_\omega(i(y), i(y)).$$

For if not, then we get

$$i(w) - \tau_\omega(i(y), i(y)) \neq 0.$$

By applying (3), upon noting that the latter is ω-closed, we get an atom $x \in A$, such that $i(x) \leq i(w)$ and $i(x) \leq -\tau_\omega(i(y), i(y)$. But $x = \tau(a,b)$ for some $a, b \in X$, thus $i(x) = \tau(i(a), i(b)) \leq \tau_\omega(i(y), i(y))$, which is a contradiction. Now we want to establish the other (harder) inclusion, namely that

$$\tau_\omega(i(y), i(y)) \leq i(w).$$

Now assume again, seeking a contradiction, that it is not the case that

$$\tau_\omega(i(y), i(y)) \leq i(w).$$

Thus we have

$$\tau_\omega(i(y), i(y)) \cdot -i(w) \neq 0.$$

By (3) there exists an atom $z \in A$ such that

$$i(z) \leq \tau_\omega(i(y), i(y)) \text{ and } i(z) \leq -i(w).$$

From the first inclusion we get that

$$i(z) \cdot s^1_\omega i(y) \cdot s^0_\omega i(y) \neq 0.$$

Since i is one to one, we obtain

$$i[z \cap (S^1_\omega y \cap S^0_\omega y)] \neq \emptyset,$$

and because $C_0 z \cap C_1 z = z$, we get

$$C_0 z \cap C_1 z \cap (S^1_\omega y \cap S^0_\omega y) \neq \emptyset.$$

From $S^j_\omega C_j z = C_j z$, and the fact that the S^j_ω's are boolean endomorphisms for each $j < \omega$, we obtain

$$S^1_\omega (C_1 z \cap y) \cap S^0_\alpha (C_0 z \cap y) \neq \emptyset.$$

By applying i, and noting that it is a homomorphism, we have

$$(4)\ s^1_\omega(i(C_1z \cap y)) \cap s^0_\omega(i(C_0z \cap y)) \neq 0.$$

Now z is a singleton in A and thus both

$$a = C_1z \cap y \text{ and } b = C_0z \cap y$$

are also singletons, hence are atoms below y. Indeed, if $z = \langle s_i : i \in \omega\rangle$ then an easy computation gives

$$C_0z \cap y = \{\langle \sum_{i>0} s_i - 1, s_i\rangle_{i>0}\}$$

and

$$C_1z \cap y = \{\langle s_0, s_0 + 1 - \sum_{i>1} s_i, s_i\rangle_{i>1}\};$$

which are indeed singletons. Being atoms below y, we thus have that

$$\tau(a, b) \leq w$$

hence

$$\tau(i(a), i(b)) \leq i(\tau(a, b)) \leq i(w).$$

But since $c_0i(x) \cap c_1i(x) = i(x)$ for $x \in \{a, b\}$, we get

$$\tau(i(a), i(b)) = \tau_\omega(i(a), i(b))$$

which is by (4) > 0. On the other hand, by appealing to definitions, the fact that $C_0z \cap C_1z = z$ and i is a homomorphism, a straightforward computation yields:

$$\tau(i(a), i(b)) = \tau(i(C_1z \cap y), i(C_0z \cap y))$$

$$= c_1(c_0(i(C_1z \cap y)) \cdot s^0_1c_1i(C_0z \cap y)) \cdot c_1i(C_1z \cap y) \cdot c_0i(C_0z \cap y)).$$

$$= i[C_1(C_0(C_1z \cap y) \cap S^0_1C_1(C_0z \cap y))] \cdot [i(C_1z \cap C_1y) \cap i(C_0z \cap C_0y).]$$

$$= i[C_1(C_0(C_1z \cap y) \cap S^0_1C_1(C_0z \cap y))] \cdot i[(C_0z \cap C_1z) \cdot (C_0y \cap C_1y).]$$

$$= i[C_1(C_0(C_1z \cap y) \cdot S^0_1C_1(C_0z \cap y))] \cdot i(z) \cdot i[(C_0y \cap C_1y).]$$

That is, we have

$$0 < \tau(i(a), i(b)) \leq i(z).$$

But $i(z)$ is an atom in $i(A)$ and $\tau(i(a), i(b) = i(\tau(a, b)$ is in $i(A)$. Being below $i(z)$ and different from 0, we get (since $i(z)$ is an atom) that

$$i(z) = \tau(i(a), i(b))$$

i.e.

$$i(z) \leq i(w).$$

But this contradicts (the second inclusion of (2) namely) that

$$i(z) - i(w) \neq 0.$$

We have proved that $i(w) = \tau_\alpha(i(y), i(y))$, thus

$$Sg^{\mathcal{B}} i(A) = Sg^{\mathcal{B}} i(A_0).$$

By this statement 3 and Theorem 3 are proved.

■

Next we prove Theorem 4 on polyadic algebras. Our positive result on polyadic algebras uses techniques that can be traced back to the landmark paper of Diagneualt and Monk [39]. There the notion of neat reducts, though, was treated following Halmos' terminology, under the name of compressions. In our treatment of (neat reducts) of polyadic algebras we follow the more current notation and terminology of [51]. (cf. Definition 5.4.1,and 5.4.16.) To prove Theorem 4, we shall prove the following:

Statement 4 . *Let $K \in \{PA, PEA\}$. Let α be infinite and $\beta > \alpha$. Then $Nr_\alpha K_\beta = K_\alpha$.*

Proof. We will assume that $K = PEA$. For PA the proof is easier and is left to the reader. We shall use the notation $f \upharpoonright X$ to denote the restriction of the function f to the set X. Also Id_X, or simply Id when X is clear from context, denotes the identity function on X. Now let α be infinite. Let $\mathcal{A} \in PEA_\alpha$. Let $\mathcal{A}'$ be its PA reduct , obtained by deleting the diagonal elements. Let $\beta > \alpha$. By [39] Theorem 3.3, there is a $\mathcal{B} \in PA_\beta$ such that $A \subseteq Nr_\alpha B$, and each element of B has the form

$s_\sigma a$ for some $a \in A$, and σ a permutation on β. Let $\mathcal{B}^+$ be the algebra of polyadic type with equality defined as follows:

$$Rd_{PA}\mathcal{B}^+ = \mathcal{B},$$

in [51] p.232, Thm 5.4.17. It is shown therein that $\mathcal{B}^+ \in PEA_\beta$ and that $\mathcal{A} \subseteq Nr_\alpha \mathcal{B}^+$. We claim that $Nr_\alpha B^+ \subseteq A$. Indeed let $x \in Nr_\alpha B$. Then by the above we have $x = s^B_\sigma y$, for some $y \in A$ and $\sigma \in {}^\beta\beta$. Let $\tau \in {}^\beta\beta$ such that (1)

$$\tau \restriction \alpha \subseteq Id \text{ and } (\tau \circ \sigma)\alpha \subseteq \alpha.$$

Such a τ clearly exists. Since $x \in Nr_\alpha B$, it follows by definition that $c_{(\beta \smallsetminus \alpha)} x = x$. From

$$\tau \restriction \beta \smallsetminus (\beta \smallsetminus \alpha) = \tau \restriction \alpha = Id \restriction \alpha = Id \restriction \beta \smallsetminus (\beta \smallsetminus \alpha)$$

we get from the polyadic axioms (cf.[51] Def. 5.4.1) that

$$s^B_\tau x = s^B_\tau c_{(\beta\smallsetminus\alpha)} x = s^B_{Id} c_{(\beta\smallsetminus\alpha)} x = s^B_{Id} x = x$$

Therefore (2)

$$x = s^B_\tau x = s^B_\tau s^B_\sigma x = s^B_{\tau\circ\sigma} y$$

Let

$$\mu = \tau \circ \sigma \restriction \alpha \text{ and } \mu^+ = \mu \cup Id \restriction (\beta \smallsetminus \alpha).$$

Since

$$\mu^+ \restriction \beta \smallsetminus (\beta \smallsetminus \alpha) = \mu^+ \restriction \alpha = \mu = \tau \circ \sigma \restriction \beta \smallsetminus (\beta \smallsetminus \alpha)$$

we have

$$s^B_{\mu^+} c_{(\beta\smallsetminus\alpha)} y = s^B_{\tau\circ\sigma} c_{(\beta\smallsetminus\alpha)} y.$$

Since $A \subseteq Nr_\alpha B$ and $y \in A$ we have $s^A_\mu y = s_{\mu^+}{}^B y$ and $c_{(\beta\smallsetminus\alpha)} y = y$. Therefore (3)

$$s^A_\mu y = s^B_{\mu^+} y = s^B_{\mu^+} c^B_{(\beta\smallsetminus\alpha)} y = s^B_{\tau\circ\sigma} c^B_{(\beta\smallsetminus\alpha)} y = s^B_{\tau\circ\sigma} y.$$

From (2) and (3) we get $x = s^A_\mu y \in A$. By this the proof is complete since x was arbitrary. ∎

By this the proofs of Theorems 1, 2, 3 and 4 and Corollaries 1,2,3 are complete. To summarize: We have proved that the class of neat reducts is not closed under forming subalgebras for SC,CA, QA and QEA, and used this to prove a negative result on amalgamation. In contrast for polyadic algebras, the class of neat reducts form a variety. In the next chapter we show that for other reducts of polyadic algebras, neat reducts commute with forming subalgebras, and this will be used to prove that such algebras have the strong amalgamation property.

1.4. Proofs of Theorems 6, 7 and 8.

In this subsection we prove our positive results on the strong amalgamation and embedding properties, i.e. we prove Theorems 6, 7 and 8. Throughout this subsection $K \in \{SA, QA, QEA, CA\}$. By our results so far, the operation of forming neat reducts commute with H, $P\,UP$, but does not commute with the operation of forming subalgebras. However when it does for a class of algebras, strong amalgamation follows for this class. More precisely if $L \subseteq RK_\alpha$ has AP and NS, then algebras in L can be *strongly* amalgamated inside RK_α. This will be the central theme of this section. We start by specifying the classes of algebras we will investigate:

Notation . *We let MnK_α stand for the class of minimal subalgebras in K_α and MgK_α stand for the class of monadic generated algebras in K_α. Note that in the absence of diagonal elements, i.e. for $K = \{SC, QA\}$, MnK_α consists only of the* 2 *element boolean algebra. In order to unify notation, we let DKc_α denote the class of dimension complemented algebras K algebras of dimension α.*

The proof of the following Lemma can be easily destilled from [113].

Lemma 1.5.1 . *Let α be any ordinal. Let*

$$L_K = \{\mathcal{A} \in K_{\alpha+\omega} : \mathcal{A} = Sg^{\mathcal{A}} Nr_\alpha A\}.$$

Then L_K has the strong amalgamation property.

Note that L_K consists of infinite dimensional algebras that are dimension complemented. We note that Lemma 1.5.1 is generalized to arbitrary systems of varieties definable by schemes in [132] and in chapter four. The following Lemma is due to Monk. A proof can be found in [104].

Lemma 1.5.2 .

(i) *Let $\mathcal{A} \in MgK_\alpha$. Then $\mathcal{A}$ is representable. Further more for any ω-dilation $\mathcal{B} \in \mathcal{K}_{\alpha+\omega}$ via i, we have the following property of neat reducts commuting with forming subalgebras.*

$$i(Sg^{\mathbf{A}}X) = Nr_\alpha Sg^{\mathbf{B}}i(X).$$

In particular, MgK_α and MnK_α have the NS.

Notation. . *Let κ be a cardinal and n be an ordinal. Then ${}_{<k}GKs_n$ stands for the class of generalized set algebras each of whose subbasis have cardinality $< \kappa$. ${}_{<n}GKs_n$ stands for the algebras of positive characteristic.*

We now restate and prove Theorem 6.

Theorem 6 .

(i) *MgK_α has the SAP.*

(ii) *For $n < \omega$, ${}_{<n}Gs_n$ has SAP.*

(i) For simplicity of notation we give the proof for the CA case. The other cases are entirely analagous. Let $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ be in $MgCA_\alpha$. Let $k : \mathcal{C} \to \mathcal{A}$, $h : \mathcal{C} \to \mathcal{B}$ be monomorphisms. We want to find a strong amalgam. Let $\mathcal{A}^+$, $\mathcal{B}^+$ and $\mathcal{C}^+$ be ω-dilations of $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}^+$, via i_C, i_B, i_A, respectively. We lift the given diagram to ω-extra dimensions. More precisely, we let $h^+ : \mathcal{C}^+ \to \mathcal{A}^+$ and $k^+ : \mathcal{C}^+ \to \mathcal{B}^+$ be such that

$$h^+ \circ i_C = i_B \circ h,$$

and

$$k^+ \circ i_C = i_A \circ k.$$

Replacing $\mathcal{C}$ by $Sg^{\mathcal{C}^+}(i_C(C)$, and similarly for $\mathcal{A}^+$, $\mathcal{B}^+$ we can and will assume that $\mathcal{A}^+$, $\mathcal{B}^+$ and $\mathcal{C}^+$ are in L_K as specified in Lemma 1.5.1. Then by Lemma 1.5.1 (ii) there exist $D^+ \in CA_{\alpha+\omega}$ $m^+ : A^+ \to D^+$ and $n^+ : B^+ \to D^+$. such that

$$m^+ \circ k^+ = n^+ \circ h^+$$

Then $\mathcal{D} = \mathcal{N}r_\alpha D^+$ is representable. Furthermore, it is absolutely straightforward to check that $\mathcal{D}$ is an amalgam via $m = m^+ \upharpoonright A$ and $n = n^+ \upharpoonright A$. We now show that $\mathcal{D}$ is actually a *strong* amalgam via m and n. For this purpose, it is enough to show that

$$(*) \text{ For all } a \in A \smallsetminus k(C) \text{ and } b \in B, m(a) \neq n(b).$$

$$(**) \text{ For all } b \in B \smallsetminus h(C) \text{ and } a \in A, m(a) \neq n(b).$$

By total symmetry, we shall prove only $(*)$.

So let $a \in A \smallsetminus k(C)$ and $b \in B$. We proceed to show that $m(a) \neq n(b)$. Now $i_A(a) \in A^+$ and $i_B(b) \in B^+$. We will show that $i_A(a) \notin k^+(C^+)$ by which we will be done because of the following: Because D^+ is a strong amalgam of A^+ and B^+, we have $i_A(a) \notin k^+(C^+)$ implies that for all $x \in B^+$ $m^+ \circ i_A(a) \neq n^+(x)$. But if $m(a) = n(b)$, then $m^+ i_A(a) = n^+ \circ i_B(b)$ which is a contradiction. Now $i_A(a) \in A^+$ and $i_B(b) \in B^+$. Suppose, seeking a contradiction, that $i_A(a) \in k^+(C^+)$. Then $i_A(a) = k^+(y)$ for some $y \in C^+$. Now $i_A(a)$ is α-dimensional, then $k^+(y)$ is α-dimensional, thus y is α-dimensional because k^+ is one to one. Thus $\Delta y \subseteq \alpha$. Thus we have

$$y \in Nr_\alpha(C^+) = Nr_\alpha(Sg^{C^+} i_C(C)).$$

Using that $MgCA_\alpha$ has the NS, we obtain that

$$Nr_\alpha(Sg^{C^+} i_C(C)) = Sg^{Nr_\alpha(C^+)}(i_C(C)) = i_C(C).$$

Then $y = i_C(c)$ for some $c \in C$. Now

$$i_A(a) = k^+ \circ i_C(c) = i_A \circ k(c).$$

But then, $a = k(c)$, which is a contradiction. Now we have found a strong amalgam of $\mathcal{A}$ and $\mathcal{B}$ over $\mathcal{C}$ that is representable. By taking $\mathcal{D}'$ to be the subalgebra of $\mathcal{D}$ generated by the images of $\mathcal{A}$ and $\mathcal{B}$, we get that D' is a monadic generated strong amalgam, hence the desired conclusion.

(ii) ${}_nGS_n$ has SAP because in this class epimorphisms are surjective [5] and it has AP, a result of Comer [33]. The result now readily follows, since I_nGs_n is a variety and in varieties AP and ES are equivalent to SAP. [92].

The following Corollary answers a question of Pigozzi in [113]. For the undefined notions the reader can consult [113].

Corollary . *The free monadic generated algebra has the strong interpolation property. In particular, $Fr_\omega CA_1$ has the strong interpolation property.*

.

Proof. This follows from Pigozzi [113] Theorem .

■

As another positive application to the NS, we now answer another question of Pigozzi concerning the strong embeding property SEP. We recall that SEP is the restriction of SAP when the common subalgebra is minimal i.e in the class MnK_ω. We now restate and prove Theorem 7.

Theorem 7 . *The classes RCA_ω and Di_ω have the SEP with respect to RCA_ω.*

Proof. That RCA_ω and Di_ω have SEP with respect to RCA_ω is the same as the proof of Theorem 6 upon observing that Mn_ω in which the base algebra lies has the NS. ■

An open question here, is whether Di_ω has SEP, i.e if the strong amalgam over the minimal base algebra can be chosen to be in Di_ω. To prove our final result in this chapter we use the following adaptation of a result of Nemeti [105] due to Sain, [114]. To formulate the Theorem we need to recall from [105] some notions:

Notation .

Let V a variety of boolean algebras with normal unary operators. Let $SigV$ denote the similarity type of V without the boolean operations.

(i) *An atom structure or a Kripke frame for V is a pair $\mathcal{F} = (F, R_f)$, where R_f is a binary relation for each $f \in SigV$. The class of all such structures is denoted by $Kripke(V)$.*

(ii) *Given an atom structure $\mathcal{F}$, the complex algebra $Cm\mathcal{F}$ is the algebra with universe $\wp(F)$ and for each $f \in SigV$ and $X \in P(\mathcal{F})$,*

$$f(X) = \{a \in \mathcal{F} : \mathcal{F} \models R_f(b, a)$$

$$\text{for some } b \in X\}.$$

(iii) *Special frames are those based on the stone space of $Bl\mathcal{A}$, i.e the set of ultrafilters of $Bl\mathcal{A}$. More precisely, let $\mathcal{A} \in \mathcal{V}$. Then theultrafilter frame of $\mathcal{A}$, or $\mathcal{A}_+$ for short, is the first order structure $(U, R_f)_{f \in SigV}$ where U is the set of ultrafilters of $\mathcal{A}$ and $R_f \subseteq U \times U$ is defined as follows*

$$F_1 R_f F_2 \text{ iff } f(F_1) \subseteq F_2.$$

(iv) *The canonical extension of $\mathcal{A}$ is the complex algebra of its ultrafilter frame, i.e the algebra $\mathcal{A}^+ = Cm(\mathcal{A}_+)$. V is cananical if it is closed under perfect extentions, i.e. if $\mathcal{A} \in \mathcal{V}$, then $\mathcal{A}^+ \in V$.* [33]

(v) *Given V, then $Cm^{-1}V = \{C \in Kripke(V) : CmC \in K\}$. Often in the literature, this class is denoted by $StrV$. $Cm^{-1}V$ is called the class of atom structures or Kripke frames corresponding to V.*

Theorem(Sain) . *Let V be a canonical variety of boolean algebras with unary operators and with no constant symbols other than the boolean meet and join in its similarity type. If $Cm^{-1}V$ is closed under finite direct products, then V has SEP.*

Proof. cf. [114] Lemma 1.3 p.4. ■

From this we obtain our final Theorem :

Theorem 8 . *Let $K \in \{SC, QA, RSCRQA\}$. Let α be an ordinal. Then K_α has SEP.*

Proof. We prove our theorem for the case of Pinter's substitution algebras. The other cases are entirely analagous. We start with the concrete case i.e. with RSC_α,

[33] *A classical result of Tarski and Johnson [69] says that any algebra A embeds into its canonical extension.*

α any ordinal. Let $L = Cm^{-1}RSC_\alpha$ be the class of atom structures corresponding to RSC_α. Then it is not hard to show that L consists of the concrete frames of the form $\langle V, T_i, S_i^j\rangle_{i,j\in\alpha}$ where V is a disjoint union of weak cartesian spaces[34] and T_i is the *accessibility (equivalence) relation* corresponding to the $i-th$ cylindrification C_i defined as follows: $s, t \in T_i$ iff they agree off i, i.e if $s(j) = t(j)$ for all $j \neq i$. S_i^j is the accessibility relation corresponding to substitutions as defined in [92] p. 37. A complex algebra of such a frame, is simply a generalized weak set algebra with unit V. The class of all such complex algebras coincide with RSC_α, up to isomorphism. Now L is closed under finite direct products, indeed it is closed under arbitrary direct products. Moreover, RSC_α is a canonical variety. To see this, we apply the Neat Embedding Theorem (NET). In more detail, if $\mathcal{A} \in \mathcal{RSC}_\alpha$ then $\mathcal{A} \subseteq Nr_\alpha\mathcal{B}$ for some $\mathcal{B} \in \mathcal{SC}_{\alpha+\omega}$. Then it can be proved similarly to [50] Thm 2.7.23 that $\mathcal{A}^+$, the canonical extension of $\mathcal{A}$, is a subreduct of $\mathcal{B}^+$, the canonical extension of $\mathcal{B}$, but then by the NET, it follows that $\mathcal{A}^+$ is representable. A direct application of Sain's Lemma gives the desired conclusion. We now turn to the abstract algebras, i.e. we prove that SC_α has SEP. But this follows by noting that SC_α is axiomatized by the so called *Sahlqvist* equations, hence by [150] it is canonical. Also by the duality theory of Goldblatt [45] $Cm^{-1}SC$ is characterized by horn formulas, thus is closed under (finite) direct products. Again Sain's Lemma finishes the proof. ∎

[34] These frames are more commonly referred to in the literature as α-frames, see e.g. [6] p.166, [92], in [51] such V's are referred to as *compressed spaces.*

Related results

(i) Let $K \in \{SC, QA, QEA, CA\}$. The following theorem relating NS and SAP, is proved in [132]. For simplicity of notation, we identify notationally algebras with their domains.

Theorem 10 . *Let $\mathbf{K} = \langle K_\alpha : \alpha \in Ord\rangle$ be as specified above. In particular, $Nr_\alpha K_\beta$ is defined for each $\alpha < \beta$, and is contained in K_α. Let α be an ordinal and let $L \subseteq SNr_\alpha K_{\alpha+\omega}$; the class of algebras having the neat embedding property; which coincides with the class of representable algebras, or RK_α for short. Let*

$$M = \{A \in K_{\alpha+\omega} : A = Sg^A Nr_\alpha A\}.$$

Then M has SAP. Furthermore, if there exists a lifting function $F : L \to K_{\alpha+\omega}$ such that $A \subseteq Nr_\alpha F(A)$ subject to:

$$(i)(\forall A \in L)(\forall X)(X \subseteq A) Sg^A X = Sg^{Nr_\alpha F(A)} X,$$

$$(ii)(\forall A \in L)(\forall X)(X \subseteq A) F(Sg^A X) = Sg^{(F(A)} X,$$

then L has AP with respect to RK_α. If further, algebras in L satisfy the NS, i.e.

$$(iii)(\forall A \in L)(\forall X)(X \subseteq A) Sg^A X = Nr_\alpha Sg^{F(A)} X,$$

then L has SAP with respect to to RK_α

In [132], it is further shown using Theorem 10 that all Pigozzi's positive results for CA's in [113] generalize to SC's, QA's and QEA's. In particular, semisimple and replacement algebras have the amalgmation property. On the other hand, our Theorem 10 is consistent with our conjecture 1 formulated on p.?, namely that when NS fails, then SAP fails.

(ii) A theme in this chapter is to show that several desirable properties of the class Lf_ω corresponding to $L_{\omega\omega}$ with its provability and semantical notions does not extend much further than Dc_ω the class of *dimension complemented* CA_ω. That Lf_ω has SAP was proved by Pigozzi and independently by Diagneault. Dc_ω

does not have SAP but it does when we seek the strong amalgam in the bigger class RCA_ω. This follows, by applying Theorem 10, upon noting that Dc_ω has NS. Another definability property addressed in 2.3.22 and 2.3.23 of [50] is that every simple, finitely generated Dc_α, α infinite, is generated by a single element. The question arose: For which *simple* CA_α's does "finitely" generated imply "generated by a single element". In terms of abstract model theory this amounts to asking for which extensions of first order logic does the property described in 2.2.23 in [50] hold. This property is roughly: In any maximal Theory any finite set of concepts is definable by a single concept. The following is proved in [132], extending the CA case proved in [17] to the algebras studied in this chapter:

Theorem . *Let $\alpha > 0$ and let $n \in \omega$. There exists a simple $\mathcal{A} \in WQES_\alpha$ such that $Rd_{SC}\mathcal{A}$ can be generated as an SC_α by $n+1$ elements, but cannot be generated by n elements using all the QEA_α operations.*

(iii) SC, QA, QEA and QEA are instances of what is known in the literature as systems of varieties definable by schemes, a concept originating with Monk from cylindric algebras, and further investigated in its own right by Andréka and Németi cf. [15], [51] p.260-263. Theorem 10 is generalized to the very general context of systems of varieties definable by schemas in [132], the exact generalization will be given in chapter four. In the next chapter, we shall use the connection between NS and SAP to prove that certain reducts of polyadic algebras introduced by Sain in [116] have SAP Such algebras were introduced by Sain as a solution to one version of the Finitization Problem. But Sain's algebras are *not* systems of varieties definable by schemes in the above sense, therefore a lot of fine-tuning will be involved, particularly in defining notions like neat reducts and neat embeddings.

(iv) Using the connection betwen NS and SAP it is proved in [122] that Németi's directed cylindric algebras studied by Sǵi in [121] which is another solution to a different version of the Finitization Problem (in certain non well-founded set theories) have SAP. This follows from the fact that in these algebras, (like

Sain's algebras) representability can be proved via a Neat Embedding Theorem, and furthermore neat reducts commute with forming subalgebras. [35] A similar result concerning SAP was proved by Sain concerning Tarski's relation algebras with a quasi-pairing element (QRA), and semi-associative relation algebras (WA). (cf. [114] Theorem 8.)

Open questions

Here we collect some open problem on neat reducts that we find interesting:

(i) Can $\mathcal{A}$ constructed in Theorem 1 be generated by a single element? We note that this is not trivial; making an algebra one generated involves complexity on its automorphism group, from many to just one, the structure becomes rigid. Note that Theorem 1 implies that the free K_α algebra on ω generators, briefly $Fr_\omega K_\alpha$ (hence $Fr_\beta K_\alpha$, for all $\beta \geq \omega$) is not in $Nr_\alpha K_{\alpha+1}$, for $\alpha > 1$. This follows from the fact that the latter is closed under homomorphic images, and $\mathcal{A}$, being countable, is a homomorphic image of $Fr_\omega K_\alpha$. An affirmative answer to (i) will imply that $Fr_\kappa K_\alpha \notin Nr_\alpha K_\beta$, for every cardinal $\kappa > 0$ and every $\beta > \alpha$, where $Fr_\kappa K_\alpha$ denotes the free K_α algebra on κ generators.

(ii) Theorems 2.6.71-2.7.72 in [50] were proved in [50] for dimension complemented algebras of infinite dimension. The proof very much depended on the fact that for dimension complemented algebras, generating subreducts are neat reducts. (cf. [50] Thm.2.6.67). Now that we showed in Corollary 2 that this is not generally true for arbitrary representable infinite dimensional algebras, we stongly conjecture that our example can be modified to show that Dc_α cannot be replaced by RCA_α in Theorems 2.6.71-72, too.

(iii) Andréka showed in [3] that the class $RQEA_\alpha$ $\alpha > 2$ is not finitely axiomatizable over RCA_α. (cf. [109], for definitions). In particular, for every such α there exits non-representable QEA_αs, whose CA reduct is representable, i.e in RCA_α. This prompts the following question: Let $\alpha > 2$. Let $\mathcal{A} \in QEA_\alpha$, such that

[35] In [122], it also proved that the version of the finitization problem addressed in [121] is independent from ZFC (Zermelo-Frankeal Set Theory) without the axiom of foundation, which adds a new set-theoretic dimension to the richness of the problem.

$Rd_{CA}\mathcal{A} \in Nr_\alpha CA_{\alpha+\omega}$. Notice that the class $Nr_\alpha CA_{\alpha+\omega}$ is strictly smaller than the class RCA_α, since $RCA_\alpha = SNr_\alpha CA_{\alpha+\omega}$, and the class of neat reducts is not closed under forming subalgebras. Notice too that $\mathcal{A}$ contains all first order definable elements, i.e elements defined by CA terms, using arbitrary extra dimensions. Does this force $\mathcal{A}$ to be representable as a quasipolyadic equality algebra? i.e. is $\mathbf{A} \in RQEA_\alpha$? A question related to (iii) is :

(iv) Let $\alpha > 2$. Is there an $\mathcal{A}$ such that its cylindric reduct, i.e. $Rd_{CA}\mathcal{A} \in Nr_\alpha CA_{\alpha+\omega}$, while $\mathcal{A} \notin Nr_\alpha QEA_{\alpha+1}$? Variations or weakenings of this can be obtained by substituting $QEA_{\alpha+1}$ by QEA_β, $\beta > \alpha + 1$. Here we require that it is only the p_{ij}s, that can "see" that $\mathcal{A}$ is *not* a neat reduct. Other variations on this question include: Let $\alpha > 2$. Let $\beta > \alpha$. Is there an $\mathcal{A} \in CA_\alpha$, such that its SC reduct, is in $Nr_\alpha SC_{\alpha+\omega}$, while $\mathbf{A} \notin Nr_\alpha CA_\beta$?. Here it is the *diagonal elements*, that would witness that $\mathcal{A}$ is not a neat reduct. We note that the diagonal free reduct of $\mathcal{A}$ constructed in statement 2, in symbols $Rd_{Df}\mathcal{A}$ is in $Nr_\alpha Df_\beta$, for all $\beta > \alpha$. But this follows from the trivial reason that $Df_\alpha = Nr_\alpha Df_\beta$, for every $\beta > \alpha$. (cf. [51] Theorem 5.1.31)

Finally, in the same flavour of (iv), but now formulated for "terms implicitly" definable, rather than terms "definable in extra dimensions" (cf. Remark 1.6):

(v) Let α be infinite. Is there $\mathcal{A}$, $\mathcal{B}$ in $RQEA_\omega$ and f a (homo)morphism from $\mathcal{A}$ into $\mathcal{B}$, such that $f : Rd_{CA}\mathcal{A} \to Rd_{CA}\mathcal{B}$ is an epimorphism in the category RCA_α, while f is *not* an epimorphism in the category $RQEA_\omega$?.

(vi) Let $1 < \alpha < \beta$. Is there an intrinsic characterization of the class $Nr_\alpha CA_\beta$?. i.e can one tell without referring to mathematical objects outside a given CA_α that it embeds neatly in a CA_β? A lot of research has been done concerning analagous characterizations for the class $RaCA_\beta$ of forming relation algebraic neat reducts.

(vii) Does Df_ω have AP?

Chapter 3

On amalgamation of reducts of polyadic algebras

Abstract

This chapter deals with certain reducts of polyadic algebras, that are relevant to the Finitization Problem in algebraic logic. Following research initiated by Tarski, Craig and Németi, and further pursued by Sain and others, we show that for certain subsets G of ${}^{\omega}\omega$, G polyadic algebras have the strong amalgamation property. G polyadic algebras are obtained by restricting (the similarity type) and axiomatization of ω-dimensional polyadic algebras to finite quantifiers and substitutions in G. Using algebraic logic, we infer that some theorems of Beth, Craig and Robinson hold for certain proper extensions of first order logic (without equality.)

Key words: neat reducts, polyadic algebras, rich semigroups, amalgamation.

Introduction: Background and History

In the 1950's I set out to revolutionize logic. I didn't succeed (not yet!), but it wasn't for want of trying.

Paul Halmos in [48].

Unless otherwise specified, algebra(s) is short for algebra(s) of infinite dimension. Polyadic algebras were introduced by Halmos to provide an algebraic reflection of the study of first order logic without equality. Later the algebras were enriched by diagonal elements to permit the discussion of equality. That the notion is indeed an adequate reflection of first order logic was demonstrated by Halmos' representation theorem for locally finite polyadic algebras (with and without equality). Daigneault and Monk [39] proved a strong extension of Halmos' theorem, namely, every polyadic algebra (without equality) is representable. The latter a typical Stone-like representation theorem, shows that the notion of polyadic algebra is indeed an adequate reflection of Keisler's predicate logic (KL).

KL is a proper extension of first order logic without equality, obtained when the bound on the number of variables in formulas is relaxed; and accordingly allowing the following as extra operations on formulas: Quantification on infinitely many variables and simultaneous substitution of (infinitely many) variables for variables. Adding equality to KL, proved problematic as illustrated algebraically by Johnson in [70]. In this paper, Johnson showed that the class of representable polyadic algebras with equality is not closed under ultraproducts, hence this class is not elementary, i.e. cannot be axiomatized by any set of first order sentences. However one can still hope for a nice axiomatization of the *variety generated* by the class of polyadic equality algebras. A subtle recent (negative) result in this direction was found by Németi and Sági: In sharp contrast to KL, the validities of KL *with equality* cannot be recaptured by any set of schemas analogous to Halmos' schemas, let alone a finite one (cf. [110]). In particular, the variety generated by the class of representable polyadic algebras with equality cannot be axiomatized by a finite schema of equations. The latter answers a question originally raised by Craig. It is interesting (and, indeed, natural) to ask for algebraic versions of model theoretic results, other than completeness, reflected algebraically by Stone-like representability results. Daigneault

[36] succeeded in stating and proving versions of Beth's and Craig's theorems. This was done by proving the algebraic analogue of Robinson's joint consistency theorem: *Locally finite* polyadic algebras (with and without equality) have the amalgamation property. Later Johnson removed the condition of local finiteness, proving a strong extention of Diagneault's result, namely that polyadic algebras *without* equality have the *strong* amalgamation property [71].[1] With this stronger result, Robinson's, Beth's and Craig's theorems hold for KL. It is open whether any of the above (interpolation) properties still holds when we add equality to KL. However KL has negative properties due to its enormous expressive power. A pecularity of KL proved by Sági [119] is: Though the set of validities can be recaptured by a finite schema, namely Halmos' schemas, this set is not recursively enumerable. Here of course, one has to be careful as to what "recursively enumerable" means in languages with uncountably many operation symbols. In this connection, we refer the reader to [124]. The above deep results of Németi and Sági point in the direction of the limited relevance of polyadic theory for "finitary" extensions of first order logic. Tarski's cylindric algebras, and for that matter Halmos' quasi-polyadic algebras, it seems, are more appropriate for studying such extensions. We will not give a precise definition of finitary, but we believe that the following is sharp enough to give an intuitive idea. A finitary extension of first order logic should be *structural* in the sense of [28]. This is necessary to admit algebraic treatment. Probably more important, is that in a finitary extension of first order logic completeness should imply recursive enumerability of the validities; an implication that does not hold for KL for non trivial reasons as demonstrated by Sági in [119] and [124], chapter 4. *Finitary logics of infinitary relations* was the name given by Andréka and Németi see e.g. [9] to the logics corresponding to the class of representable cylindric algebras and variants thereof like Halmos' quasi-polyadic algebras, and Pinter's substitution algebras. In these logics, formulas of infinite length *are* allowed (recall that a formula is a sequence of symbols); the operations though

[1] In locally finite algebras, amalgamation is equivalent to strong amalgamation, see e.g. [113] Thm 2.2.8 and Remark 2.2.9. In the general case, however strong amalgamation is indeed stronger. The class of diagonal cylindric algebras, or Di_ω for short , for instance has the amalgamation property as proved by Pigozzi in [113], but it does not have strong amalgamation property as shown in the previous chapter. An even stronger unpublished result of Judit Madarasz is that Di_ω does not have the amalgamation property even with respect to the class RCA_ω.

of quantification and substitutions are only restricted to finitely many variables at a time. Here we emphasize that it is only (atomic) formulas that are allowed to be infinite, while the scope of the logical connectives, like in the case of first order logic, is finite. In particular, infinite conjunctions (hence disjunctions) and infinite quantifiers are not permitted. These logics are referred to as *typeless logics* in [51] sec 4.3., the adjective typeless referring to abstracting away the arity of relation symbols in the formalism. Andréka, Gergely and Németi prove in [9] that the set of validities of such logics is recursively enumerable. Unfortunately, it turns out that the set of validities of these logics cannot be captured by a finite schema in the sense of Monk [95]. This follows from Monk's celebrated algebraic result [95]: The class of representable cylindric algebras is *not* finite schema axiomatizable. Monk's result was generalized to Halmos' quasipolyadic algebras (with and without equality) by Sain and Thompson [118], the latter being preceded by a relevant result of Johnson [70].[2] These negative results were considerably sharpened by Andréka [3] when she showed that any axiomatization of several variants and extensions of the above classes, besides being "essentially infinite", has to be extremely complex. Recently, Venema showed [152] that these varieties cannot be axiomatized by the so-called *Sahlqvist* equations. Hodkinson and Mikulas [66] show that such negative results cannot be avoided when we pass to certain reducts. All this suggests that the notion of representability for cylindric algebras and quasipolyadic algebras is unruly, subtle and difficult to capture. Any such impression may be reinforced by a look at existing axiomatizations of such classes, for example in [51] and [3]: they are not easy to understand. Indeed [50] p. 461 identifies one of the two outstanding problems of the representation theory as '.... the problem of providing a simple intrinsic characterization for all representable cylindric algebras...'

Also the theorems of Robinson, Beth and Craig fail for finitary logics of infinitary relations. This was proved algebraically much earlier though by Pigozzi in his landmark paper [113] when he showed that: Any class of cylindric algebras, containing the class of representable ones, does not have the amalgamation property. Most questions

[2]In [70], Johnson proved non-finite axiomatizability of finite dimensional representable polyadic algebras, of dimension > 2. An open question here, is whether this class can be axiomatized by a set of first sentences containing only finitely many variables [118]. The analagous result for quasipolyadic *equality* algebras is proved by Andreka [3].

concerning strong amalgamation for classes of representable cylindric algebras were left as open problems, cf. the tables on p. 366-367 of [113]. In the previous chapter and in [80], all Pigozzi's remaining open questions concerning strong amalgamation are answered negatively.

The above (both positive and negative) results prompted the search for classes of algebras that lie "somewhere" between the cylindric paradigm and the polyadic one, so to speak, and at the same time enjoy the positive properties of both.

This problem can be seen as one instance of the so-called *Finitization Problem* [3] in algebraic logic which very roughly states: Find a structural extension of first order logic where the class of algebras associated to this logic is axiomatizable by a nice (simple, preferably finite), set of mathematically elegant schema of equations or quasi-equations. A harder version of the problem which appeared in Henkin-Monk [49] asks for a strictly finite axiomatization. We quote Henkin-Monk's formulation of this problem.

Devise an algebraic version of predicate logic in which the class of representable algebras forms a finitely based variety.

Here finitely based means finitely axiomatizable (by equations.) According to Henkin and Monk [49] and Németi (cf. [109] p. 9), positive solutions to the finitization problem have implications also for (non-algebraic) logic.

Indeed, the Finitization Problem has initiated a lot of fruitful research and is still creating employment to the present day. There are (at least) five different appoaches to this problem to circumvent or sidestep the unavoidable complexity of the hitherto existing axiomatizations to the algebras corresponding to the structural algebraisable modifications of first order logic (i.e. the n-variable restricted version of first order logic, and the typless logics or finitary logics of infinitary relations).

(1) Providing non-orthodox rules, coming from modal logic, to obtain completeness

[3] The Finitization Problem, briefly discussed in chapter 1, can be traced back to the works of Pierce and Schróder in the 19'th century. Tarski put it on a modern footing in around 1940. The book [149] carries out the project of finitizing set theory by formalizing it in the calculus of relation algebras (RA's). Németi carried out a similar task for (SA's), the class of semi-associative relation algebras introduced by Maddux. The Finitization Problem is the more ambitious attempt to "finitize" first order logic, not only set theory. The reader is referred to [109] and [116] and [124] for a comprehensive survey of the Finitization Problem. We should point out that [124] is devoted to this problem alone.

theorems. This typically involves providing Gabbay style -rules. In this connection we mention the references [150],[151],[148], [94] and [92]. One algebraic form of such Gabbay rules is the notion of density invesigated in e.g. [7], [151] and [94].

(2) Changing the semantics, broadening the permissable models. Instead of seeking (classical) representations on cartesian squares of the form ${}^{\alpha}U$, or disjoint unions of those, one allows only a part $V \subseteq {}^{\alpha}U$. From the modal point of view, in contrast to what happens in the semantics of first order logic, here the quantification over the model is restricted to an *accessible* part of the structure. This approach is referred to in the literature as *relativization* and is related to *dynamic logic*, cf.[83], [24] , [92] and [90].[4] Relativizing or "modalizing" set algebras yield variants of the n-variable fragment of first order logic differing from the classical Tarskian view because the unit of the algebras in question may not be of the form ${}^{n}U$, but could be any subset thereof. These mutant logics are under intensive study at the present time and we cite [10] and [93] as sources. [5]

(3) To get positive results one changes the ontology, i.e. the underlying set theory. This is done by dropping the axiom of foundation, and possibly adopting weaker axioms. This approach is due to Maddux. In this context, Németi proves that solutions to (several versions of) the finitization problem is independent from Aczel set theory, i.e. ZF (Zermelo Fraenkel set theory) minus foundation, cf. [106], [117] and [121]. It is open whether any of these solutions is independent of ZF.

(4) A more adventureous approach is not to change either the existing class of algebras nor the underlying set theory, but instead try to provide an axiomatization of the class of representable cylindric algebras that is intuitive and

[4]In [90] shows how relativization leads to finite axiomatizability decidability and amalgamation.

[5]In this connection, we mention that Henkin generalized semantics for second (or higher) order logics , in order to sidestep Godels's incompleteness results, making the set of validities (in this new semantics) recursively enumarable. A remarkable result of Simon, in this connection, is that any CA_3 can be obtained by twisting and dilating a Cs_3. This shows that every CA_3 can be represented in a generalized (but concrete) way. This sheds light on the "distance" between the abstract and concrete algebras and can be seen as a dual to the Finitization Problem.

comprehensible inspite of the restrictions of the above mentioned negative results. A poineering paper worthy of note in this connection, is Lyndon's [?] where Lyndon initiates a step-by-step approach to such investigations, in the context of relation algebras. [6] This approach was considerably generalized and enhanced by Ian Hodkinson and Robin Hirsch to encompass more complex situations. In their recent book [60], Hirsch and Hodkinson axiomatize almost all pseudo-elementary classes (i.e. reducts of elementary classes) existing in the literature of algebraic logic, via synthesing axioms by games, cf. [59].

Yet, another research direction initiated by Craig [34], and further pursued by Németi, Sain, Thompson and others, cf. [118] and [116], is the following:

Search for the desired "nice" class of algebras among the uncountably [7] *many reducts of polyadic algebras of dimension α, α an infinite ordinal. It is this line of research that we will be concerned with throughout this chapter.*

Let us be more concrete. Craig observed that for any subset G [8] of transformations on α, we have a potentially interesting reduct of α-dimensional polyadic algebras obtained by restricting the similarity type of the latter to substitutions indexed by elements in G, and to finite quantifiers on α. Such algebras are called α-dimensional G polyadic algebras. (The precise definition will be given below.)

Here the ordinal α specifies the length of the sequence of variables available in the corresponding logic. Let it be the least infinite, i.e. assume that the sequence of variables in our logic is of order type ω. Fixing the dimension at ω, such algebras will be referred to simply as G algebras. Sain showed in [116] that the Daigneault-Monk strong representation theorem (for polyadic algebras) still holds for various G algebras, pending on the structure of $[G]$, the semigroup generated by G. Abstracting away such properties, Sain called such semigroups *rich.* An example of a rich semigroup is the full semigroup ${}^{\omega}\omega$. But ${}^{\omega}\omega$ algebras are amenable to Sági's negative

[6]We should mention that Lyndon was also the first to construct non-representable relation algebras based on projective goemetries. Such counterexamples were very influential. Monk soon used these to show that RRA is not finitely axiomatizable. In [124] certain refinements of Lyndon's algebras were used to settle a rather general form of the finitization problem.

[7]*Having 2^{α} many operations, there are $2^{2^{\alpha}}$ many reducts of polyadic algebras for infinite α.*

[8]Craig though considered *semigroups* of transformations i.e sets of transformations closed under the operation of composition; for any G Craig rather considered the semigroup generated by G. His approach is therefore often referred to in the literature as the *semigroup approach* cf. [109].

results. Another far more interesting example, at least in so far as it survives Sági's negative results, is the semigroup generated by the set of finite transformations on ω together with the successor and predecessor functions on ω. What Sain proved in [116] is that Halmos' finite schemas restricted to the similarity type of G algebras, *does capture* the infinity of the validities of the corresponding logic, as long as $[G]$ is rich. The non-boolean operations of G algebras are all unary normal operators or modalities, i.e. they distribute over the boolean join. Like KL, the logic corresponding to G algebras - referred to as KL_G (KL restricted to G) - is a proper extension of first order logic *without* equality. In this logic, formulas can be of length ω. In addition, to the boolean connectives and usual quantification, (i.e. quantification on finitely many variables), the additional (possibly infinitary) operations of simultaneous substitutions of variables for variables corresponding to the transformations in $[G]$, are allowed. Also at least one infinitary substitution should exist in such languages, for else the logic considered reduces to an equality free version of the typless logic studied in [51] sec 4.3., and this logic, as well known, is not complete [118]. But again infinite conjunctions and quantification on infinitely many variables are not allowed. However KL_G is strictly stronger in expressive power than first order logic without equality since it allows infinitary atomic formulas. Moreover, Sain proved that when $[G]$ happens to be also *finitely presented*, then Halmos' schemas can be truncated to be strictly finite. In this case, KL_G admit a finite complete Hilbert style axiomatization which involves only *type- free* valid formulas in the sense of [51] 4.3.65. The notion of type-free valid formulas is a natural new logical notion coming from algebraic logic. (We will further elaborate on the notion of type-free valid formulas in Section 2.) The problem of truly "finitizing" first order logic without equality, thus reduces to the following problem in semigroups:

Find a "simple" rich finitely presented semigroup on ω.

In [116] a rich finitely presented semigroup is given, and the question to find simpler such semigroups is posed. We refer the reader to [116] and [99] p. 732 for several candidates of such semigroups. As pointed out by Monk [99], the confirmation that any of these semigroups, is finitely presented, would yield a rather nice satisfactory solution to one of the central and indeed classic problems of algebraic logic.

The main results

In this chapter, we show that for countable and rich $[G]$ satisfying an extra "finitary" condition, the class of G algebras also have the strong amalgamation property, i.e. if two G algebras have a given common subalgebra then they can be embedded in another G algebra in such a way that the intersection of the images of the two algebras is the image of the given common subalgebra. We will first prove our result for a specific rich semigroup, namely the one generated by all finite transformations together with the successor and predecessor functions on ω. This semigroup is rich, but not finitely presented. This special case, we believe, is easier to visualize and is general enough to illustrate the main ideas. We show how our proof generalizes to the case when the semigroup considered is countable and what we call *strongly rich.* Using the methods of algebraic logic (see e.g. [11] or [113]), we infer that for a given countable set of transformations G, as long as $[G]$ is strongly rich, the theorems of Craig, Beth and Robinson hold for KL_G. Also, for "good" choices of G, KL_G can be chosen to be a finitary extension of first order logic as well as an algorithmic reduct of KL. Compactness and completeness of KL_G was proved by Sain, cf. [115] and [116]. We do not know whether any of the above interpolation properties holds when we enrich our language with equality. The work in this chapter is based on the article [130] and the talk [131].[9]

Organization

The layout of this chapter is as follows. In section 0 we formulate our results. Proofs are given in section 1. Finally in section 2 we discuss related results and put forward some open questions.

0. Formulation of the main results

In preparation of formulating our main result algebraically, we review the basic notions to be used in the sequel. We start off by defining certain concrete set-theoretic

[9] In fact, this chapter as is will appear in Algebra Universalis [130].

operations on sets of sequences, then we specify our algebras. Our notation throughout this chapter is mostly standard (see e.g. [50] or [31]). Less usual needed notations will be introduced at their first occurrence in the text.

Let X be a set. Then $\wp(X)$ denotes the set of all subsets (the powerset) of X and $B(X)$ denotes the boolean set algebra with universe $\wp(X)$. That is

$$B(X) = \langle \wp(X), \cap, \cup, \smallsetminus_X, \emptyset, X\rangle.$$

When the unit X is clear from context, we write $\smallsetminus$ instead of $\smallsetminus_X$. Let α be an ordinal. Then ${}^{\alpha}U$ denotes the set of all U termed sequences of length α. Elements in ${}^{\alpha}\alpha$ will be referred to as *transformations* on α.

The non-boolean operations we deal with are the usual unary operations of cylindrifications and substitutions indexed by arbitrary transformations. More precisely:

Let $V \subseteq {}^{\alpha}U$, $i < \alpha$ and $\tau \in {}^{\alpha}\alpha$. Then C_i^V and S_τ^V are unary functions from $\wp(V)$ into itself defined for $X \subseteq V$ as follows:

$$C_i^V(X) = \{s \in V : \text{ there exists } t \in X \text{ such that } s(j) = t(j) \text{ for all } j \neq i\};$$

and

$$S_\tau^V(X) = \{s \in V : s \circ \tau \in X\}.$$

Here, and elsewhere throughout this chapter , composition of functions is defined so that the righthand most function acts first. In particular, $s \circ \tau(x) = s(\tau(x))$. When no confusion is likely to ensue, superscripts are usually omitted. For $i \in \alpha$, C_i is called the i-th *cylindrification*. For $\tau \in {}^{\alpha}\alpha$, S_τ is called the *substitution* operation corresponding to τ, or simply a *substitution*.

Let U be a set and $p \in {}^{\alpha}U$. Then ${}^{\alpha}U^{(p)}$ denotes the set

$$\{s \in {}^{\alpha}U : \{i \in \alpha : s_i \neq p_i\} \text{ is finite }\}.$$

${}^{\alpha}U^{(p)}$ is more commonly referred to in the literature as *the weak space* of dimension α, determined by p. U is called its *base*. (cf. [50] Def. 3.1.5. and chapter two.) $\diamondsuit$

Now we turn to define certain expansions of boolean set algebras, that are also reducts of Halmos' polyadic algebras. The units of such algebras are unions of weak spaces of dimension α, and their elements are thus α-ary relations. The operations, besides the familiar boolean set operations, are the extra (unary) operations of cylindrifications and substitutions indexed by a fixed set G of transformations on α. In our subsequent discussion we will impose further restrictions on α and G.

Definition 0.1. Concrete G algebras .

Let α be an ordinal and $G \subseteq {}^{\alpha}\alpha$. By a G polyadic set algebra of dimension α, a $GPSA_{\alpha}$ for short, we understand a subalgebra:

$$\mathcal{A} \subseteq \langle B(V), C_i, S_{\tau}\rangle_{i\in\alpha,\tau\in G}$$

where V, the unit of $\mathcal{A}$, is a union of weak spaces having a common base U, i.e. V is of the form $\cup_{i\in I}{}^{\alpha}U^{(p_i)}$, for a non-empty set I, and U termed sequences p_i $(i \in I)$. $\diamondsuit$

Following the conventions of the monograph [50], $GPSA_{\alpha}$ also stands for the class of all $GPSA_{\alpha}$'s. When α is clear from context, we refer to $GPSA_{\alpha}$'s as G set algebras or concrete G algebras. Here "concrete" refers to the fact that elements of the algebras are *genuine* α-ary relations and the operations are set theoretic ones, utilizing the form of the elements as α-ary sequences. We refer the reader to [109] for a discussion of the notion "set theoretic". Later on (cf. Definition 1.2 below) we will define abstract G algebras. These will be defined by a set of equations that are valid in the (more) concrete class of G set algebras.

Remarks on Definition 0.1.

(1) When G is empty then the class of G set algebras reduces to the case of the so-called diagonal free cylindric algebras, the theory of which is well developed, see e.g. [51]. We will be primarily concerned with the case when $\alpha = \omega$ (the least infinite ordinal) and a specific set of transformations G on ω, to be specified shortly. We should mention that Definition 0.1, in this general form, covers almost all CA-like (diagonal-free) algebras of relations, where CA abbreviates cylindric algebras. These include the concrete versions of Pinter's algebras and Halmos' quasipolyadic algebras (without equality), the so-called representable algebras. Here by representable we mean isomorphic copies of subdirect products of set algebras, cf. [118],[116] and chapter two.

(2) The unit of G algebras as indicated in Definition 0.1 is one instance of a *compressed space* in the sense of [51] Def. 3.1.5. This choice of units as unions of weak spaces (having a common base) is not haphazard. The reader is referred to [116] for a fuller discussion of the reasons of this choice. One important reason, but not the only one, is that this choice of units makes our algebras boolean algebras with normal unary operators. Actually in [116] the definition of G set algebras is apparently broader allowing units of the form $V = \cup_{i\in I}{}^{\omega}U_i^{(p_i)}$ for some sets I, U_i and functions $p_i \in {}^{\omega}U_i$ $(i \in I)$ such that $U_i \cap U_j = \emptyset$ for all distinct $i, j \in I$. These units together with the compressed spaces addressed in our Definition 0.1 are referred to in [116] as $\mathbf{G}ws_{\omega}$ units. Here $\mathbf{G}ws_{\omega}$ denotes the class of *generalized weak set cylindric algebras* in the sense of [51] Definition 3.1. But for G algebras considered herein, any G set algebra in the sense of [116] is isomorphic to one having a unit as specified in Definition 0.1. (A proof of this can be easily recovered from our proof of Proposition 1.)

(3) Let $\mathcal{A}$ be a G set algebra, and let $\langle [G], \circ\rangle$, or simply $[G]$ be the semigroup generated by G. Then every $\tau \in [G]$ induces a unary operation s_{τ} on $\mathcal{A}$. (cf. Definition 1.1 below). These unary operations are defined uniformly so that $GSPA_{\alpha}$ is term-definitionally equivalent to $[G]SPA_{\alpha}$. Therefore it is no surprise that problems about axiomatizing $GPSA_{\alpha}$ often reduces to problems about the semigroup $[G]$ as was first observed by William Craig. This approach, the so-called *the semigroup approach to finitizing first order logic* was further pursued by Németi [109], Sain, Thompson [118], [116] and others, and is still an active part of research in algebraic logic. In this connection, the reader is referred to [109], [99] and [116] for open questions concerning (the semigroup approach to) the finitization problem.

(4) Let $\mathcal{A}$ be a G set algebra with base U and $\tau \in [G]$. Then the unary operation s_{τ} is *permutation invariant*, cf. [116]. That is if $p : U \to U$ is a permutation then p commutes with s_{τ} in the following sense:

$$\bar{p}(s_{\tau}(X)) = s_{\tau}\bar{p}(X) \text{ for any } X \in A.$$

We recall from [51] that for any permutation $p : U \to U$ and $Y \subseteq {}^{\omega}U$,

$\bar{p}(Y) = \{p \circ q : q \in Y\}$. Tarski called such (permutation invariant) operations *logical* [149]. This requirement - that the newly added operations to set algebras be logical in the above sense - was explicitly indicated among the conditions in the book Tarski-Givant [149] when formulating problems related to the finitization problem (e.g. on p. 60 there). We refer the reader to [109] and [116] for further elaboration on this requirement, in connection to the finitization problem. The logical counterpart of this requirement is that in the corresponding logic, isomorphic models satisfy the same formulas. This is one of the basic axioms of abstract Model Theory. In [109] it is shown that without this requirement there are rather easy solutions to the finitization problem.

(End of Remarks on Definition 0.1.) $\diamondsuit$

ω denotes the least infinite ordinal. We will show that for certain subsets $G \subseteq {}^{\omega}\omega$, the class $GPSA_{\omega}$ has the strong amalgamation property. To formulate our main result, we recall from [113] the notion of strong amalgamation, [10] then we specify a set G (of transformations on ω) that we shall be mostly dealing with.

Definition 0.2 . *Let K be a class of algebras. Then K is said to have the amalgamation property, if for any $\mathcal{A}, \mathcal{B}, \mathcal{C} \in K$, and any monomorphisms f, g of $\mathcal{A}$ into $\mathcal{B}, \mathcal{C}$ respectively, there exists $\mathcal{D}$ and a monomorphism m from $\mathcal{B}$ into $\mathcal{D}$ and a monomorphism n from $\mathcal{C}$ into $\mathcal{D}$ such that $m \circ f = n \circ g$. If, in addition, m and n can be chosen such that $m(B) \cap n(C) = m \circ f(A)$ is satisfied, then K is said to have the strong amalgamation property.* $\diamondsuit$

Concerning the transformations we shall mostly deal with, we need:

Notation .

(i) *Let α be an ordinal and $i, j \in \alpha$. Then $[i|j]$ denotes the replacement that maps i to j and leaves the rest of the elements of α fixed. $[i, j]$ denotes the transposition that interchanges i and j and leaves the rest of the elements fixed.*

[10]The definition of strong amalgamation was given in chapter two. To keep every chapter self-contained we recall the definition again. We apolgize to the reader for such and similar repititions (in definitions in different chapters). We (very) occasionally do this, to stick to our promise that every chapter is completely self-contained and can be read independently.

(ii) *suc is an abbreviation for the successor function on ω. That is $suc(i) = i + 1$, for every $i \in \omega$. pred denotes the predecessor function on ω, i.e. $pred(0) = 0$ and $pred(i) = i - 1$ for $i \in \omega \smallsetminus \{0\}$.* $\diamondsuit$

Now we are ready to formulate our main result:

Theorem 1 (Main Theorem) . *Let $G = \{[i|j], [i, j] :, i, j \in \omega\} \cup \{suc, pred\}$. Then $GPSA_\omega$ has the strong amalgamation property.*

Theorem 1 is proved in section 1. Let G be as specified in Theorem 1. Then, $[G]$ the semigroup generated by G, is *rich* in the sense of [116]. A rich semigroup is, loosely speaking, one that contains all finite transformations together with an infinitary non-surjective transformation , a *suc*-like element, and its quasi-inverse, a *pred*-like one. We now define certain rich semigroups, by imposing an extra (finitary) condition on the elements acting like *suc* and *pred*. We call such semigroups *strongly rich*. In preparation of defining strongly rich semigroups we fix some needed notation.

Notation. . *For a set X, $|X|$ stands for the cardinality of X. Id_X, or simply Id when X is clear from context, denotes the identity function on X. For functions f and g and a set H, $f[H|g]$ is the function that agrees with g on H, and is otherwise equal to f. $Rng(f)$ denotes the range of f. For a transformation τ on ω, the support of τ, or $supp(\tau)$ for short, is the set:*

$$supp(\tau) = \{i \in \omega : \tau(i) \neq i\}.$$

Let $i, j \in \omega$, then $\tau[i|j]$ is the transformation on ω defined as follows:

$$\tau[i|j](x) = \tau(x) \text{ if } x \neq i \text{ and } \tau[i|j](i) = j.$$

For a function f, f^n denotes the composition $f \circ f \ldots \circ f$ n times.

Definition 0.3. Strongly Rich Semigroups .

(a) Let $T \subseteq \langle {}^\omega\omega, \circ\rangle$ be a semigroup. We say that T is rich if T satisfies the following conditions:

$$(1) \qquad (\forall i, j \in \omega)(\forall \tau \in T)\tau[i|j] \in T.$$

(2) There exists $\sigma, \pi \in T$ such that

$$(\pi \circ \sigma = Id,\ Rng(\sigma) \neq \omega),$$

and

$$(\forall \tau \in T)(\sigma \circ \tau \circ \pi)[(\omega \smallsetminus Rng(\sigma))|Id] \in T.$$

(b) Let $T \subseteq \langle {}^{\omega}\omega, \circ\rangle$ be a rich semigroup. Let σ and π be as in (a) above. If σ and π satisfy (i), (ii) below:

$$(i)(\forall n \in \omega)|supp(\sigma^n \circ \pi^n)| < \omega.$$

$$(ii)(\forall n \in \omega)[supp(\sigma^n \circ \pi^n) \subseteq \omega \smallsetminus Rng(\sigma^n)];$$

then we say that T is a strongly rich semigroup. $\diamondsuit$

The notion of a rich semigroup, i.e. a semigroup satisfying Definition 0.3 (a) above, is due to Sain. The simplest example of a rich semigroup is ${}^{\omega}\omega$. $[G]$ as specified in Theorem 1 is also rich by taking $\sigma = suc$ and $\pi = pred$. Indeed, both semigroups are strongly rich. We refer the reader to [115] and [116] for a comprehensive and detailed discussion of the important novel notion of rich semigroups in (connection to) the (representation) theory of polyadic algebras.

Generalizing Theorem 1 to the case when [G] is countable and strongly rich, we shall also prove:

Theorem 1^* . *Let $G \subseteq {}^{\omega}\omega$ be countable. Assume that $[G]$, the semigroup generated by G, is strongly rich. Then $GPSA_{\omega}$ has the strong amalgamation property.*

Note that a strongly rich semigroup contains necessarily a non-surjective transformation, namely σ, which in our concrete case formulated in Theorem 1 is the successor function on ω. Compare with the contrasting result formulated as statement 5 in the previous chapter , cf. [80]. Namely, when G consists only of surjective maps, then strong amalgamation *fails* for G (set) algebras.

From now on, unless otherwise specified, G is as specified in Theorem 1^*. [11] To prove Theorem 1^*, we shall show that the countable $\mathbf{I}GPSA_\omega$ free algebras satisfy the algebraic version of Craig's interpolation property when G is strongly rich. Here $\mathbf{I}$ stands for the operation of taking all isomorphic copies. To make this precise we need:

Notation .

(i) Given a class K and a set X, $Fr_X K$ denotes the K algebra freely generated by X. Recall that $Fr_X K$ is uniquely determined up to isomorphism by $|X|$ and K. Henceforth, however, when we write $Fr_X K$, we shall be tacitly assuming that $X \subseteq Fr_X K$.

(ii) Let $\mathcal{A}$ be an algebra and $X \subseteq A$. Then $Sg^{\mathcal{A}}X$ or simply SgX, when $\mathbf{A}$ is clear from context, denotes the subalgebra of $\mathcal{A}$ generated by X. $\diamondsuit$

Definition 0.4. . *Let K be a class of algebras with a boolean reduct* [12] *and X be a set. Then $Fr_X K$ is said to have the interpolation property if whenever $X_1, X_2 \subseteq X$, $a \in SgX_1$ and $c \in SgX_2$ are such that $a \leq c$, then there exists $b \in Sg(X_1 \cap X_2)$ such that $a \leq b \leq c$. Here $\leq$ is the usual boolean ordering.* $\diamondsuit$

As previously noted, the above is the algebraic counterpart of the Craig interpolation property in logic. Accordingly we refer to b as an interpolant of a and c.

Fact 0.6 below, essentially due to Maksimova [87], (see also [76]) is the cornerstone on which our proof relies. But first we recall from [87] the notion of super amalgamation.

Definition 0.5. . *Let K be a class of algebras with a boolean reduct. Let $\leq$ denote the boolean order. We say that K has the super amalgamation property, or $SUPAP$ for short, if for any $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ in K and for any monomorphisms f, g of $\mathcal{A}$ into $\mathcal{B}, \mathcal{C}$ respectively, there exists $\mathcal{D}$ and a monomorphism m from $\mathcal{B}$ into $\mathcal{D}$ and a*

[11] In passing we note that adding transpositions to G as specified in Theorem 1, is a redundancy since $[i,j] = pred \circ [0|j+1] \circ [j+1|i+1] \circ [i+1|0] \circ suc$.

[12] *The greatest and least elements are explicit in the similarity type of boolean algebras. In particular, SgX - containing at least these elements - is not empty for any X.*

monomorphism n from $\mathcal{C}$ into $\mathcal{D}$ such that $m \circ f = n \circ g$ and such that for all $x \in B$ for all $y \in C$

$$m(x) \leq n(y) \implies (\exists z \in A)(x \leq f(z) \text{ and } g(z) \leq y)$$

and

$$n(y) \leq m(x) \implies (\exists z \in A)(y \leq g(z) \text{ and } f(z) \leq x). \ \diamondsuit$$

The notion of super amalgamation as introduced by Maksimova is actually stronger than that of strong amalgamation, when algebras considered have a boolean reduct and the order is the natural boolean order [87]. For arbitrary orders though, this might not be true [77]. However for cylindric like algebras of relations - including G algebras - the two notions seem to coincide. Under favarouble conditions on K, super amalgamation turns out to be equivalent to the interpolation property on the free K algebras [87]. More precisely (the form we need is) :

Fact 0.6 (Maksimova) . *Let K be a variety with a countable signature. Suppose further that every $\mathcal{A} \in K$ is a boolean algebra with normal unary operators, briefly a BAO. Then the following two conditions are equivalent:*

(i) $Fr_X K$ has the interpolation property, for every countable X.

(ii) K has the super amalgamation property.

The fact that $\mathbf{I}GPSA_\omega$ is a variety of BAO's was proved by Sain in [116]. In fact, in Definition 1.2 below we shall exhibit a set of equations (also due to Sain) axiomatizing the class $\mathbf{I}GPSA_\omega$. To show that $\mathbf{I}GPSA_\omega$ has the strong (and indeed the super) amalgamation property it is enough, in view of Fact 0.6 to prove the following:

Proposition 1 . *Let X be countable. Let G be as specified in Theorem 1^*. Then $Fr_X(\mathbf{I}GPSA_\omega)$ has the interpolation property.*

A historical Remark

We should mention that Fact 0.6 can be traced back to Pigozzi [[113], Theorem 1.2.8] where he relates the local property of the conguence extension property of the free algebras of a given class V, to the global one of amalgamation. Also Pigozzi [[113], Definition 2.1.16] introduced the notion of interpolation property for cylindric algebras, which he called *strong interpolation.* The interpolation property was reserved for a different weaker form of interpolation [[113], Definition 2.1.8 (i).] Pigozzi proved in [[113], Theorem 2.1.20] that for a variety K of cylindric algebras (i) in Fact 0.6 implies that K has the strong amalgamation property. Whether the converse holds (i.e. whether (ii) implies (i) for varieties of cylindric algebras), is posed as an open question in Pigozzi [113] cf. Remark 2.1.21 on p. 313 therein. In view of the equivalence established for BAO's by Maksimova, this question amounts (in today's terminology) to asking for a variety V of cylindric algebras that has SAP but not $SUPAP$. We note that Maksimova's characterization for super amalgamation of BAO's is substantially generalized in [76],[77]. In the latter references Madárasz deals with the super amalgamation for arbitrary discriminator varieties the algebras of which are endowed with some partial order. $\diamondsuit$.

The next section, section 1, is devoted to proving Proposition 1 (and Theorem 1*). The proof of Proposition 1 is preceded by an informal outline stressing on its metalogical source, as an instance of Robinson's methods of finite forcing in Model Theory.

1. Proofs

In this section we prove Proposition 1. From Fact 0.6 Theorems 1 and 1^* readily follow. First we define an abstract class of G algebras satisfying a set of equations denoted by $(P_1 - P_{10})$ below which are valid in the corresponding class of concrete G algebras. When G is as specified in our Theorems 1 and 1^* - in fact when $[G]$ is rich (and not necessarily strongly so) - then this set of equations turns out to be an adequate axiomatisation of $\mathbf{I}GPSA_\omega$, in the sense that if $\mathcal{A} \models (P_1 - P_{10})$, then $\mathcal{A}$ is isomorphic to a G set algebra. The latter, a typical Stone-like representability result, was proved by Sain [116]. Actually, our proof of Proposition 1 will be carried out for the variety $Mod(P_1 - P_{10})$, where Mod is an abbreviation for "the class of all models of".

In our formulation of abstract G algebras of dimension α, we allow as extra (yet term-definable) operations those substitutions which are indexed by transformations in the sub-semigroup of $\langle {}^\alpha\alpha, \circ\rangle$ generated by G. So before postulating $(P_1 - P_{10})$, we recall from [116] some basic semigroup theoretic notation.

In the following notation, and Definition 1.1 no conditions whatsoever are assumed on α nor on G. Restrictions on G will be imposed in due time. In harmony with the notation of [51], we use $+, ., -, 0, 1, c_i, s_\tau (\tau \in G)$ to denote the function symbols of the similarity type of G set algebras. Here, of course, $+, ., -, 0, 1, c_i, s_\tau$, correspond, respectively, to $\cup, \cap, \setminus, \emptyset, V, C_i^V, S_\tau^V$ in any G set algebra with unit V.

Notation .

(i) *Let α be an ordinal and $G \subseteq {}^\alpha\alpha$. Then $\langle [G], \circ\rangle$, or simply $[G]$ denotes the subsemigroup of $\langle {}^\alpha\alpha, \circ\rangle$ generated by G.*

(ii) *G^* denotes the set of all finite words on G, i.e. $G^* = \cup_{n\in\omega}{}^nG$. For $u, w \in G^*$, $u^\frown w$, or simply uw, denotes the concatenation of u and w. Recall that G^* is the free semigroup generated by G. We let $\hat{}: G^* \to [G]$ denote the unique anti-homomorphism extending the identity inclusion $Id : G \to [G]$.* $\diamondsuit$

Definition 1.1 .

Let $w \in G^*$. Suppose that $w = \langle w_0, \cdots, w_{n-1}\rangle$, with $w_i \in G$. Then $s_w(x)$ denotes the unary term $s_{w_{n-1}}(\cdots(s_{w_0}(x))\cdots)$ of the language of $GPSA_\alpha$. Σ_G denotes the following set of equations (easily seen to be) valid in $GPSA_\alpha$:

$$\Sigma_G := \{s_u(x) = s_w(x) : u, w \in G^* \text{ and } \hat{u} = \hat{w}\}.$$

When Σ_G holds in a class K of algebras similar to $GPSA_\alpha$, then for any $\mathcal{A} \in K$ and $\tau \in [G]$, we can correlate a unary operation s_τ on A determined by the following stipulation:

If $\tau = \tau_0 \circ \cdots \circ \tau_{n-1}$, with $\tau_i \in G$, then

$$s^{\mathcal{A}}_\tau(x) = s^{\mathcal{A}}_{\tau_0}(\cdots(s^{\mathcal{A}}_{\tau_{n-1}}(x))\cdots).$$

By Σ_G holding in K and $\mathcal{A} \in K$, $s^{\mathcal{A}}_\tau(x)$ is well defined. $\diamondsuit$

Accordingly, in what follows, we do not distinguish between an algebra in the class $\mathbf{I}GPSA_\alpha$, and its expansion obtained by allowing substitutions corresponding to elements in $[G] \setminus G$ which are, as indicated above, *term definable*. By the same token, we may (and will occasionally) write $[G]PSA_\alpha$ instead of $GPSA_\alpha$. Now we turn to define the abstract class of G algebras.

In the forthcoming Definition 1.2 we assume that α is an arbitrary infinite countable ordinal and that G is a set of transformations on α containing all replacements. That is $\{[i|j] : i, j \in \alpha\} \subseteq G$. Recall that $[G]$ denotes the semigroup generated by G.

Definition 1.2. Abstract G algebras .

By an abstract G polyadic algebra of dimension α, a GPA_α for short, we understand an algebra of the form

$$\mathcal{A} = \langle A, +, ., -, 0, 1, c_i, s_\tau\rangle_{i\in\alpha,\tau\in[G]}$$

where c_i $(i \in \alpha)$ and s_τ $(\tau \in [G])$ are unary operations on A such that postulates $(P_1 - P_{10})$ below hold for $\tau, \sigma \in [G]$ and all $i, j, k \in \alpha$.

(P_1) $\langle A, +, ., -, 0, 1\rangle$ is a boolean algebra, henceforth denoted by $Bl\mathcal{A}$.

(P_2) $x \leq c_ix = c_ic_ix,\ c_i(x + y) = c_ix + c_iy,\ c_i(-c_ix) = -c_ix,\ c_ic_jx = c_jc_ix.$

In other words c_i is an additive closure operator, and c_i, c_j commute.

(P_3) s_τ is a boolean endomorphism.

$(P_4)\Sigma_G$. *In particular* $s_\tau s_\sigma x = s_{\tau\circ\sigma}x$ *and* $s_{Id}x = x$.

$(P_5) s_\tau c_i x = s_{\tau[i|j]} c_i x$.

Recall that $\tau[i|j]$ *is the transformation that agrees with* τ *on* $\alpha \setminus \{i\}$ *and* $\tau[i|j](i) = j$.

(P_6) $s_\tau c_i x = c_j s_\tau x$ *if* $\tau^{-1}(j) = \{i\}$.

(P_7) $c_i s_{[i|j]} x = s_{[i|j]} x$.

(P_8) $s_{[i|j]} c_i x = c_i x$.

(P_9) $s_{[i|j]} c_k x = c_k s_{[i|j]} x$ *whenever* $k \notin \{i, j\}$.

(P_{10}) $c_i s_{[j|i]} x = c_j s_{[i|j]} x$.

In conformity with standard notation, we often write $s^i_j x$ instead of $s_{[i|j]}x$. Also for better readability, or if there is likelihood of confusion, we write $s_\tau(x)$ instead of $s_\tau x$. We note that the set of equations Σ_G, postulated for abstract G algebras, permits us to consider GPA_α as $[G]$ algebras. (cf. Definition 1.1.) $\diamondsuit$

Remarks on Definition 1.2.

(1) We choose to restrict ourselves to infinite α, because in this chapter we are only interested in infinite dimensional algebras. Nevertheless, Definition 1.2 makes perfect sense for finite α. For example when G consists only of replacements on α, then the G algebras coincide with Pinter's substitution algebras. (Comapre with Definition 0.1 in the previous chapter.) When G consists of replacements and transpositions, then the G algebras coincide with Halmos' quasipolyadic algebras. In the latter case $[G]$ is the set of all finite transformations. Such equivalences are worked out systematically for finite as well as for infinite α in [118].

(2) We should point out that $(P_1 - P_{10})$ are not independent. In fact, $(P_8 - P_{10})$ are easily derivable from the rest of the equations, as the reader is invited to show. Else he can consult [118]. Also note that Σ_G is in fact an infinite set of equations. When G consists only of replacements, or G consists of finite transformations, then Σ_G can be, and is indeed considerably simplified in [118]. In [118] G algebras, when G consists soley of finite transformations, are also

presented as *systems of varieties definable by Monk's schemes.* (See also [51] p.260-263.)

(3) Algebras satisfying $(P_1 - P_{10})$ will be hereafter referred to as abstract G algebras, being abstractions from the more concrete $GPSA_\alpha$'s. That is if $\mathcal{A} \in \mathbf{I}GPSA_\alpha$, then $\mathcal{A} \models (P_1 - P_{10})$. This is a *soundness* condition in the sense of [11] for the logics corresponding to GPA_α. We recall that a logic is sound if every derivable formula is logically valid. In passing, we note that if the units of the algebras in Definition 0.1 were broadened to allow any α-ary relations (better known in the literature as Crs_α units, see e.g [109]), then it is easy to construct examples violating the above set of axioms.

(4) Loosely speaking, the axiomatization given in $(P_1 - P_{10})$ is the restriction of Halmos' schemas (cf. [51] Def.5.4.1) to the similarity type of G algebras, i.e. to finite quantifiers and substitutions indexed by transformations in G. These schemas are actually designed to *enforce* algebras satisfying them to be representable as concrete G algebras. Pending on G of course, it turns out that the relation of the abstract to the concrete is unfortunately not as close as we wish. A result of Sain and Thompson (cf. [118]) (extending a result of Johnson [70]) is that when G consists of finite transformatons only, then G set algebras are not even finite schema axiomatizable in the sense of [95] over Σ_G. In particular, there are abstract G algebras that are not representable. However when we start putting infinitary substitutions, i.e. substitutions indexed by infinitary transformations [13] in G, switching to the polyadic paradighm, so to speak, we get strong representability results as shown by Sain in [116].

(5) In fact our research in this chapter was inspired by the following strong representability result of Sain [115]. When G is as specified in Theorem 1, i.e. $G = \{[i|j], [i,j] : i,j \in \alpha\} \cup \{suc, pred\}$, then every abstract G algebra is representable, i.e. is isomorphic to a set algebra. In [115] the axiomatization $(P_1 - P_{10})$ is polished and a simpler (equivalent) set of axioms is given. In

[13] At least one such transformation should be non-surjective, else the Sain-Thompson non-finite axiomatizability result applies. In other words if G consists of surjective transformations only, then the class of concrete G algebras is not finite-schema axiomatizable.

[116] the logical implications of the representability of G algebras when $[G]$ is rich is worked out in a manner parallel to the treatment carried out in [51] sec 4.3. Strong completeness and compactness of KL_G is proved in [116], where KL_G is the corresponding logic. Our Theorem 1^*, will show that the classical interpolation, definability and consistency theorems of Craig, Beth and Robinson respectively also, hold for KL_G, when G is further strongly rich. In [116] a specific G, for which $[G]$ is *finitely presented* and rich is exhibited. In this case, the corresponding logic is not only complete, but admits a *strictly finite* Hilbert style axiomatization that captures the logically valid formula schemas. It is unknown to us whether this semigroup is also strongly rich. In particular, it remains to be seen whether the logic introduced in [116] as a possible solution of the instance of the finitization problem of Henkin and Monk quoted in the introduction - for which all operations of set algebras are required to be logical in the sense of Tarski- has the interpolation property. In this connection, an interesting question that suggests itself is: *Find a simple* **strongly** *rich finitely presented semigroup on ω.* This is stronger than the question posed by Sain in [116] (and quoted in the introduction) in connection to the finitization problem, because here we require that the semigroup - besides being finitely presented - is also *strongly* rich. We will return to this question in Section 2, where we give a candidate of such a semigroup.

(End of Remarks on Definition 1.1. $\diamondsuit$)

Sain's proof of representing G algebras (as G set algebras) in [116] appeals to a *Neat Embedding Theorem* and has a wide range of applicability. In fact, Sain's proof applies to the case when $[G]$ is rich. The techniques of Sain will be used and further elaborated upon in the proof of Proposition 1. We first give an informal outline of the idea of the proof, stressing on its metalogical source. A novelty that occurs here is the likening of the Neat Embedding Theorem to Henkin's methods of constructing models from constants. To emphasize the strong tie with logic, we borrow terminology and concepts from Model Theory. We start off with the algebraic counterpart of the notion of satisfiability. This is folklore (see [51] sec 4.3.) in algebraic logic.

Definition 1.3 . *Let $\mathcal{A}$ be an abstract G algebra, and a a non-zero element in A.*

A model of $(\mathcal{A}, a)$ is a homomorphism h from $\mathcal{A}$ into a G set algebra such that $h(a)$ is non-empty. $\diamondsuit$

Note that an abstract $\mathcal{A}$ is representable, i.e. isomorphic to a set algebra, iff for every non-zero $a \in A$, $(\mathcal{A}, a)$ has a model. We next define the notion of α-dimensional elements for G algebras. The definition of neat reducts for G algebras turns out to be a little bit more tricky than that of CA's, and the related algebras addressed in chapter one. Though true, it is not evident that neat reducts are subalgebras of reducts becuase of the presence of infinitary substitutions.

Definition 1.4 .

Let α be an infinite ordinal, and let $G \subseteq {}^{\alpha}\alpha$ be an arbitary subset containing all replacements on α. Let $\mathcal{A}$ be an abstract G algebra.

(i) Let $a \in A$. Then Δa, referred to as the dimension set of a, (cf. [50] Def. 1.6.1) is the set $\{i \in \alpha : c_i a \neq a\}$. Let $\beta \leq \alpha$, then a is β-dimensional if $\Delta a \subseteq \beta$.

(ii) Let $Y \subseteq A$. Then $Nr_\omega Y$ is the set of ω-dimensional elements in Y, that is

$$Nr_\omega Y = \{a \in Y : \Delta a \subseteq \omega\}.$$

Note that in (ii) we defined only the set consisting of ω-dimensional elements. We do not know (yet) when such sets are algebras, i.e closed under the "natural" induced operations. In fact we have not even specified those natural operations; though the operations of finite substitutions and cylindrifications are defined in a manner entirely analagous to the quasipolyadic case. Restricting infinitary substitutions to ω-dimensional elements proves to be more problematic as will be illustrated below.

The following Definition singles out certain ultrafilters of (the boolean reduct of) G algebras. These ultrafilters correspond to the notion of Henkin's complete rich theories (cf. [128] chapter 4, or [35]), hence the notation.

(iii) Let F be a boolean ultrafilter of $\mathcal{A}$. Then F is called a Henkin ultrafilterif whenever $c_i a \in F$, then there exists $j \notin \Delta a$ such that $s_{[i|j]} a \in F$. $\diamondsuit$

Remark

Henkin ultrafilters, never in this name though, have appeared frequently in the literature in the context of proving a representability result. To mention a few references, see e.g. [13], [9], [113], [39]. More recent references include the mongraph [51], [128], [42], and [116]. This is not surprising, since representability corresponds to completeness in algebraic logic (cf. [51] sec 4.3 and 5.6 [9] and [11]). In what follows we give a proof outline of the main result in this chapter.

An outline of the proof of Proposition 1.

As is standard practice among algebraists, from now on we do not distinguish between an algebra $\mathcal{A}$ and its universe A, and denote both of them by latin letters. Unless otherwise indicated, G is as specified in Theorem 1^*.

Let X be a countable set. For the sake of brevity we set $A = Fr_X(GPA_\omega)$. Let $X_1, X_2 \subseteq X$. Let $a \in SgX_1$ and $c \in SgX_2$ be such that $A \models a \leq c$. We must show that there exists an interpolant of a and c, i.e. some $b \in Sg(X_1 \cap X_2)$ such that $A \models a \leq b \leq c$.

The proof goes by contradiction. We assume no such b exists. Then we construct a model of $(A, a-c)$. This contradicts that $A \models a \leq c$. We divide the proof into three parts (i), (ii) and (iii). Using Hodge's terminology [67], in parts (i) and (ii) we *force* with A, preparing for constructing the model. This is done by adding enough supply of witnesses, in preparation of defining a notion of forcing. Algebraically we *neatly embed* A into another algebra in ω extra dimensions. The *neat embedding* part is done in part (i). In part (ii) we define a notion of forcing and construct appropriate Henkin ultrafilters. In part (iii) we construct the desired model from these ultrafilters.

Part (i): Adding witnesses.

In this part we follow [116]. We *neatly embed* A into an algebra in ω extra dimensions. That is we view A as "living" in a bigger $\omega + \omega$ -dimensional algebra A^+, in the sense that A can be embedded in an appropriate subreduct of A^+. This subreduct referred to $Rd_\omega A^+$ below, is loosely speaking obtained by restricting the similarity type of A^+ to the first ω dimensions. Moreover A can be *neatly embedded* into A^+, or A is embeddable in the *neat ω-reduct* of A^+, namely into $Nr_\omega A^+$. A novelty that occurs here is that we show that A^+ is a *minimal* extension of A in the following sense: No new ω-dimensional elements of A^+ are definable from A even in the presence of the ω extra operations of A^+. Formally:

$$(*)\ A \subseteq Nr_\omega A^+ \text{ and for all } X \subseteq A \text{ we have } Sg^A X = Nr_\omega Sg^{A^+} X.$$

[14] Note that according to Definition 1.4 (b), Nr_ω is an operator, which when applied to subsets of A^+ only keeps the ω-dimensional elements and discards the rest. We point out that (*) is crucial. The proof of (*) depends on the *strong richness* of $[G]$, i.e. on condition (b) in Definition 0.3, and it seems that it does so essentially. [15] By (*) we are sure that no interpolant of a and c exists even in $Sg^{A^+}(X_1 \cap X_2)$. This will be the base of defining a notion of forcing in part (ii). (*) will be further elaborated in Section 2. Also the algebraic property of neat embeddability will be compared to the metalogical process of adding constants (or witnesses) in Remarks 1.12-13 below.

Part (ii): Defining a notion of forcing.

Using the ω extra dimensions, we extend a and $-c$ to Henkin ultrafilters F_1 and F_2 of the bigger algebra A^+. In forcing terminology[16] such ultrafilters are called generic. F_1 and F_2 are constructed in a step-by-step fashion to agree on $Sg^{A^+}(X_1 \cap X_2)$, with (*) providing a base of this construction. That is for all $x \in Sg^{A^+}(X_1 \cap X_2)$ we have $x \in F_1$ if and only if $x \in F_2$. Borrowing terminology from [[35] p. 88], F_1 and F_2, so constructed, may be called *inseparable* Henkin extensions of a and $-c$. In this part of the proof we closely follow Pigozzi [113] Thm 2.2.6.

Part (iii): Forming the desired model of $(A, a - c)$.

In this last part of the proof we build the desired model of $(A, a - c)$. We assume without loss of generality that $X_1 \cup X_2 = X$. Again following [116], we form a model

[14]Note that is nothing more than what we called in the previous chapter the NS property.

[15]We thank the referee of ur article [130] for drawing our attention to the fact that condition (a) in Definition 0.3 (i.e the richness of $[G]$) might not be enough to ensure that (*) holds. This made us explicitly formulate condition (b). In the first version of this paper we thought that our proof generlizes to arbitray rich semigroups.

[16]By forcing we mean Robinson's finite forcing in Model theory.

of $(Sg^A X_1, a)$ corresponding to F_1 and another of $(Sg^A X_2, -c)$ corresponding to F_2. In the construction of such models, we make explicit use of the extra ω dimensions, or the ω new operations of A^+. This consists of forming the so called *canonical models* of the rich complete theories corresponding to F_1 and F_2. In these models genericity and satisfaction coincide. This is standard practice in Model Theory. Now the *freeness* of A comes in. Together with the fact that F_1 and F_2 agree on $Sg^{A^+}(X_1 \cap X_2)$, we are able to paste these models, constructing a model of $(Sg^A(X_1 \cup X_2), a-c) = (A, a-c)$.

(End of outline of proof of Proposition 1.) $\diamondsuit$

The proof of Proposition 1 given here leads rather easily to both the representability of G abstract algebras, when $[G]$ is rich, and to the strong amalgamation for G algebras when $[G]$ is strongly rich. Because we are only interested in the latter result, we shall not formulate the former one explicitly. However, the reader will certainly have no difficulty in discerning the representatibility result just below the surface of our proof. Else he can consult [116]. Now we turn to implementing the details.

Details of the Proof

Now we turn to implementing the outline of the proof of Proposition 1. Throughout the proof, unless otherwise specified, G stands for the set

$$\{[i|j], [i,j] : i, j \in \omega\} \cup \{suc, pred\}.$$

In parts (i) and (ii) A is an arbitrary countable G abstract algebra. It is only in part (iii) that we assume that A is the free algebra on some countable set.

The proof of Proposition 1, is rather long, so we break it into several Claims. These Claims are interrupted occasionally by Remarks. The reasons of including such Remarks, are one of three things:

(1) To give detailed proofs for statements that we skip during Claims (not to interrupt the main stream).

(2) To point out certain analogies. We hope this makes certain ideas clearer.

(3) To generalize certain parts of the proof from the concrete case dealing with G algebras to the more general case of T algebras when T is an arbitrary (strongly) rich semigroup.

The following notation will be used extensively in the proof:

Let $f : B \to C$ be a function from B into C. Let $X \subseteq B$. Then $f \upharpoonright X$ denotes the *restriction* of f to X. $f(X)$ denotes the image of X under f, i.e. $f(X) = \{f(x) : x \in X\}$. Recall that $Rngf$ denotes $f(B)$, the range of f. Let $Y \subseteq C$. Then $f^{-1}Y = \{x \in B : f(x) \in Y\}$. We write $f^{-1}y$ instead of the more cumbersome $f^{-1}(\{y\})$. Assume that $B = C$. Then f^n stands for the function $f \circ f \cdots \circ f$, n times. Recall that for a set X, Id_X or simply Id, when X is clear from context, denotes the identity function on X and $|X|$ stands for the cardinality of X. If A is an algebra with a boolean reduct, then BlA denotes the boolean reduct of A. $\diamondsuit$

The details of part (i).

In part (i) we follow almost literally [116] sec 2.2. The strategy is: We *neatly embed* A into algebras in arbitrary finite extra dimensions, then using ultraproducts we neatly embed A into an algebra in ω extra dimensions. We should warn the reader, though, that we deliberately choose to deviate from [116] in two respects:

(1) We consider mostly a specific strongly rich semigroup namely the one generated by G. While in [116] the proof is carried out for any *rich* semigroup. We choose to do this because we believe that the concrete case is general enough to illustrate the main ideas and is far more tangible. The proof , adopted herein, adapts easily to the more general context of countable *strongly* rich semigroups as will be shown in Remarks 1.4 and 1.19. We do not know how much "strong richness" is needed. In particular, we do not know whether our proof can be pushed through for arbitrary rich semigroups but it seems highly unlikely that it can. Richness is discussed quite lengthly in connection to *representability* in [116]. In particular, it is shown therein that condition (1) and (2) in Definition 0.3 (a) are really needed in the sense that T abstract algebras may not be representable if T does not satisfy either of these conditions.

(2) Sain was primarily concerned with choosing a minimal set of operations generating all the rest; in fact she was concerned with choosing only finitely many non-boolean operations. (This is necessary for any concievable solution to the

finitization problem, and Sain was concerned with finitizing first order logic.) However in our formulation of G algebras we allow term definable operations (like substitutions indexed by transformations in $[G]\smallsetminus G$) as basic ones. By this the proofs which we include in some detail - even when easily recovered from those available in [116] - are usually simpler and sometimes even slightly different. Proofs that have appeared in [116] in the more general context of rich semigroups, will of course be credited to Sain. By doing this, we hope that the present chapter is self contained and that the reader with no access to [116], should have no difficulty figuring out what is going on.

We start off our proof by fixing some notation:

Notation . *Unless otherwise specified, $T=[G]$, the semigroup generated by G. Let $n\leq\omega$. Then $\alpha_n=\omega+n$ and $M_n=\alpha_n\smallsetminus\omega$. Note that when $n\in\omega$, then $M_n=\{\omega,\cdots,\omega+n-1\}$. Let $\tau\in T$. Then $\tau_n=\tau\cup Id_{M_n}$. T_n denotes the subsemigroup of $\langle{}^{\alpha_n}\alpha_n,\circ\rangle$ generated by $\{\tau_n:\tau\in T\}\cup\cup_{i,j\in\alpha_n}\{[i|j],[i,j]\}$. For $n\in\omega$, we let $\rho_n:\alpha_n\to\omega$ be the bijection defined by $\rho_n\restriction\omega=suc^n$ and $\rho_n(\omega+i)=i$ for all $i<n$.*

Claim 1.1 . *(a) Let $n\in\omega$. For $v\in T_n$ let $v'=\rho_n\circ v\circ\rho_n^{-1}$. Then $v'\in T$.*

(b)For $\tau\in T_\omega$, let $D_\tau=\{m\in M_\omega:\tau^{-1}(m)=\{m\}=\{\tau(m)\}\}$. Then $|M_\omega\smallsetminus D_\tau|<\omega$.

Proof. The proofs of (a) and (b) are given in [116] sec 2.2, in the more general context of rich semigroups. (a) and (b) are also straightforward to check from the definitions. Here we give a sketch of (a). Let $n\in\omega$. Since $(v\circ u)'=v'\circ u'$, it suffices to check that $v'\in T$ for every v in a generating set of T_n. Let $i,j\in\alpha_n$. Then $[i|j]'=[\rho_n(i)|\rho_n(j)]$. Similarly for transpositions. Let $v=suc^n\cup Id_{M_n}$. Let $x\in\omega$. Then it is not hard to check that $v'(x)=x$ if $x<n$ and $v'(x)=suc^n(x)$ otherwise. Thus $v'=[n+n|n]\circ\cdots\circ[n|0]\circ suc^n\in T$. The case when $v=pred^n\cup Id_{M_n}$ is similar. QED (Claim 1.1.) $\square$

From now on, unless otherwise specified, n is a fixed finite ordinal, α_n, M_n, T_n, ρ_n are as indicated above, and A is an arbitrary countable G algebra. Now let A_n be the algebra defined as follows:

$$A_n = \langle A, +, ., -, 0, 1, c_i^{A_n}, s_v^{A_n} \rangle_{i \in \alpha_n, v \in T_n}$$

where for each $i \in \alpha_n$ and $v \in T_n$,

$$c_i^{A_n} := c_{\rho_n(i)}^A \text{ and } s_v^{A_n} := s_{v'}^A.$$

Note that A_n has the same universe as A. In fact $BlA_n = BlA$. Each non-boolean operation of A_n, though, is obtained by renaming an operation of A. For example for $n > 0$ we have $c_\omega^{A_n} = c_0^A$. Following the terminology of [50] used in Definition 2.6.1 for cylindric algebras, A_n may be called the $\rho_n - reduct$ of A. It is tedious but absolutely straightforward to show that A_n, so defined satisfies $(P_1 - P_{10})$ of Definition 1.1 by replacing G by T_n and α by α_n. Let $Rd_\omega A_n$ be the following reduct of A_n obtained by restricting the type of A_n to the first ω dimensions:

$$Rd_\omega A_n = \langle A_n, +, ., -, 0, 1, c_i^{A_n}, s_{\tau_n}^{A_n} \rangle_{i \in \omega, \tau \in T}.$$

Now $Rd_\omega A_n$ has the same similarity type and universe as the algebra A. The boolean operations are the same; however the extra non-boolean ones are ρ_n shifted, so to speak.

For $x \in A$, let $e_n(x) = s_{suc^n}^A(x)$. Then $e_n : A \to A_n$.

Claim 1.2 . *(Sain) e_n is an isomorphism from A into $Rd_\omega A_n$.*

Proof. (i) e_n preserves the boolean operations because by (P_3), we have $s_{suc^n}^A$ is a boolean endomorphism, and $BlA = BlA_n$.

(ii) e_n preserves cylindrifications.

Let $x \in A$ and $i \in \omega$. Then, by definition

$$e_n(c_i^A x) = s_{suc^n}^A c_i^A x.$$

Let $j = suc^n(i)$. Then $j = suc^n(k)$ iff $k = i$, because suc, hence suc^n is one to one. Applying (P_6) we get

$$e_n(c_i^A x) = s_{suc^n}^A c_i^A x = c_j^A s_{suc^n}^A x.$$

By $\rho_n \upharpoonright \omega = suc^n$ and $i \in \omega$ we have $\rho(i) = suc^n(i) = j$. It follows from the definition of e_n and the above that

$$e_n(c_i^A x) = c_j^A s_{suc^n}^A x = c_{\rho_n(i)}^A e_n(x) = c_i^{A_n} e_n(x) = c_i^{Rd_\omega A_n} e_n(x).$$

(iii) e_n preserves substitutions.

First of all note that

$$suc^n \circ \tau = (\tau_n)' \circ suc^n \text{ for all } \tau \in T.$$

In fact this can be derived from the general definition of a rich semigroup T, cf. [116] Claim 2.15.1. However for the sake of completeness, we include a proof in the concrete case when $T = [G]$. Let $x \in \omega$. Then $suc^n \circ \tau(x) = \tau(x) + n$. On the other hand, a direct computaton gives:

$$(\tau_n)' \circ suc^n(x) = \rho_n \circ \tau_n \circ \rho_n^{-1} \circ suc^n(x)$$

$$= \rho_n \circ \tau_n(x) = \rho_n \circ \tau(x) = suc^n(\tau(x)) = \tau(x) + n.$$

Now let $\tau \in T$ and $x \in A$. Computing we get by the definition of e_n, (P_4) and the above that

$$e_n(s_\tau^A x) = s_{suc^n}^A(s_\tau^A x) = s_{suc^n \circ \tau}^A(x)$$

$$= s_{\tau_n' \circ suc^n}^A(x) = s_{\tau_n'}{}^A(s_{suc^n}^A x) = s_\tau^{Rd_\omega A_n} e_n(x).$$

We have proved that e_n preserves substitutions.

(iv) e_n is one to one.

Let $x \in A$. Assume that $e_n(x) = 0$. We will show that $x = 0$.

By (P_3) and (P_4), upon noting that $pred^n \circ suc^n = Id$, we get

$$0 = s_{pred^n}^A e_n(x) = s_{pred^n}^A s_{suc^n}^A(x) = s_{pred^n \circ suc^n}^A(x) = s_{Id} x = x.$$

QED (Claim 1.2.) □

Claim 1.3 . *$e_n(Sg^A Y) = Nr_\omega(Sg^{A_n} e_n(Y))$ for all $Y \subseteq A$.*

Proof. Recall that $Nr_\omega Y = \{a \in Y : c_i^{A_n} a = a \text{ for all } \omega \leq i < \omega + n\}$. Since $c_{\omega+i}^{A_n} a = c_i^A a$ for all $i < n$ and $a \in A$, it follows thus that $Nr_\omega Y = \{a \in Y : c_i^A a = a$ for all $i < n\}$. Now let $Y \subseteq A$. First we show that $e_n(Sg^AY) \subseteq Nr_\omega(Sg^{A_n}e_n(Y))$. This part of the proof is due to Sain. The formulation in [116] is slightly different, though. for example in [116] the notation Nr_ω which we borrow from cylindric algebra, cf. [50] Def.2.6.28, is never used. Other than this formal distinction the proof which we include herein is literally the same. Towards this end, let $y \in e_n(Sg^AY)$. Then $y = e_n(a)$ say with $a \in Sg^A(Y)$. We will show that $y \in Nr_\omega A_n$, i.e. $c_m^{A_n}y = y$ for all $m \in M_n$, by which we will be done because

$$Nr_\omega A_n \cap e_n(Sg^AY) = Nr_\omega e_n(Sg^AY).$$

So let $m \in M_n = \{\omega, \dots, \ \omega + n - 1\}$. From $\rho_n(m) \in \{0, \dots, n-1\}$, i.e. $\rho_n(m) \notin Rng(suc^n)$, it follows that

$$[\rho_n(m)|j] \circ suc^n = suc^n \text{ for any } j \in \omega.$$

Let $j \in \omega$. Then from (P_4) and (P_7) we get

$$c_m^{A_n}y = c^A_{\rho_n(m)}s^A_{suc^n}(a) = c^A_{\rho_n(m)}s^A_{[\rho_n(m)|j]\circ suc^n}(a)$$

$$= s^A_{[\rho_n(m)|j]\circ suc^n}(a) = s^A_{suc^n}(a) = e_n(a) = y.$$

We have shown that $c_m^{A_n}y = y$ for each $m \in M_n$, thus y is as required. Now we prove the other inclusion, namely that,

$$Nr_\omega(Sg^{A_n}e_n(Y)) \subseteq e_n(Sg^AY).$$

This part of the proof, to the best of our knowledge, is new.

Let $y \in Nr_\omega Sg^{A_n}e_n(Y)$. Then $y \in Sg^{A_n}e_n(Y)$. Let $a = s^A_{pred^n}(y)$. We will show that

$$e_n(a) = y \text{ and that } a \in Sg^AY$$

by which we will be done. We start by showing that $e_n(y) = a$.

For the sake of brevity, let $c_{(n)}x$ abbreviate $c_0 \dots c_{n-1}x$. Since $y \in Nr_\omega Sg^{A_n}e_n(Y)$, we get $c_i^Ay = y$ for all $i < n$, i.e. $c^A_{(n)}y = y$. Together with the definition of e_n and (P_4) we obtain

$$e_n(a) = s^A_{suc^n}(s^A_{pred^n}y) = s^A_{suc^n\circ pred^n}c^A_{(n)}y.$$

Note that $suc \circ pred = [0|1]$. Also $suc \circ [i|j] = [i+1|j+1] \circ suc$ for all $i, j \in \omega$. This is absolutely straightforward. Then it follows, implementing a straightforward induction, that

$$suc^n \circ pred^n = [n-1|n] \circ \cdots \circ [0|1].$$

Thus by (P_4) we have

$$e_n(a) = s^A_{[n|n-1]}(\cdots(s^A_{[0|1]}(c^A_{(n)}y)\cdots).$$

Now using (P_8) and (P_9) we get that

$$e_n(a) = c^A_{(n)}y = y.$$

It thus remains to show that $a = s_{pred^n}y \in Sg^AY$. Towards this end, we define inductively the sequence $\langle Y_k : k \in \omega\rangle$ as follows:

$$Y_0 = e_n(Y) \cup \{0, 1\}$$

and for $k \geq 0$, we set

$$Y_{k+1} = Y_k \cup \{f^{A_n}(x) : \ f \text{ is an } i\text{-ary operation of } A_n, i \in \{1, 2\} \text{ and } x \in {}^iY_k\}.$$

We distinguish between two cases:

Case (i).

$y \in Y_0$, then either $y = e_n(y')$ for some $y' \in Y$ or $y \in \{0, 1\}$. If $y \in \{0, 1\}$, then by (P_3) we have $a = s^A_{pred^n}y \in \{0, 1\}$, and, hence, $a \in Sg^AY$. If $y = e_n(y')$ for some $y' \in Y$, then by (P_4) and while noting that $pred^n \circ suc^n = Id$, we get

$$a = s^A_{pred^n}e_n(y') = s^A_{pred^n \circ suc^n}(y') = s_{Id}y' = y' \in Y \subseteq Sg^AY.$$

Case (ii).

$y \in Y_{k+1}$ for some $k \in \omega$. If $y \in Y_k$, then $y \in Sg^AY$ by induction hypothesis, and so $a = s^A_{pred^n}y \in Sg^AY$, as well.

Let us now assume that $y = f^{A_n}(x)$, where $x \in {}^iY_k$ and $i \in \{1, 2\}$. If f is a boolean operation of A_n then $a = s^A_{pred^n}f^{A_n}(x) = s^A_{pred^n}f^A(x)$ since $BlA = BlA_n$. The latter is in Sg^AY since by the induction hypothesis we have $f^A(x) \in Sg^AY$.

We are left with the case when f is a unary non-boolean operation of A_n. We only check the case when f is a cylindrification. The case when f is a substitution is entirely analogous, the idea being that in both cases, the operations of A_n are obtained by renaming those of A.

Towards this end, assume $y = c_i^{A_n}x$, $i < \omega + n$. Then $y = c^A_{\rho_n(i)}x$. A direct application of the definitions gives:

$$a = s^A_{pred^n}y = s^A_{pred^n}c^A_{\rho_n(i)}x.$$

By the induction hypothesis we have $x \in Sg^AY$, thus $a \in Sg^AY$, as well. By this Claim 1.3 is proved. Taking $Y = A$ we get that $e_n(A) = Nr_\omega A_n$. QED (Claim 1.3.)

□

In accordance with [50] Definition 2.6.28, we refer to $Nr_\omega A_n$ as *the neat*$-\omega$ *reduct of* A_n. Like in the cylindric algebra case, we have proved that $Nr_\omega A_n$ is a subalgebra of $Rd_\omega A_n$. Therefore Claim 1.3 can be written in the form:

$$e_n(Sg^AY) = Sg^{Nr_\omega A_n}e_nY = Nr_\omega Sg^{A_n}e_nY \text{ for all } Y \subseteq A.$$

We note that the above is the finite version of the property abbreviated by (*) in the informal outline of the proof. (*) will be proved in Claim 1.6 below. We have also shown that A has the so-called *neat embedding property* in [50], Remark 2.6.26. That is for every $n \in \omega$ there exists an $\omega + n$-dimensional algebra A_n such that A can be embedded into the set of ω- dimensional elements of A_n. Using polyadic terminology, cf. [39], A_n may be called an $n-$ *dilation* of A and A is a *compression* of A_n. In fact, we have proved more, namely that A can be embedded *onto* the set of ω-dimensional elements of A_n, for each n. This follows from Claim 1.3 which expresses more succintly that A_n is a minimal n dilation of A. In the next Remark we show how this generalizes to the case when T is *any* strongly rich semigroup.

Remark 1.4. The generalization to an arbitrary rich semigroup.

We show how the proofs of Claims 1.2-1.3 adapt to the more general case of a strongly rich semigroup. It turns out that the "strong richness" is used only in the proof of Claim 1.3. Recall that $T \subseteq \langle {}^\omega\omega, \circ\rangle$ is rich if T satisfies the following conditions:

$$(1)(\forall i,j\in\omega)(\forall\tau\in T)\tau[i|j]\in T.$$

(2) There exists $\sigma,\pi\in T$ such that

$$(\pi\circ\sigma=Id,\ Rng(\sigma)\neq\omega),$$

and

$$(\forall\tau\in T)(\sigma\circ\tau\circ\pi)[(\omega\smallsetminus Rng(\sigma))|Id]\in T.$$

From now on, throughout Remark 1.4, we assume that T is a given rich subsemigroup of $\langle{}^{\omega}\omega,\circ\rangle$. Now the idea is simply to replace *suc* and *pred* everywhere in Claims 1.2 and 1.3 by the special elements σ and π provided by (2) in the definition of a rich semigroup and check that the proofs survive.

For $n\in\omega$, let

$$R_n=\omega\smallsetminus Rng(\sigma^n).$$

Then we have

$$0\subseteq R_1\subseteq R_2\subseteq\cdots R_n\subseteq\omega$$

by the hypothesis that $Rng(\sigma)\neq\omega$. Also, because $Rng(\sigma)\neq\omega$, we have $|R_n|\geq n$. For the special case treated in Claims 1.2-3, i.e. when $T=[G]$, we have that

$$R_n=n=\{0,1\cdots n-1\}.$$

However in the general case, R_n may not be even finite. To make the passage to the proof of the general case as close as possible to the case when $T=[G]$, in what follows we use the same notation as in Claim 1.1. Let $n\in\omega$ be arbitrary but fixed. Then we define

$$\alpha_n=\omega+|R_n|,$$

$$M_n=\alpha_n\smallsetminus\omega,\ \tau_n=\tau\cup Id_{M_n}\text{ for every }\tau\in T.$$

We let T_n denote the subsemigroup of $\langle{}^{\alpha_n}\alpha_n,\circ\rangle$ generated by

$$\{\tau_n:\tau\in T\}\cup\cup_{i,j\in\alpha_n}\{[i|j],[i,j]\}.$$

For $n\in\omega$, we let $\rho_n:\alpha_n\to\omega$ be a bijection such

$$\rho_n\restriction\omega=\sigma^n\text{ and }\rho_n\restriction M_n\to R_n\text{ is a bijection.}$$

This is now possible because σ^n is one to one, $|M_n| = |R_n|$ and $Rng(\sigma^n) \cap R_n = \emptyset$. The following easy consequence of the definition of a rich semigroup, of which we already formulated (a) and (b) in Claim 1.1 above for $[G]$, is proved by Sain [116]:

(+) . *Let everything be as above. Then the following hold:*

(a) Let $n \in \omega$. For $v \in T_n$ let $v' = \rho_n \circ v \circ \rho_n^{-1}$. Then $v' \in T$.

(b)For $\tau \in T_\omega$, let $D_\tau = \{m \in M_\omega : \tau^{-1}(m) = \{m\} = \{\tau(m)\}\}$. Then $|M_\omega \setminus D_\tau| < \omega$.

(c) $(v \circ u)' = v' \circ u'$

(d) $\sigma^n \circ \tau = (\tau_n)' \circ \sigma_n$

We note that the proofs of (a) and (d) depend on the second condition of (2) in the definition of a rich semigroup [116]. (b) and (c) are immediate.

Let n be a fixed finite ordinal, α_n, M_n, T_n, and ρ_n as indicated above, and now let A be an arbitrary countable T algebra. Then we define A_n and $Rd_\omega A_n$ exactly as before. That is, we let

$$A_n = \langle A, +, ., -, 0, 1, c_i^{A_n}, s_v^{A_n} \rangle_{i \in \alpha_n, v \in T_n}$$

where for each $i \in \alpha_n$ and $v \in T_n$,

$$c_i^{A_n} := c_{\rho_n(i)}^A \text{ and } s_v^{A_n} := s_{v'}^A;$$

and we let $Rd_\omega A_n$ be the following reduct of A_n:

$$Rd_\omega A_n = \langle A_n, +, ., -, 0, 1, c_i^{A_n}, s_{\tau_n}^{A_n} \rangle_{i \in \omega, \tau \in T}.$$

By (a) of (+), the definition of A_n is sound. Neatly embedding A into A_n is done via s_{σ^n}. In more detail, for $x \in A$ we let $e_n(x) = s_{\sigma^n}^A(x)$. Then $e_n : A \to A_n$.

We need to check the following two conditions (++) and (+++):

(++) e_n is an isomorphism from A into $Rd_\omega A_n$.

(+++) For any $Y \subseteq A$ we have

$$e_n(Sg^A Y) = Nr_\omega(Sg^{A_n} e_n(Y)).$$

The proof of (++) is exactly like the proof of Claim 1.2. Here only the richness of T is used. In more detail: That e_n preserves cylindrification, follows from σ^n is one to one, (P_6) and the definition of e_n. That e_n preserves substitutions, follows from (d) of (+), i.e. from

$$\sigma^n \circ \tau = (\tau_n)' \circ \sigma_n,$$

together with (P_4) and the definition of e_n. That e_n is one to one follows from it's definition, together with that $\pi^n \circ \sigma^n = Id$. Now we prove (+++), i.e. that

$$e_n(Sg^A Y) = Nr_\omega(Sg^{A_n} e_n(Y)) \text{ for all } Y \subseteq A.$$

Note that for $Y \subseteq A$, $Nr_\omega Y$ is now equal to the the set

$$\{a \in Y : c_i^{A_n} a = a \text{ for all } i \in M_n\}.$$

Let Y be given. Then $e_n(Sg^A Y) \subseteq Nr_\omega Sg^{A_n} e(Y)$ is the same as before (cf. [116].) Again here it is only the richness of T that is used. For the other inclusion, i.e. to prove that $Nr_\omega Sg^{A_n} e(Y) \subseteq e_n(Sg^A Y)$ we *need* that T is *strongly* rich, i.e. that, in addition, σ and π satisfy (i), (ii) below:

$$(i)(\forall n \in \omega)|supp(\sigma^n \circ \pi^n)| < \omega.$$

$$(ii)(\forall n \in \omega)[supp(\sigma^n \circ \pi^n) \subseteq \omega \smallsetminus Rng(\sigma^n).]$$

To see this, let $y \in Nr_\omega(Sg^{A_n}(Y))$. We want to find

$$a \in Sg^A Y \text{ such that } e_n(a) = y.$$

Let $a = s_\pi y$. We show that a is as required. Since $y \in Nr_\omega(Sg^{A_n}(Y))$, we have by definition that

$$c_i^{A_n} y = y \text{ for all } i \in M_n \text{ and so } c_{\rho_n(i)}^A y = y \text{ for all } i \in M_n.$$

Since ρ_n maps R_n onto M_n, it follows thus that

$$c_j^A y = y \text{ for all } j \in R_n.$$

Let

$$\Gamma = supp(\sigma^n \circ \pi^n).$$

Then by (i) we have Γ is finite. Suppose that $\Gamma = \{j_0, \dots j_{m-1}\}$. For the sake of brevity, we let $c_{(\Gamma)} = c_{j_0} \dots c_{j_{m-1}}$. Now by (ii) we have $\Gamma \subseteq R_n$, and so it follows that $c^A_{(\Gamma)} y = y$. Let us now compute

$$e_n(a) = s^A_{\sigma^n}(s^A_{\pi^n} y).$$

We then have

$$s^A_{\sigma^n}(s^A_{\pi^n} y) = s^A_{\sigma^n \circ \pi^n}(y) = s^A_{\sigma^n \circ \pi^n} c_{(\Gamma)} y.$$

We now use the following easy consequence of (P_8). If τ is a transformation on ω having finite support Δ, say, then

$$s_\tau c_{(\Delta)} a = c_{(\Delta)} a.$$

Here $c_{(\Delta)} = c_{i_0} \cdots c_{i_{m-1}}$, where $\Delta = \{i_0, \dots, i_{m-1}\}$. From this it follows that

$$s^A_{\sigma^n \circ \pi^n} c_{(\Gamma)} y = c_{(\Gamma)} y = y;$$

and so

$$e_n(a) = y.$$

$e_n(a) \in Sg^A Y$ can be proved exactly as before. (QED. Claims 1.2-1.3 for a strongly rich semigroup.) □(End of Remark 1.4. ◇)

Neat embeddability of an algebra in algebras having arbitrary finite extra dimensions, usually implies neat embeddability into an algebra having ω extra dimensions. Fortunately, this is the case with G algebras, as we proceed to show. In fact, this is a typical compactness argument. Using ultraproducts, we paste the A_n's forming an expansion and extension A^+ of A. A^+ is an $\omega + \omega$ dimensional algebra, in which A is *neatly embedded*. We construct A^+ as a limit of the A_n's, $(n \in \omega)$. First we expand the A_n's to be all of the same similarity type, then we form an ultraproduct of the resulting expanded algebras. [17] This is standard practice in algebraic logic (see e.g. [50], Thms 2.6.7 and 2.6.35.)

For the sake of brevity, let $\alpha = \alpha_\omega = \omega + \omega$. Recall that T_ω is the semigroup generated by the set

$$\{\tau_\omega : \tau \in T\} \cup_{i,j \in \alpha} \{[i|j], [i,j]\}.$$

[17] An ultraproduct of a family of algebras may seem different than forming a limit. However in [40] it is shown how reduced products (and in particular, ultraproducts) can be defined via the categorial notion of a direct limit.

For $\sigma \in T_\omega$, and $n \in \omega$, let $[\sigma]_n = \sigma \upharpoonright \omega + n$. Note that for fixed n, $[\sigma]_n$ is not necessarily in T_n, however according to Claim 1.1.(b), there exists $l \in \omega$ such that for all $n \geq l$ $[\sigma]_n \in T_n$. For each $n \in \omega$ let

$$A_n^+ = \langle A, +, -, 0, 1, c_i^{A_n^+}, s_\sigma^{A_n^+} \rangle_{i \in \alpha, \sigma \in T_\omega}$$

be an expansion of A_n such that

$$BlA_n^+ := BlA (= BlA_n);$$

and for each $\sigma \in T_\omega$ and $i \in \alpha$,

$$s_\sigma^{A_n^+} := s_{[\sigma]_n}^{A_n} \text{ iff } [\sigma]_n \in T_n,$$

and

$$c_i^{A_n^+} := c_i^{A_n} \text{ iff } i < \omega + n.$$

This means that if $[\sigma]_n \notin T_n$ $(i \geq \omega + n)$, and $a \in A$, then $s_\sigma^{A_n^+} a$ $(c_i^{A_n^+} a)$ is an arbitary but fixed element of A.

Fix F to be any non-principal ultrafilter on ω. Now forming the ultraproduct of the A_n^+'s relative to F, let

$$A^+ = \mathbf{P}_{n \in \omega} A_n^+ / F.$$

We shall henceforth refer to A^+ as an ω dilation of A. This is, as previously noted, a term we borrow from polyadic algebras (cf. [39]). We point out that, given $n \in \omega$, A_n^+ may (and actually does) not satisfy $(P_1 - P_{10})$. But, by taking T_ω for G in Definition 1.2, the limit A^+ does. This will be partially justified in Remark 1.10 below and will be used thereafter without any further notice. A proof of this can be easily recovered from fact 0.5.15 (ii) in [50], formulated for ultraproducts of general algebras.

Next we neatly embed A into A^+. Towards this end, for $x \in A$ we let

$$e(x) = \langle e_n(x) : n \in \omega \rangle / F.$$

Then of course $e : A \to A^+$. Now we let $Rd_\omega A^+$ be the following reduct of A^+, obtained by restricting its similarity type to the first ω dimensions:

$$Rd_\omega A^+ = \langle A^+, +, ., -, 0, 1, c_i^{A^+}, s_{\tau_\omega}^{A^+} \rangle_{i < \omega, \tau \in T}.$$

Recall that for $\tau \in T$, $\tau_\omega = \tau \cup Id_{\alpha\smallsetminus\omega} \in T_\omega$. In particular, $s_{\tau_\omega}^{A^+}$ is meaningful. Now that $Rd_\omega A^+$ has the same similarity type as A , we can compare the two algebras. And indeed we have:

Claim 1.5 . *(Sain) e is an isomorphism from A into $Rd_\omega A^+$.*

Proof. Though the proof follows directly from the basic properties of ultraproducts, we include it for the sake of completeness.

First of all, e is a boolean homomorphism because every e_n is a boolean homomorphism. Also e is one to one because every e_n is one to one. Next we check cylindrifications. Let $x \in A$ and $i \in \omega$. Then computing we get

$$e(c_i^A x) = \langle e_n(c_i^A x) : n \in \omega\rangle/F$$

$$= \langle c_i^{Rd_\omega A^+} e_n(x) : n \in \omega\rangle/F (\text{by } e_n \text{ preserving cylindrifications for all } n)$$

$$= \langle c_i^{A_n} e_n(x) : n \in \omega\rangle/F \text{ (by definition)}$$

$$= \langle c_i^{A_n^+} e_n(x) : n \in \omega\rangle/F \text{ (by } i < \omega \leq \omega + n \text{ and by definition of } A_n^+)$$

$$= c_i^{A^+} \langle e_n(x) : n \in \omega\rangle/F \text{ (by definition of ultraproducts)}$$

$$= c_i^{A^+} e(x).$$

Finally we show that e preserves substitutions: Let $x \in A$ and $\tau \in T$. Then

$$e(s_\tau^A x) = \langle e_n(s_\tau^A x) : n \in \omega\rangle/F \text{ (by definition)}$$

$$= \langle s_\tau^{Rd_\omega A_n} e_n(x) : n \in \omega\rangle/F \text{ (by } e_n \text{ preserving } \tau_n \text{ for all } n \in \omega)$$

$$= \langle s_{\tau_n}^{A_n} e_n(x) : n \in \omega\rangle/F \text{ (by definition)}$$

$$= \langle s_{[\tau_\omega]_n}^{A_n^+} e_n(x) : n \in \omega\rangle/F \text{ (Since } [\tau_\omega]_n = \tau_n = \tau \cup Id_{M_n} \in T_n \text{ for all } \tau \in T)$$

$$= \langle s_{\tau_\omega}^{A_n^+} e_n(x) : n \in \omega\rangle/F \text{ (by definition of } A_n^+)$$

$$= s_{\tau_\omega}^{A^+} \langle e_n(x) : n \in \omega\rangle/F (\text{by definition of ultraproducts})$$

$$= s_{\tau_\omega}^{Rd_\omega A^+} e(x) \text{ (by definition) .}$$

By this the proof of Claim 1.5 is complete. QED (Claim 1.5.) $\square$

Next we show that A^+ is also a *minimal* dilation of A. This means that A can be embedded into $Nr_\omega A^+$, via e, and that A^+ satisfies Claim 1.6 below. Loosely speaking, Claim 1.6 can be seen as a limiting case of Claim 1.3 , obtained by letting n go to infinity, i.e. grow without bound[18]

Claim 1.6 . *$e(Sg^A Y) = Nr_\omega Sg^{A^+} e(Y)$ for all $Y \subseteq A$.*

Proof. Let $Y \subseteq A$. First we show (the easier incusion namely) that

$$e(Sg^A Y) \subseteq Nr_\omega Sg^{A^+} e(Y).$$

This part of the proof is the same as Sain's (cf. [116]), but we include it for the sake of completeness. Let $y \in Sg^A(Y)$ and $m \in \alpha \smallsetminus \omega$. We have to show that $c_m^{A^+} e(x) = e(x)$. Computing we get

$$c_m^{A^+} e(x) = c_m^{A^+} \langle e_n(x) : n \in \omega \rangle / F = \langle e_n(x) : n \in \omega \rangle / F.$$

since $c_m^{A_n^+} e_n(x) = c_m^{A_n} e_n x = e_n(x)$ for all n for which $m \in \omega + n$ and since $J = \{n \in \omega : m \in \omega + n\} \in F$, the latter containing the cofinite filter. We have shown that $c_m^{A^+} e(x) = e(x)$ whenever $m \in \alpha \smallsetminus \omega$, i.e. that $e(x) \in Nr_\omega(A^+)$. Since $e(x)$ is clearly in $Sg^{A^+} e(Y)$, we get the desired conclusion.

Before proving the other inclusion, namely that

$$Nr_\omega Sg^{A^+} e(Y) \subseteq e(Sg^A Y),$$

we shall further need:

Claim 1.7 . *(i) (Sain) For the sake of brevity, let $M = \alpha \smallsetminus \omega$. Let $Y \subseteq Nr_\omega A^+ = \{x \in A^+ : \Delta x \subseteq \omega\}$. Then for all $x \in Sg^{A^+} Y$ we have $|M \cap \Delta x| < \omega$.*

(ii) Let $0 \leq j < \omega$. For $\sigma \in T_j$, let $\sigma_\omega = \sigma \cup Id_{\alpha \smallsetminus \alpha_j}$, i.e. for $x \in \alpha$, $\sigma_\omega(x) = \sigma(x)$ if $x \in \omega + j$ and otherwise $\sigma_\omega x = x$. Then $\sigma_\omega \in T_\omega$. Now let $Rd_{\omega+j} A^+$ be the following reduct of A^+ having the same similarity type as A_j:

$$Rd_{\omega+j} A^+ = \langle A, +, ., -, c_i^{A^+}, s_{\sigma_\omega}^{A^+} \rangle_{i \in \omega+j, \sigma \in T_j}.$$

[18]Claim 1.6 says that G algebras have the NS.

Then

$$Sg^{A^+}Y = \cup\{Sg^{Rd_{\omega+j}A^+}Y : j \in \omega\}.$$

Proof of Claim 1.7 (i):

Since all operations are evaluated in A^+ we omit the superscript A^+ for better readability. Let $Y \subseteq Nr_\omega A^+$. For the sake of brevity, we let $R = Sg^{A^+}Y$. Since for all $y \in Y$ we have $\Delta y \subseteq \omega$, i.e. $M \cap \Delta x = \emptyset$, it suffices to show that for any $x \in R$ and any (unary) non boolean operation f of A^+, the following condition holds:

$$\text{If } |M \cap \Delta x| < \omega \text{ then } |M \cap \Delta(fx)| < \omega.$$

Of course, we should check that the above holds for the boolean operations as well, but this is absolutely straightforward.

This time we start with the case when f is a substitution. So let $\tau \in T_\omega$. Let D_τ be as defined in Claim 1.1, i.e $D_\tau = \{m \in M : \tau^{-1}(m) = \{m\} = \{\tau(m)\}\}$. Then by Claim 1.1 (b) we have$|M \smallsetminus D_\tau| < \omega$. For the sake of brevity, let C_τ denote the finite set $M \smallsetminus D_\tau$. By $|M \cap \Delta x| < \omega$, we have $|(M \cap \Delta x) \cup C_\tau| < \omega$. We will show that $M \cap \Delta(s_\tau x) \subseteq (M \cap \Delta x) \cup C_\tau$ by which we will be done. So assume that $i \in M \smallsetminus \omega$, and that $i \notin (M \cap \Delta x) \cup C_\tau$. Then $i \in D_\tau \smallsetminus \Delta x$, so $\{\tau(i)\} = \{i\} = \tau^{-1}(i)$. By (P_6) holding in A^+, we get that $c_i s_\tau x = s_\tau c_i x$ proving that $i \notin M \cap \Delta s_\tau x$. Now assume that $f = c_j$ with $j \in \alpha$. If $i \in M$ and $i \notin \Delta x$ then by (P_2) holding in A^+ , we have $c_i c_j x = c_j c_i x = c_j x$. [19], i.e. $i \notin \Delta c_j x$. QED (Claim 1.7 (i).) □

Note that, we have actually proved the following (only implicit in [116]):

Fact 1.8 . *Given any finite subset X of R, then $\alpha \smallsetminus \cup\{\Delta x : x \in X\}$ is infinite.*

Proof. Let $X = \{x_0, \cdots x_{n-1}\}$. Then for every $i < n$ there exists $l_i \in \omega$ such that $c_m x_i = x_i$ for every $m \geq l_i$. Let $l = max\{l_0, \cdots l_{n-1}\}$. Then $\{\omega + k : k \geq l\}$ is an infinite subset of $\alpha \smallsetminus \cup\{\Delta x : x \in X\}$. QED (Fact 1.8). □

Remark 1.9

Fact 1.8 says that the $\omega + \omega$ dimensional algebra R is *dimension complemented*, a term used for cylindric algebras satisfying Fact 1.8 (cf. [50] Def 1.11.1 (ii) and Thm

[19] Here we only use that cylindrifications commute.

1.11.4 (iii)). Dimension complemented cylindric algebras are generalizations of the locally finite ones. While in locally finite algebras the dimension sets of elements are finite, in dimension complemented ones, this need not be the case. What they share with locally finite algebras is rather: The complement (relative to the dimension) of dimension sets is infinite. Here, of course we are speaking about infinite dimensional algebras.

(End of Remark 1.9.) $\diamondsuit$

Remark 1.10

In the proof of Claim 1.7 (i) we have assumed that the "cylindrifications commute" part of (P_2) and (P_6) hold in A^+. This follows from the basic properties of ultraproducts. However for the sake of completeness we give a self-contained justification of this. The non-sceptical reader can skip this remark, and go on straight away with the proof of Claim 1.7.

We put back superscripts. Let $i,j\in\alpha$ and $x=\langle x_n : n\in\omega\rangle/F\in A^+$. That $c_i^{A^+}c_j^{A^+}x=c_j^{A^+}c_i^{A^+}x$ can be proved as follows: Let

$$J_i=\{n\in\omega : i\in\omega+n\}, J_j=\{n\in\omega : j\in\omega+n\} \text{ and } J_{ij}=J_i\cap J_j.$$

Since F is closed under finite intersections, it follows that $J_{ij}\in F$, because J_i and J_j are clearly in F. Let $n\in J_{ij}$. Then

$$c_i^{A_n^+}c_j^{A_n^+}x_n=c_i^{A_n}c_j^{A_n}x_n=c^A_{\rho_n(i)}c^A_{\rho_n(j)}x_n=c^A_{\rho_n(j)}c^A_{\rho_n(i)}x_n=c_j^{A_n}c_i^{A_n}x_n=c_j^{A_n^+}c_i^{A_n^+}x_n.$$

It then follows that $c_i^{A^+}c_j^{A^+}x=c_j^{A^+}c_i^{A^+}x$.

Now we justify (P_6) holding in A^+. Let $\tau\in T_\omega$, x, D_τ and C_τ be as above. Recall that $C_\tau=M\smallsetminus D_\tau$. Let $i\in C_\tau\smallsetminus\Delta x$. Then by Claim 1.1 (b) there exists $l\in\omega$ such that for all $n\geq l$ we have $[\tau]_n\in T_n$. We can obviously assume that $i\in\omega+l$. Let $J_l=\{n\in\omega : n\geq l\}$. Then $J_l\in F$. Also for any $n\in J_l$ we have

$$s^{A_n^+}_{[\tau]_n}c_i^{A_n^+}x_n=s^{A_n}_{[\tau]_n}c_i^{A_n}x_n=c_i^{A_n}s^{A_n}_{[\tau]_n}x_n=c_i^{A_n^+}s^{A_n^+}_{[\tau]_n}x_n.$$

This proves that $c_i^{A^+}s_\tau^{A^+}x=s_\tau^{A^+}c_i^{A^+}x$. $\square$

(End of Remark 1.10.) $\diamondsuit$

In parts (ii) and (iii) of the proof, we shall extensively use the fact that $(P_1 - P_{10})$ hold in the algebra A^+, by taking T_ω for G. As previously noted, this is routine, following the same reasoning as above.

We go back to proving Claim 1.7, then we complete the proof of the inclusion remaining of Claim 1.6.

Proof of Claim 1.7 (ii).

Let everything be as in the hypothesis of Claim 1.7 (ii). We define inductively the sequence $\langle Y_k : k \in \omega \rangle$ as follows:

$$Y_0 = Y \cup \{0, 1\}$$

and

$$Y_{k+1} = Y_k \cup \{f^{A^+} y : y \in {}^iY_k : i \in \{1, 2\} \text{ and } f \text{ is an } i \text{ ary operation of } A^+\}.$$

Then of course we have

$$Sg^{A^+} Y = \cup \{Y_k : k \in \omega\}.$$

We will show, by induction on k, that for all $y \in Y_k$ there exists $j \in \omega$, such that $x \in Sg^{Rd_{\omega+j} A^+} Y$, by which we will be done.

The case when $k = 0$ is trivial. Ignoring the boolean operations as usual, assume then that $y = f(x)$, where f is a non boolean (hence unary) operation, and inductively that $x \in Sg^{Rd_{\omega+j} A^+} Y$ for some $j \in \omega$.

Assume that $f = s_\tau$ is a substitution. Let D_τ be as in Claim 1.1 (b) above and let $C_\tau = M \smallsetminus D_\tau$. Let $n \geq j$ be such that $(M \cap \Delta x) \cup C_\tau \subseteq \omega + n$. Then $x \in Sg^{Rd_{\omega+n} A^+} Y$. Let $\sigma = \tau \restriction \omega + n$. Then $\sigma \in T_n$ and $\sigma_\omega = \tau$. Also

$$y = s_\tau^{A^+} x = s_{\sigma_\omega}^{A^+} x = s_\sigma^{Rd_{\omega+n} A^+} x,$$

proving that y is as required. The case when f is a cylindrification is entirely analogous, and is left to the reader. QED (Claim 1.7 (ii)). $\square$

Now we are ready to prove the remaining inclusion of Claim 1.6, namely that

$$Nr_\omega Sg^{A^+} e(Y) \subseteq e(Sg^A Y).$$

We write $f \in Ism(A,B)(Is(A,B))$ to abbreviate that f is an isomorphism from A into (onto) B.

Let $y \in Nr_\omega Sg^{A^+}e(Y)$. Then $y \in Sg^{A^+}e(Y)$ and $\Delta y \subseteq \omega$. By $e(Y) \subseteq Nr_\omega A^+$, we have by Claim 1.7 (ii) that there exists $j \in \omega$ such that $y \in Sg^{Rd_{\omega+j}A^+}e(Y)$. Fix any such j among the infinitely many. Identifying j with $\{0, 1, \ldots, j-1\}$, let

$$F_j = \{\Gamma \cap (\omega \smallsetminus j) : \Gamma \in F\}.$$

Then F_j is a non-principal ultrafilter on $\omega \smallsetminus j$. For $x \in A_j = A$, let

$$e^j(x) = \langle e_n(x) : j \leq n < \omega \rangle / F_j;$$

and let

$$A_j^+ = \mathbf{P}_{j \leq n < \omega} A_n^+ / F_j.$$

Then it is not hard to see that $e^j : A_j \to A_j^+$ is one to one. For $x = \langle x_n : n \in \omega \rangle / F \in A^+$, let

$$f(x) = \langle x_n : j \leq n < \omega \rangle / F_j.$$

Then clearly $f \in Is(A^+, A_j^+)$. Let g be the inverse of f. Then

$$g \circ e^j \in Ism(A_j, Rd_{\omega+j}A^+).$$

Moreover for any $a \in A$ we have

$$e(a) = g \circ e^j \circ e_j(a).$$

Now

$$y \in Sg^{Rd_{\omega+j}A^+}e(Y) = Sg^{Rd_{\omega+j}A^+}(g \circ e^j \circ e_j(Y)).$$

Let $y' = f(y)$. Then $g(y') = y$ and $\Delta y' \subseteq \omega$ because $\Delta y \subseteq \omega$ and g is an isomorphism. Also

$$y' \in f(Sg^{Rd_{\omega+j}A^+}(g \circ e^j \circ e_j(Y)) = Sg^{Rd_{\omega+j}A_j^+}(e^j \circ e_j(Y)).$$

Recall that $e^j : A_j \to e^j(A_j)$ is an isomorphism. Let $h : e^j(A_j) \to A_j$ denote its inverse. Note that $y' \in e^j(A_j)$ since

$$Sg^{Rd_{\omega+j}A_j^+}(e^j \circ e_j(Y)) \subseteq e^j(A_j).$$

Now let $a' = h(y')$. Then $e^j(a') = y'$ and

$$a' \in h(Sg^{Rd_{\omega+j}A_j^+}(e^j \circ e_j)(Y)) = Sg^{A_j}e_j(Y).$$

Also, by the same reasoning as above, $\Delta a' \subseteq \omega$, i.e $a' \in Nr_\omega Sg^{A_j}e_j(Y)$. By Claim 1.3 we get an $a \in Sg^A Y$ such that $e_j(a) = a'$. Thus

$$e(a) = g \circ e^j \circ e_j(a) = g \circ e^j(a') = g(y') = y.$$

By this the proof of Claim 1.6 is complete. QED (Claim 1.6). □

From now on we assume that A is a subalgebra of $Rd_\omega A^+$, i.e. we identify A with $e(A)$, regarding $e : A \to Rd_\omega A^+$ as the inclusion map. In particular, we have $A \subseteq Nr_\omega A^+$ and for all $Y \subseteq A$ we have (*)

$$Sg^A Y = Sg^{Nr_\omega A^+} Y = Nr_\omega Sg^{A^+} Y.$$

Also we fix sets X,X_1 and X_2 such that X generates A, and $X_1, X_2 \subseteq X$.

Claim 1.11 . *Let $a \in Sg^A X_1$, $b \in Sg^A X_2$ such that $a \leq c$. Assume that there is no $b \in Sg^A(X_1 \cap X_2)$ such that $A \models a \leq b \leq c$. Then there is no $b \in Sg^{A^+}(X_1 \cap X_2)$ such that $A^+ \models a \leq b \leq c$.*

First we recall a useful notation. Let Γ be a finite subset of α, $\Gamma = \{j_0, \ldots, j_{n-1}\}$ say, and $x \in A^+$. If $n = 0$, i.e $\Gamma = \emptyset$ then $c_\emptyset x = x$. Else $n > 0$, in which case we set $c_{(\Gamma)}x = c_{j_0} \cdots c_{j_{n-1}}x$. $c_{(\Gamma)}$ as defined above is referred to as generalised cylindrification (cf., [50], Def 1.7.1).

Now we prove Claim 1.11:

Assume, seeking a contradiction, that there exists $b \in Sg^{A^+}(X_1 \cap X_2)$ such that

$$A^+ \models a \leq b \leq c.$$

By Claim 1.7 (i) we have $\Delta b \smallsetminus \omega = \Gamma$, say, is finite. Set $b' = c_{(\Gamma)}b$. Then

$$A^+ \models a \leq b \leq c_{(\Gamma)}b = b'.$$

Since $c \in Sg^A X_2 \subseteq Nr_\omega Sg^{A^+} X_2$, we get that $c_{(\Gamma)}c = c$, hence $A^+ \models a \leq b' \leq c$. Since $b' \in Nr_\omega Sg^{A^+}(X_1 \cap X_2)$, it then follows by Claim 1.6 , that $b' \in Sg^A(X_1 \cap X_2)$. This contradicts our assumption that no interpolant of a and c exists in $Sg^A(X_1 \cap X_2)$. QED (Claim 1.11.) □

End of details of part (i).

In the next two remarks we elaborate on representing G algebras as algebras of sentences in first order languages endowed with individual constants which, to the best of our knowledge, is a novel interpretation of such algebras originating with Amer [2]. The number of individual constants determines the dimension of the algebra in question. (The standard interpretation uses algebras of formulas, in which case the dimension is determined by the number of variables available, see e.g. [51] sec 4.3.) Then we go back to the details of part (ii) of the proof of Proposition 1.

Remark 1.12

We have mentioned in the informal outline of proof that the algebraic property of neat embedding can be viewed as adding constants. Let us dwell on this a little longer. We illustrate this by an example, cf. [2] and [128]:

(a) Let L be an ordinary first order language with two propositional constants T,F, and a list $\langle c_k\rangle$ of individual constants of order type α. L has no operation symbols, but as usual, the list of variables is of order type ω. Denote by Sn^L the set of all L sentences. For $\phi \in Sn^L$ and $k<\alpha$ let $\exists_k\phi := \exists x\phi(c_k|x)$ where x is the first variable not occuring in ϕ. Here $\phi(c_k|x)$ is the formula obtained from ϕ by replacing all occurences of c_k, if any, in ϕ by x. For $\tau \in {}^{\alpha}\alpha$, and $\phi=\phi(c_0,\cdots c_{n-1})\in Sn^L$ we let $s_{\tau}\phi$ be the sentence obtained from ϕ by the simultaneous substitution of the constants $c_{\tau(0)},\cdots,c_{\tau(n-1)}$ for the constants $c_0,\cdots c_{n-1}$ respectively. Let Σ be a set of sentences in which no constants occur. Define the equivalence relation $\equiv_{\Sigma}$ on ${}^2Sn^L$ as follows

$$\phi\equiv_{\Sigma}\psi \text{ iff } \Sigma\models\phi\equiv\psi.$$

Then, as easily checked, for any $G\subseteq {}^{\alpha}\alpha$, $\equiv_{\Sigma}$ is a congruence relation on the algebra

$$\mathbf{S}n=\langle Sn,\land,\lor,\neg,T,F,\exists_k,s_{\tau}\rangle_{k<\alpha,\tau\in G}$$

We let $\mathbf{S}n^L/\Sigma$ denote the quotient algebra. For the sake of conveniance we let $\alpha=\omega$ and fix G to be the set of all finite transformations on ω. In this case, $\mathbf{S}n^L/\Sigma$ is a locally finite polyadic algebra of dimension ω [cf. [128] Chapter 4]. Here locally finite means that the dimension set of any element is finite. This follows from the simple observation that the number of constants occuring in a given sentence is finite.

(b) Let L be as described in (a) above. But now we denote it L_α, the subsript α indicating that we have α-many individual constants. Now enrich L_α with constants (and nothing else) obtaining L_β, with $\beta > \alpha$. Recall that both languages have a list of ω variables. Let Σ be a set of sentences in which no constants occur. For $\kappa \in \{\alpha, \beta\}$ let $\mathbf{A}_\kappa = \mathbf{S}n^{L_k}/\Sigma$. (Here we are also assuming that G is the set of all finite transformations on κ). For $\phi \in Sn^{L_\alpha}$, let $f(\phi/\Sigma) = \phi/\Sigma$. Then, as easily checked f is an embedding of A_α into A_β. Moreover f has the additional property that it maps A_α, into (and onto) the neat α reduct of A_β, (i.e. the set of α dimensional elements of A_β). In short, $A_\alpha \cong Nr_\alpha A_\beta$.

(End of Remark 1.12.) $\diamondsuit$

Remark 1.13

Let A be a countable abstract G algebra. In preparation of representing A (as a concrete set algebra), Sain [116] proved that A has the *the neat embedding property*. This, we mimmicked in our construction of the ω dilation A^+ of A, which was constructed in a step-by-step fashion. First we neatly embedded A into A_n, for every n, then we neatly embedded A into an ultraproduct of (expansions of) the A_n's. We delay defining the representation for a while. This will be done in part (iii). We point out that the novelty that occurs here is that A^+ is a minimal extension of A. This was expressed in Claim 1.6. With Remark 1.12 at hand, we see that neatly embedding A into A^+ step-by-step corresponds to the following metalogical fact: The process of adding ω new constants, can be broken into infinitely many steps, or tasks, each of which consists of adding one new constant. Neatly embedding A into A_n means we are adding n new constants. Claim 1.6 expresses the rather simple (and expected) fact that when we add constants and nothing else then we do not add to the expressive power of our logics. No "new elements" are defined from old ones; borrowing yet another concept from model theory: A^+ is a conservative extension of A, so to speak. (End of Remark 1.13.) $\diamondsuit$

Details of Part (ii)

From now on we let everything be as in the hypothesis of Claim 1.11.

That is X is a generating set of A, and $X_1, X_2 \subseteq X$. Further fix $a \in Sg^A(X_1)$ and $c \in Sg^A(X_2)$, for which no interpolant exists in $Sg^A(X_1 \cap X_2)$ (equivalently (in light of Claim 1.11) no interpolant exists in $Sg^{A^+}(X_1 \cap X_2)$.)

We work with the big algebra A^+. Using the supply of witnesses (the extra ω dimensions of A^+), we extend a and $-c$ to *Henkin* ultrafilters. In preparation of defining the desired model, which will be done in part (iii) , we make sure that these ultrafilters, constructed in a step-by-step manner agree on $Sg^{A^+}(X_1 \cap X_2)$. The first step is already taken care of by Claim 1.11. Now for the details.

Recall that $\alpha = \omega + \omega$. For $Y \subseteq A$, let $A^+(Y)$ stand for $Sg^{A^+}Y$. Recall that we assumed that $A \subseteq A^+$, that A is generated by X (not freely yet) and that $X_1, X_2 \subseteq X$. Now arrange $\alpha \times A^+(X_1)$ and $\alpha \times A^+(X_2)$ into ω-termed sequences:

$$\langle (k_i, x_i) : i \in \omega \rangle \text{ and } \langle (l_i, y_i) : i \in \omega \rangle \text{ respectively.}$$

This is possible since both $A^+(X_1)$ and $A^+(X_2)$, being subalgebras of A^+, are countable and $|\alpha| = \omega$.

Using Fact 1.8, we can define by recursion (or step by step) ω-termed sequences of witnesses:

$$\langle u_i : i \in \omega \rangle \text{ and } \langle v_i : i \in \omega \rangle$$

such that for all $i \in \omega$ we have:

$$u_i \in \alpha \smallsetminus (\Delta a \cup \Delta c) \cup \cup_{j \leq i} (\Delta x_j \cup \Delta y_j) \cup \{u_j : j < i\} \cup \{v_j : j < i\}$$

and

$$v_i \in \alpha \smallsetminus (\Delta a \cup \Delta c) \cup \cup_{j \leq i} (\Delta x_j \cup \Delta y_j) \cup \{u_j : j \leq i\} \cup \{v_j : j < i\}.$$

To formulate our next Claim we need to fix some notation.

Notation. For a boolean algebra B and $Y \subseteq B$, we write $fl^B Y$ to denote the boolean filter generated by Y in B. We let

$$Y_1 = \{a\} \cup \{-c_{k_i} x_i + s^{k_i}_{u_i} x_i : i \in \omega\},$$

$$Y_2 = \{-c\} \cup \{-c_{l_i} y_i + s^{l_i}_{v_i} y_i : i \in \omega\},$$
$$H_1 = fl^{BlA^+(X_1)} Y_1,\ H_2 = fl^{BlA^+(X_2)} Y_2,$$

and

$$H = fl^{BlA^+(X_1 \cap X_2)}[(H_1 \cap A^+(X_1 \cap X_2) \cup (H_2 \cap A^+(X_1 \cap X_2)].$$

Claim 1.14 . *H is a proper filter of $A^+(X_1 \cap X_2)$.*

To prove this it is sufficient to consider any pair of finite, strictly increasing sequences of finite ordinals

$$\eta(0) < \eta(1) \cdots < \eta(n-1) < \omega \text{ and } \xi(0) < \xi(1) < \cdots < \xi(m-1) < \omega$$

and to prove that the following condition holds:

(1) For any b_0, $b_1 \in A^+(X_1 \cap X_2)$ such that

$$a.\Pi_{i<n}(-c_{k_{\eta(i)}} x_{\eta(i)} + s^{k_{\eta(i)}}_{u_{\eta(i)}} x_{\eta(i)}) \leq b_0$$

and

$$(-c).\Pi_{i<m}(-c_{l_{\xi(i)}} y_{\xi(i)} + s^{l_{\xi(i)}}_{v_{\xi(i)}} y_{\xi(i)}) \leq b_1$$

we have

$$b_0.b_1 \neq 0.$$

We prove this by induction on $n+m$. If $n+m=0$, then (1) simply expresses the fact that no interpolant of a and c exists in $A^+(X_1 \cap X_2)$ as already proved in Claim 1.11.

In more detail: if $n+m=0$, then $a_0 \leq b_0$ and $-c \leq b_1$. So if $b_0.b_1 = 0$, we get $a \leq b_0 \leq -b_1 \leq c$ which contradicts Claim 1.11.

Now assume that $n+m > 0$ and for the time being suppose that $\eta(n-1) > \xi(m-1)$.

Apply $c_{u_{\eta(n-1)}}$ to both sides of the first inclusion of (1).

By $u_{\eta(n-1)} \notin \Delta a$, i.e. $c_{u_{\eta(n-1)}} a = a$, and by noting that $c_i(c_i x.y) = c_i x.c_i y$, we get (2)

$$a.c_{u_{\eta(n-1)}} \Pi_{i<n}(-c_{k_{\eta(i)}} x_{\eta(i)} + s^{k_{\eta(i)}}_{u_{\eta(i)}} x_{\eta(i)}) \leq c_{u_{\eta(n-1)}} b_0.$$

Let $c_i^\partial(x) = -c_i(-x)$. c_i^∂ is the algebraic counterpart of the universal quantifier $\forall x_i$.

Now apply $c^\partial_{u_{\eta(n-1)}}$ to the second inclusion of (1). By noting that c_i^∂, the dual of c_i, distributes over the boolean meet and by $u_{\eta(n-1)} \notin \Delta c = \Delta(-c)$ we get (3)

$$(-c).\Pi_{j<m}(c^\partial_{u_{\eta(n-1)}}(-c_{l_{\xi(j)}}y_{\xi(j)} + s^{l_{\xi(j)}}_{v_{\xi(j)}}y_{\xi(j)})) \leq c^\partial_{u_{\eta(n-1)}}b_1.$$

Before going on, we formulate a claim that will enable us to eliminate the quantifier $c_{u_{\eta(n-1)}}$ (and its dual) from (2) (and (3)) above.

In order not to interrupt the main stream, we shall assume that the claim is true and proceed with proving that H is indeed a proper filter, as claimed in 1.14 , then return to proving the claim.

Claim 1.15 . *For the sake of brevity set for each $i < n$ and each $j < m$:*

$$z_i = -c_{k_{\eta(i)}}x_{\eta(i)} + s^{k_{\eta(i)}}_{u_{\eta(i)}}x_{\eta(i)}$$

and

$$t_j = c_{l_{\xi(j)}}y_{\xi(j)} + s^{l_{\xi(j)}}_{v_{\xi(j)}}y_{\xi(j)}.$$

Then (i) and (ii) below hold:

$$(i)\ c_{u_{\eta(n-1)}}z_i = z_i \text{ for } i < n-1 \text{ and } c_{u_{\eta(n-1)}}z_{n-1} = 1.$$

$$(ii)\ c^\partial_{u_{\eta(n-1)}}t_j = t_j \text{ for all } j < m.$$

Assume that Claim 1.15 is true. Then

$$c_{u_{\eta(n-1)}}\Pi_{i<n}z_i = c_{u_{\eta(n-1)}}[\Pi_{i<n-1}z_i.z_n] = c_{u_{\eta(n-1)}}\Pi_{i<n-1}z_i.c_{u_{\eta(n-1)}}z_{n-1} = \Pi_{i<n-1}z_i.$$

Here we are using that $c_i(c_ix.y) = c_ix.c_iy$, which follows from (P_2). Combined with (2) we obtain

$$a.\Pi_{i<n-1}(-c_{k_{\eta(i)}}x_{\eta(i)} + s^{k_{\eta(i)}}_{u_{\eta(i)}}x_{\eta(i)}) \leq c_{u_{\eta(n-1)}}b_0.$$

On the other hand, from Claim 1.15 (ii) and (3), it follows that

$$(-c).\Pi_{j<m}(-c_{l_{\xi(j)}}y_{\xi(j)} + s^{l_{\xi(j)}}_{v_{\xi(j)}}y_{\xi(j)}) \leq c^\partial_{u_{\eta(n-1)}}b_1.$$

Now making use of the induction hypothesis we get

$$c_{u_{\eta(n-1)}}b_0.c^\partial_{u_{\eta(n-1)}}b_1 \neq 0;$$

and hence that

$$b_0.c^{\partial}_{u_{\eta(n-1)}}b_1 \neq 0.$$

From

$$b_0.c^{\partial}_{u_{\eta(n-1)}}b_1 \leq b_0.b_1$$

we reach the desired conclusion, i.e. that

$$b_0.b_1 \neq 0.$$

The other case, when $\eta(n-1) \leq \xi(m-1)$ can be treated analgously and is therefore left to the reader.

We have proved that H is a proper filter, modulo Claim 1.15. To complete the proof of Claim 1.14, we must now prove Claim 1.15.

Proof of Claim 1.15 (i):

Proof of $c_{u_{\eta_{n-1}}}z_i = z_i$ *for* $i < n-1$.

Let $i < n-1$. Then by the choice of witnesses we have

$$u_{\eta(n-1)} \neq u_{\eta(i)}.$$

Also it is easy to see that for all $i, j \in \alpha$ we have

$$\Delta c_j x \subseteq \Delta x \text{ and that } \Delta s^i_j x \subseteq \Delta x \smallsetminus \{i\} \cup \{j\},$$

In particular,

$$u_{\eta(n-1)} \notin \Delta c_{k_{\eta(i)}} x_{\eta(i)} \text{ and } u_{\eta(n-1)} \notin \Delta(s^{k_{\eta(i)}}_{u_{\eta(i)}} x_{\eta(i)}).$$

It thus follows that

$$c_{u_{\eta(n-1)}}(-c_{k_{\eta(i)}}x_{\eta(i)}) = -c_{k_{\eta(i)}}x_{\eta(i)} \text{ and } c_{u_{\eta(n-1)}}(s^{k_{\eta(i)}}_{u_{\eta(i)}}x_{\eta(i)}) = s^{k_{\eta(i)}}_{u_{\eta(i)}}x_{\eta(i)}.$$

Finally, by $c_{u_{\eta(n-1)}}$ distributing over the boolean join, we get

$$c_{u_{\eta(n-1)}}z_i = z_i \text{ for } i < n-1.$$

Proof of $c_{u_{\eta(n-1)}}z_{n-1} = 1$.

Computing we get, by $u_{\eta(n-1)} \notin \Delta x_{\eta(n-1)}$ and by (P_8) and (P_{10}), the following:

$$c_{u_{\eta(n-1)}}(-c_{k_{\eta(n-1)}}x_{\eta(n-1)} + s^{k_{\eta(n-1)}}_{u_{\eta(n-1)}}x_{\eta(n-1)})$$

$$= c_{u_{\eta(n-1)}} - c_{k_{\eta(n-1)}} x_{\eta(n-1)} + c_{u_{\eta(n-1)}} s^{k_{\eta(n-1)}}_{u_{\eta(n-1)}} x_{\eta(n-1)}$$

$$= -c_{k_{\eta(n-1)}} x_{\eta(n-1)} + c_{u_{\eta(n-1)}} s^{k_{\eta(n-1)}}_{u_{\eta(n-1)}} x_{\eta(n-1)}$$

$$= -c_{k_{\eta(n-1)}} x_{\eta(n-1)} + c_{u_{\eta(n-1)}} s^{k_{\eta(n-1)}}_{u_{\eta(n-1)}} c_{u_{\eta(n-1)}} x_{\eta(n-1)}$$

$$= -c_{k_{\eta(n-1)}} x_{\eta(n-1)} + c_{k_{\eta(n-1)}} s^{u_{\eta(n-1)}}_{k_{\eta(n-1)}} c_{u_{\eta(n-1)}} x_{\eta(n-1)}$$

$$= -c_{k_{\eta(n-1)}} x_{\eta(n-1)} + c_{k_{\eta(n-1)}} c_{u_{\eta(n-1)}} x_{\eta(n-1)}$$

$$= -c_{k_{\eta(n-1)}} x_{\eta(n-1)} + c_{k_{\eta(n-1)}} x_{\eta(n-1)} = 1.$$

With this the proof of Claim 1.15 (i) is complete.

Proof of Claim 1.15 (ii).

Let $j < m$. Then we have

$$c^{\partial}_{u_{\eta(n-1)}}(-c_{l_{\xi(j)}} y_{\xi(j)}) = -c_{l_{\xi(j)}} y_{\xi(j)}$$

and

$$c^{\partial}_{u_{\eta(n-1)}}(s^{l_{\xi(j)}}_{v_{\xi(j)}} y_{\xi(j)}) = s^{l_{\xi(j)}}_{v_{\xi(j)}} y_{\xi(j)}.$$

Indeed, computing we get

$$c^{\partial}_{u_{\eta(n-1)}}(-c_{l_{\xi(j)}} y_{\xi(j)}) = -c_{u_{\eta_{n-1}}} - (-c_{l_{\xi(j)}} y_{\xi(j)}) = -c_{u_{\eta(n-1)}} c_{l_{\xi(j)}} y_{\xi(j)} = -c_{l_{\xi(j)}} y_{\xi(j)}.$$

Similarly we have

$$c^{\partial}_{u_{\eta(n-1)}}(s^{l_{\xi(j)}}_{v_{\xi(j)}} y_{\xi(j)}) = -c_{u_{\eta(n-1)}} - (s^{l_{\xi(j)}}_{v_{\xi(j)}} y_{\xi(j)})$$

$$= -c_{u_{\eta(n-1)}}(s^{l_{\xi(j)}}_{v_{\xi(j)}} - y_{\xi(j)}) = -s^{l_{\xi(j)}}_{v_{\xi(j)}} - y_{\xi(j)} = s^{l_{\xi(i)}}_{v_{\xi(j)}} y_{\xi(j)}.$$

By $c^{\partial}_i(c^{\partial}_i x + y) = c^{\partial}_i x + c^{\partial}_i y$ we get from the above that

$$c^{\partial}_{u_{\eta(n-1)}}(t_j) = c^{\partial}_{u_{\eta(n-1)}}(c_{l_{\xi(j)}} y_{\xi(j)} + s^{l_{\xi(j)}}_{v_{\xi(j)}} y_{\xi(j)})$$

$$= c^{\partial}_{u_{\eta(n-1)}} c_{l_{\xi(j)}} y_{\xi(j)} + c^{\partial}_{u_{\eta(n-1)}} s^{l_{\xi(j)}}_{v_{\xi(j)}} y_{\xi(j)}$$

$$= c_{l_{\xi(j)}} y_{\xi(j)} + s^{l_{\xi(j)}}_{v_{\xi(j)}} y_{\xi(j)} = t_j.$$

With this the proof of Claim 1.15 is complete. Therefore H is a proper filter of $A^+(X_1 \cap X_2)$. QED (Claim 1.15) $\square$

Now let H^* be a (proper boolean) ultrafilter of $A^+(X_1 \cap X_2)$ containing H. From BA theory[20] we get ultrafilters F_1 and F_2 of $A^+(X_1)$ and $A^+(X_2)$, respectively such that

$$H^* \subseteq F_1, \quad H^* \subseteq F_2$$

and

$$F_1 \cap A^+(X_1 \cap X_2) = F_2 \cap A^+(X_1 \cap X_2).$$

Then by their construction, it is easy to see that F_1 and F_2 are Henkin ultrafilters and that for all $x \in A^+(X_1 \cap X_2)$ we have $x \in F_1$ iff $x \in F_2$.

End of details of part (ii).

Now we are not too far away from defining the desired model. We try to make it even closer by drawing another analogy from Model Theory in the next Remark.

Remark 1.16

Let L_α and Σ be as described in Remark 1.12, except that now we assume that everything is countable. For simplicity we further assume that $\alpha = \omega$. Let F be an ultrafilter of $\mathbf{S}n^{L_\omega}/\Sigma$. Then it is not hard to check that F is a Henkin ultrafilter iff $T = \cup F$ is a complete rich extension of Σ. Rich here means that T has witnesses in L_ω, i.e. whenever $\exists x\phi \in T$, then $\phi(c)$ is in T, where c is a constant not occurring in ϕ, called a witness to $\exists x\phi$, and $\phi(c)$ is the sentence obtained from ϕ by substituting all free occurrences of x , if any, by c. The reader is referred to [128], chapter 4, for more on such connections. It is known how canonical models corresponding to complete rich theories are defined. (cf. [35] Lemma 2.1.2 p.63.) This is precisely what we do next.

(End of Remark 1.16.) $\diamondsuit$

Details of Part(iii)

In this last part of the proof we form the desired model of $(A, a - c)$. First we define models of $(Sg^A X_1, a)$ and $(Sg^A X_2, -c)$. Then, using the freeness of A, we amalgamate

[20] BA abbreviates boolean algebra.

these models to get the desired one.

Let U be a set and $p \in {}^{\omega}U$ be a U termed sequence. Then to keep notation simpler, we write ${}^{\omega}U^{p}$ instead of ${}^{\omega}U^{(p)}$. Recall that the latter denotes the ω-dimensional weak space with base U defined as follows:

$$\{s \in {}^{\omega}U : |\{i \in \omega : s_i \neq p_i\}| < \omega\}.$$

A generalized weak space is a union of weak spaces having a common base. We will define our models using a generalized weak space having a common base $\alpha = \omega + \omega$.

Claim 1.17 . *(Sain) (i) Let V be the generalized weak space $\cup_{\tau \in T}{}^{\omega}\alpha^{\tau}$. For $v \in V$, let $v^* = v \cup Id \upharpoonright (\alpha \smallsetminus \omega)$. Then $v^* \in T_\omega$.*

(ii)For $j \in \{1,2\}$, let f_j be the function with domain $A_j = Sg^{A}(X_j)$ such that

$$f_j(a) = \{v \in V : s_{v*}^{A^+(X_j)}a \in F_j\}.$$

Then f_j is a homorphism from A_j into $\langle B(V), C_i, S_\tau\rangle_{i\in\omega,\tau\in T}$. Also $Id \in f_1(a) \cap f_2(-c)$.

Proof of (i). Follows from the definitions (see also [116]).

Proof of (ii).

$Id \in f_1(a) \cap f_2(-c)$ follows from $a \in F_1$ and $-c \in F_2$. It is easy to check that f_j is a boolean homomorphism for $j \in \{0,1\}$. It is clearly sufficient, by total symmetry in the definition to show that f_1 preserves the extra non-boolean operations. This is proved in [116], but for the sake of completeness we include Sain's proof.

For the sake of brevity let

$$f = f_1, F = F_1 \text{ and } B = Sg^{A}X_1.$$

Then, in view of the identification $A \subseteq Nr_\omega A^+$, we have $B \subseteq Nr_\omega A^+$ and for all $b \in B$ and $\tau \in T$ we have $s_\tau^{B}b = s_{\tau_\omega}^{A^+}b$.

Also for all $\sigma \in T_\omega$ and $b \in B$, since $A^+(X_1) \subseteq A^+$ we have $s_\sigma^{A^+(X_1)}b = s_\sigma^{A^+}b$.

(i) f preserves cylindrifications

Let $b \in B$ and $i \in \omega$. We want to show that $C_i f(b) = f(c_i b)$. We first show that $C_i f(b) \subseteq f(c_i b)$. Let $\tau \in C_i f(b)$. Then $\tau_j^i \in f(b)$ for some $j \in \alpha$, where $\tau_j^i(x) = \tau(x)$

if $x \neq i$ and $\tau^i_j(i) = j$. Let $v = \tau^i_j$. Then, by definition, we have $s^{A^+}_{v*} b \in F$. By $s^{A^+}_{v^*} b \leq s^{A^+}_{v^*} c^{A^+}_i b$ and since F is a filter we get $s^{A^+}_{v^*}(c^{A^+}_i b) \in F$.

Now τ^* differs from v^* in at most the i-th place, and so we get by (P_5) that

$$s^{A^+}_{v^*} c^{A^+}_i b = s^{A^+}_{\tau^*} c^{A^+}_i b,$$

and so

$$s^{A^+}_{\tau^*} c^{A^+}_i b = s^{A^+}_{\tau^*} c^{B}_i b \in F.$$

Hence $\tau \in f(c_i b)$. We have shown that $C_i f(b) \subseteq f(c_i b)$. Now for the opposite inclusion, namely that $f(c_i b) \subseteq C_i f b$. Since everything is evaluated in A^+ we omit superscripts for better readability.

Let $v \in f(c_i b)$. Let $\tau = v^*$. Then $s_\tau c_i b \in F$.

We will show that there exists $j \in \alpha$ such that $s_{\tau[i|j]} b \in F$, by which we will be done, because of the following:

$v[i|j]^* = \tau[i|j]$ and so $v[i|j] \in f(b)$ thus $v \in c_i f(b)$.

Let $m \in \alpha$ such that $c_m b = b$ and $\tau^{-1} m = \{m\} = \{\tau(m)\}$. Such an m exists by $|M \cap \Delta x| < \omega$ and $|C_\tau| < \omega$. Let $l \in \alpha \smallsetminus \{i, m\}$. Let $k = \tau(l)$. Then $k \neq m$, otherwise we would have that $\tau(l) = m$, i.e. $l \in \tau^{-1} m$, contradicting that $\tau^{-1} m = \{m\}$ since $l \neq m$. Recall that $\tau[i|m](x) = \tau(x)$ if $x \neq i$ and $\tau[i|m](i) = m$. By the choice of m, it follows that, $\tau[i|m] = \tau \circ [i|m]$. Now by (P_5) and the above we have

$$s_\tau c_i b = s_{\tau[i|m]} c_i b = s_{\tau \circ [i|m]} c_i b.$$

By $c_m b = b$, we have by (P_8) that $s^m_l c_m b = c_m b$. By the commutativity of cylindrifications we get that

$$s_{\tau \circ [i|m]} c_i b = s_{\tau \circ [i|m]} s^m_l c_m c_i b,$$

which in turn is equal to $s_{\tau \circ [i|m] \circ [m|l]} c_i b$.

Let $\sigma = \tau \circ [i|m] \circ [m|l]$. Then it is not difficult to check that $\sigma(x) = m$ iff $x = i$. By (P_6), we have $s_\sigma c_i b = c_m s_\sigma b$. We have shown that $s_\tau c_i b = c_m s_\sigma b$, thus $c_m s_\sigma b \in F$. Since F is a Henkin ultrafilter, it follows that there exists $j \in \alpha$ such that $s^m_j s_\sigma b \in F$. Let $\sigma' = [m|j] \circ \sigma$. Then it is not difficult to check that $\sigma'(x) = \tau[i|j]x$ for all $x \in \alpha \smallsetminus \{m\}$. By the above we get

$$s^m_j s_\sigma b = s^m_j s_\sigma c_m b = s_{[m|j] \circ \sigma} c_m b = s_{\tau[i|j]} c_m b = s_{\tau[i|j]} b,$$

and so $s_{\tau[i|j]}b \in F$, which is the desired conclusion.

(ii) f preserves substitutions.

Let τ be a substitution and $b \in B$. Then

$$v \in f(s^B_\tau b) \text{ iff } s^{A^+}_{v^*}(s^B_\tau b) \in F \text{ iff } s^{A^+}_{v^*} s^{A^+}_{\tau_\omega} b \in F \text{ iff } s^{A^+}_{v^* \circ \tau_\omega} b \in F.$$

By noting that

$$(v \circ \tau)^* = v^* \circ \tau_\omega$$

this is equivalent to

$$v \circ \tau \in f(b) \text{ i.e. } v \in S_\tau(f(b)).$$

QED (Claim 1.17) $\square$

Now suppose that A is freely generated by X. Assume further, without any loss of generality, that $X_1 \cup X_2 = X$.

Claim 1.18 . *Let $g = f_1 \cup f_2 \upharpoonright X$. Then g is a function. Moreover g extends to a homomorphism from A into $\langle B(V), C_i, S_\tau \rangle_{i \in \omega, \tau \in T}$. In particular, we have $g(a-c) \neq 0$.*

Proof: To prove that g is a function, we have to show that f_1 and f_2 agree on $X_1 \cap X_2$. But this follows immediately from how f_1 and f_2 are defined, by noting that F_1 and F_2 agree on $A^+(X_1 \cap X_2)$. Since A is freely generated by $X = X_1 \cup X_2$, it follows that g extends to a homomorphism on A. Also by

$$g(a - c) = g(a) \cap g(-c) = f_1(a) \cap f_2(-c),$$

and Id belonging to $f_1(a)$ and $f_2(-c)$, we get that

$$Id \in g(a - c) \text{ i.e. } g(a - c) \neq 0.$$

But this means that g defines a model of $(A, a - c)$, which contradicts that $a \leq c$. By this the proof of Proposition 1 when G is specified as in Theorem 1 is complete and therefore Theorem 1 holds, as well.

The proof in the general case

Now we check that our proof survives the case when the semigroup T is *strongly rich*, i.e. we prove Theorem 1*. We have shown in Remark 1.4 way above the following:

Fact 1.20 . *Let $T \subseteq {}^{\omega}\omega$ be rich. Let $A \in TPS_{\omega}$ be countable. Then for every $n \in \omega$, there exists a semigroup $T_n \subseteq {}^{\omega+n}\omega + n$, an $n-$ dilation A_n of A, i.e. an $A_n \in T_nPA_{\omega+n}$, and an embedding e_n of A into $Nr_{\omega}A_n$. Furthermore if T is strongly rich then the following condition (of neat reducts commuting with forming subalgebras) holds:*

$$\text{For all } Y \subseteq A,\ e_n(Sg^{A}Y) = Nr_{\omega}Sg^{A_n}e_n(Y).$$

Recall that e_n is defined as s_{σ^n}. We have included the proof of Fact 1.20 in a separate Remark 1.4, to show that Claims 1.2-3, do not use any special properties of G, other than the richness of [G] in Claim 1.2, and its strong richness in Claim 1.3. The rest of the proof goes through by undergoing the obvious modifications as follows: replace in the hypothesis of any Claim *suc* by σ and *pred* by π. Here σ and π are the special elements of the rich T, as described in Definition 0.3. In more detail: A_n is defined in Remark 1.4. Next define A_n^+, from A_n as before. The definition of A^+, as an ultraproduct of the A_n^+ relative to any non-principal ultrafilter F on ω, is also the same. The proofs of the analogues of Claims 1.5, 1.7 and Fact 1.8 go through without any difficulty. A careful reading of these claims will reveal that it is only the *richness* of T that is needed, see also [116]. In particular, $e = \langle e_n(x) : n \in \omega\rangle/F$, is an isomorphism from A into $Rd_{\omega}A^+$, the latter also defined as above. The proof of Claim 1.6. uses Claim 1.3, so here *strong* richness is needed. Identifying A with $e(A)$, we thus have (*)

$$Sg^{A}Y = Sg^{Nr_{\omega}A^+}Y = Nr_{\omega}Sg^{A^+}Y.$$

The proof of Claim 1.11 is the same. (Here also strong richness is used, since this depends on the last equality of (*).) By this the details of part (i) of the proof of Proposition 1 - for any strongly rich T - is complete. Finally the details of part (ii) and (iii) of the proof of Proposition 1 for $[G] = T$ generalize verbatim to any rich (not necessarily strongly rich) semigroup. We have proved Proposition 1 and Theorem 1*. □

(End of Remark 1.19.) ♢

End of details of part (iii), and end of proof.

Concluding Remarks

(1) When we have a Neat Embedding Theorem, or NET for short, to the effect that algebras having the neat embedding property are representable, then we have a completeness theorem in the corresponding logics, cf. [9], [51] sec 4.3-5.6, and [128] chapters 4-5 . Sain showed that the class of G algebras when $[G]$, the semigroup generated by G, is rich, enjoys a NET. The interpretation of G algebras as algebras of sentences in languages endowed with individual constants, cf. Remarks 1.12-13, highlights the similarity with Henkin's celebrated completeness proof. Henkin's method of constructing models from constants (cf. [35] chapter 2), have a potential of applications proving a diversity of results, like interpolation theorems and omitting types (cf. [35] chapter 3). So when we have extensions of first order logic (like logics corresponding to Sain's algebras), for which the NET applies on the algebra side, it is natural to ask the reverse question: *How far can we use Henkin's methods (already available, in a disguised algebraic form as a NET) to prove other model-theoretic results like interpolation and omitting types, which we already know that can be proved using a "Henkin construction" for first order logic?* This chapter shows that the algebraization of Henkin's methods, (outlined and highlighted in the sketch of proof of proposition 1), works for G algebras when $[G]$ is countable and *strongly* rich, to prove interpolation results. We note that we only proved that the corresponding logics have the Craig interpolation property. Beth-definability and Robinson's joint consistency theorems follow by applying the general methods in [11] or [113]. Other metalogical properties in first order logic, that can be proved using Henkin's methods like Henkin-Orey's omitting types Theorem, is investigated for Sain's logics in [133], see also [23]. The metalogical property of omitting types is investigated in connection to the algebraic properties of complete representations and atom-canonicity in chapter five.

(2) (a) The following Theorem was proved by Sain (cf. [116] Corollary 3.3), providing algebraic logic with a finitely based variety as its class of representable algebras:

Theorem 2.1 . *Assume that $T \subseteq \langle {}^{\omega}\omega, \circ \rangle$ is a finitely presented rich semigroup. Let G be a finite generator set for T. Then (i) and (ii) below hold:*

(i) $\mathbf{I}GPSA_{\omega}$ *is a finitely based variety.*

(ii) The class of quasi-polyadic reducts of $IGPSA_{\omega}$ coincides with the class of representable quasi-polyadic algebras.

A semigroup $\mathbf{T}$ satisfying the hypothesis of Theorem 2.1 is given in [116]. We do not know whether this semigroup $\mathbf{T}$ is also *strongly* rich. In particular, we do not know whether $I\mathbf{T}PSA_{\omega}$ has the strong amalgamation property. At the end of this section we will give another (concrete) semigroup that could satisfy Theorem 2.1. While this semigroup is strongly rich, it is not known whether it is finitely presented. As pointed out in the introduction the problem to find simpler (strongly) rich finitely presented semigroups is open.

When looking for solutions to the finitization problem, it might be a good idea to look for finitely axiomatizable *quasivarieties* too, and not only for varieties. Blok and Pigozzi [28] indicate that quasivarieties are just as good vehicles for algebraic logic as varieties. [115] contains results in this direction. For a class K, we let SK stand for the class of all subalgebras of algebras in K. We recall from [115] the following result which is joint with Istvan Németi.

Theorem 2.2 . *(Sain-Gyuris-Németi [115] Thm 2.5. [109] Thms 17-18) Let G be as specified in Theorem 1, herein. Let RPA_G be the following class of set algebras.*

$$PPA_G = S\{\langle B({}^{\omega}U), C_i, S_{\tau}\rangle_{\tau\in G, i\in\omega} : U \text{ a set }\}.$$

Then (i),(ii) and (iii) below hold:

(i) $RPA_G = Mod(Qax)$ for some finite schema Qax of quasiequations.

(ii) The class of quasipolyadic reducts of $Mod(Qax)$ coincides with the class of representable quasi-polyadic algebras.

(iii) RPA_G expanded with $c_{(\omega)}$ forms a finite schema axiomatizable variety.

Here $c_{(\omega)}$ - the algebraic counterpart of existential closure - is a discriminator term, making the resulting variety a discriminator variety. The class RPA_G is tighter than G algebras treated herein and in [116]. This follows frpm the fact that the greatest element of any RPA_G is a *cartesian space*, i.e. a set of the form ${}^{\omega}U$, and not a weak one. The price paid for these improvements is that in Theorem 2.2 only a finite schema of axioms is available while in Theorem 2.1 we have a truly finite set of axioms. It remains an open problem to improve Theorem 2.2 in the style of Theorem 2.1, i.e. to finitely axiomatize the quasivariety RPA_G. Since any cartesian space ${}^{\omega}U$ is a union of weak cartesian spaces namey, $\cup_{p\in {}^{\omega}U}{}^{\omega}U^{(p)}$, it follows that every full set algebra of the form $\langle B({}^{\omega}U), c_i, S_\tau\rangle_{i\in\omega,\tau\in G}$ is in $GPSA_\omega$. Since $GPSA_\omega$ is closed under forming subalgebras we have $RPA_G\subseteq GPSA_\omega$, in fact $GPSA_\omega$ is the variety generated by RPA_G. Our Theorem 1 shows that RPA_G has the strong amalgamation property with respect to $GPSA_\omega$. That is for any $A, B, C\in RPA_G$, and any monomorphisms f, g of A into B, C respectively, there exists $D\in GPSA_\omega$ and a monomorphism m from B into D and a monomorphism n from C into D such that $m\circ f=n\circ g$ and such that $m(B)\cap n(C)=m\circ f(A)$ is satisfied. We do not know whether the strong amalgam D remains inside RPA_G. In particular, we do not know whether RPA_G and for that matter $RPA_G+c_{(\omega)}$, as described in Theorem 2.2, have the strong amalgamation property. We refer the reader to [109] and [115] for other semigroups that satisfy Theorem 2.2.

(b) A weaker solution for the finitization problem for first order logic *with* equality is given in [115] and in [116] section 5. In [116] the following is proved: Let T be a rich subsemigroup of ${}^{\omega}\omega$. $TPSA_\omega+d_{01}$ denotes the class $TPSA_\omega$, as introduced in Definition 0.1 above, but allowing for units any $\mathbf{G}ws_\omega$ units, expanded with the extra constant d_{01}. d_{01} is interpreted in any T set algebra with unit V as $D_{01}=\{s\in V: s_0=s_1\}$. It turns out that $TPSA_\omega+d_{01}$ is not closed under ultraproducts in a rather strong sense see [116] Lemma 5.2. [21] In particular,

[21]When one adds the set $D_{01}=\{s\in V: s_0=s_1\}$ as the "natural" interpretation of equality to $GPSA_\omega$, G a set, then in the presence of just one infinitary substitution in G, i.e. one that moves infinitely many points like for example the successor function, then the resulting class of algebras is not closed under ultraproducts, cf.[116]. This was first observed by Johnson [70]; in [116] Sain

this class is not first order axiomatizable. However one can still ask how the variety generated by this class behaves. It turns out that when T is rich, then the variety generated by $TPSA_\omega + d_{01}$, or $V(TPSA_\omega + d_{01})$ for short, is axiomatizable by a finite schemata of equations that is really simple and transparent cf. [115]. Furthermore, when T is finitely presented, then $V(TPSA_\omega + d_{01})$, is axiomatizable by a *finite* set of equations. The logical counterpart of this as presented in [116] is: There is an expansion L_1 of first order logic with equality obtained by adding finitely many connectives, that admits a truly finite *weaky* complete inference system $\vdash_1$ say in the sense of [11]. Weakly complete here means completeness without compactness. We always have $\models \phi$ implies $\vdash_1 \phi$, but there is a set of formulas Σ such that $\Sigma \models \phi$ but not $\Sigma \vdash_1 \phi$, i.e. ϕ is not derivable from Σ. It is open whether (the variety generated by) $TPSA_\omega + d_{01}$ has the strong amalgamation property, i.e whether L_1 has the Craig interpolation property. We strongly conjecture that our proof can be pushed through to show that $V(TPSA_\omega + d_{01})$ has the strong amalgamation property when T is *strongly* rich.

(3) A subtle very delicate difference between Gödel's classical completeness theorem for first order logic and the completeness theorem proved by Sain (for extensions of first order logic without equality) in [116], in favour of the latter is the following: Let L_{FOL}^{ranked} denote the *ranked* version of first order logic defined in [18]p. 110. The set of connectives of L_{FOL}^{ranked} is given by

$$Cn = \{\wedge, \neg, \exists v_i, v_i = v_j : i, j < \omega\},$$

with $\wedge$ binary, $\neg$, $\exists v_i$ unary, and $v_i = v_j$ zero-ary. Let $\mathbf{R}$ be a set (the set of relation-symbols) and let $\rho : \mathbf{R} \to \omega$ be a given function (the *rank function*, $\rho(R)$ is the *rank* of R.) Atomic formulas are of the form $R(v_0, \cdots v_{\rho(R)-1})$ for $R \in \mathbf{R}$ with the variables occurring only in their natural order. But since the individual variables following relation symbols are uniquely determined, one can just as well dispense with variables altogether. Accordingly, we write

pushes Johnson's observation to the limit.

R instead $R(v_0 \cdots v_{\rho(R)-1})$. [22] The set of formulas F^ρ are built up from the atomic formulas using the connectives the usual way. Now let $\vdash_r$ stand for the proof system given in [51] sec 4.3., which is the same as that recalled in [18] p.103. Let Ax^ρ denote the set of formulas defined in [18] p.111 as follows:

$$R \to \forall v_i R \text{ if } \rho(R) \leq i < \omega \text{ and } R \in \mathbf{R}.$$

Consider the axiomatization $\vdash^\rho$ of first order logic roughly obtained from $\vdash$ by adding Ax^ρ as new axoims. Gödel's completeness theorem is equivalent to the following: For every first order $\phi \in F^\rho$, if ϕ is logically valid (here validity is defined the usual way) then ϕ is $\vdash^\rho$ provable. A formula schema is an element of F^ρ. An instance of a formula schema is obtained by substituting formulas for the formula variables, i.e for atomic formulas, in this formula schema. A formula schema is called type-free valid if all its instances are valid. This is a new notion of validity defined in [51] Remark.4.3.65. A drawback, at least from the algebraic point of view for first order logic, is the following: There are type-free valid formula schemas ψ, say of first order logic that are not *uniformy* provable. Though each instance of ψ is $\vdash^\rho$ provable, these proofs vary from one instance to the other. We cannot give a uniform proof of all these instances in spite of there being a uniform cause ψ of their validity. The reason for this phenomena is that this formalism of first order logic is *not* structural in the sense of [28] if $\rho \neq 0$. In fact, this formalism is not even *substitutional* in the sense of [[18], Def 4.7 (ii) p. 72.] This means that a formula resulting from *substituting* formulas for atomic formulas in any valid formula, may not be valid. [23] To remedy this "defect" one can give a *structural formalism* of first order logic. One way to do this is to drop the rank function ρ and recover it later from the semantics. Although every relation symbol is a finitary relation, we do not know its arity in advance. The actual arity of any relation symbol is given by its interpretation in models [[18], p.112], i.e. after specifying its meaning and this

[22] Algebraic logicians - like combinatory logicians- tend to get rid of variables [109] p. 56. In algebraic logic, variables are treated like logical connectives. For example $\exists$ is not a logical connective, but $\exists x_i$ is a logical connective for every variable x_i. In this connection we refer to [146].

[23] For example: Let R be an n-ary relation symbol in $\mathbf{R}$ and let ϕ be the formula $v_0 = v_1 \wedge \cdots v_0 = v_n$. Then $R \implies (\forall v_n)R$ is valid while $\phi \implies (\forall)v_n\phi$ is not.

comes later. So these logics are *typeless* in the sense of [51], the adjective typless referring to abstracting away the notion of arity in the formalism. However any such structural formalism of first order logic is amenable to Andréka's negative results concerning the complexity of the equational theory of RCA_ω [3]. In particular, any strongly sound and weakly complete inference system for such a logic must involve an infinite set of formula variables, cf. [109] and [[18], Cor 7.10]. The finitization problem thus asks for *structural extensions* of first order logic that admit a truly finite Hilbert-style axiomatization capturing uniformy the type-free valid *schemas.* This is problem 4.16 in [51] which has inspired a lot of fruitful research. [24] In contrast, for Sain's logics (which are indeed structural extensions of first order logic *without* equality), valid formula schemas are uniformly derivable and indeed this can be achieved in a thorough finitary manner. To see this, let T be a finitely presented rich semigroup and G a finite set of generators of T. We recall from [116] that Sain's logics, or KL_G for short, are roughly obtained from L_{FOL}^{ranked} by dropping equality and the rank function ρ - so that KL_G is also typeless - and adding as unary connectives a finite number of substitutions corresponding to substitutions in G. A point of deviation from first order logic is that in KL_G, relation symbols are *not* finitary, they may be interpreted in models to have rank ω and therefore KL_G are expansions of the equality free versions of the finitary logics of infinitary relations studied in [9]. Now let f be a computable function associating GPA terms with formula schemas ψ of KL_G, cf. [[51] Definition 4.3.55] such that

$$GPSA_\omega \models f(\psi) = 1 \text{ iff } \models \psi.$$

Translating [25] the finite set of equations axiomatizaing the class of G set algebras to the logic side, one obtains a finite Hilbert style axiomatization involving only type-free valid formula schemas. Moreover given any ψ that is type-free valid

[24] Inspite of Andréka's negative results concerning the complexity of various structural extensions of first order logic with equality, there are partial positive results which somehow circumvent or side-step these negative results. For positive results in this direction, we refer the reader to [150], [53], [109], [115] and [146]. It seems, however, that for one reason or another none of these results is considered final, cf. [109].

[25] The translation of valid formula schemas in a logic L to equations or quasi-equations in the corresponding class of algebras $Alg(L)$, is a standard practise in algebraic logic, cf. e.g [11].

there is a uniform proof for all instances of this ψ. [26] Also in [116] there is a translation function tr mapping the "normal" formulas without equality into her set of formulas showing that her logic is as expressive as first order logic without equality. We refer the reader to [109] for further elaboration of the notion of formula schemas, as the "natural" logical counterparts of equations.

(4) Let L be an abstract logic in the sense of [[35], Def. 2.5.1.] L is *(countably) compact* if every (countable) finitely satisfiable set of formulas has a model. The *Lowenheim number* of L is the least cardinal α such that every formula of L which has a model has a model of power at most α. First order logic is compact. The Downward Lowenheim Skolem Theorem shows that first order logic has Lowenheim number ω. It is known from Lindström's (abstract) characterization of first order logic cf. e.g [35] that first order logic is the only abstract logic L that satisfies

  (i) L is countably compact and has Lowenheim number ω. Also first order logic can be characterized as the abstract logic L that satisfies

  (ii) L has Lowenheim number ω and the Robinson consistency theorem holds for L.

Sain's logic satisfies (i) and (ii), and is more expressive than first order logic (without equality). It would be interesting to try and work out some form of a "Lindström's characterization" for this logic, and more generally for logics that allow infinitary relation symbols, like those studied in [9]. [27]

(5) Let V be a variety of boolean algebras with operators. Let $SigV$ denote the the similarity type of V without the boolean operations. Let $A \in V$ be atomic. Then the *atom structure* of A is the structure

$$(AtA, R_f)_{f\in SifV}$$

[26] Using the terminology in problem 4.16 of [51], KL_G admit a complete proof calculus for type-free valid formula which involves only type-free valid formulas.

[27] Could it be possible that Lindström's characterization conflicts a solution to the finitization problem for (first order) logic *with* equality? In passing we note that working out an algebraic counterpart of Lindström's characterization Theorem for first order logic is an open problem in algebraic logic. (cf. [99].

where R_f is an $n+1$ ary relation defined for each n-ary $f \in SigV$ by: For all atoms $a_1 \cdots a_n, b \in A$

$$R_f(a_1 \ldots a_n, b) \longleftrightarrow f(a_1 \ldots a_n) \leq b.$$

$At(V)$ is the class of all such structures. More generally an *atom structure* or a *Kripke frame* of type $Sig(V)$ is a pair $\mathcal{F} = (F, R_f)$ where F is a set and R_f is an $n+1$ ary relation symbol for each n-ary function $f \in SigV$. Given an atom structure $\mathcal{F}$, the complex algebra $Cm\mathcal{F}$ is the algebra with universe $P(F)$ for each $n-ary$ $f \in SigV$ and $X_1 \cdots X_n \in P(\mathcal{F})$,

$$f(X_1 \ldots X_n) = \{a \in \mathcal{F} : \mathcal{F} \models R_f(a_1, \ldots a_n, b)$$

$$\text{for some } a_1 \in X_1, \cdots a_n \in X_n\}.$$

Cm and At are therefore in some way dual to each other. If we apply $AtCm$ to an atom structure we get back the original atom structure. As for $CmAt$ applied to a given atomic boolean algebra with operators we may not get the original algebra but possibly a bigger one, in fact we get its *completion* in the sense of [58]. It is possible for two non-isomorphic (infinite) boolean algebra with operators to have the same (or isomorphic) atom structures. These differences can be non-trivial as in the case for example of the variety of representable cylindric algebras [65]. However this cannot occur for G algebras considered herein by a result of Goldblatt [46], namely that the variety of GPA_ω is *atom-canonical.* This means that for any G algebra A, we have $Cm(AtA)$ is again a G algebra. (This is not the case for the class of representable cylindric algebras of dimension > 2. For the finite case, this is due to Hodkinson [65], while the infinite case is an unpublished result of the author [143].) This means that GPA_ω is generated by its atom structures in the sense that

$$GPA_\omega = SCmAt(GPA_\omega).$$

Here $At(GPA_\omega) = \{At(\mathcal{A}) : \mathcal{A} \in GPA_\omega\}$. In particular, for algebras $\mathcal{A}, \mathcal{B}$ having the same signature as G algebras, if their boolean reducts are atomic boolean algebras, and $At(\mathcal{A}) \cong At(\mathcal{B})$ as first order structures, then $\mathcal{A} \in GPA_\omega$

implies that $\mathcal{B} \in GPA_\omega$. We should mention that an important question in modal logic is which varieties of BAO's are generated by their atom structures in the above sense, since using the terminology of modal logic, this enables one to work at the frame level.

(6) We conjecture that the condition of countability of G can be omitted from 1^*. In fact, the following weaker version of amalgamation, namely that the class of G algebras for arbitrary rich $[G]$ has the *strong embedding property*, easily follows from [114] Lemma 1.3. The strong embedding property - dealt with in the previous chapter - is roughly the restriction of the strong amalgamation property to the case when the algebras required to be strongly amalgamated are required only to have isomorphic *minimal* subalgebras. Minimal subalgebras are the subalgebras generated by the constants in the algebra in question. In the case of Sain's algebras, these are (all isomorphic to) the 2 element boolean algebra. We note that the class K of G algebras with rich $[G]$, forms a *canonical variety*, i.e. K is closed under forming *perfect or canonical extensions.* The reader is referred to [50] Remark 2.7.7 or [114] for these notions. This follows from the fact that the equations defining K as in Definition 1.2 above, can be formulated as *positive equations in the wider sense*, which are a special case of the so-called *Sahlqvist* equations. By a result of Venema [150], it follows that K is also closed under the so-called *minimal completions* and the atomic algebras are completely representable in the sense of [54]. K, however violates the conditions of Lemma 1.2 in [114], namely the corresponding class of atom structures [28] denoted by

[28] Such atom structures are sometimes referred to in the literature [6] as ω- *frames* or *concrete frames*. An ω-frame in the case of G algebras, is simply a first order structure of the form $\langle V, T_i, R_\tau \rangle_{i\in\omega,\tau\in G}$, where V is a union of weak ω-spaces having a common base, and T_i and R_τ are the (accessability) binary relations corresponding to cylindrifications and substitutions defined as in the proof of Theorem 8 in the previous chapter. For example, given $s, t \in V$ and $i \in \omega$, then $(s,t) \in T_i$ if s and t agree except for i i.e. $s(j) = t(j)$ for all $j \neq i$. Given any such frame, forming its complex algebra, gives a concrete G set algebra. We note that working with frames (using terminology of modal logic) or equivalently atom structures (using terminology of algebraic logic), is a *dual* in the sense of Goldblatt [45] to working with algebras; and in fact has proved to be very fruitful. See also [6] for applying the duality of Goldblatt to several varieties of cylindric like set algebras. We refer the reader to [150], for a survey on this modal-approach to algebraic logic. Marx reep the harvest of this duality by proving decidability, interpolation and completeness of fragments of the so called *arrow logic.* Fragments of arrow logic has been studied quite extensively lately; their algebraic counterpart turns out to be reducts and variants of Tarski's relation algebras. Marx's main

$Cm^{-1}K$ in [92] - is *not* closed under forming substructures; not even under forming *finite zig-zag products*, in the sense of [92], and yet K, as proved herein, has the strong amalgamation property. In fact by Fact 0.6, K has the super amalgamation property in the sense of [87].[29] Compare here with problem 1.4 in [114] which asks for relaxing the condition of closure under substructures. [92] 5.2.6 gives a partial answer. Our main result shows that the conditions in Lemma 1.2 in [114] and in Lemma 5.2.6 in [92] , though sufficient for proving strong amalgamation for classes of boolean algebras with operators, are not necessary. The proofs of the two above mentioned Lemmas are generalizations of the techniques of Németi developed in [105], for proving SAP for various subclasses of boolean algebras with operators and relativized versions of set algebras like for example D_α and NCA_α. The reader is referred to [91] for the definition of such subclasses. The latter is the class of non-commutative cylindric algebras. A purely "modal approach" of Németi's results formulated for boolean algebras with operators in [105] can be found in [153]. In [153] the argument used is a back and forth constuction to establish a modal analogue of Robinson's joint consistency Theorem. In the next chapter we compare the techniques of Németi in [105] to those adopted herein which can be traced back to the techniques of Pigozzi in [113] conjuncted with a model-theoretic forcing argument.

Open problem. We end this paper by the following question from [99], which originates with Sain and Thompson [116]:

Instead of considering semigroups on ω, one can instead use any countable indexing set. In fact, there is no reason to stick to ordinals as possible index set, see e.g [146]. Now let Z denote the set of intergers. Considering Z as an indexing set, cf. [116], we ask for a semigroup of ${}^ZZ$ satisfying the condition of Theorem 2.1 above and that, in addition, is *strongly rich.*

results in [92], and other results concerning amalgamation in cylindric-like algebras of relations are streamlined in [90] and [91].

[29] As previously noted, it is not known, at least to the author, whether there exists a variety V of CA-like algebras of relations that separates the notions of strong and super amalgamation.

Let $T \subseteq \langle {}^Z Z, \circ \rangle$ be the semigroup generated by the following five transformations on Z:

$shift = \langle z+1 : z \in Z \rangle$, $shift^{-1}$, $suc = \langle n+1 : n \leq 0 \rangle$ $pred = \langle n : n \leq 0 \rangle \cup \langle n-1 : n \geq 1 \rangle$ and $[0, 1]$ (the transposition that interchange 0 and 1). Does T have a finite presentation in terms of these generators?

Motivation: A positive solution would yield a truly finitary extension of first order logic without equality that admits a finite Hilbert style axiomatization capturing all the valid schemas. This logic is also compact and has the properties of Beth definability and Craig interpolation. The latter follows from our Theorem 1^* since Z is strongly rich. The reader is referred to [99] p. 723-733, [116], and [133] for further elaboration on this problem in the wider context of one of the central classic problems of algebraic logic.

Chapter 4

Amalgamation proper

Abstract

Here we review and in the process amalgamate two techniques for proving strong amalgamation in algebraic logic.

keywords neat reducts, amalgamation, adjoint situations

A note on amalgmation

It was Roland Fraisse who first called attention to an amalgmation diagram. Everything that has happened in Model theory during the last thirty years has confirmed how important this diagram is. Hodges in [67].

Unless otherwise specified, SC_α, CA_α, QA_α and QEA_α stand for the classes of Pinter's substitution algebras, Tarski's cylindric algebras, Halmos' quasipolyadic algebras, and quasipolyadic equality algebras of dimension α, respectively. These are defined in chapter two. In the literature of algebraic logic two techniques for proving amalgamation results, which we intend to compare, are dominant. Pigozzi's techniques used in [113] which seems to be special to CA-like algebras since it appeals to notions like neat reducts and neat embeddings. In fact, all positive results in section 2.2 of [113] follow from Lemma 2.2.12 therein. The latter is slightly generalized and formulated as Theorem 10 in chapter two. Another technique is invented by Németi and is more general since it addresses boolean algebras with operators. This technique was refined and used by Sain [114], Lemma 1. 2-3, and Marx [92] Lemma 5.2.2. In fact we used Sain's adaptation in chapter two to prove that several classes have the strong embedding property, or SEP for short, cf. Theorem 9 in chapter two. The novelty of Marx's approach in [92] chapter 3, is basically the stressing on the modal aspect of this technique, and pushing it to the limit for proving amalgamation for relativized versions of Cs's like for example Crs_α, and certain CA-like algebras in which commutativity fails like the classes D_α and NCA_α (cf. [91] for definitions.) [1] A purely modal approach of Nemeti's results formulated for BAOs in [105] can be found in [153]. In [153] the argument used is a back and forth construction to establish a modal analogue of Robinson's joint consistency. On the other hand, Pigozzi's techniques were adapted and used (together with a model-theoretic forcing argument) in the previous chapter to show that various reducts of Halmos' polyadic algebras have the strong amalgmation property, or SAP for short. We believe that it is a natural to compare the techniques of Pigozzi used in [113] and in the previous chapter , on the one hand, to the techniques adopted by Németi in [105] (and by Weaver and Welaish in [153]), on the other. To do this, we recall Németi's main result formulated as Lemma 3 in [105]. This entails that we review some definitions.

Definition .

(i) *Bo_α stands for the class of BAOs having the same signature as CA_α.*

(ii) *$Mod(t_\alpha)$ is the corresponding class of Kripke frames or atom structures. Thus $\mathbf{M} \in Mod(t_\alpha)$ if it is a first order structure $\langle M, Z_iR_i, D_{ij}\rangle_{i,j\in\alpha}$, where Z_i and D_{ij} are unary relations while R_i is binary for all i, j. cf. [105] Definition 2 (i).*

(iii) *A modal representation of $\mathcal{A} \in Bo_\alpha$ is a $\mathbf{M} \in Mod(t_\alpha)$ such that $\mathcal{A} \subseteq \mathcal{C}b(\mathbf{M})$, where $\mathcal{C}b(\mathbf{M})$ is the complex algebra of $\mathbf{M}$ as defined in chapter two or in [105], Def 2, (ii).*

We refer the reader to [105], Def 2 (v) for the definition that a representation $\mathbf{M}$ of $\mathcal{A}$, is a *compact saturated* representation of $\mathcal{A}$. $SUPAP$ is short for the super amalgamation property. Replacing the strong amalgamation property by the super

[1] For a review of those aspects of varieties of boolean algebras with operators or BAOs for short, that emphasize connections with modal logic and structural properties (like canonicity, complexity) related to natural properties of logical systems (like completeness, compactness) we refer the reader to [45].

amalgmation property as shown by Marx [92], Németi's main result [105], Lemma 3, is the following:

Proposition 1 . *Let $K \subseteq Bo_\alpha$ and $L \subseteq Mod(t_\alpha)$. Assume that (i), (ii) and (iii) below hold:*

(i) L is closed under finite direct products and substructures. (Here substructures is understood in the usual first order sense; recall that L is a class of first order structures.)

(ii) $\{\mathbf{C}b\mathbf{M} : \mathbf{M} \in L\} \subseteq K$

(iii) Every element in K is isomorphic to one, which has a compact and saturated representation from L.

Then K has the super amalgamation property.

Sain [114], Lemma 1.2-3 generalizes Nemeti's result to arbitary classes of boolean algebras with normal operators ,while Marx [92] Lemma 5.2.6 (i) relaxes the the conditions of closure under substructures and finite direct products to the single condition of closure under *zig-zag morphic* images. [2] By this, Marx answers a question of Sain [114], problem 1.4 and provides (new) proofs that $SUPAP$ holds for several classes of BAO's, like WA the weakly associative algebras of Maddux and D_α. Pigozzi's techniques, on the other hand generalize, to the context of systems of varieties definable by schemes in the sense of [51] Def 5.6.12. All algebras considered so far, with the exception of Sain's algebras in chapter three, are instances of varieties definable by schemes. We do not have to remember the exact definition. What is important in this juncture is to know that in systems of varieties definable by schemes the notion of neat reducts makes sense. Indeed for such a system $\mathbf{K} = \langle K_\alpha : \alpha \in Ord\rangle$, the operation of forming α neat reducts, or Nr_α for short, is an operator from K_β to K_α for any pair of ordinals $\alpha < \beta$. The following generalization highlighting the link between $SUPAP$ and what we called the NS property in chapter two formulated as

[2] This is a term borrowed from the well developed duality theory [44][45] between modal logic and boolean algebras with operators.

condition (iii) is proved in [132]. [3]

In our formulation we do not distinguish between an algebra $\mathcal{A}$ and its domain and we denote both by A. $Sg^A X$ denotes the subalgebra of A generated by $X \subseteq A$.

Proposition 2 . *Let $\mathbf{K} = \langle K_\alpha : \alpha \in Ord \rangle$ be a system of varieties definable by schemes. Let α be an ordinal and let $L \subseteq SNr_\alpha K_{\alpha+\omega}$; the class of algebras having the neat embedding property; which we denote by RK_α. Let*

$$M = \{A \in K_{\alpha+\omega} : A = Sg^A Nr_\alpha A\}.$$

Assume that M has $SUPAP$. If there exists a lifting function $F : L \to K_{\alpha+\omega}$ such that $A \subseteq Nr_\alpha F(A)$ subject to:

$$(i)(\forall A \in L)(\forall X)(X \subseteq A) Sg^A X = Sg^{Nr_\alpha F(A)} X,$$

$$(ii)(\forall A \in L)(\forall X)(X \subseteq A) F(Sg^A X) = Sg^{(F(A)} X,$$

then L has AP with respect to RK_α. If further, algebras in L satisfy the NS, i.e.

$$(iii)(\forall A \in L)(\forall X)(X \subseteq A) Sg^A X = Nr_\alpha Sg^{F(A)} X,$$

then L has $SUPAP$ with respect to to RK_α.

If the reader compares Proposition 2 with Theorem 10 in chapter two, he will notice an important difference. Then in "Then M has SAP", is replaced by the conditional "if M has SAP." If follows from this modification that Proposition 2 can be proved on a purely syntactical level, relying only on the abstract definition of systems of varieties definable by schemes (cf. [51] 5.6.14, 5.6.16).

When we include semantics in the sense of [9], i.e. set algebras corresponding to models, see example [51] sec 4.3 [4], and we happen to have a Neat embedding Theorem stipulating that the representable algebras coincide with these algebras having the neat embedding property then RK_α is the class of representable algebras of dimension α, hence the notation. Here by representable, we understand representability in

[3]The proof of proposition 2 was presented by the author in Logic Colloquium 2000. A less mature form was presented also by the author in the universal algebra conference held in Szeged Hungary 1998.

[4]For first order logic, these are the so-called *regular locally finite* set algebras, more commonly denoted by $Cs_\omega^{reg} \cap Lf_\omega$. This is the standard notation adopted in the monograph [51].

the sense of [50] i.e. as subdirect products of set algebras. This is the case for $K \in \{Df, SC, CA, QA, QEA\}$. An application of proposition 2, is that all of Pigozzi's positive results formulated for infinite dimensional cylindric algebras in section 2.2 generalize to SC, QA and QEA (and possibly other expansions and reducts thereof) which together with cylindric algebras, are prime examples of systems of varieties definable by schemes. The logic corresponding to M in the concrete cases of SC and CA is studied under the name of *finitary logic of infinitary relations* in [9], and indeed is proved (cf. [9] proposition 6.7), to have the Craig interpolation and Beth definability properties, therefore in most applications M does have $SUPAP$, and this is not so severe a condition. An algebraic proof that M has $SUPAP$ (which is the K analogue of [113] 2.2.10 for SC, QA, and QEAs follows from the K-analogues of Pigozzi's Lemma 2.2.5 and Theorem 2.2.6 in [113] which hold for such algebras. Theorem 2.2.6 proving that the dimension-restricted free algebra has the so called strong interpolation property when it is in Dc_ω, is the central theorem in [113] section 2.2. Indeed, from this single theorem (together with one or two others of a non-trivial but still auxiliary character) there follows via the general results summarized in [113] section 2, all positive results in Pigozzi's paper. Though the proof of Thm 2.2.6 in [113] itself is rather involved, its generalization to SC, QA and QEA is absolutey straightforward. The proof, in essence, is an algebraization of Henkin's methods of constructing models out of constants proving completeness and interpolation for first order logic. This is further illustrated in the previous chapter where a similar result is proved for the countable free algebras introduced by Sain in [116], which we refer to as Sain's algebras. In chapter two, cf. [145], it is shown that in contrast to Sain's algebras, for $K \in \{SC, CA, QPA, QPEA\}$, the class of the so-called weak set algebras of dimension ω, or WKs_ω for short, does not have the NS property. WKs_ω are representable K_ω's having units weak spaces of dimension ω [51]. This result - announced in [21] - is strengthened in chapter two , cf. [22] and [80], to show that any class L such that $WKs_\omega \subseteq L \subseteq RK_\omega$ fails to have the strong amalgamation property, or SAP for short, even with respect to RK_ω, i.e. even if we look for the strong amalgam in the (possibly) bigger class RK_ω. An even stronger negative result due to Madarász, is that if we expand the signature of $QPEA_\omega$ by generalized cylindrifications and any set of substitutions indexed by *surjective* transformations on ω, then in any L between this class and its representables, epimorphisms in the categorial sense fail to be surjective. The condition of surjectiveness is essential here, i.e. cannot be dropped as illustrated by our main result in chapter three. It is known that if V is a quasivariety that has the amalgamation property, then V has SAP if and only if in V epimorphisms (in the categorial sense) are surjective. or, V has ES for short.

In chapters two and three, when dealing with systems of varieties, not necessarily definable by schemes, we have replaced ES by the NS property, which loosely speaking says: "term functions that are definable with extra variables are already term definable without extra variables". Very roughly in quasivarities we have $AP + ES = SAP$, while for systems of vareities we have $AP + NS = SAP$, as established herein.

The unification of propositions 1 and 2 as adjoint situations.

In what follows, for undefined categorial notions we refer the reader to [1]. In [132] it is further shown that both of Pigozzi's and Németi's techniques of amalgamation can be seen as instances of the use of the Keisler-Shelah ultrapower theorem (resorting to special models) of Robinsons joint consistency theorem, cf.[35] Theorem 2.2.23. To get a grasp of this, let us look closely at condition (iii), in both propositions 1 and 2 In proposition 2 condition (iii) appeals to neat reducts and the neat embedding theorem, it says that the class in question has the NS, i.e algebras embed into ω extra dimensions in such a way that neat reducts commute with forming subalgebras. On the other hand, Németi's condition (iii) appeals to compact saturated representations (a notion borrowed from model theory). However both can be seen as *saturation* conditions. Indeed, if an algeba A has the NS then $A \in Nr_\alpha CA_{\alpha+\omega}$ and A is saturated, in the sense that it contains all first order definable operations. [5] A first order definable operation is roughly a $CA_{\alpha+\omega}$ term, for an exact definition we refer the reader to [29] or the introduction of [124]. Németi's definition of saturation [105] Def. 2(v) on the other hand, is the algebraization of the usual definition of a saturated model. (cf. [35] 5.1). Also in the actual application of condition (iii) of both

[5]To construct set algebras that are neat reducts, one constructs models that are saturated, cf. Theorem 1.2 in the next chapter and our construction in chapter six.

propositions 1 and 2 in proving $SUPAP$ for the class of algebras in question, there is a similar pattern. In both cases we start off with a diagram (of algebras) required to be strongly amalgamated. We embed the given algebras in another diagram - in Nemeti's case a diagram of frames [105] Lemma 3, in Pigozzi's case a diagram of algebras in ω extra-dimensions [113], the proof of 2.2.12 - the (relevant) components of which are *saturated*, and thus this new diagram can be strongly, and indeed superly amalgamated. [6] This can be seen as embedding given models of a two theories consistent on the common (complete) theory, into *special thus saturated* models which are isomorphic (by the Keisler-Shelah) ultrapower theorem (cf. the proof of 2.2.23 in [35], and these can be indeed amalgamated over the base model. [7] We find a super amalgam in the new diagram and then return to our original diagram by an *inverse operator*. Both of these techniques can be given a functorial formulation; making the notion of inverse operator precise, and highlighting their adjoint character. The inverse operator is in fact an *inverse functor.* In Pigozzi's case the lifting function F is a functor, and indeed an inverse to the *neat reduct functor.* We denote the latter by Nr. Indeed the operation of forming α neat reducts can be easily seen as a functor from K_β to K_α for any $\alpha < \beta$ as follows: For $A \in K_\beta$ $Nr(A) = Nr_\alpha A$, and for any morphism f between algebras in K_β, $Nr(f)$ is the natural ristriction.

Thus in Pigozzi's case we find the super amalgam by applying the neat reduct functor to the super amalgam we found in ω extra dimensions. Without the NS, the amalgam obtained this way, surely enough an amalgam, may not be a *super* amalgam nor even a strong one; it is precisely the NS property on the base algebra that guarantees that the super amalgam remains so when go back to our original diagram via the neat reduct functor (cf. the proof of [113], Lemma 2.2.12, starting item (9), and our proofs of Theorem 6, 7 in chapter 2.) An elegant concise functorial formulation of NS i.e condition (iii) in proposition 4 herein, reads that $F \circ Nr$, the

[6] In Pigozzi's case (cf. (iii) of [113] 2.2.12 and propositions 2-4 (iii) herein) the base algebra is a neat reduct, in Németi' s case the three components are saturated modal representations of the given algebras [cf. the proof of Lemma 4 and Lemma 7, in [105].]

[7] In fact, in the proof of Lemma 7 of [105] showing that Crs_α has SAP, Nemeti explicitly constructs ultrapowers relative to β^+ good ω incomplete ultrafilters, β a large enough cardinal. An easier proof found by the author uses that Crs_α is a canonical variety, the latter is a result of Goldblatt. Also in [113] 2.2.19, Pigozzi essentially uses an "ultrapower functor" in neatly embbeding a given algebra into ω extra dimensions, cf. Lemma 1.3.3 herein.

composition of F with the neat reduct functor, is in fact *naturally isomorphic* to Id, the identity functor when the latter is restricted to the class L. Németi, on the other hand, is implicitly dealing with the functor that takes $\mathcal{A}$ to a suitable atom structure (or a representation) that is compact and saturated; when $\mathcal{A}$ is merely a BAO, this is simply its *ultrafilter frame*(cf. [105] Lemma 4.) Its inverse is the functor that sends every atom structure to its complex algebra. So in Németi's case we go back to our original diagram by forming the complex algebra of the super amalgam frame we found. Nemeti's technique is an instance of the well developed triple duality exiting between BAOs, relational frames (sometimes referred to as Kripke frames) and abstract modal logic. (cf. [44], [6] and [45].) We note that Pigozzi's functor, denoted by F in proposition 1 above, is a neat embedding functor, and it can be seen as sending every algebra to a classical representation, i.e a (generalized) set algebra that can be built up of the ω-extra dimensions, in analogy to constructing models out of constants. This analogy is elaborated upon in the previous chapter. The second functor addressed implicitly in [105] Lemma 3 sends a given algebra to its ultrafilter frame, which is a representation of the given algebra in the modal "less goemetric" sense. In both situations, each functor and its inverse is an adjoint situation in the categorial sense as shown in [132]. It is interesting to note that both techniques can be applied to show that several classes of algebras have the strong embedding property. This is case for example with RQA_α, RSC_α and Sain's algebras.

Chapter 5

Neat reducts and complete representations

Abstract

In this chapter we give a new characterization of the class of completely representable algebras, and we show that the class of all neat n-reducts of β dimensional such algebras for $2 < n < \beta \cap \omega$ is not elementary. For the first result we use an omitting types construction which is a Baire category argument at heart, and for the second we use Fraisse's majestic method of constructing universal homogenoues models.

Key words: neat reducts, complete representations, Martin's axiom, omitting types.

Introduction: Background and History

The cardinal number ω is the only infinite cardinal which is a limit of finite cardinals. This gives us two reasons why countable structures are good to build. First, a countable structure can be built as the union of a chain of finite pieces. And second we have infinitely many chances to make sure that the right pieces goes in. No other cardinal allows this amount of controlle. So its not surprising that Model Theory has a rich array of methods of constructing countable structures. Pride of place goes to Roland Fraisse's majestic construction, next comes the omitting types construction: it is one of these happy ideas that was discovered in several forms by several poeple.

Hodges in [67].

There are two kind of representations in boolean algebras with operators and of interest in this chapter are the *complete* representations as opposed to *ordinary* representations. Ordinary representations, or just representations are isomorphisms from a boolean algebra with operators to a more concrete structure whose elements are sets, in which the boolean operations are interpreted as intersection, union and complementation. The other non-boolean operations have certain set-theoretically definable interpretations. For example, in Tarski's relation algebras the binary operation ; becomes interpretable as composition of binary relations, while in Tarski's cylindric algebras, the $i-th$ cylindrification becomes interpreted as the projection on the $i-th$ axis. Complete representations- originally called strong- by Scott has the additional property that they preserve arbitrary supremums (hence infimums) too, whenever they exist. Historically there has been some confusion here, when arbitrary and complete representations have been mistaken for each other, cf. [74]. Representation classes - i.e. classes of all structures of the appropriate type that posses a representation - form *varieties* in almost every case in algebraic logic. This means that they can be characterized by a set of equational axioms, although in most cases it has been shown that infinitely many axioms are required. This is indeed the case for the class of representable cylindric of dimension > 2 and relation algebras. The class of Sain's algebras investigated in the previous chapter is a rare exception, and therefore it is considered as a breakthrough as far as the *Finitization Problem* is concerned, at least in Nemeti's opinion [109].

Now consider instead the the class of *completely representable* structures of some type of algebraic logic. These are the algebras that have a *complete* representation, i.e a representation that preserves *infinite* meets (and joins) carrying them to set-theoretic intersections and unions. In most cases, these are not closed under substructures, and therefore cannot be characterized by universal axioms and cannot be a variety. Hirsch and Hodkinson prove that the class of completey representable relation and cylindric algebras of dimension > 2 is not elementary and they give a characterization of the elementary closure of this class, using games [54]. [1] It turns out , rather

[1]The class of completely representable algebras is psuedo-elementary, i.e. a PC_{Δ} class in the sense of Hodges [67]. In [60] a general method is introduced to axiomatize the elementary closure of

surprisingly, that the so-called *Lyndon conditions* introduced by Lyndon far back in the fifties of the 20th century, in [74] characterize the elementary closure of the class of completely representable relation algebras [54]. An analogous characterization of the elementary closure of the class of completely representable cylindric algebras is given in [54]. Here we give a new characterization of both classes of completey representable algebras via special neat embeddings. However our characterization only works for countable algebras. We show that for countable and atomic algebra $\mathcal{A}$, $\mathcal{A}$ is completely representable, if and only if it embeds *completely* and neatly into an algebra in ω extra dimensions. Our proof of this fact is inspired by the Rasowa-Sikorski use of the Baire Category Theorem in proving the Completeness of first order logic. It is a typical omitting types construction. The connection between the Baire Category approach to forcing arguments is well known. Having said that, it seems that one can make the translation from the "Baire category" approach adopted herein to Hirsch - Hodkinson's games, both being instances of Robinson's finite forcing, by way of the Banach-Mazur Theorem (cf [67] p. 358). It has been for a long time that not every (countable) atomic cylindric algebra has a complete representation. We use this result to show that Henkin-Orey's omitting types Theorem fail for the finite variable fragments of first order logic, a result that to the best of our knowledge is new. In fact, we show that omitting types fail in a rather strong sense. We also give counterexamples to questions that suggest themselves. We show that countability is essential in the sense that it cannot be omitted from our characterization. Finally, using a Fraisse-type contruction we show that the class Nr_nCA_β for $2 < n < \beta \cap \omega$ is not elementary. The work in this chapter is based on the articles [136] and [142].

Organization and The Main Results

This chapter has two sections. In section one we give a new characterization of the classes of countable completely representable cylindric and relation algebras (Theorem 1.3). We formulate a statement concerning existence of representations of cylindric algebras, preserving $< {}^{\omega}2$ set of infinite meets. This statement turns out to be equivalent to an instance of (the famous independent) Martin's axiom restricted to countable

any psuedoelementary class.

boolean algebras (Theorem 1.4). The connection of this statement to omitting types for $L_{\omega\omega}$ is discussed. We apply our results to the so called finite-variable fragments of first order logic. More precisely, using examples that abound in the literature of algebraic logic, but oriented to different purposes, we show that Henkin-Orey's celebrated omitting types Theorem fails for L_n the first order logic restricted to the first n variables when $\omega > n > 2$ (Theorem 1.5). Also we discuss what implications our result could have for omitting types for usual first order logic. (Conjecture (1), and the immediately following remark.) In section 2, we prove that that the class of finite dimensional neat reducts is not elementary if the dimension > 1. Finally we discuss related results and pose some questions. The work in chapter 1 is based on the article [136] while that in section 2 is based on the article [142]. [2]

1. Complete representations

Notation

We fix our notation adopted in this chapter and recall the basic concepts to be used in the sequel. Throughout this chapter, we follow the more or less standard terminolgy and notation of [50]. In particular, CA_n stands for the class of cylindric algebras of dimension n. Cs_n (Ws_n) stands for the class of cylindric (weak) set algebras of dimension n. RCA_n stands for the class of representable cylindric algebras of dimension n. RA stands for the class of relation algebras and RRA stands for the class of representable relation algebras. These are define in e.g. [51] 5.3.1-5.3.2. Let $\beta > n$ be an ordinal. Then Nr_nCA_β denotes the class of neat n reducts of CA_β's and $RaCA_\beta$ stands for the class of the relation algebra reducts of CA_β's. We recall from [50] that $Nr_nCA_\beta \subseteq CA_n$ and that $RaCA_\beta \subseteq RA$ if $\beta > 3$. The latter result is a classical result of Henkin and Tarski [51] 5.3.8. According to the Neat Embedding Theorem of Henkin, we have $RCA_n = SNr_nCA_{n+\omega}$ and that $RRA = SRaCA_\omega$. In this chapter we frequently, but not always, identify (notationally) algebras with their domains. From now on, unless otherwise specified, n denotes a finite ordinal. Let A, B be algebras (having the same signature), then $Ho(A, B)$ stands for the set of

[2] After the paper [136], on which section 1 of this chapter is based, was refereed, Hajnal Andréka and Robin Hirsch provided solutions to two questions posed herein. We did not change the structure of the chapter. Instead we add their results (without proofs) at a separate section (further work).

all homomorphisms from A to B. If A is an algebra with a boolean reduct, and $X \subseteq A$, then $\sup X$ $\inf X$ denotes the *supremum* (*infimum*) of X, if it exists. We start by defining the central notion in this chapter, namely that of complete representation. Roughly, complete representation are representations that have the additional property that they preserve arbitrary disjunctions (hence conjunctions, too), carrying them to set-theoretic unions (intersections), whereever these are defined; formally:

Defintion 1.1 . *Let $A \in CA_n$. A is completely representable if for all non-zero $a \in A$, there exists $C \in Cs_n$ and $f \in Ho(A, C)$ such that $f(a) \neq 0$ and such that*

$$f(\sup X) = \cup_{x \in X} f(x)$$

for all $X \subseteq A$ whenever $\sup X$ exists. $\diamondsuit$

If A is completely representable, then there exists a generalized set algebra, a $C \in Gs_n$ for short, and an isomorphism f from A to C such that

$$f(\sup X) = \cup_{x \in X} f(x)$$

whenever $\sum X$ exists. We recall from [50] that $C \in Gs_n$ if its unit is a disjoint union of cartesian squares, i.e. a set of the form $\cup_{i \in I} {}^n U_i$, where $U_i \cap U_j = \emptyset$ for distinct i and j. We call such a C a *complete representation* of A. When A is simple (i.e. has no proper ideals), then a complete representation of A can be taken to be a Cs_n, i.e. a Gs_n with unit of the form ${}^n U$. Not every RCA_n is completely representable. In fact, Hirsch and Hodkinson prove that if $A \in CA_n$ is completely representable, then A is atomic (cf. [53] Thm 5). [3] Further if $C \in Gs_n$ is a complete representation of A with unit V say, then B is an *atomic* representation of A in the sense of [54]. That is, there exists an isomorphism $f : A \to C$ such that

$$\cup \{f(x) : x \text{ is an atom of A } \} = V.$$

Conversely, ([53], Thm 5) an atomic representation of A, is also a complete representation. Atomicity, though necessary for the existance of a complete representation, is not sufficient for $n > 2$. In the case of $n > 2$, not every atomic representable CA_n is

[3] Recall that a boolean algebra is atomic if every non-zero element contains an atom i.e. a minimal non-zero element.

completely representable. Maddux [82] proves that not every atomic relation algebra is completely representable. Maddux's result is strengthened by Hirsch and Hodkinson in [54] for both relation and cylindric algebras. In [54], Hirsch and Hodkinson show that the class of completely representable RA's and CA_n's, $n > 2$ cannot be characterized by any first order theory. Now we give a new characterization of the class of *countable* completely representable CA_n's by appealing to special neat embeddings. The RA analogue is also formulated; its proof is identical to the CA case and therefore will be omitted.

Definition 1.2 . *For K a class with a boolean reduct, we define*

$$S_cK = \{A : \exists B \in K : A \subseteq B, \text{ and whenever } \sup X = 1 \text{ in } A$$

$$\text{then } \sup X = 1 \text{ in } B \text{ for all } X \subseteq A\}. \ \diamondsuit$$

The subscript c in S_c abbreviates that we are forming complete subalgebras. Let n be a countable ordinal. In [50] the class $SNr_nCA_{n+\omega}$ is referred to as the class of algebras having the neat embedding property. Recall that this class coincides with RCA_n. Accordingly, we sometimes refer to algebras in the class $S_cNr_nCA_{n+\omega}$ as algebras having the *strong* neat embeding property. (In Corollary (?) below we shall show that the strong neat embedding property is indeed stronger than the neat embedding property.) In passing, we note that $Nr_nCA_{n+\omega} \subseteq S_cNr_nCA_{n+\omega}$, and that the inclusion is proper. Now we are ready to formulate and prove our first main result in this chapter, which relates algebras having the strong neat embedding property to completely representable algebras.

Theorem 1.3 .

(i) Let $A \in CA_n$ be countable. Then A is completely representable if and only if A is atomic and $A \in S_cNr_nCA_\omega$.

(ii) Let $A \in RA$ be countable. Then A is completely representable if and only if A is atomic and $A \in S_cRaCA_\omega$.

We only prove the non-trivial implication, the if part of (i). We prove something stronger namely:

(*) . *If $A \in S_cNr_nCA_{n+\omega}$ is countable, $n \leq \omega$ (note that here n is allowed to be infinite) and $\{X_i : i < \omega\}$ is a family of subsets of A such that $\inf X_i = 0$ for all $i < \omega$, then for every non-zero $a \in A$ there exists $C \in Ws_n$, with countable base, and $f \in Ho(A, C)$ such that $f(a) \neq 0$ and for all $i \in \omega$ we have $\bigcap_{x \in X_i} f(x) = \emptyset$.*

We recall that a weak set algebra has unit of the form ${}^nU^{(p)} = \{s \in {}^nU : |\{i \in n : s_i \neq p_i\}| < \omega\}$, for some $p \in {}^nU$. U is called its base. To emphasize the connection with the classical omitting types Theorem , we refer to the X_i's as *non-principal types* and to C as a representation *omitting* these types. Note that for $n < \omega$ we have $Ws_n = Cs_n$, so that a unit of a Ws_n is simply of the form ${}^nU$. We show how the only if part of Theorem 1.3 (i) follows from (*). Assume that (*) is proved and let $n < \omega$. Let $A \in S_cNr_nCA_\omega$ be countable and atomic. We can assume, without loss of generality, that A is simple, for if not then we could replace it by its simple components. Then taking $X_i = Y = \{-b : \ b \text{ is an atom of } A\ \}$, and applying (*) for any non-zero a in A, upon noting that $\inf Y = 0$ since A is atomic, we get an atomic representation, hence a complete representation, of A.

Now we prove (*). In the proof we use the following form of the celebrated Rasiowa-Sikorski Lemma:

(**) . *Let B be a boolean algebra. Let J and K be countable sets. Let $\{S_j : j \in J\}$ and $\{T_k : k \in I\}$ be families of subsets of B such that $\sup S_j$ and $\inf T_k$ exist for each $j \in J$ and $k \in K$. Then for every non-zero $a \in B$, there exists an ultrafilter F containing a such that for all $j \in J$ if $\sup S_j \in F$ then $S_j \cap F \neq \emptyset$ and for $k \in K$ if $T_k \subseteq F$ then $\inf T_k \in F$.*

We note that such an ultrafilter F is referred to as a *generic* ultrafilter in forcing constructions, see e.g [67]. Let $A \in CA_n$ and $\{X_i : i < \omega\}$ be as in the hypothesis of (*) and let $a \in A$ be non-zero. For brevity, let $\beta = n + \omega$. Note that $\beta = \omega$ when n is finite. We want to find a $C \in Ws_n$ and $h : A \to C$ such that

$$h(a) \neq 0 \text{ and } \bigcap_{x \in X_i} h(x) = \emptyset \text{ for all } i \in \omega.$$

Using Hodge's terminology [67], p.369 , we *force* with A.[4] Towards this end, let $B \in CA_\beta$ such that $A \subseteq Nr_nB$, A is a generating set for B and $\inf{}^BX_i = 0$ for each

[4]Strictly speaking, Hodges forces with theories, here we force with their algebraic counterpart.

$i < \omega$. This is possible since A has the strong neat embedding property. In particular, B, having a countable signature and generated by a countable set, is also countable and by [50] Theorem 2.1.5 (ii) is *dimension complemented* in the sense of [50] 1.11.1 (ii), that is the following holds

$$(\forall x \in B)(|\Delta x \smallsetminus n| < \omega.)$$

Recall that Δx the *dimension set* of x is defined to be the set $\{i \in \beta : c_i x \neq x\}$. Note that when $n < \omega$ then B is locally finite. We have by [50] Theorem 1.11.6 that

$$(1)\quad (\forall j < \beta)(\forall x \in B)(c_j x = \sup_{i \in \beta} s_i^j x.)$$

Now let V be the weak space ${}^{\beta}\beta^{(Id)}$, where Id is the identity function on β, i.e.

$$V = {}^{\beta}\beta^{(Id)} = \{s \in {}^{\beta}\beta : |\{i \in \beta : s_i \neq i\}| < \omega\}.$$

For each $\tau \in V$ for each $i \in \omega$, let

$$X_{i,\tau} = \{s_{\tau}^B x : x \in X_i\}.$$

Here s_{τ}^B is the unary operation as defined in 1.11.9 in [50] [5] , which is a complete boolean endomorphism on B by [50] Theorem 1.11.12 (iii). It thus follows that

$$(2)\quad (\forall \tau \in V)(\forall i \in \omega) \inf{}^B X_{i,\tau} = 0.$$

The superscript B indicates that these meets are evaluated (and indeed preserved) in the big algebra B. Now using (**), let F be the (generic) ultrafilter preserving the set of infinite meets and joins expressed in (1) and (2) and has $a \in F$. Then it is not difficult to check, by the very construction of F, that

$$(3)\quad (\forall k < \beta)(\forall x \in B)(c_k x \in F \implies (\exists i \notin \Delta x)(s_k^i x \in F).$$

and that

$$(4)\quad (\forall \tau \in V)(\forall i \in \omega)(\exists x \in X_i)(s_{\tau} x \notin F).$$

[5] s_{τ} is the algebraic counterpart of the metalogical operation of the simultaneous *substitution* of variables (indexed by the range of τ) for variables (indexed by its domain), cf. [50], the remark after Theorem 1.11.8.

Next we form the *canonical* representation corresponding to F in which satisfaction coincide with genericity. To handle equality we define

$$R = \{(i,j) \in {}^2\beta : d_{ij} \in F\}.$$

Then, as easily checked, R is an equivalence relation on β. Let $\sigma, \tau \in V$. Then we write $\sigma \equiv_R \tau$ iff

$$(\forall i \in \beta)(\sigma(i), \tau(i)) \in R,$$

and we set

$$R^+ = \{(\sigma, \tau) \in {}^2V : \sigma \equiv_R \tau\}.$$

Then the following conditions hold:

$$R^+ \text{ is an equivalence relation on } V,$$

the set

$$C = \{X \subseteq V : (R^+)X = X\}$$

is a boolean field of sets, i.e it is closed under set-theoretical intersection $\cap$, and complementation with respect to V ($\smallsetminus_V$), and it is also closed under the i th cylindrification C_i, for each $i < \beta$. Finally, the algebra

$$\langle C, \cup, \cap, \smallsetminus_V,\ C_i, R(d_{ij} \cap V)\rangle_{i,j<\beta}$$

is a weak set algebra with unit V, and it is not difficult to check that the map $h : A \to C$ defined by

$$h(x) = \{\sigma \in V : s_\sigma^B x \in F\}$$

is the desired homomorphism. Indeed, that h is a homomorphism follows from (3) (cf. the appendix of [9] p. 39 for a similar proof), that $h(a) \neq 0$ follows from that Id, the identity function on β, is in $h(a)$ since $s_{Id}a = a \in F$, and finally, that h omits the non-principal X_i's follows directly from (4). $\square$

Remark.

The above proof is an instance of Robinson's finite forcing forcing in Model Theory and is indeed a Baire Category argument at heart. In fact (**) is equivalent in ZFC (Zermelo-Fraenkel set with choice) to that every compact Hausdorf space is Baire. Indeed, instead of invoking (**), one can instead find the desired ultrafilter F[6], by noting that the set X say, of such ultrafilters is dense in the Stone space of A. This follows from the fact that X - consisting of a set of ultrafilters *respecting* a countable set of infinite meets and joins in the sense of [43] p.414 - can be written as the intersection of a countable family of dense open sets in the (compact Hausdorff) Stone space of A. Therefore by the Baire Category Theorem for compact Hausdorff spaces X is dense. We refer in this connection to the topological proof in [43] p. 414 of the Rasiowa-Sikorski Lemma. An interesting result of Goldblatt [43] is that the Rasiowa-Sikorski Lemma is equivalent in ZF (Zermelo-Fraenkel set theory *without* choice) to the conjunction of the Ultrafilter Theorem and the Baire Category Theorem for compact Hausdorff spaces. In Theorem 1.4 below we shall show that when we ask for omission of $< {}^{\omega}2$ many non-principal types, then we are led to an instance of the famous Martin's axiom of Set Theory. Very much used in the proof, we do not know how much countability is needed the hypotheses of Theorem 1. In this connection we pose the following question.

Question . *Is there a similar characterization for uncountable algebras? How much is countability needed? Is there (an uncountable) atomic $A \in S_cNr_nCA_\omega$ that is not completely representable? If so, can A be chosen to be in Nr_nCA_ω?*

We note that the characterization established in Theorem 1.3 of a completely representable algebra in terms of neat reducts, is not intrinsic, in the sense that it involves mathematical objects (in fact, certain cylindric algebras) from "outside" of the algebra involved. But this is the case too with Henkin's classical Neat Embedding Theorem. In fact, one of the major open problems of algebraic logic is to give a simple intrinsic characterization of the class of representable cylindric algebras, cf. [109], and variants thereof [116] and chapter 3. Solutions providing axiomatizations of RCA_n,

[6] F is what we called a Henkin ultrafilter in the previous chapter

$n \leq \omega$, can be found in [150] and [54]. The former uses techniques inspired by modal logic while the latter uses game-theoretic techniques, that again is in essence an instance of Robinson's finite forcing in Model theory. Sain [116] gives a strictly finite axiomatization to a variant of RCA_ω, side-stepping the negative results of Andreka [3] These algebras we studied in the previous chapter where we proved that they further have the strong amalgmation property. It thus follows that Sain's algebras are solutions to the stronger version of the Finitization Problem posed in [131]. [7]

On the other hand, if the (non-intrinsic) characterization established in Theorem 3.1 works for uncountable algebras, then this gives a new proof that $A \in RCA_n$ if and only if A^+ its *perfect extention* in the sense of [50] 2.7.7 is completely representable, since the latter, by [50] 2.7.23 and the immediately following remark, is always in $S_cNr_nCA_\omega$. We recall from [50] that the boolean reduct of A^+ is the complete boolean algebra, whose domain is the powerset of the Stone space of A, thus is atomic. That A^+ is *completely* representable, for representable A, is due to Monk. Another (model-theoretic) proof of this can be found in [54] Theorem 22 p.258. In [54], A^+ is referred to as the *ultrafilter or perfect extension* of A. In contrast, it is known that RCA_n when $\omega > n > 2$, is not closed under *minimal* or *Dedekind* completions in the sense of [50] 2.7.21 , a result of Hodkinson [65]. Unlike perfect extensions of infinite algebras, a minimal completion A^*, say, of $A \in CA_n$ (based on the minimal completion of its boolean reduct), [8] preserves meets and joins already existing in A. However, if $\omega > n > 2$ and $A \in RCA_n$, then A^*, though still a cylindric algebra of dimension n, may not be representable, as indeed shown by Hodkinson. Hodkinson's result is sharpened by Andreka [4]. The examples in [4] and [65] show that RCA_n, though a

[7] We should mention in this connection that Hirsch and Hodkinson prove that the so called *Lyndon conditions* characterize the elementary closure of the class of completely representable RA's, [53]. A subtlty that occurs here is that for atomic relation algebras, the Lyndon conditions though necessary for the existance of a complete representation are not sufficient. On the other hand, the Lyndon conditions are sufficient but not necessary for the existance of ordinary representations. We could say thus that the Lyndon conditions are complete but not sound over RRA. For finite relation algebras, though all such notions coincide: For $A \in RA$ finite, A is representable if and only if A is completely representable if and only if A satisfies the Lyndon conditions.

[8] For a boolean algebra B, its minimal completion is defined to be the boolean algebra of regular open sets in the stone space of B. The boolean meet is set-theoretic intersection, while the complement of an element is defined to be the *interior* of its usual set-theoretic complement. For an atomic cylindric algebra its minimal completion can be recovered from its atom structure U, say ; it is simply the complex algebra of U[65].

canonical variety i.e closed under perfect extensions, [9] cannot be characterized by the so-called *Sahlqvist* equations when $2 < n < \omega$. This follows from a result of Venema that says that any Sahlqvist variety i.e a variety that is axiomatized by Sahlqvist equations is closed under minimal completions. A recent unpublished result of the author extends Andreka's and Hodkinson's result to infinite dimensions. In particular, RCA_ω is not a Sahlqvist variety. These examples for the finite dimensional case will be used below for an entirely different purpose, namely to show that the omitting types Theorem fail for the so called finite-variable fragments of first order logic as long as the number of variables available are > 2. We refer the reader to [73] for a detailed treatment of minimal completions of boolean algebras with operators. There another example is given to show that Sahlqvist equations need not be preserved in such completions, but in the more general context of boolean algebras with operators.

A connection with Martin's Axiom

Viewing the X_i's in (*) in the proof of Theorem 1 as *non-principal types*, then the proof of (*) shows that ω non-principal types in a countable algebra that has the strong neat embedding property can be omitted. Now we ask for omission of $< {}^\omega 2$ non-principal types. Consider the following statement which we abbreviate by $MA_c(n, \kappa)$. The subscript c indicates that algebras considered are countable. We have two other parameters. κ stands for the cardinal indexing the family of non -principal types while n is the dimension of the algebra, which is restricted to be $\leq \omega$.

$MA_c(n, \kappa)$. *Let $A \in CA_n$ be countable. Assume that A has the strong neat embedding property. Let $\{X_i : i < \kappa\}$ be a family of subsets of A such that $\bigwedge X_i = 0$ for all $i < \kappa$. Then for every non-zero $a \in A$ there exists $B \in Ws_n$ with countable base, and $f \in Ho(A, B)$ such that $f(a) \neq 0$ and $\bigcap_{x \in X_i} f(x) = \emptyset$ for all $i < \kappa$.*

Note that fixing κ and varying n over ω we have infinitely many statements. That they are all equivalent in ZFC, follows from our next Theorem. To formulate it,

[9] A proof of this follows from the Neat Embedding Theorem (NET) In more detail, if $A \in RCA_n$ then $A \subseteq Nr_n B$ for some $B \in CA_{n+\omega}$. Then it can be proved (cf. [50] Thm 2.7.23) that A^+, the perfect extension of A, is a subreduct of B^+, the perfect extension of B, but then by the NET, it follows that A^+ is representable. In fact A^+ is completely representable, a proof of the latter using a saturation argument can be found in [54].

we need to recall that a boolean algebra satifies the countable chain condition, CCC for short, if for any partition X of the unit, we have $|X| \leq \omega$. In particular, any countable boolean algebra satisfies the CCC. Recall that Martins axiom for κ, κ a cardinal, $MA(\kappa)$ for short, is (equivalent to) the following assertion about boolean algebras.

$MA(\kappa)$. *Let B be a boolean algebra satisfying the CCC. Let $\{X_i : i < \kappa\}$ be a family of subsets of B such that $\bigwedge X_i = 0$ for all $i \in \kappa$. Then for every non-zero $b \in B$, there exists an ultrafilter F containing b such that for all $i \in \kappa$ there exists $x \in X_i$ whose complement is in F.*

We let $MA_c(\kappa)$ be the weaker statement obtained from $MA(\kappa)$ by replacing boolean algebras satisfying CCC, with countable boolean algebras. Note that in this case we have at most ${}^{\omega}2$ many distinct X_i's. CH is short for the continuum hypothesis, and MA is short for Martins axiom. We recall (see e.g. [26]) that MA is the assertion that $\forall\kappa < {}^{\omega}2 \to MA(\kappa)$.

Theorem 1.4 . *Let $n \leq \omega$. Let κ be a cardinal. Then the following holds:*

(i) $MA_c(n, \kappa)$ is equivalent (in ZFC) to $MA_c(\kappa)$.

(ii) $M_c(n, \kappa)$ implies that $\kappa < {}^{\omega}2$. Furthermore the statement $\forall\kappa < {}^{\omega}2 \to MA_c(n, \kappa)$ is independent of ZFC+$\neg CH$.

Proof.

(i) That $MA_c(n, \kappa) \to MA_c(\kappa)$ can be seen by taking a countable boolean algebra and giving it the discrete structure, i.e. defining cylindrifications and diagonal elements as in [50] 1.3.10, that is define $d_{ij} = 1$ and $c_i x = x$ for all $i, j < n$. The resulting CA_n satisfies all the hypothesis in $MA_c(n, \kappa)$, in particular it has the strong neat embedding property. Also in this case a *cylindric* representation of A stipulated by $MA_c(n, \kappa)$ is nothing more than a boolean representation respecting the κ given infinite meets, carrying them to set-theoretic intersection. The construction of the desired ultrafilter readily follows. The other direction is harder. But a moment's reflection shows that $MA_c(\kappa) \to MA(n, \kappa)$ is the same as the proof of (*), but appealing to $MA_c(\kappa)$ instead of (**).

(ii) There is a model of ZFC, due to Miller [103], in which $\omega_1 < {}^{\omega}2$ and the Cantor space ${}^{\omega}2$ can be covered by ω_1 nowhere dense closed sets. But this gives that the countable free boolean algebra B has ω_1 non-principal types that *cannot* be omitted, i.e in this model we have $\omega_1 < {}^{\omega}2$ and $MA_c(\omega_1)$ is false. Conversely, let M be a countable transitive model of ZFC in which $MA + \neg CH$ holds. Then $\forall \kappa < {}^{\omega}2 \to MA_c(n, \kappa)$ also holds in this model. We have proved (ii).

■

Remark

For a Polish space X, $K(X)$ denotes the ideal of meager subsets of X. If X is the real line, or the Baire space ${}^{\omega}\omega$, or the Cantor set ${}^{\omega}2$, which are the prime examples of Polish space, we write K instead of $K(X)$. For an ideal I on a set X, we set

$$covI = min\{|C| : C \subseteq I,\ \cup C = X\}.$$

A Polish space is a complete separable metric space; the above three spaces are sometimes referred to as *real* spaces since they are all Baire isomophic to the real line, cf. [26] p.223. Miller's model referred to above is constructed to show that $covK = \omega_1$. $covK$ plays a key role in omitting types. Newelski [112] shows that one can omit at most $< covK$ non-principal types in countable first order complete theories. That $< covK$ many types can be omitted for countable (not necessarily complete) theories could be readily obtained from the proof of Theorem 1 above, by taking A to be a countable Lf_ω and taking $< covK$ many "finitary" meets, instead of just countably many, upon noting that $Lf_\omega \subseteq S_c Nr_\omega CA_{\omega+\omega}$ and that the jump from countably many to $< covK$ many types can be made because $MA_c(\kappa)$ is provable in ZFC for any $\kappa < covK$. By a finitary meet we mean a subset X of $Nr_m A$, for some $m \in \omega$, such that $\bigwedge^A X = 0$. But then, one has to replace the weak space ${}^{\beta}\beta^{(Id)}$ by the cartesian space, or square ${}^{\beta}\beta$ to get a regular locally finite set algebras, or a $Cs_\omega^{reg} \cap Lf_\omega$ [51] sec 4.3. for short, with countable base β, which is the underlying set of the desired standard model omitting the $< covK$ many non-principal types. This can be done because every $\tau \in {}^{\beta}\beta$ induces a unary operation s_τ on A, since A is locally finite. (cf. [50] Thm 1.11.14.) Although not, in general, a complete endomorphism (cf. [50] the example immediately preceding 1.11.15), s_τ still preserves the finitary m types, in the sense that if $A \in Lf_\omega$ and $X \subseteq Nr_m A$ is such that $\inf X = 0$ then $\inf s_\tau X = 0$. Other than that, the proof is the same. The above argument is essentially the one used in [32] Theorem 2.1 p. 241, to extend Newelski's result to non-complete theories, by taking into consideration the correspondance established in [51] Thm 4.3.28 (iii) between countable Lf_ω and countable $L_{\omega\omega}$ theories. We note that if we consider infinitary non-principal types (i.e arbitrary subsets) of $A \in Lf_\omega$, then it is not always possible to "square" the weak space on which the representation provided by (*) is based. An interesting question in this connection is whether we could replace Lf_ω by the bigger class Dc_ω, and still get a Cs_ω in place of a Ws_ω that omits $< covK$ many *finitary* non-principal meets. Here Dc_ω denotes the class of dimension complemented CA_ω's in the sense of [50] Def.1.11.1. By [50] 2.6.67 $Dc_\omega \subseteq Nr_\omega CA_{\omega+\omega}$ so that members of Dc_ω have the strong neat embedding property. Thus (*) applies. We should mention that MA implies that $covK = {}^{\omega}2$, but according to the model of Miller referred to above, it is also consistent that $covK = \omega_1 < {}^{\omega}2$. In [112] and [32] Theorem 2.2, it is shown that for countable theories without a prime model, $covK$ many non-principal types *may not* be omitted simultaneously. This implies that $MA_c(\omega, covK)$ is false. A result of Shelah [126] is that if the finitary types are maximal m types - algebraically they are ultrafilters in $Nr_m A$ - then one can omit $< {}^{\omega}2$ non-principal types. For a different algebraic proof of the omitting types Theorem, we refer the reader to [97] and [125]. The latter reference studies countable *neatly atomic* locally finite cylindric algebras. $A \in Lf_\omega$ is neatly atomic if $Nr_n A$ is atomic for all $n \in \omega$. These are the algebraic counterparts of atomic theories. A deep result concerning non-uniqueness of atomic models for uncountable atomic first order theories, due to Shelah, can be found in [127] . Another counterexample showing the non-existence of atomic models for uncountable atomic theories can be found in [30]. $\diamondsuit$

Omitting types for logics with finitely many variables

Now we apply our results to finite-variable first order logic. We turn our attention to omitting types in first order logic restricted to the first n variables (L_n). L_n has been studied in the literature rather extensively, both as a restricted version of first order logic for purely logical reasons cf. ([51], [116], [9]), and as a multi-modal logic, with emphasis on its relation with temporal logics in computer science, cf. [72], [10] and [150].

From now on, unless otherwise specified, n denotes a finite ordinal. Let $\mathbf{L}$ be a countable first order language. Then we let $Fm^{\mathbf{L}_n}$ stand for the set of $\mathbf{L}$-formulas using the first n variables only. Let $T \subseteq Fm^{\mathbf{L}_n}$ be a theory, and let $\Gamma \subseteq Fm^{\mathbf{L}_n}$ be consistent over T, i.e. assume that $T \cup \Gamma$ has a model. For $M \models T$ and $\phi \in Fm^{\mathbf{L}_n}$ we let ϕ^M be the set of all assignments satisfying M, that is

$$\phi^M = \{s \in {}^nM : \phi[s]\}.$$

Definition 1.5 .

(i) We say that Γ is implicitly principal over T, or simply implicitly principal when T is clear from context, if Γ cannot be omitted in any model M of T, that is for every $M \models T$ we have

$$\cap_{\phi\in\Gamma}\phi^M \neq \emptyset.$$

(ii) Let $k \in \omega$. We say that Γ is explicitly k-principal over T, or simply explicitly k-principal, if there exists a formula using $n+k$ variables with at most the first n variables free, such that ϕ isolates Γ, in the sense that

$$T \models \phi \to \psi, \ \forall\psi \in \Gamma.$$

We write simply *explicitly principal* instead of explicitly 0-principal. The Henkin-Orey omitting types Theorem, hereafter OTT for short, implies that for a countable L_n theory T, if Γ is implicitly principal, then this is witnessed by some formula possibly using more than n variables, i.e. it is k-explicitly principal for some $k \in \omega$. The usual terminolgy for OTT (see e.g. [32]) refers to such Γ as a non-principal or isolated type.

We now show that the OTT fails for L_n when $n > 2$, in the sense that in some cases n-variables are not enough to isolate non-principal types; we have to “step outside” L_n. To prove our next Theorem, we use the correspondence established in Thm 4.3.28 (ii) in [51] sec 4.3. between L_n theories and RCA_n.

Theorem 1.6 . *Let $2 < n < \omega$. Then there exists a countable language $\mathbf{L}$, a theory $T \subseteq Fm^{\mathbf{L}_n}$ and $\Gamma \subseteq Fm^{\mathbf{L}_n}$ (consistent over T) such that Γ is implicitly principal but not explicitly principal. In other words, Γ is a non-principal type that cannot be omitted.*

Proof. Let $2 < n < \omega$. Let $A \in RCA_n$ be any atomic, countable, simple representable algebra, that is not completely representable. The C_n constructed by Hirsch and Hodkinson in [53] p 841. works. C_n is atomic and representable, but not completely representable. Further, we can assume that C_n is generated by its atoms (cf. [53] line 4 on p. 841) which are countable, thus C_n is also countable. Other examples of such algebras abound in the literature of algebraic logic. See for example [82], p. 154-173, [65] or [4]. Then by [51] Theorem 4.3.28 (ii) there is a countable language $\mathbf{L}$, such that $A \cong Fm^{\mathbf{L}_n}/T$ [10] for some (countable) L_n theory $T \subseteq Fm^{\mathbf{L}_n}$. Here $Fm^{\mathbf{L}_n}/T$ is the Lindenbaum-Tarski representable n-dimensional cylindric algebra corresponding to T. Fix θ an isomorphism from $Fm^{\mathbf{L}_n}/T$ to A and let AtA denote the set of atoms of A. Put

$$\Gamma = \cup\{\neg\phi/T : \theta(\phi/T) \in AtA\}.$$

Here ϕ/T denotes the equivalence class of ϕ, consisting of all formulas equivalent to ϕ modulu T. Since A is atomic, it follows that $\sup AtA = 1$, thus (+)

$$\inf\{\theta(\psi/T) : \psi \in \Gamma\} = 0.$$

Now we check that Γ is indeed a non-principal type that is realizable in every model of T, i.e Γ cannot be omitted. To see that Γ is non-principal, assume to the contrary

[10] Strictly speaking, $\mathbf{L}_n$ would consist only of *restricted* formulas in the sense of [51] p. 152. In restricted formulas the only admissable *atomic* formulas are of the form $x_i = x_j$ and $R(x_0 \cdots x_{n-1})$, i.e the variables occur in a fixed order. However the algebras referred to above are all *symmetric* in the sense that they are closed under the polyadic operations of substitutions, i.e they are reducts of quasipolyadic representable algebras. Thus, cf. [109], $Fm^{\mathbf{L}_n}$ is the set of *all* formulas built up using n variables.

that there exists $\phi \in Fm^{\mathbf{L}_n}$ such that ϕ is consistent with T and ϕ isolates Γ. But then we get that for all $\psi \in \Gamma$

$$0 < \theta(\phi/T) \leq \theta(\psi/T).$$

This contradicts (+). To see that Γ cannot be omitted, assume to the contrary that $M \models T$ is such that $\cap_{\phi\in\Gamma}\phi^M = \emptyset$. Then $\cup_{\phi\in\Gamma}(\neg\phi^M) = {}^nM$. But then $\{\phi^M : \phi \in Fm^{\mathbf{L}_n}\}$ would be (the domain of) an atomic representation of A in the sense of [53], Definition 4, thus by Theorem 5 of [53], it is a complete representation of A, contradiction.

■

Remark

In [4] and [65], Andreka and Hodkinson construct, countable, atomic, representable simple relation and cylindric algebras of dimension $2 < n < \omega$, with a non representable minimal completion. The algebras constructed in [4] and [65] cannot be completely representable. At a certain time, though, it was thought that every atomic, representable, relation algebra (satisfying the so called Lyndon conditions) has a complete representation; this is more commonly referred to in the literature of algebraic logic as *Lyndon's error.* [11] But it quickly transpired that this is not the case. In this connection, the relation algebra Maddux constructs (cf. [82] p.154-173) is an atomic, *non-integral*, representable relation algebra, with no complete representation. It is related to an example of Lyndon [**?**]. Maddux [82] asks whether such algebras can be integral. We recall that an integral relation algebra is one for which the composition of two non-zero elements is non-zero. In showing that the class of completely representable relation algebras is not axiomatizable, Hirsch and Hodkinson construct an integral (hence simple), representable relation algebra, that is not completely representable. This answers Maddux's question. The cylindric algebra of dimension $n > 2$, constructed for the same purpose, and denoted by C_n in the proof of Theorem 3 above is simple. This means that the corresponding theory T, that does

[11] Lyndon in [75] constructed an example of an RA that does not satisfy the Lyndon conditions, and mistakenly inferred that it has no representation at all. However this algebra *is* representable, the point is it has no *complete* representation.

not omit the non-principal type denoted by Γ in Theorem 3, is complete, and so for every $M \models T$ the map: $\phi/T \to \phi^M$, is actaully an isomorphism from $Fm^{\mathbf{L}_n}/T$ to the cylindric set algebra $\{\phi^M : \phi \in Fm^{\mathbf{L}_n}\}$. In [4] , using a combinatorial trick of Maddux [82], the representable algebra whose minimal completion is not representable is countable, simple and can be generated by a single binary relation. This means that we can choose our counterexample to OTT in a countable complete L_n theory having only one binary relation symbol. [12]

We conjecture the following substantial strengthening of Theorem 3[13]:

Conjecture 1 . *Let $2 < n < \omega$. Then for every $k \in \omega$, there exists a countable language $\mathbf{L}$, an L_n theory $T \subseteq Fm^{\mathbf{L}_n}$, and $\Gamma \subseteq Fm^{\mathbf{L}_n}$ (consistent over T) such that Γ is implicitly principal but not k- explicitly principal. Furthermore, the language of T can be chosen to have only one binary relation symbol.*

On a proof. Let $2 < n < \omega$. If there exists for any $k \in \omega$, a countable atomic, simple $A_k \in S_cNr_nRCA_{n+k}$, that is not completely representable and generated by a single 2-dimensional element, then this will confirm conjecture 4. To see this, let T be the L_n theory corresponding to A_k, and Γ the type as defined in Theorem 3 above. Then T has only one binary relation symbol. By the same reasoning used in Theorem 3 we get that Γ is implicitly principal but not k-explicitly principlal (Note that by the usual OTT, Γ is isolated by some $\phi \in L_m$, for some $m > n + k$.) $\square$

Conjecture 1 says that, though there are always formulas isolating or witnessing realizable types, we cannot control the number of variables occuring in such formulas. They grow without bound. If true, then this adds to the list of deep (negative) results known for L_n concerning the complexity of its proof theory for $n > 2$ ([95], [3], and

[12]The constructions of Hajnal Andreka and Ian Hodkinson go via *atom structures* (in the terminology of algebraic logic), [65] or Kripke frames (in the terminolgy of modal logic[**?**] Ian Hodkinson [65] constructs an atom structure U such that the *term algebra* over U, or TmU for short, is representable while the full complex algebra, or CmU for short, is not. Such a U is an example of a *weakly representable* atom structure that is not *strongly representable* in the sense of [58]. Andreka's atom structure has the same properties, except that it contains enough information to construct atom structures for higher arity relations; namely n dimensional cylindric algebras and therefore is a good candidate for confirming conjecture four. Technically, the complex algebra based on the atom structure U constructed by Andreka posses n-dimensional cylindric basis in the sense of Maddux [84].

[13]Our conjecture 4 was confirmed by an unpublished result of Hajnal Andreka, cf. the subsection further work herein for more details.

[56], [62]), (failure of) interpolation for $n > 1$ ([114], [91],) and undecidabilty of the validities for $n > 2$ ([51] Thm 4.2.18.) Also this adds to our knowledge of omitting types for usual first order logic, since it says that there is no bound on the number of variables needed to isolate non-omissable types. What this entails in general terms is that in isolating types, all ω variables are needed. This is indeed analogous to the situation with provability and interpolation. In contrast, we should mention that the OTT holds for L_1, and more generally for countable L_n theories, $n \leq \omega$, with only (countably) many *unary* relation symbols. This follows from Nemeti's algebraic result [104] that *monadic generated* cylindric algebras are neat reducts. That is if $1 < n \leq \omega$ and $A \in CA_n$, is generated by a set X such that every $x \in X$ $|\Delta x| \leq 1$, [14] then $A \in Nr_nCA_{n+\omega}$, hence A has the strong neat embeddding property. The proof of this can be easily recovered from the proof of "the only if part" of (ii) of Theorem 1 in [104] p. 404-405. While every atomic RCA_2 is completely representable; this follows easily from [51] Theorem 3.2.65, we do not know whether OTT holds for L_2, but it is likely that it does.

In the next section we give a model-theoretic solution to the long-standing open problem 4.4 in the monograph Henkin Monk Tarski [51].

[14] Recall that Δx is the dimension set of x, which is defined to be $\{i \in n : c_i x \neq x\}$.

A model Theoretic solution to a problem of Tarski

In 1954 Roland Fraisse published a paper which has become a classic of model theory. He pointed out we can think of the class of finite linear orderings as a set of approximations to the ordering of the rationals, and he described a way of building the rationals out of these approximations. Fraisse's construction is important because it works in many other cases too. Starting from a set of finite structures, we can build their "limit", and some of the structures built in this way have turned out to be remarakbly interesting. We shall use the Fraisse construction to prove:

Theorem 2.1 . *Let $2 < n < \omega$, and let β be an ordinal $> n$. Then the class Nr_nCA_β is not closed under elementary subalgebras. In particular, Nr_nCA_β is not first order axiomatizable.*

In other words, if we start with the class of algebras that arise from a first order logic with β-many variables, and then restrict our attention to those algebras that concern formulas with n free variables with $2 < n < \omega \cap \beta$, then this class cannot be axiomatized by means of first order sentences. This solves the finite part of the long-standing open problem 4.4 in the monograph [51] on cylindric algebras by Henkin, Monk and Tarski. Theorem 1 also confirms a conjecture of Németi formulated as conjecture 1 in [104]. In this paper, Németi proves that for any pair of ordinals $n < \beta$, the class Nr_nCA_β is closed under ultraproducts. By the celebrated Keisler-Shelah ultrapower theorem, cf. Theorem 6.15 of [35] our Theorem 1 implies that for $1 < n < \beta \cap \omega$, the class Nr_nCA_β is not closed under ultraroots. Various refinements of our result summarized in Corollary (?) below also solve the finite part of problem 18 on p. 312 in [52]. We note that Andréka and Németi prove that Nr_2CA_β is not elementary in 8.6 of Part II of [52] and that the analagous result for several other algeras was proved in chapter two. Theorem 1 was announced in [134], and a different proof of Theorem 2.1 adressing also the infinite dimensional case is given in [135]. A novelty that occurs here is the proof method, which - as indicated in the title - uses genuinely model-theoretic arguments inspired by a theorem of Fraïssé. As previously pointed out in chapter 2, the problem as to whether the class of neat reducts is closed under (elementary) subalgebras has a long history and dates back to problem 2.11 in

[50] and to related questions raised by Pigozzi in [113](cf. [113], remark 2.2.21.) An implicit form of this problem appears as item (v) in the introduction of the monograph [51] as a conjecture of Tarski whose proof could not be reconstructed, cf. Theorem 1, Theorem 2, Corollary 1 and Corollary 2 in Chapter two.

We now prove the Theorem 2.1. For simplicity of notation we give the proof for $n = 3$. We start by formulating a Lemma (Lemma 2.2) stipulating the existence of a first order structure M. This first order structure M, among other conditions, enjoys quantifier elimination, meaning that every formula in the first order language of M is equivalent to a quantifier free one. Because M has quantifier elimination, the cylindric set algebra $\mathcal{A}$, consisting of all those relations definable in M using only 3 variables, will be a neat reduct. Using the other conditions, we then extract an elementary subalgebra $\mathcal{B}$ of $\mathcal{A}$ that is not a neat reduct. We proceed as follows. First we prove (modulo Lemma 2.2) that the class of neat reducts is not elementary, then we prove our Lemma at the end of this section. To formulate Lemma 2.2 we need:

Notation . *S_3 denotes the set of all permutations of 3. ${}^XY$ denotes the set of functions from X to Y. For $u, v \in {}^33$, $i < 3$ we write u_i for $u(i) < 3$, and we write $u \equiv_i v$ if u and v agree except for i i.e. if $u_j = v_j$ for all $j \in 3 \smallsetminus \{i\}$. We identify (notationally) a structure M with its domain. For a symbol R of the signature of M we write R^M for the interpretation of R in M.*

Now we are ready to formulate our Lemma.

Lemma 2.2 . *Let L be a signature consisting of the unary relation symbols P_0, P_1, P_2 and uncountably many 3-ary predicate symbols. For $u \in {}^33$, let χ_u be the formula $\bigwedge_{i<3} P_{u_i}(x_i)$. Then there exists an L-structure M with the following properties:*

(1) M has quantifier elimination, i.e. every L-formula is equivalent in M to a boolean combination of atomic formulas.

(2) The sets P_i^M for $i < 3$ form a partition of M,

(3) $M \models \forall x_0x_1x_2(R(x_0, x_1x_2) \longrightarrow \bigvee_{u\in S_3} \chi_u)$, for all $R \in L$,

(4) $M \models \exists x_0x_1x_2(\chi_u \wedge R(x_0, x_1, x_2) \wedge \neg S(x_0, x_1, x_2))$ for all distinct ternary $R, S \in L$, and $u \in S_3$,

(5) For $u \in S_3$, $i < 3$, $M \models \forall x_0 x_1 x_2(\exists x_i \chi_u \longleftrightarrow \bigvee_{v \in {}^33, v \equiv_i u} \chi_v)$,

(6) For $u \in S_3$ *and any* L*-formula* $\phi(x_0, x_1, x_2)$, *if* $M \models \exists x_0 x_1 x_2(\chi_u \wedge \phi)$ *then* $M \models \forall x_0 x_1 x_2(\exists x_i \chi_u \longleftrightarrow \exists x_i(\chi_u \wedge \phi))$ *for all* $i < 3$.

Proof outline.

We now explain the idea behind the construction of such an M. In the process we give a sketchy outline of the proof that the class of neat reducts is not elementary in order to pave the way for a smooth (formal) proof of Theorem 1. From now on fix M as in Lemma 2. Property (1) of quantifier elimination postulates that the set of atomic formulas

$$J = \{R(y_0, y_1, y_2) : \{y_0, y_1.y_2\} = \{x_0, x_1, x_2\} \text{ and } R \in L \text{ is a ternary relation}\}$$

$$\cup \{P_i(x_j) : i, j < 3\} \cup \{x_i = x_j : i, j < 3\}$$

is an elimination set for M. This means that every formula $\phi \in L$ is equivalent in M to a boolean combination of formulas in J. This implies that the cylindric set algebra consisting of those relations definable using only the first three variables is a neat reduct. In more detail, for an L-formula ϕ, let ϕ^M be the set of all assignments satisfying ϕ in M i.e.

$$\phi^M = \{s \in {}^\omega M : M \models \phi[s]\}.$$

For an ordinal α, let Cs_α denote the class of cylindric set algebras of dimension α. Let $\mathcal{A}_\omega$ be the Cs_ω with domain

$$\{\phi^M : \phi \in L\}$$

and operations (well-) defined by (cf. [51])[15]

$$\phi^M \cdot \psi^M = \phi^M \cap \psi^M = (\phi \wedge \psi)^M;$$

$$-\phi^M = (\neg\phi)^M;$$

[15] In [51], sec 4.3, cf. Definition 4.3.4 $\mathcal{A}_\omega$ would be denoted by Cf_3^M, which is the set algebra based on M. In this connection we note that $\mathcal{A}$ is a *regular locally* finite ω-dimensional cylindric set algebra. The notion of a regular locally finite set algebra will be recalled below.

and for $i, j < \omega$

$$\mathsf{d}_{ij} = (x_i = x_j)^M;$$

and

$$\mathsf{c}_i(\phi^M) = (\exists x_i \phi)^M.$$

Now write L_3 for the set of all L-formulas using only the first three variables. A moment's reflection reveals that property (1) equates the Cs_3 $\mathcal{A}$ having domain

$$\{\phi^M : \phi \in L_3\}$$

with the (possibly bigger) Cs_3 having domain

$$\{\phi^M : \phi \in L \text{ and } \phi \text{ contains } x_0, x_1, x_2 \text{ as free variables}\},$$

with the cylindric operations defined for both as for the operations defined for $\mathcal{A}_\omega$. But the latter, as easily checked, is isomorphic to $\mathcal{N}r_3\mathcal{A}_\omega$, so property (1) guarantees that $\mathcal{A} \in Nr_3CA_\omega$. The rest of the conditions are designed to extract an elementary subalgebra of $\mathcal{A}$ that is not in Nr_3CA_4. But let us first understand the (abstract) structure of $\mathcal{A}$ based on M. Condition (2) postulates that

$$\{\chi_u^M : u \in {}^33\}$$

is a partition of ${}^3M$, the unit of A, that is

$$\cup_{u \in {}^33}(\chi_u)^M = {}^3M,$$

and for distinct $u, v \in {}^33$ we have

$$(\chi_u)^M \cap (\chi_v)^M = \emptyset.$$

Conditions (3) and (4) single out the χ_u^M's that are indexed by permutations $u \in S_3$. Note that for any such u, if $\langle a_0, a_1, a_2\rangle \in \chi_u^M$, then the a_i's are distinct because $a_i \in P_{u_i}$ for $i < 3$ and by condition (2), these are disjoint. Condition (3) says that for any ternary $R(x_0, x_1, x_2) \in L$, $R(x_0, x_1, x_2)^M \subseteq \cup_{u \in S_3}(\chi_u)^M$ with $u \in S_3$, so this means that if $\langle a_0, a_1, a_2\rangle \in R(x_0, x_1, x_2)^M$, then the a_i's must be distinct too. Condition (4) says that below every such $(\chi_u)^M$ with $u \in S_3$, there are *uncountably* many pairwise distinct non-empty elements, namely, the $R(x_0,x_1,x_2)^M\cap(\chi_u)^M$, for ternary $R\in L$. Condition (5) tells us how the $(\chi_u)^M$'s behave with respect to cylindrifications. It simply says that for $u\in S_3$ and $i<3$ we have

$$\mathsf{c}_i(\chi_u)^M=\cup_{v\in{}^33,v\equiv_i u}(\chi_v)^M.$$

which follows from conditions (3) and (4). Finally, condition (6) says that elements below χ_u^M are *big*, as far as cylindrifications are concerned, that is for any ϕ such that

$$(\phi\cap\chi_u)^M\neq\emptyset$$

and any $i<3$, we have

$$c_i(\phi^M\cap\chi_u^M)=c_i(\chi_u^M)=\cup_{v\in{}^33,v\equiv_i u}(\chi_v)^M.$$

To summarize the above, for the sake of brevity, let 1_u denote χ_u^M. Then it follows from property (2) that the set $\{1_u:u\in{}^33\}$ forms a partition of ${}^3M$, the unit of $\mathcal{A}$. If $u\in S_3$, i.e. u is a permutation, then, by property (4), below every 1_u there are uncountably many pairwise distinct non-empty elements, namely the $R(x_0,x_1,x_2)^M$'s intersected with 1_u. Such elements are *big* as far as the cylindrifications are concerned, that is for $i<3$ we have by properties (3), (6)

$$\mathsf{c}_i[R(x_0,x_1,x_2)^M\cap 1_u]=\mathsf{c}_i(1_u)=\cup\{1_u:u\equiv_i v\}.$$

Having explained the idea behind the conditions of Lemma 2, we explain how we can obtain an elementary subalgebra of $\mathcal{A}$ that is not a neat reduct. For $u\in{}^33$, let A_u stand for the *relativisation* of A to 1_u i.e.

$$A_u=\{x\in A:x\leq 1_u\}.$$

A_u is the domain of a boolean set algebra which we denote by $\mathcal{A}_u$. Then for $u\in S_3$, A_u is uncountable. Because $\{1_u:u\in{}^33\}$ is a partition of the unit of $\mathcal{A}$, it follows that the boolean reduct of $\mathcal{A}$ is isomorphic to the boolean product $\prod_{u\in{}^33}\mathcal{A}_u$. Moreover, we can (and will) expand the language of boolean algebras by diagonal elements and the constants 1_u in such a way that the cylindric algebra $\mathcal{A}$ becomes *interpretable* in this product. Then we extract an elementary subalgebra $\mathcal{B}$ of $\mathcal{A}$ by an infinite cardinality twist. $\mathcal{B}$ is obtained from $\mathcal{A}$ by keeping only countably many elements below 1_{Id}, where Id is the identity function on 3, and discarding the rest of the elements below 1_{Id}. In the product, this corresponds to replacing the component $\mathcal{A}_{Id}$ by an elementary countable boolean subalgebra $\mathcal{B}_{Id}$ of $\mathcal{A}_{Id}$ and giving the resulting algebra the interpretation given to the the boolean product $\prod_{u\in {}^33}\mathcal{A}_u$. This will not be witnessed by first order logic and will ensure that the resulting structure $\mathcal{B}$ is *not* a neat reduct. In fact, $\mathcal{B}$ will not be even in Nr_3CA_4. The idea is that if $\mathcal{B}$ were a neat reduct, then using a substitution term definable in one extra dimension, we generate uncountably many elements in the component B_{Id}, which contradicts that the latter (by its very construction) is countable. Now we implement the details of the above sketch.

Notation . *From now on, we follow the conventions of algebraists. We do not distinguish between an algebra $\mathcal{A}$ and its domain A, and denote both by A. If C is a cylindric algebra, we let $bool(C)$ denote its boolean reduct. For a non-empty set I, and a family of algebras $\langle A_i : i \in I\rangle$ (of the same signature) indexed by I, we let $\prod_{i\in I} A_i$ denote the product of the A_i's. For a boolean algebra $\langle B, \cdot\ , -\rangle$ and X a finite subset of B, we write $\bigwedge X$ and $\sum X$ for the product and sum of elements of X, respectively. In particular $+$ denotes the boolean join.*

Details of Proof.

Fix L and M as in Lemma 2.2. Let $A_\omega \in Cs_\omega$, and $A \in Cs_3$ be as specified above. That is $A_\omega = \{\phi^M : \phi \in L\}$ and $A = \{\phi^M : \phi \in L_3\}$. Then $A \cong Nr_3A_\omega$ via the map $i : A \to A_\omega$ defined by:

$$i(\phi^M) = \phi^M.$$

This map is of course well defined. Furthermore, quantifier elimination in M guarantees that this map is onto Nr_nA_ω. For $u \in {}^33$, let A_u denote the relativisation of A to χ_u^M, i.e.

$$A_u = \{x \in A : x \le \chi_u^M\}.$$

Then A_u is uncountable for every $u \in S_3$ because by property (4) of Lemma 2.2, the sets $(\chi_u \wedge R(x_0, x_1, x_2))^M$, for $R \in L$ are distinct elements of A_u. Define a map

$f : A \to \prod_{u\in {}^33} bool(A_u)$ by

$$f(a) = \langle a \cdot \chi_u\rangle_{u\in {}^33}.$$

We will expand the language of the boolean algebra $\prod_{u\in {}^33} bool(A_u)$ in such a way that the cylindric algebra A becomesinterpretable in the expanded structure. For this we need:

Definition 2.3 . *Let P denote the following structure for the signature of boolean algebras expanded by constant symbols 1_u for $u \in {}^33$ and d_{ij} for $i, j \in 3$:*

(1) The boolean part of P is the boolean algebra $\prod_{u\in {}^33} bool(A_u)$,

(2) $1^P_u = f(\chi^M_u) = \langle 0, \dots 0, 1, 0, \dots\rangle$ (with the 1 in the u^{th} place) for each $u \in {}^33$,

(3) $d^P_{ij} = f(d^A_{ij})$ for $i, j < 3$.

We now show , cf. 5.3 p.212 of [67], that A is interpretable in P. For this it is enough to show that f is one to one and that $Rng(f)$ (Range of f) and the f-images of the graphs of the cylindric algebra functions in A are definable in P. Since the χ^M_u's form a partition of the unit of A, each $a \in A$ has a unique expression in the form $\sum_{u\in {}^33}(a \cdot \chi^M_u)$, and it follows that $f : bool(A) \to \prod_{u\in {}^33} bool(A_u)$ is a boolean isomorphism.

So the f-images of the graphs of the boolean functions on A are trivially definable. f is bijective so $Rng(f)$ is definable by $x = x$. For the diagonals, $f(\mathsf{d}^A_{ij})$ is definable by $x = \mathsf{d}_{ij}$. Finally, we consider cylindrifications. For $S \subseteq {}^33$, $i < 3$, let t_S be the closed term

$$\sum\{1_v : v \in {}^33, v \equiv_i u \text{ for some } u \in S\}.$$

Let

$$\eta_i(x, y) = \bigwedge_{S\subseteq {}^33} (\bigwedge_{u\in S} x.1_u \neq 0 \wedge \bigwedge_{u\in {}^33\smallsetminus S} x.1_u = 0 \longrightarrow y = t_S).$$

We claim that for all $a \in A$, $b \in P$, we have

$$P \models \eta_i(f(a), b) \text{ iff } b = f(\mathsf{c}^A_i a).$$

To see this, let $f(a) = \langle a_u\rangle_{u\in {}^33}$. So in A we have $a = \sum_{u\in {}^33} a_u$. Let u be given. Then a_u has the form $(\chi_i \wedge \phi)^M$ for some $\phi \in L^3$, so that $\mathsf{c}^A_i(a_u) = (\exists x_i(\chi_u \wedge \phi))^M$.

By property (6) of lemma 2.2, if $a_u \neq 0$, this is $(\exists x_i \chi_u)^M$; by property (5), this is $(\bigvee_{v\in {}^33, v\equiv_i u} \chi_v)^M$. Let $S = \{u \in {}^33 : a_u \neq 0\}$. By normality and additivity of cylindrifications we have,

$$\mathsf{c}_i^A(a) = \sum_{u\in {}^33} \mathsf{c}_i^A a_u = \sum_{u\in S} \mathsf{c}_i^A a_u = \sum_{u\in S}\Big(\sum_{v\in {}^33, v\equiv_i u} \chi_v^M\Big)$$

$$= \sum \{\chi_v^M : v \in {}^33, v \equiv_i u \text{ for some } u \in S\}.$$

So $P \models f(\mathsf{c}_i^A a) = t_S$. Hence it follows that

$$P \models \eta_i(f(a), f(\mathsf{c}_i^A a)).$$

Conversely, if $P \models \eta_i(f(a), b)$, we require that $b = f(\mathsf{c}_i a)$. Towards this end, let S be the unique subset of ^{33} such that

$$P \models \bigwedge_{u\in S} f(a)\cdot 1_u \neq 0 \wedge \bigwedge_{u\in {}^33\smallsetminus S} f(a)\cdot 1_u = 0.$$

So we obtain

$$b = t_S = f(\mathsf{c}_i^A a).$$

We have proved that P is interpretable in A. [16] Next, we extract an elementary subalgebra B of A that is not a neat reduct i.e. B is not in Nr_3CA_4. This will imply that $B \notin Nr_3CA_\beta$ for all $\beta \geq 4$, since for any such β it is easy to see that $Nr_3CA_\beta \subseteq Nr_3CA_4$. Let $Id \in {}^33$ be the identity map on 3. Choose any countable elementary boolean subalgebra of $bool(A_{Id})$, B_{Id} say. Thus $B_{Id} \preceq bool(A_{Id})$. By the Feferman-Vaught theorem [67] corollary 9.6.5

$$Q = ((B_{Id} \times \prod_{u\in {}^33\smallsetminus Id} bool(A_u)), 1_u, d_{ij})_{u\in {}^33, i,j<3} \preceq$$

$$(\prod_{u\in {}^33} bool(A_u)), 1_u, d_{ij})_{u\in {}^33, i,j<3} = P.$$

Let B be the result of applying the interpretation given above to Q. By [67] Theorem 5.3.2 and Remark 3, $B \equiv A$ as cylindric algebras. Indeed $B \preceq A$. Therefore $B \in RCA_3$. In fact, it is easy to see that B is simple (since simplicity is expressible by a

[16] Furthermore it is easy to see that the interpretation is one dimensional and quantifier free in the sense of [67].

first order formula) so that B is isomorphic to a Cs_3. Now we show that B cannot be a neat reduct, in fact we show that $B \notin Nr_3CA_\beta$ for any $\beta > 3$. Assume, seeking a contradiction, that $B = Nr_3D$ for some $D \in CA_\beta$ with $\beta > 3$. Note that D may not be representable. Now $\chi_u^M \in B$ for each $u \in {}^33$. As is common practise, identifying functions with sequences, we let

$$v = \langle 1, 0, 2 \rangle \in {}^33.$$

Let $t(x)$ be the unary CA_2 term $\mathsf{s}_1^0\mathsf{c}_1x \cdot \mathsf{s}_0^1\mathsf{c}_0x$, where $\mathsf{s}_i^jx = \mathsf{c}_i(\mathsf{d}_{ij} \cdot x)$ for $i \neq j$. Then we claim that

$$t^B(\chi_v^M) = \chi_{Id}^M.$$

For better readability, in our calculations we omit the superscript B. For the sake of brevity, we denote χ_v^M by 1_{10} and χ_{Id}^M by 1_{01}. Then, by definition, we have

$$t(1_{01}) = \mathsf{c}_0(\mathsf{d}_{01} \cdot \mathsf{c}_1 1_{10}) \cdot \mathsf{c}_1(\mathsf{d}_{01} \cdot \mathsf{c}_0 1_{10}).$$

Computing we get

$$\mathsf{c}_0(\mathsf{d}_{01} \cdot \mathsf{c}_1 1_{10}) = \mathsf{c}_0(\mathsf{d}_{01} \cdot (\sum\{1_u : u \equiv_1 1_{10}\})$$

$$= \mathsf{c}_0(d_{01} \cdot 1_{112}) = 1_{01} + 1_{112}.$$

Here 1_{112} denotes $\chi_{\langle 1,1,2 \rangle}$. Note that we are using that the evaluation of the term $\mathsf{c}_1 1_{10}$ in B is equal to its value in A. This is so because B inherits the interpretation given to $\prod A_u$. A similar computation gives

$$\mathsf{c}_1(\mathsf{d}_{01} \cdot \mathsf{c}_0 1_{01}) = 1_{002} + 1_{01},$$

where 1_{002} denotes $\chi_{\langle 0,0,2 \rangle}$. Therefore, as claimed,

$$t^B(1_{10}) = 1_{01}.$$

Now let ${}_3s(0, 1)$ be the unary substitution term as defined in [50] 1.5.12, that is

$${}_3\mathsf{s}(0, 1)x = \mathsf{s}_0^3\mathsf{s}_1^0\mathsf{s}_3^1(x).$$

Then for any $\beta > 3$ we have

$$CA_\beta \models {}_3\mathsf{s}(0, 1)\mathsf{c}_3x \leq t(\mathsf{c}_3x).$$

Indeed, by [50] 1.5.12, 1.5.8 and 1.5.10 (ii), we get

$${}_3\mathsf{s}(0,1)\mathsf{c}_3x \leq {}_3\mathsf{s}(0,1)\mathsf{c}_1\mathsf{c}_3x = \mathsf{s}_0^3\mathsf{s}_1^0\mathsf{s}_3^1\mathsf{c}_1\mathsf{c}_3x = \mathsf{s}_0^3\mathsf{s}_1^0\mathsf{c}_1\mathsf{c}_3x$$

$$= \mathsf{s}_0^3\mathsf{s}_1^0\mathsf{c}_3\mathsf{c}_1x = \mathsf{s}_0^3\mathsf{c}_3\mathsf{s}_1^0\mathsf{c}_1x = \mathsf{c}_3\mathsf{s}_1^0\mathsf{c}_1x = \mathsf{s}_0^1\mathsf{c}_1\mathsf{c}_3x.$$

Similarly

$${}_3\mathsf{s}(0,1)\mathsf{c}_3x \leq \mathsf{s}_1^0\mathsf{c}_0\mathsf{c}_3x.$$

Therefore

$${}_3\mathsf{s}(0,1)\mathsf{c}_3x \leq t(\mathsf{c}_3x).$$

It thus follows that

$$D \models {}_3\mathsf{s}(0,1)(1_{10}) \leq \mathsf{s}_1^0\mathsf{c}_1 \cdot \mathsf{s}_0^1\mathsf{c}_0(1_{10}) = 1_{01}.$$

Now ${}_3\mathsf{s}(0,1)$ preserves $\leq$ and permutes Nr_3D. In fact ${}_3\mathsf{s}(0,1)$ is a boolean automorphism of Nr_3D, with inverse ${}_3\mathsf{s}(1,0)$. Now let

$$B_v = \{x \in B : x \leq 1_{10}\}$$

and

$$A_v = \{x \in A : x \leq 1_{10}\}.$$

Then $B_v = A_v$, and it follows (by property (4) of Lemma 2) that B_v is uncountable. Since ${}_3\mathsf{s}(0,1)$ is one to one, it follows that

$${}_3s(0,1)B_v = \{{}_3s(0,1)x : x \in B_v\}$$

has the same cardinality as B_v. Now let $x \in B_v$. Then $x \leq 1_{10}$. Since ${}_3\mathsf{s}(0,1)$ preserves order we have

$${}_3\mathsf{s}(0,1)x \leq_3 \mathsf{s}(0,1)1_{10} = 1_{01}.$$

Thus

$${}_3\mathsf{s}(0,1)B_v \subseteq B_{Id} = \{b \in B : b \leq 1_{01}\},$$

and so it follows that B_{Id} is uncountable too. But by construction, we have that $B_{Id} = \{b \in B : b \leq 1_{01}\}$ is countable. This contradiction shows that $B \notin Nr_3CA_\beta$ for any $\beta > 3$. ■

We now conclude:

Theorem 2.1 . *Let $\beta > 3$. Then Nr_3CA_β is not closed under forming elementary subalgebras, hence is not elementary.*

Now before proving Lemma 2.2, we consider certain refinements of Theorem 1, obtained by narrowing the class CA_β.

Definition 2.4 . *Let Ws_β stand for the class of weak set algebras of dimension β. We recall from [51] that these are CA_β's whose units are of the form*

$$^{\beta}U^{(p)} = \{s \in {}^{\beta}U : |\{i \in \beta : s_i \neq p_i\}| < \omega\}$$

for some $p \in {}^{\beta}U$. Let β be an infinite ordinal. Then Cs_β^{reg} and Lf_β stand for the classes of regular set algebras, and locally finite algebras of dimension β, respectively. We recall from [51] Def.3.1.1 (viii) that $\mathcal{A} \in Cs_\beta$ with unit ${}^{\beta}U$ is regular if for all $x \in A$, for all $f, g \in {}^{\beta}U$, whenever $f \upharpoonright \Delta x \subseteq g$ and $f \in x$ then $g \in x$. $\mathcal{A} \in K_\beta$ is locally finite if Δx is finite for every element in A. Here Δx, more commonly referred to as the dimension set of x is the set $\{i \in \beta : \mathsf{c}_i(x) \neq x\}, cf.$[50].

Here I stands for the operation of taking all isomorphic copies. We recall from [51] that RCA_β denotes the class of representable CA_β. For $\beta > 2$, RCA_β is a variety that coincides with the class of all isomorphic copies of subdirect products of weak set algebras. From the above we immediately get:

Corollary 2.5 . *Let $\beta > 3$. Let $IWs_\beta \subseteq L \subseteq CA_\beta$. Then $Nr_3L = \{Nr_3A : A \in L\}$ is not elementary. In particular, Nr_3RCA_β is not elementary.*

Corollary 2.5 answers problem 18 on p. 312 in [52] but only for the finite dimensional case. The infinite version of Corollary 5 is proved in the next chapter and in [135].

The rest of this section is devoted to proving Lemma 2.2. The construction is model-theoretic, so we need to review some model-theoretic notions from [67].

Definition 2.6 .

(1) *Let L be a signature and D an L structure. The age of D is the class $\mathbf{K}$ of all finitely generated structures that can be embedded in D. A class $\mathbf{K}$ is the age*

of D if the structures in **K** *are up to isomorphism, exactly the finitely generated substructures of D.*

(2) *Let* **K** *be a class of structures.*

(i) **K** *has the Hereditary Property, HP for short if whenever $A \in$* **K** *and B is a finitely generated substructure of A then B is isomorphic to some structure in* **K**.

(ii) **K** *has the Joint Embedding Property, JEP for short if whenever $A, B \in$* **K** *then there is a $C \in$* **K** *such that both A and B are embeddable in C.*

(iii) **K** *has Amalgamation Property, or AP for short if $A, B, C \in K$ and $e : A \to B$, $f : A \to C$ are embeddings, then there are D in* **K** *and embeddings $g : B \to D$ and $h : C \to D$ such that $g \circ e = h \circ f$.*

(3) *We call a structure D homogeneous if every isomorphism between finitely generated substructures extends to an automorphism of D.*

We recall from [67] Thm 7.1.2, a theorem of Fraïssé that puts the above definitions in the context we need.

Fact 2.7(Fraïssé) . *Let L be a countable signature and let* **K** *be a non-empty finite or countable set of finitely generated L-structures which has HP, JEP and AP. Then there is an L structure D, unique up to isomorphism, such that*

(1) *D has cardinality $\leq \omega$,*

(2) **K** *is the age of D, and*

(3) *D is homogeneous.*

Following Hodges [67], we refer to D as *Fraïssé limit* of the class **K**. The next theorem, also due to Fraïssé, gives a sufficient condition for when the Fraïssé limit D of a class **K** of finitely generated structures has quantifier elimination. This condition is that **K** is *uniformly locally finite*, a condition that is satisfied when the signature of **K** is finite and has no function symbols, which is the form we need. Recall that L-structure M

has *quantifier elimination* if every L formula $\phi(\bar{x})$ is equivalent in M to a boolean combination of quantifier free formulas. We now recall from [67] the form of Theorem 7.4.1 that we shall apply to construct our desired model.

Fact 2.8 (Fraïssé) . *Suppose that the signature L is finite and has no function symbols. Suppose that $\mathbf{K}$ is a countable set of finite L structures with HP, JEP and AP. Let M be the Fraïssé limit of $\mathbf{K}$. Then M has quantifier elimination.*

Now we have all the necessary machinery to prove Lemma 2.2.

Proof of Lemma 2.2.

Throughout the proof, we use the notation $\bar{x}, \bar{a}$ for finite sequences, or tuples $\langle x_0, \cdots x_{m-1}\rangle, \langle a_0, \cdots a_{m-1}\rangle$. Given a structure M and a tuple $\bar{a}$, we often write, with a slight abuse of notation, $\bar{a} \in M$ instead of $\bar{a} \in {}^mM$, where m is the length of the tuple $\bar{a}$. The length of tuples will be clear from context. Let $\mathcal{L}$ be the relational signature containing unary relation symbols $P_0, \ldots, P_3$ and a 4-ary relation symbol X. Let $\mathbf{K}$ be the class of all finite $\mathcal{L}$-structures D satsfying

$$(1) \qquad \forall x \bigvee_{i<j<4} (P_i(x) \wedge \bigwedge_{j\neq i} \neg P_j(x)).$$

$$(2) \qquad \forall x_0 \ldots x_3 (X(x_0, \ldots, x_3) \longrightarrow P_3(x_3) \wedge \bigvee_{u \in S_3} \chi_u).$$

Note that (1) simply says that the P_i's are disjoint. Then $\mathbf{K}$ contains countably many isomorphism types, because for each $n \in \omega$, there are countably many isomorphism types of finite L structures (satifying (1) and (2)) having cardinality $\leq n$. Also, it is easy to check that $\mathbf{K}$ is closed under substructures and that $\mathbf{K}$ has the AP. From the latter it follows that it has the JEP, since $\mathbf{K}$ contains the one element structure that is embeddable in any structure in $\mathbf{K}$ [17]. By Fact 2.7 there is a countably infinite homogeneous $\mathcal{L}$-structure $\mathcal{M}$ with age $\mathbf{K}$. By Fact 2. 8, $\mathcal{M}$ has quantifier elimination, and obviously, so does any elementary extension of $\mathcal{M}$. $\mathbf{K}$ contains structures with arbitrarily large P_3-part, so $P_3^{\mathcal{M}}$ is infinite. Let $\mathcal{M}^*$ be an elementary extension of $\mathcal{M}$ such that $|P_3^{\mathcal{M}^*}| = |L|$, and fix a bijection $*$ from the set of ternary relation symbols of

[17] It is not always true that AP implies JEP; think of fields.

L to $P_3^{\mathcal{M}^*}$. Define an L-structure M with domain $P_0^{\mathcal{M}^*} \cup P_1^{\mathcal{M}^*} \cup P_2^{\mathcal{M}^*}$, by: $P_i^M = P_i^{\mathcal{M}^*}$ for $i < 3$ and for ternary $R \in L$,

$$M \models R(a_0, a_1, a_2) \text{ iff } \mathcal{M}^* \models X(a_0, a_1, a_2, R^*).$$

If $\phi(\bar{x})$ is any L-formula, let $\phi^*(\bar{x}, \bar{R})$ be the $\mathcal{L}$-formula with parameters $\bar{R}$ from $\mathcal{M}^*$ obtained from ϕ by replacing each atomic subformula $R(x, y, z)$ by $X(x, y, z, R^*)$ and relativizing quantifiers to $\neg P_3$. In more detail, replace $(\exists x)\phi(x)$ and $(\forall x)\phi(x)$ by $(\exists x)(\neg P_3(x) \wedge \phi(x))$ and $(\forall x)(\neg P_3(x) \wedge \phi(x))$ respectively. A straightforward induction on complexity of formulas gives that for $\bar{a} \in M$

$$M \models \phi(\bar{a}) \text{ iff } \mathcal{M}^* \models \phi^*(\bar{a}, \bar{R}).$$

We show that M is as required, i.e. that M so constructed satisfies conditions (1) to (6) of Lemma 2.2 For quantifier elimination, if $\phi(\bar{x})$ is an L-formula , then $\phi^*(\bar{x}, \bar{R}^*)$ is equivalent in $\mathcal{M}^*$ to a quantifier free $\mathcal{L}$-formula $\psi(\bar{x}, \bar{R}^*)$. Then, replacing ψ's atomic subformulas $X(x, y, z, R^*)$ by $R(x, y, z)$, replacing all $X(t_0, \ldots t_3)$ not of this form by $\perp$, replacing subformulas $P_3(x)$ by $\perp$, and $P_i(R^*)$ by $\perp$ if $i < 3$ and $\top$ if $i = 3$, gives a quantifier free L -formula ψ equivalent in M to ϕ.

Now we check that property (2) holds. For this end, let

$$\sigma = \forall x(\neg P_3(x) \longrightarrow \bigvee_{i<3}(P_i(x) \wedge \bigwedge_{j \neq i} \neg P_j(x))).$$

Then $K \models \sigma$, so $\mathcal{M} \models \sigma$ and $\mathcal{M}^* \models \sigma$. It follows from the definition that M satisfies (2). Checking property (3) is similar.

For property (4), let $u \in S_3$ and let $r, s \in P_3^M$ be distinct. Take a finite $\mathcal{L}$-structure D with points $a_i \in P_{u_i}^D (i < 3)$ and distinct $r', s' \in P_3^D$ with

$$D \models X(a_0, a_1, a_2, r') \wedge \neg X(a_0, a_1, a_2, s').$$

Then $D \in K$, so D embeds into $\mathcal{M}$. By homogeneity, we can assume that the embedding takes r' to r and s' to s. Therefore

$$\mathcal{M} \models \exists \bar{x}(\chi_u \wedge X(\bar{x}, r) \wedge \neg X(\bar{x}, s)),$$

where $\bar{x} = \langle x_0, x_1, x_2 \rangle$. Since r, s were arbitrary and $\mathcal{M}^*$ is an elementary extension of $\mathcal{M}$, we get that

$$\mathcal{M}^* \models \forall yz(P_3(y) \wedge P_3(z) \wedge y \neq z \longrightarrow \exists \bar{x}(\chi_u \wedge X(\bar{x}, y) \wedge \neg(X(\bar{x}, z))).$$

The result for M now follows. Note that it follows from properties (3) and (4) of Lemma 2.2, that $P_i^M \neq \emptyset$ for each $i < 3$. So it is clear that

$$M \models \forall x_0 x_1 x_2 (\exists x_i \chi_u \longleftrightarrow \bigvee_{v \in {}^3 3, v \equiv_i u} \chi_v);$$

giving (5).

Finally consider (6). Clearly, it is enough to show that for any $\mathcal{L}$-formula $\phi(\bar{x})$ with parameters $\bar{r} \in P_3^M, u \in S_3, i < 3$, we have

$$\mathcal{M} \models \exists \bar{x}(\chi_u \wedge \phi) \longrightarrow \forall \bar{x}(\exists x_i(\chi_u \longrightarrow \exists x_i(\chi_u \wedge \phi)).$$

In order to simplify notation assume that $i = 2$. Let $\bar{a}, \bar{b} \in \mathcal{M}$ with

$$\mathcal{M} \models (\chi_u \wedge \phi)(\bar{a}) \text{ and } \mathcal{M} \models \exists x_2(\chi_u(\bar{b})).$$

We require

$$\mathcal{M} \models \exists x_2(\chi_u \wedge \phi)(\bar{b}).$$

It follows from the assumptions that

$$\mathcal{M} \models P_{u_0}(a_0) \wedge P_{u_1}(a_1) \wedge a_0 \neq a_1, \text{ and } \mathcal{M} \models P_{u_0}(b_0) \wedge P_{u_1}(b_1) \wedge b_0 \neq b_1.$$

These are the only relations on $a_0 a_1 \bar{r}$ and on $b_0 b_1 \bar{r}$ (cf. property (3) of Lemma 2), so

$$\theta^- = \{(a_0, b_0)(a_1, b_1)(r_l, r_l) : l < |\bar{r}|\}$$

is a partial isomorphism of $\mathcal{M}$. By homogeneity, this partial isomorphism is induced by an automorphism θ of $\mathcal{M}$. Let $\bar{c} = \theta(\bar{a}) = (b_0, b_1, \theta(a_2))$. Then $\mathcal{M} \models (\chi_u \wedge \phi)(\bar{c})$. Since $\bar{c} \equiv_2 \bar{b}$, we have $\mathcal{M} \models \exists x_2(\chi_u \wedge \phi)(\bar{b})$ as required. ■

By proving Lemma 2.2, our proof of Theorem 2.1 and Corollary 2.5 is now complete.

Remarks

(i) Let SC stand for the class of Pinter's substitution algebras. Let $\beta > 3$. In chapter two we proved that Nr_3SC_β is not closed under forming subalgebras. It follows from the above construction that Nr_3SC_β is not closed under forming *elementary* subalgebras. This can be seen by passing to reducts. That is, let $Rd_{SC}C$ be the SC reduct of a given $C \in CA_3$. Then $Rd_{SC}A \in Nr_3SC_\omega$ and $Rd_{SC}B \notin Nr_3SC_4$. In fact, the complete analogue of Corollary 3 holds for SCs, since reducts of weak set algebras are weak set algebras. However, the construction adopted herein (in its present form) does not settle the QA (quasipolyadic) and QEA (quasipolyadic equality) cases. This follows from the fact that the term ${}_3s(0,1)$ which played the key role in showing that B is not a neat reduct, is a basic operation in these algebras, whereas the proof of Theorem 1 depended on the fact that this term is not even term definable in cylindric algebras. The reader interested in extending theorem 2.1 to the quasipolyadic case, could go back to chapter two where it is proved that when K is QA or QEA, and $\beta > 2$, then Nr_2K_β is not first order axiomatizable.

(ii) It seems likely that property (3) in Lemma 2.2 can be strengthened to show that the relations in question are *disjoint* rather than distinct. In this case, the resulting algebras A and B would be atomic and completely representable in the sense of [54]. Also, for each $u \in S_3$, the set of atoms below 1_u would be an *uncountable splitting* of 1_u in the sense of [3]. That is, letting AtA denote the set of atoms of A , we get that for all $u \in S_3$ and for all $x \in AtA$ such that $x \leq 1_u$, x is *cylindrically equivalent* to 1_u, formally

$$(\forall i < 3)(\mathsf{c}_i x = \mathsf{c}_i 1_u = \sum\{1_v : v \equiv_i u\}.)$$

Andréka [3] refers to such atoms as "big atoms". The method of splitting elements (which is an instance of Henkin's method of *dilation* in the sense of 3.2.69 of [51], see also [147], chapter 4) [18]) was used by Andréka in [3] to show that for $\omega > n > 2$ and $\beta > n+1$ the class SNr_nCA_β cannot be axiomatized

[18]Henkin's own methods of splitting, introduced in [50] lemma 2.6.12 to construct non-representable CAs , is different than Andréka's splitting.

by a set of universal formulas containing only finitely many variables.

(iii) We have proved that first order logic cannot distinguish between the algebras $A \in Nr_3CA_\omega$ and $B \notin Nr_3CA_4$. It actually follows from our construction that stronger logics like $L_{\kappa\omega}$, where κ is a regular cardinal, also cannot distinguish between the algebras A and B.

The following Corollary addresses closure of neat reducts under forming special subalgebras.

Corollary 2.9 . *Let $2 < n < \omega$. Then we have*

(i) The class $S_cNr_nCA_\omega$ is not elementary.

(ii) The following inclusions are proper

$$Nr_nCA_\omega \subset S_cNr_nCA_\omega \subset SNr_nCA_\omega$$

Proof. For a class K we let ElK stand for the least elementary class containing K. Let $2 < n < \omega$. Let C_n be the countable cylindric algebra used in Theorem 1.4 above. By [53] Theorem 34 C_n is *elementary equivalent* to a completey representable algebra. It follows from the characterization established in Theorem 1 that $C_n \in ElS_cNr_nCA_\omega \smallsetminus S_cNr_nCA_\omega$. This proves (i) and takes care of the second inclusion of (ii), since C_n is representable. The first inclusion of (ii) follows from Theorem 1.2. because of the following: $A \in Nr_nCA_\omega$, B is a complete elementary subalgebra of A, and $B \notin Nr_nCA_{n+1}$. □

Concluding Remarks

(i) Problem 2.11 in [50] asks whether for $1 < \alpha < \beta$ the class $Nr_\alpha CA_\beta$ is closed under forming subalgebras. In [104], Németi proves that it is not. In Theorem 1.2 we sharpen Nemeti's result by showing that the class $Nr_\alpha CA_\beta$ with $1 < \alpha < \beta \cap \omega.$, is not even closed under elementary subalgebras, i.e. is not elementary. (Being closed under ultraproducts, this is equivalent to that the class $Nr_\alpha CA_\beta$ for $1 < \alpha < \beta$ is not closed under ultraroots.) The significance of the closure of the class of neat reducts under forming subalgebras in connection to interpolation properties in the corresponding logics is investigated in chapters two, three and four.

(ii) We note that the counterexamples of Andreka [4], Hodkinson [65] and Hirsch and Hodkinson [53], used in the proof of Theorem 3 to violate OTT, also show that Vaught's Theorem on existence of atomic (or prime) models for countable atomic theories fails for L_n when $\omega > n > 2$. In fact, Hirsch and Hodkinson prove that a cylindric algebra has a complete representation if and only if it has an atomic representation. (cf. Theorem 5 on p. 819 of [53]). They also show that the class RCA_n for $n > 2$ is not *atom-canonical* in the sense of Venema [152]. Conversely, when $n \leq 2$ RCA_n is closed under minimal completions and is atom-canoincal.

(iii) Let V be a variety of completely additive boolean algebras with operators. Let $SigV$ denote the signature of V without the boolean operations. A completely additive variety is one for which the the non-boolean operations in $SigV$ distribute over arbitrary joins. Examples include the so-called *conjugated varieties* in the sense of [50], and more concretely RCA_α, α finite, and the class of representable relation algebras, or RRA for short, cf. [65]. Now let $A \in V$ be atomic. Then the *atom structure* of A is the structure

$$(AtA, R_f)_{f \in SigV}$$

where R_f is an $n+1$ ary relation defined for each n-ary $f \in SigV$ by: For all atoms $a_1 \cdots a_n, b \in A$

$$R_f(a_1, a_n, b) \longleftrightarrow f(a_1 \cdots a_n) \leq b.$$

$At(V)$ is the class of all such structures. More generally an *atom structure* or a *Kripke frame* of type $Sig(V)$ is a pair $\mathcal{F} = (F, R_f)$ where F is a set and R_f is an $n+1$ ary relation symbol for each n-ary function $f \in SigV$. Given an atom structure $\mathcal{F}$, the complex algebra $Cm\mathcal{F}$ is the algebra with universe $P(F)$ for each $n - ary$ $f \in SigV$ and $X_1 \cdots X_n \in P(\mathcal{F})$,

$$f(X_1 \cdots X_n) = \{a \in \mathcal{F} : \mathcal{F} \models R_f(a_1, \cdots a_n, b)$$

for some $a_1 \in X_1, \cdots a_n \in X_n\}$.

$Str(V)$ is the (possibly smaller) class of atom structures of full complex algebras in V. That is $\mathcal{F} \in StrV$ iff $Cm\mathcal{F}$ is in V. It is known that both $At(V)$ and $Str(V)$ are closed under ultraroots, cf. [58]. The examples of Hodkinson and Andreka in [4] and in [65] referred to above show that for $3 \leq n < \omega$ and $V \in \{RRA, RCA_n\}$, $At(V) \neq Str(V)$. While $At(V)$ is closed under ultraproducts as well- hence is an elementary class a result of Venema - a recent result of Hirsch and Hodinson [58] shows that $Str(RRA)$, the so called *strongly representable* atom structures of relation algebras is not closed under ultraproducts, hence is not elementary. In [58] it is proved that there exists a sequence of strongly representable atom structures whose ultraproduct is (only weakly representable but) not strongly representable. The analagous question for RCA_n with $4 \leq n < \omega$ makes sense, and to the best of our knowlege is still open.[19]

(iv) Let $MA_c(n, \kappa)$ be as in Theorem 2 and let $covK$ be the cardinal defined in the remark following Theorem 2. Let $n < \omega$. Then the statement

$$(\forall l < covK)(MA_c(n, l))$$

is an omitting types theorem for L_n, while the statement

$$(\forall l < covK)(MA_c(\omega, l))$$

is an omitting types Theorem for the logics studied in [9] [150] and [51] sec 4.3. These are referred to as *typless* logics in [51] sec 4.3. Both statements are provable in ZFC. In the former (finite dimensional) case, models omitting types are standard models i.e. they are squares, while in the latter (infinite-dimensional) case, models, i.e, the units of the set algebras are non-square (in the sense of [150]); they are weak cartesian spaces. An interesting question here, is when (other than the case of $L_{\omega\omega}$), can we "square" the units provided by (*) in Theorem 1 when $n = \omega$?

[19] An unpublished result of the author shows that $Str(RCA_3)$ is not elementary.

(v) The characterization established in Theorem 1.2 herein works for all algebras studied in chapter two. These include Halmos' quasipolyadic algebras and Pinter's substitution algebras [109], and various reducts and extensions thereof. We note that Sain's algebras, or SA_ω for short, studied in chapter three have the strong neat embeding property. This can be easily destilled from our main Theorem in chapter three. Moreover, (in contrast to the CA case) atomic algebras are *completely* representable. The same result holds for Halmos' polyadic algebras without equality. (This is not proved in chapter three, but can be found in [141]). Also Sain's algebras can be axiomatized by *Sahlqvist* equations, so that (unlike the CA case, [65]) the (complete) representabilty of an atomic algebra in SA_ω can be pinned down to its atom structure. That if for algebras $\mathcal{A}, \mathcal{B}$ having the same signature as SA_ω, if their boolean reducts are atomic boolean algebras, and $At(\mathcal{A}) \cong At(\mathcal{B})$, then $\mathcal{A} \in \mathcal{SA}_\omega$ implies that $\mathcal{B} \in SA_\omega$. In particular, SA_ω is atom-canonical. So these algebras are less complex than $(R)CA_\omega$ in many respects. Beth-definability was proved for the corresponding logic in the previous chapter. This logic was introduced as one possible solution to the so called *Finitization Problem* [109] in algebraic logic. Omitting types for this logic is investigated in [133].

(vi) Besides the confirmation of Conjecture four, we find the following an interesting (open) question: Is the strong neat embedding property stronger than the neat embedding property for the infinite dimensional case? That is, is the second inclusion of Corollary 5 (ii) proper when $n = \omega$? $\diamondsuit$

Futher work

(1) Let $n \geq 4$. Hajnal Andréka constructs a countable symmetric integral (hence simple) representable relation algebra R_n such that its completion, i.e. the complex algebra of its atom structure is not representable, with the additional property that the (countable) set B_nR_n of all n by n basic matrices over R_n constitutes an n-dimensional cylindric basis in the sense of Maddux [84] Definition 4, p. 953 , so that B_nR_n is a cylindric atom structure. This means that the full complex algebra $Ca(B_nR_n)$ with universe the power set of B_nR can be

turned into an n dimensional cylindric basis in a natural way. Since R_n is not completely representable, then $Ca(B_nR_n)$, sure enough a CA_n is not a representable one. However the *term algebra* over the atom structure B_nR_n, which is the subalgebra of $Ca(B_nR_n)$ generated by the countable set of n by n basic matrices, $Tm(B_nR_n)$ for short, is a countable representable CA_n, and further it is simple so that it is in fact (isomorphic to) a Cs_n. $Tm(B_nR_n)$ is representable, but it does not have a complete representation. (If it did, then R_n will also have a complete representation, which is not the case.) Now let $3 \leq n < m$. Then Andreka further shows that $Tm(B_nR_n)$ embeds densely and neatly in $Tm(B_mR_m)$. Thus $Tm(B_nR_n)$ is an example of a countable atomic algebra in $S_cNr_nRCA_m$, with no complete representations, confirming conjecture 4. Since the algebra constructed by Andreka is binary generated i.e generated by its two dimensional elemnents, it follow that the diagonal free reduct of this algebra is not completely representable. This gives that OTT fails for the equality free version of L_n, as well, and the multi-modal logic $S5^n$ studied in [151]. [20]

(2) In a joint work with Robin Hirsch [64], we provide an example showing that countability is indeed needed in the characterization of the completely representable RAs and CAs given in Theorem 1. We construct an uncountable RA in $RaCA_\omega$ that has an ω-dimensional amalgamation base in the sense of [84] Definition 3, p.953, (which roughly means that the the set of ω by ω basic matrices can be glued together to give a CA_ω of which the original algeba is an RA reduct). In particular it has an n-dimensional cylindric base for every finite $n \in \omega$, which gives an uncountable $A_n \in Nr_nCA_\omega$ for each $3 \leq n < \omega$ with no complete representation. This algebra is a simplified version of the rainbow relation algebra $A_{R,Q}$, defined in [60]p. 270 where R denotes the set of ordered real numbers and Q the ordered rationals. [21] The author has used this example to provide $L_{\omega\omega}$ with an uncountable *atomic* theory having no atomic models,

[20] What is multi-modal logic or cylindric modal logic? It has n modalities $\diamond_0, \cdots \diamond_{n-1}$ and a countable set of propositional variables. Formulas are built from the propositional variables using conjunction, disjunction, negation and diamonds the usual way. Frames for n modal logic, or n frames for short, are first order structures of the form $\mathcal{F} = (F, R_0 \cdots R_{n-1})$ where R_i is a binary relation on F. A model on an n-frame $\mathcal{F}$ is a pair $\mathcal{M} = (\mathcal{F}, v)$ where

$$v\omega \to \wp(F).$$

The inductive definition of a formula ϕ true at point $x \in F$ is the usual one. That is:

$$\mathcal{M}, x \models \diamond_i \phi \text{ iff } \exists y (x R_i y \land \mathcal{M}, y \models \phi.$$

Given an n-formula ϕ and a frame $\mathcal{F}$ we say that ϕ is satisfiable in $\mathcal{F}$ if $\mathcal{M}, x \models \phi$ for some $\mathcal{M}$ based on F and some $x \in F$. ϕ is valid if $\mathcal{M}, x \models \phi$ for all such x and $\mathcal{M}$. $\mathcal{F}$ is a frame for a set L of modal formulas if all formulas of L are valid in $\mathcal{F}$. L is called a Kripke complete n-modal logic if there is some class C of frames such that every formula in L is valid in every frame in C. We also say that L is the logic for C. Well known complete unimodal logics are K and $S5$ the logic of (W, R) where R is an equivalence relation. Special n frames are *product* frames. These are defined as follows: Given n-frames, $\mathcal{F}_0 = (F_0, R_0) \cdots \mathcal{F}_{n-1} = (F_{n-1}, R_{n-1})$, their product is $\mathcal{F}_0 \times \mathcal{F}_{n-1}$ is defined to be the relational structure

$$W_0 \times W_{n-1}, (R_0)*.$$

where $(R_0)*$ is the relation:

$$u_i R v_i$$

iff $u_0 R v_0$ and $u_k = v_k$. Define a product of logics to be the logic corresponding to product of frames. Thus K^n is the logic corresponding to the product any unimodal logic while $S5^n$ corresponds to the produst of $S5$ and the complex algebras of its Kripke frames are just RDf_n. By Andreka's result omitting types fail for this logic as well.

[21] In this particular example we do not need that Q is dense in R, in fact this algebra can be just as well constructed from the graphs ω and ω_1)

giving a different proof of the result of Biro and Shelah [30]. (This theory is roughly the algebraic counterpart of the direct limit of the A_n's which is an uncountable $Cs_\omega^{reg} \cap Lf_\omega$.)

(3) Let $n \geq 4$. Let B_n be the class of relation algebras having n-relational basis in the sense of Maddux [84]. Then B_n coincides with the class CRA_n of relation algebras having n-square complete representations in the sense of [55]. In [55] it is pointed out that CRA_n is an n-analogue of the class CRA of completely representable relation algebras. Also it is proved that the class CRA_n, $n \geq 5$ is not elementary. By Theorem 1.2 S_cRaCA_ω and CRA the class of completely representable relation algebras coincide on countable algebras. In view of this characterization, a candidate - other than the class CRA_n- for an n-approximation or n-analogue of CRA is the class of atomic algebras in S_cRaCA_n, obtained by truncating the dimension ω to n. It turns out in [64], that for $n \geq 4$, this class coincides with the class of relation algebras having n-dimensional hyperbasis in the sense of [57]. Now let $RA_n = SB_n$. While RA_n and $SRaCA_n$ are n-analogues of RRA, CRA_n and atomic algebras in S_cRaCA_n are n-analogues of the class of completely representable relation algebras. Recently we prove jointly with Robin Hirsch [63], that like CRA_n, $n \geq 5$, the class of relation algebras having an n-dimensional hyperbasis is not elementary either. In [63] we construct an algebra $B \notin RaCA_n$ such that an ultrapower of B is in $RaCA_n$. [22] This shows that $RaCA_n$, the class of relation algebra reducts of CA_n, $n \geq 4$ is not closed under (elementary) subalgebras. When $n = \omega$ the proof is the same as that of Hirsch and Hodkinson in [53] showing that the class of completely representable relation algebras is not elementary. In passing we note that the class $RaCA_n$ for $n \geq 3$ was proved not be closed under forming subalgebras by Németi and Simon [111] and independently by Roger Maddux [82].

[22] We also show that this B has an m-dimensional cylindric basis for every $m > 4$, from this we give a different proof to Theorem 1.2. The proof seems to generalize to QA and QEA's but we did not check that.

(4) Let $1 < \alpha < \beta$. It is not hard to show that the class $Nr_\alpha CA_\beta$ is a *psuedo-elementary* class, i.e a PC_Δ class in the sense of [67] p.207. A psuedo-elementary class is roughly a reduct of an elementary class. Let $ElNr_\alpha CA_\beta$ denote the *elementary closure* of $Nr_\alpha CA_\beta$, i.e $ElNr_\alpha CA_\beta = UpUrNr_\alpha CA_\beta$ where Up and Ur stand for the operations of forming ultraproducts and ultraroots respectively. By Theorem 1 in chapter 2, Theorem 1 in [135] (and in the next chapter) and Theorem 1.2 herein, all the following inclusions are proper for $1 < \alpha < \beta$:

$$Nr_\alpha CA_\beta \subset ELNr_\alpha CA_\beta \subset SNr_\alpha CA_\beta.$$

In particular, $ELNr_\alpha CA_\beta$ is not closed under forming subalgebras hence is not (psuedo-) universal. By [59] one can synthesize (and consequently obtain) an axiomatization of the elementary class $ELNr_\alpha CA_\beta$ by games, in the spirit of Robinson's finite forcing in Model theory. It seems plausible that the class $ELNr_\alpha CA_\beta$ is finitely axiomatizable over the variety $SNr_\alpha CA_\beta$. The latter is finitely axiomatizable if and only if $\beta \leq \alpha + 1$, a result of Andreka[3]. In particular, we conjecture that $ElNr_\alpha CA_\beta$ is finitely axiomatizable if and only if $\beta \leq \alpha + 1$.

Open questions

(1) Let $n \geq 4$. Is the class of strongly representable stom structures of 4-dimensional cylindric algebras, or $Str(RCA_4)$ for short, an elementary class? Equivalently, is it closed under elementary equivalence? Is it set theoretically absolute?

It can be shown that for $\omega > n > 2$, $Str(RCA_n)$ is not finitely axiomatizable in first order logic nor in $L^\omega_{\infty\omega}$. In [54] on complete representations, there is a set of first order axioms characterizing the elementary closure of the class of completely representable cylindric algebras. All quantifiers in these conditions are already relativized to atoms, so they can be easily re-written as conditions on cylindric algebra atom structures. It can be shown that any cylindric atom structure satisfying these conditions is in $Str(RCA_n)$. The example of Hodkinson in [65] shows that the converse fails.

(2) Let V be a variety of boolean algebras with operators. Recall that $At(V)$

consists of those atom structures $\mathcal{F}$ for which there exists $\mathcal{A} \in \mathcal{V}$ such that $At\mathcal{A} = \mathcal{F}$. If V is further a completely additive variety, then it is not hard to check that $Str(V)$ consists of all those atom structures $\mathcal{F}$ such that $\mathcal{A} \in V$ *for all* $\mathcal{A}$ based on $\mathcal{F}$. Thus, generally we have $StrV \subset AtV$. A natural question in this context is: for which varieties of boolean algebras with operators V, is $Str(V)$ an elementary class? Closed under ultraroots, as proved by Goldblatt in e.g [46], this amounts to asking when is it closed under ultrapowers, or equivalently under elementary equivalence?

Partial results in this direction are known. One is that when V is atom- canonical in the sense of Venema [152], another is that when V is Sahlqvist, i.e can be characterized by Sahlqvist equations. In these two cases we actually have $AtV = StrV$. Since AtV is always elementary, a result of Venemna [152] the desired result follows. A recent generalization due to Goldblatt is that $StrV$ is still equal to AtV, hence is elementary, when V is only *single-persistent* [46]. Single persistance and atom-canonicity coincide in the case of completely additive varieties of boolean algebras with operators, but are generally distinct.

(3) More generally, a long standing open problem in the literature of both modal logic and algebraic logic is the following: Is every canonical variety V, i.e a variety closed under perfect extentions, generated by the complex algebras of an elementary class of frames? That is, is it true that if V is canonical, then there exists an elementary class K of frames such that V is the variety generated by CmK? This is an important question in modal logic. Equivalently it can be rephrased by asking whether any canonical modal logic (one validated by its canonical frames) is characterized by an elementary class of Kripke frames, cf. [45]. (The converse - a classical result in the literature of modal logic- [**?**] was proved by Van Banthem and independently by Goldblatt. In fact Van Banthem proves that if K is closed under ultraproducts then the varietry generated by CmK is canonical.) We note that in a series of publications, Goldblatt gives a partial anwser to this question. When $AtV = StrV$, then the (canonical) variety in question is generated by the elementary class $StrV$, since it is always the case that AtV is elementary and $HSPSCmStrV \subseteq V$. In contrast, it is

not always true that $CmAtV \subseteq V$. Natural examples are RRA and RCA_n, $n > 3$. These varitieties are not single-persistent, least atom-canonical. In this case $Str(V)$ would be a candidate. But this class is not always elementary. This is shown by Hirsch and Hodkinson in [55] for RRA and by the present author in [137] for RCA_3. However, for $V \in \{RRA, RCA_n\}$, the canonical vareities V *can be* generated by an elementary class of frames that lie between $At(V)$ and $Str(V)$. In fact, several such classes are exhibited for the class of representable cylindric and relation algebras in [60]. But this means that the condition of single persistance, introduced by Goldblatt, is only sufficient.

(4) By the construction of [65] and [3], it follows there is an atomic RCA_3, such that $CmAtA \notin RCA_3$. What is the least k such that $CmAtA \notin SNr_3CA_{3+k}$?

(5) Due to Németi's result in [105] recalled in the previous chapter it turns out that canonicity is related to strong amalgamation which in turn is related to interpolation. The results in this chapter seem to point out that atom-canonicity is related to omitting types. Roughly, we have shown that when V is not atom-canonical, then the omitting types Theorem fails for the corresponding logic. It would be interesting to work out a context that relates the logical property of omitting types to the algebraic one of atom-canonicity.

(6) Is RCA_ω closed under completions? In particular, is it single-persistent (equivalently) atom-canonical? Is it Sahlqvist?. Unpublished results of Andreka and Givant and Nemeti show that not every atomic RCA_ω has a complete representation.

(7) It is known that the class of cylindric algebras of dimension $n > 2$ possesing homogeneous representation in the sense of [53] is not closed under substructures. Let us denote this class by $HRCA_n$. It is not hard to show that $Nr_nCA_\omega \subseteq HRCA_n$ and that the inclusion is proper. Is $HRCA_n$ closed under elementary substructures? Recently Robin Hirsch proved that $RaCA_\omega$ coincides with algebras having 2 homogenous representations.

Chapter 6

The class of neat reducts is not elementary

Abstract

In this final chapter we show that for any pair of infinite ordinals $\alpha < \beta$, the class $Nr_\alpha CA_\beta$ is not elementary. This extends our Theorem 2.1 in the previous chapter to infinite dimensions.

Introduction

There are problems in which various classes of cylindric algebras - or rather formalized theories of such classes - appear not as tools, but as subjects of metamathematical investigations. Given a class of algebras (e.g. the class of neat reducts) the formalized theories which we discuss are the elementary theory or equational theory; the former is the set of all first order sentences, i.e sentences formulated in first order logic, which hold in every algebra of the given class, while the latter is the narrower set of all equations which hold in every such algebra. In chapters two, three, four, and five we discussed the closure of the class of neat reducts as tools in connection to interpolation properties and omitting types, cf. [145], [130],and [136]. But in chapters two and five we also had the occasion to study the class of neat reducts in its own right. We showed that the class of neat reducts is not a variety, i.e. cannot be axiomatized by any set of equations. We proved this by showing that it is not closed under forming subalgebras.

In chapters two and four, we showed that the class of *finite* dimensional neat reducts is not elementary as long as the dimension is > 1. This was done by proving that this class is not even closed under elementary subalgebras. In this chapter, we return to the metamathematical investigation of the class of neat reducts. We extend our results, concerning the closure of the class of neat reducts under forming elementary suybalgebras, in chapter two and five to infinite dimensions. Recall that CA_β denotes the class of cylindric algebras of dimension β and Ws_β denotes the class of weak set algebras of dimension β. In this chapter we shall prove:

Theorem 1 (Main result) . *Let $\alpha < \beta$ be infinite and let $Ws_\beta \subseteq K \subseteq CA_\beta$. Then $Nr_\alpha K$ is not closed under forming elementary subalgebras, hence is not elementary.*

Our proof in chapter four for the finite dimensional case appealed to a Fraisse construction. Because an analogous Fraisse theorem is missing; we now construct our desired model (on which our algebras are based) in a standard step-by-step argument, which can be seen as an infinite analogue of Fraisse's Theorem for certain infinitary logics, the so-called finitary logics of infinitary relations. However the algebras constructed are weak set algebras, so that we retain the "finitistic" character of our proof. The rest of this chapter is devoted to proving Theorem 1. The work in this chapter is based on the article [135].

The proof of Theorem 1.

As usual, we start off by fixing the notation adopted throughout in this chapter. We try to use conventional notation. We introduce less usual needed notations at their first occurrence in the text. ω denotes the least infinite ordinal, which is also the set of all finite ordinals. More generally an ordinal is the set of smaller ordinals. Cardinals, as usual, are (identified with) initial ordinals. If X, Y are sets then ${}^XY$ denotes the set of all functions from X into Y. Composition of two functions f and g, $f \circ g$ is defined so that $f \circ g(x) = f(g(x))$. Id_X, or simply Id when X is clear from context, denotes the identity function on X. $|X|$ denotes the cardinality of X, and $\wp(X)$ denotes the set of all subsets of X. We write $X \subseteq_\omega Y$ to denote that X is a *finite* subset of Y. Let f be a function. Then $dom(f)$ denotes the domain of f and

$range(f)$ denotes its range. The following notation will be used extensively:

For a function f and $X \subseteq domf$ (the domain of f), $f \upharpoonright X$ and $f(X)$ denote *the restriction of f to X* and *the image of X under f*, respectively. We identify notationally algebras with their domains. Given an algebra A and $X \subseteq A$, then $Sg^A X$ or simply SgX, when A is clear from context, denotes the subalgebra of A generated by X. Let B be an algebra having the same similarity type as A. Then we write $f \in Ism(A, B)$ to abbreviate that f is an isomorphism from A into B. For $A \in CA_\alpha$, BlA denotes its boolean reduct.

Throughout this chapter α is a fixed arbitrary (infinite) ordinal $\geq \omega$ and R is an uncountable set with $|R| > |\alpha|$.

Notation. . *Let s be an α-ary sequence and $i < \alpha$. Then $s(i|w)$ denotes the α-ary sequence for which: $s(i|w)(i) = w$ and $s(i|w)(x) = x$ for $x \neq i$.*

For a set W, $w \in W$ and β an ordinal, we let

$$\mathbf{w} = \beta \times \{w\} = \langle w : i < \beta \rangle.$$

The arity of $\mathbf{w}$ will be clear from context.

The proof of the following Lemma is an easy step-by-step construction and is therefore omitted. However its proof follows from that of the stronger Lemma 6 formulated below. We recall that ${}^\alpha W^{(\mathbf{w})}$ denotes the weak space

$$\{t \in {}^\alpha W : |\{i \in \alpha : t_i \neq w\}| < \omega\}.$$

Lemma 1 . *There are a set W, $w \in W$ and α-ary relations $D_r \subseteq {}^\alpha W^{(\mathbf{w})}$ for all $r \in R$ such that conditions (i) and (ii) below hold:*

(i) (Bigness) For all $r \in R$, for all $i < \alpha$, for all $t \in {}^\alpha W^{(\mathbf{w})}$, there exists $y \in W \smallsetminus Rng(t)$ such that $t(i|y) \in D_r$.

(ii) (Disjointness) The D_r's are pairwise disjoint.

Notation . *Let $\mathbf{W} = \langle W, D_r \rangle_{r \in R}$ $w \in W$, and $D_r \subseteq {}^\alpha W^{(\mathbf{w})}$ as specified above. Let*

$$U = W \times \alpha = \cup_{i \in \alpha} W \times \{i\}.$$

U is $|\alpha|$ disjoint copies of W. Now let s be the following one to one sequence:

$$s = \langle\langle w, 0\rangle, \cdots \langle w, j\rangle \cdots\rangle_{j<\alpha}.$$

Then $s \in {}^{\alpha}U$. Let V be the weak space ${}^{\alpha}U^{(s)}$, and let C be the full Ws_{α} with unit V.

Next we define $B \subseteq A \in Ws_{\alpha}$ each with unit V, such that B is an elementary subalgebra of A, A is a neat reduct and B is not a neat reduct.

Defining our algebras

Let $\mathbf{W} = \langle W, D_r\rangle_{r\in R}$, $U = W \times \alpha$, $s \in {}^{\alpha}U$ and $V = {}^{\alpha}U^{(s)}$ be as specified above. Let S_{α} denote the set of all one to one functions (equivalently permutations) which agrees cofinitely with the identity function on α, that is S_{α} is the set of permutations in ${}^{\alpha}\alpha^{(Id)}$. For $u = \langle u_j\rangle_{j\in\alpha} \in S_{\alpha}$ and $r \in R$, we set

$$p(u, r) = \{\langle\langle a_j, u_j\rangle\rangle_{j<\alpha} \in V : \langle a_j\rangle_{j<\alpha} \in D_r\},$$

$$P(u) = \{p(u, r) : r \in R\}.$$

Notice that $P(u)$ is an uncountable set of α-ary relations. Let N be a fixed countable subset of R. That is $N \subseteq R$ and $|N| = \omega$. Let

$$P_{\omega}(u) = \{p(u, r) : r \in R\}.$$

The subscript ω is to indicate that $P_{\omega}(u)$ is a countable set of α-ary relations.

Now we define $A \in Ws_{\alpha}$ and $B \subseteq A$ as follows:

$$A = Sg^{C}(\cup\{P(u) : u \in S_{\alpha}\});$$

and

$$B = Sg^{C}((\cup\{P(u) : u \in S_{\alpha} \smallsetminus \{Id\}) \cup P_{\omega}(Id)).$$

Now we have our algebras. We shall show that B is an elementary subalgebra of A, that B is not a neat reduct , and finally that A is a neat reduct. Accordingly, we devide the proof to three parts. We should mention that the proofs in parts 1 and 2 are very similar to the proofs of parts 2 and 3 of Theorem 2 in chapter 2. The real hard work starts at part 3, where we formulate and prove a Fraisse theorem for certain infinitary logics.

Part 1. B **is an elementary subalgebra of** A.

Notation. .

Let $u = \langle u_0, u_1 \cdots u_j \cdots \rangle \in {}^{\alpha}\alpha^{(Id)}$. *Then* 1_u *denotes the following* α*-ary relation on* V*:*

$$\{\langle\langle a_0, u_0\rangle, \langle a_1, u_1\rangle \cdots \langle a_j, u_j\rangle \cdots\rangle_{j<\alpha} \in V : \langle a_0 \cdots a_j \cdots\rangle_{j\in\alpha} \in {}^{\alpha}W^{(\mathbf{w})}\}.$$

L *stands for* 1_{Id} *where* Id *is the identity function, i.e.*

$$L = \{\langle\langle a_0, 0\rangle, \langle a_1, 1\rangle \cdots \langle a_j, j\rangle \cdots\rangle_{j<\alpha} \in V : \langle a_j\rangle \in {}^{\alpha}W^{(\mathbf{w})}\}.$$

Claim 2 .

(i) For all $u \in S_\alpha$, $1_u \in A$.

(ii) Let $Rl_L A = \{x \in A : x \leq L\}$. *Then* $Rl_L A$ *is an atomic boolean algebra and its set of atoms is equal to* $P(Id)$. *Furthermore,* $Rl_L A$ *is generated as a boolean algebra by* $P(Id)$.

(iii) For all non-zero $a \in Rl_L A$, *for all* $i < \alpha$, $C_i a = C_i L$.

(iv) For all $a \in A$, *for all* $i \in \alpha$, $C_i a \cap L \in \{0, L\}$.

(v) Let $P = \cup\{P(u) : u \in S_\alpha\}$. *Let* p *be a permutation of* $P(Id)$. *Then there exists an automorphism* f *of* A *such that* $p \subseteq f$ *and* $f(x) = x$ *for each* $x \in P \smallsetminus P(Id)$.

Proof.

(i) Let $u \in S_\alpha$. Let $r_1, r_2 \in R$. Then by Lemma 1 (i), we have $C_0 p(u, r_1) \cap C_1 p(u, r_1) = 1_u$.

(ii) Since the D_r's are pairwise disjoint, we have $p(Id, r_1) \cap p(Id, r_2) = \emptyset$ for distinct r_1 and r_2. Thus the set of atoms of $Rl_L A$ is precisely $P(Id)$. By $C_i(p(Id, r)) \cap 1_{Id} = 1_{Id}$ and $p(Id, r) \cap D_{ij} = \emptyset$ for $i < j < \alpha$, it follows that $Rl_L A$ is indeed generated as a boolean algebra by the set $P(Id)$.

(iii) Let $a \in Rl_L(\alpha)$ be non-zero. By (ii) there exists an atom $p(Id, r)$, say, below a. Let $i < \alpha$. Then

$$C_i(p(Id, r)) = \cup\{1_u : u \in {}^\alpha\alpha^{(Id)} \text{ and } u(j) = j \text{ for all } j \neq i\} = C_iL.$$

From $p(Id, r) \leq a$, (iii) readily follows.

(iv) follows from (iii).

(v) Let p be a given permutation of $P(Id)$. By Claim 2 (ii), Rl_LA is generated by the set $P(Id)$ of atoms. It then follows that there exists a *boolean* automorphism p', say, of the boolean reduct of Rl_LA such that $p \subseteq p'$. For $x \in A$, define

$$f(x) = p'(x \cap L) \cup (x \cap -L).$$

We will show that f is the desired automorphism. Clearly f is a boolean automorphism of A such that $p' \subseteq f$ and $f \upharpoonright P(Id)$ is the identity function. So all we have to check is that f preserves the extra non-boolean cylindric operations.

(1) Diagonals

Let $i < j < \alpha$. Then

$$f(D_{ij}) = p(D_{ij} \cap L) \cup (D_{ij} \cap -L) = D_{ij}.$$

(2) Cylindrifications

Let $x \in A$ and $i < \alpha$. On the one hand we have:

$$C_if(x) = C_i[p'(x \cap L) \cup (x \cap -L)] = C_ip(x \cap L) \cup C_i(x \cap -L)$$

$$= C_i(x \cap L) \cup C_i(x \cap -L) = C_ix.$$

On the other hand,

$$f(C_ix) = p'(C_ix \cap L) \cup (C_ix \cap -L) = (C_ix \cap L) \cup (C_ix \cap -L) = C_ix.$$

We have shown that for all $x \in A$ and $i < \alpha$,

$$C_if(x) = f(C_ix) = C_ix.$$

The proof is now complete.

■

Claim 3 . *B is an elementary subalgebra of A.*

Proof. We shall use the Tarski-Vaught criterion to show that B is an elementary submodel (i.e. subalgebra) of A. So let $\exists x_n \phi(x_0, \ldots, x_n)$ be any first order formula of the language of CA_α. Suppose that

$$b_0, \ldots, b_{n-1} \in B \text{ and that } A \models \phi(b_0, \ldots b_{n-1}, a) \text{ for some } a \in A.$$

We want to find $b_n \in B$ such that

$$A \models \phi(b_0, \ldots, b_n).$$

Since A is generated by $P = \cup\{P(u) : u \in S_\alpha\}$, there exists a finite subset H_0, say, of P such that $a \in Sg^A H_0$. Let

$$X = \cup\{P(u) : u \in S_\alpha \smallsetminus \{Id\}\} \cup P_\omega(Id).$$

Since $b_0, \ldots b_{n-1} \in B$ and B is generated by X, then there exists a finite $H_1 \subseteq X$, such that

$$(H_0 \cap B) \cup \{b_0, \ldots, b_{n-1}\} \subseteq Sg^B H_1.$$

Let p be a permutation of $P(Id)$ such that

$$p \restriction P(Id) \cap H_1 \subseteq Id \text{ and } p(H_0 \smallsetminus H_1) \subseteq P(Id) \cap B.$$

Such a permutation clearly exists. By Claim 2 (v), p extends to an automorphism f of A such that

$$p \subseteq f \text{ and } f \restriction (P \smallsetminus P(Id)) = Id$$

but

$$A \models \phi(b_0, \ldots, b_{n-1}, a)$$

and so

$$A \models \phi(f(b_0), \ldots, f(b_{n-1}), f(a)).$$

By $f \restriction H_1 \subseteq Id$ and $\{b_0, \ldots b_{n-1}\} \subseteq Sg^B H_1$, it follows that $f(b_i) = b_i$ for all $i < n$. Since $f(H_0) \subseteq B$ and $a \in Sg^A H_0$ we get that $f(a) \in B$. This completes the proof. ■

Part 2. B is not a neat reduct.

We have kept only countably many atoms below 1_{Id} in B. In our next Claim, we show that the set of *all* elements in B below 1_{Id} remains $\leq |\alpha|$.

Claim 4 . *Let $Rl_L B = \{x \in B : x \leq L\}$. Then $|Rl_L B| \leq \alpha$.*

Proof. First of all note that by Claim 2 (i) we have $L \in B$. Now let

$$X = \cup\{P(u) : u \in S_\alpha \smallsetminus \{Id\}\} \cup P_\omega(Id),$$

$$Y = \{C_i a : a \in A\} \cup \{D_{ij} : i < j < \alpha\},$$

and

$$D = Sg^{BlA}(X \cup Y).$$

D is the boolean subalgebra of A generated by $X \cup Y$. Then $|Y| = \alpha$. This can be seen by noting that for every $x \in Y$, if x is non-zero, and x is not a diagonal element, then by Lemma 1 (ii), there exists

$$S \subseteq {}^\alpha\alpha^{(Id)} \text{ such that } x = \cup\{1_u : u \in S\}.$$

Also D is a subuniverse of A because of the following: D is closed under the boolean set operations, contains the diagonal elements, and for $a \in D$ and $i < \alpha$, we have $a \in A$ and so $C_i a \in D$, so that D is closed under cylindrfications, as well. Since $X \subseteq D$, X generates B and $D \in CA_\alpha$, we have $B \subseteq D$. Let

$$rl_L(a) = a \cap L.$$

Then we have

$$Rl_L D = \{x \in D : x \leq L\} = Sg^{BlA} rl_L(X \cup Y).$$

But $|rl_L(X \cup Y)| = \alpha$, because $P_\omega(Id)$ is countable and $|Y| = \alpha$. Therefore

$$|rl_L(X \cup Y)| = \alpha \text{ and so } |Rl_L(D)| = \alpha.$$

By $Rl_L(B) \subseteq Rl_L(D)$ we are done. ■

Claim 5 . *Let $t(x)$ be the CA_2 term $s_1^0 x c_1 x \cap s_0^1 c_0 x$, where $s_i^j(x) = c_i(d_{ij} \cdot x)$, for $i \neq j$. Let $v = \langle 1023 \cdots \rangle$. That is $v(0) = 1$, $v(1) = 0$ and $v(i) = i$ for all $i \in \alpha \smallsetminus \{0, 1\}$.*

Let ${}_\alpha s(0,1)$ *be the substitution term*

$${}_\alpha s(0,1)x = s_0^\alpha s_1^0 s_\alpha^1(x).$$

Then the following hold:

(i) $t^B(1_v) = 1_{Id}$

(ii) For all $\beta > \alpha$, $CA_\beta \models {}_\alpha s(0,1)C_\alpha x \leq t(C_\alpha x)$.

(iii) For all $\beta > \alpha$ *and* $A \in CA_\alpha$, ${}_\alpha s(0,1)$ *is boolean endomorphism of* $Nr_\alpha A$.

Proof.

(i) For the sake of brevity we write 1_{ij} for 1_u where $u(0) = i$, $u(1) = j$ and $u(k) = k$ for all $k > 1$. In particular, we have

$$1_{10} := 1_v \text{ and } 1_{01} := 1_{Id}.$$

We also write $u \equiv_i v$ if $u(k) = v(k)$ for all $k \neq i$. By [50] 1.5.1, we have

$$t^B(1_{10}) = C_0(D_{01} \cap C_1 1_{10}) \cap C_1(D_{01} \cap C_0 1_{10}).$$

Computing we get

$$C_0(D_{01} \cap C_1 1_{10}) = C_0(D_{01} \cap (\cup\{1_u : u \equiv_1 1_{10}\})$$

$$= C_0(D_{01} \cap 1_{11}) = 1_{01} \cup 1_{11}.$$

An entirely analagous computation gives

$$C_1(D_{01} \cap C_0 1_{01}) = 1_{00} \cup 1_{01}.$$

Therefore $t^B(1_{10}) = 1_{01}$.

(ii) By [50], 1.5.12, 1.5.8 and 1.5.10 (ii), we have

$${}_\alpha s(0,1)C_\alpha x \leq_\alpha s(0,1)C_1 C_\alpha x = S_0^\alpha S_1^0 S_\alpha^1 C_1 C_\alpha x = S_0^\alpha S_1^0 C_1 C_\alpha x$$

$$= S_0^\alpha S_1^0 C_\alpha C_1 x = S_0^\alpha C_\alpha S_1^0 C_1 x = C_\alpha S_1^0 x = S_0^1 C_1 C_\alpha x.$$

Similarly ${}_\alpha s(0,1)C_\alpha x \leq S_1^0 C_0 C_\alpha x$. Therefore ${}_\alpha s(0,1)C_\alpha x \leq t(C_\alpha x)$.

(iii) Let $C \in CA_\beta$ with $\beta > \alpha$ and $0 < x \in Nr_\alpha C$. By [50], 1.5.12 and 1.5.1, we have

$$_\alpha s(0,1)C_\alpha x = S_0^\alpha S_1^0 s_\alpha^1 C_\alpha x = C_\alpha(D_{\alpha 0} \cap C_0(D_{01} \cap C_1(D_{01} \cap C_1(D_{1\alpha} \cup C_\alpha x))).$$

By [50] 1.3.8, $0 < x$ implies $0 < D_{ij} \cap C_j x$ for all $i, j \in \beta$. We have shown that if $x > 0 \in Nr_\alpha C$ then $_\alpha s(0,1)x > 0.$, i.e. that $_\alpha s(0,1)$ is one to one. since the S_i^js are boolean endomorphisms, it follows from [50] 1.5.12 that $_\alpha s(0,1)$ is an isomorphism from $Nr_\alpha C$ to C. It remains to show that $_\alpha s(0,1)x \in Nr_\alpha C$ for all $x \in Nr_\alpha C$. But this follows easily from [50] 1.6.13.

■

We next show that had B been a neat reduct then $|RL_{Id}B| = |R| > \alpha$ which contradicts what we established in Claim 4. We conclude thus that B is not a neat reduct.

Claim 6 . $B \notin Nr_\alpha CA_{\alpha+1}$.

Proof.

Suppose, seeking a contradiction that $B = Nr_\alpha C$, say, with $C \in CA_{\alpha+1}$. We write 10 for $v = \langle 1023\ldots\rangle$. and 01 for the identity. Then

$$P(10) \subseteq Rl_{1_{10}} Nr_\alpha C$$

and so

$$|Rl_{1_{10}} Nr_\alpha C| > \alpha.$$

By Claim 5 (iii), we have

$$|_\alpha s(0,1)(Rl_{1_{10}} Nr_\alpha C)| > \alpha.$$

On the other hand ,

$$_\alpha s(0,1)^C(1_{10}) =_\alpha s(0,1)C_\alpha(1_{10}) \leq t(c_\alpha(1_{10}) = t(1_{10}) = 1_{01}.$$

This implies that

$$_\alpha s(0,1)(Rl_{1_{10}} Nr_\alpha C) \subseteq Rl_{1_{01}} Nr_\alpha C = Rl_{1_{01}} B.$$

By Claim 4, we conclude that

$$|_{\alpha}s(0,1)(Rl_{1_{10\cdots n-1}}Nr_{\alpha}C| \leq \alpha.$$

Contradiction. ■

Part 3. A **is a neat reduct**

The algebra A constructed so far may or may not be a neat reduct. We now impose further restrictions on the structure $\mathbf{W} = \langle W, D_r \rangle$ described in Lemma 1, that will enforce A to be in $\cap_{k\in\omega}Nr_{\alpha}Ws_{\alpha+k}$. These restrictions are described in Lemma 7, to be formulated in a while. Roughly we add to the conditions in Lemma 1 a saturation condition and a symmetry condition. (The symmetry condition though is not essential; it is added to make the other conditions easier to formulate.) The saturation condition will imply that the new relativized model $\mathbf{W} = (W, D_r)$ is in a sense $|R|$-saturated. This, in turn, will imply that the algebra A based on $\mathbf{W}$ is a neat reduct. Forming a limit out of the $Ws_{\alpha+k}$'s ($k \in \omega$), the neat -α reduct of which is A we, will show that $A \in Nr_{\alpha}Ws_{\alpha+\beta}$ for infinite β as well. All of this should be done, of course, in such a way that does not interfere with what we have already established, namely that B is not a neat reduct and that further it is an elementary subalgebra of A.

To formulate Lemma 7 we fix some notation:

Notation. . *Let* $1 \leq k < \omega$. *Then* $S(\alpha, \alpha+k)$ *or simply*

$$S(\alpha, k) = \{i \in {}^{\alpha}(\alpha+k) : \alpha+k-1 \in Rg(i) : \text{ and } \{m \in \omega : |\{i(m) \neq m\}| < \omega\}.$$

More consicely

$$S(\alpha, k) = \{i \in {}^{\alpha}(\alpha+k)^{(Id)} : \alpha+k-1 \in Rg(i)\}.$$

$Cof^{+}R$ denotes the set of all non empty finite or cofinite subsets of R. Let D_r be an α-ary relation symbol for all $r \in R$. For any $X \subseteq R$, X finite we define the infinitary formulas:

$$\eta(X) = \bigvee\{D_r(x_0, \cdots x_j \cdots)_{j<\alpha} : r \in X\},$$

$$\eta(R \smallsetminus X) = \neg\eta(X) = \bigwedge\{\neg D_r(x_0 \cdots x_j \cdots)_{j<\alpha} : r \in X\}.$$

Now we are ready to formulate and prove our Lemma. Recall that $|R| > \alpha$.

Lemma 7 . *There are a set U, an equivalence relation E with α-many blocks on U, $s \in {}^{\alpha}U$ with $s_i \in U_i$, the ith block, $i \in \alpha$ and α-ary relations D_r on U for all $r \in R$ such that conditions (i)-(v) below hold:*

(i) $D_r(w_0 \cdots w_i \cdots)_{i<\alpha}$ *implies* $D_E(w_0 \cdots w_i)_{i<\alpha}$. *Here* $D_E(w_0, w_2, \cdots)_{i<j<\alpha}$ *means that for any $i < j < \alpha$, w_i, w_j are not E-equivalent i.e they are in distinct blocks.*

(ii) For all $f \in {}^{\alpha}U^{(s)}$, for all $r \in R$, for all permutations $\pi \in S_\alpha$, if $s \in D_r$ then $f \circ \pi \in D_r$.

(iii) For all $r \in R$, for all $i < \alpha$ and $f \in {}^{\alpha}U^{(s)}$ such that $D[f_i]_{i<\alpha}$ there exists $x \in U$ such that $f[i|x] \in D_r$.

(iv) For all $1 \leq k < \omega$, for all $v \in {}^{\alpha+k-1}U^{(s)}$ one to one, for all $x \in U$, for any function $g : S(\alpha, k) \to Cof^+R$ for which $|\{i \in S(\alpha, k) : \{g(i) \neq R\}| < \omega$, there is a $v_{\alpha+k-1} \in U \smallsetminus Rng(v)$ such that $xEv_{\alpha+k-1}$, i.e. $v_{\alpha+k-1}$ is in the same block as x, and

$$\bigwedge\{D(v_{i_j})_{j<\alpha} \implies \eta(g(i))[\langle v_{i_j}\rangle] : i \in S(\alpha, k)\}.$$

(v) The D_r's are pairwise disjoint.

Proof. We construct the structure $\langle U, D_r\rangle$ in a standard step-by-step fashon. Let $k \geq 1$. For the sake of brevity, set

$$Y(\alpha, R, k) = \{g \in {}^{S(\alpha,k)}Cof^+R : |\{i \in S(\alpha, k) : \{g(i) \neq R\}| < \omega\}.$$

Let

$$Q = \cup\{{}^{\alpha+k-1}|R| \times Y(\alpha, R, k) : 1 \leq k < \omega\}.$$

Roughly Q represents the set of all tasks that we have to exhaust. Now $|Q| = |R| = \mu$, say. We shall construct the underlying set U of the desired model with cardinality μ. Let ρ be an enumeration of Q such that: For all $l < \mu$, for all $q \in Q$, there exists j with $l < j < \mu$ such that $\rho(j) = q$. Such a ρ clearly exists. Fix a well ordering $\prec$ of R. Let $l < \mu$ and suppose that for all $i < l$ we have already defined the element v_i

and the α-ary relation $D_r^i \subseteq {}^{\alpha}U_i^{(s)}$ where $U_i = \{v_j : j < i\}$. In the l th-step we will make the $\rho(l)$'th instance of (iv) true. Assume that

$$\rho(l) = \langle\langle\beta_j\rangle_{j<\alpha+k-1}, g\rangle.$$

Then

$$\beta_j \in \mu \text{ for all } j < \alpha + k - 1 \text{ and } g \in Y(\alpha, R, k)$$

for some $k \in \omega$. ($k = 1$ is allowed too, and this reduces to the bigness condition (iii).) Let v_l be a brand new element not in $\cup\{U_i : i < l\}$. If there exists $i < \alpha + k - 1$ such that $l \leq \beta_i$, then for all $r \in R$ we define

$$D_r^l = \cup\{D_r^i : i < l\}.$$

Else, $l > \beta_i$ for all $i < \alpha + k - 1$. In this case, let

$$u_i = v_{\beta_i} \text{ for all } r \in R \text{ and } u_{\alpha+k-1} = v_l.$$

For all $r \in R$ and $i \in S(\alpha, k)$ we let

$$\langle u_{i_j}\rangle \in X_r^l \text{ iff } r \text{ is the } \prec\text{- least element of } g(i).$$

and

$$D_r^l = \cup\{D_r^i \cup \{\langle s_{\pi(0)}, s_{\pi(1)}, \cdots\rangle : s \in X_r^l, \text{ and } \pi \in S_\alpha\}.$$

Finally we set

$$U = \cup\{U_l : l < \mu\} \text{ and } D_r = \cup\{D_r^l : l < \mu\}.$$

Now we are going to check that the structure $\mathbf{U} = (U, D_r)$ satisfies conditions (i)-(iv). To this end, for any $l < \mu$ and $r \in R$ let

$$Y_r^l = \{\langle s_{\pi(0)} \cdots s_{\pi(1)}, \cdots\rangle : s \in X_r^l \text{ and } \pi \in S_\alpha\}.$$

and

$$D_r^l = \cup\{D_r^j : j < l\}.$$

Then for all $r \in R$ we have

$$D_r^l = Y_r^l \cup D_r^l.$$

Also the following is straighforward to check:

(1) $w_l \in Rgs$ if $s \in Y^l_r$ and $w_l \notin Rgs$ if $s \in D^l_r$.

(2) $D^j_r \subseteq D^m_r$ if $j \leq m < \mu$.

It is easy to show by induction on $l < \mu$ that for all $r \in R$, D^l_r is symmetric, and so (ii) holds. Let $r, p \in R$ be distinct. We want to show by induction on $l < \mu$ that D^l_r and C^l_p are disjoint. Now D^l_r and D^l_p are disjoint by the induction hypothesis and by (2). By (1) it is therefore enough to show that Y^l_r and Y^l_p are disjoint. Assume $s \in Y^l_r$ and let

$$\rho(l) = \langle \langle \beta_j \rangle_{j<\alpha+k-1}, g \rangle$$

and let $z : \alpha + k \to U$ be defined as follows:

$$z_j = v_{\beta_j} \text{ for } j < \alpha + k - 1 \text{ and } z_{\alpha+k} = v_l.$$

Then z is one to one, by the choice of v_l and since v_{β_i} and v_{β_j} are in distinct blocks for $i < j < \alpha + k - 1$ are distinct. There is a *unique* $i \in S(\alpha, k)$ and permutation π of α such that $s = \langle t_{\pi(0)}, t_{\pi(1)}, \cdots \rangle$, where

$$t = \langle v_{i_j}, \cdots \rangle_{j<\alpha} \in X^l_r.$$

Thus r is the $\prec$-least element of $g(i)$, by $t \in X^l_r$. Since $p \neq r$ we get that $t \notin X^l_p$, and so $t \notin Y^l_p$. We have shown that D^l_r and C^l_p are disjoint. By (2) condition (v) holds.

Finally we check condition(s) (iv) (and bigness (iii)). Let $\langle v_{\beta_i} \rangle_{i<\alpha+k-1}$ be a U termed sequence, $x \in U$ and let $g \in Y(\alpha, R, k)$. Recall that $g : S(\alpha, k) \to Cof^+R$ is such that $|\{i \in S(\alpha, k) : \{g(i) \neq R\}| < \omega$. Let $l < \mu$ be such that

$$\rho(l) = \langle \langle \beta_j \rangle_{j<\alpha+k-1}, g \rangle \text{ and } l > i_j.$$

Such an l exists by properties of ρ. Then it is not difficult to check that we constructed v_l so that given that the $v'_{i_j}s$ are in distinct blocks, then it satisfies

$$\phi = \bigwedge \{\eta(g(i))[v_{\beta_{i_j}}, v_{\beta_{i_1}}, \cdots v_{\beta_{i_{n-2}}}, x_{i_{n-1}}) : i \in S(\alpha, k)\}$$

in $(U_l, D^l_r)_{r\in R}$. By (1) and (2) we have $D^l_r =^n U_l \cap D_r$, and so ϕ is satisfied in $\langle U, D_r \rangle_{r\in R}$, as well. □

■

Let $\mathbf{U} = (U, D_r)$ and s be as specified in Lemma 8. Then U is partitioned via E into $|\alpha|$ many blocks. We let U_i denote the ith copy. Thus, we have

$$U = U_0 \cup U_1 \cdots \cup U_i : i < \alpha.$$

Recall that $s \in {}^{\alpha}U$ such that $s(i) \in U_i$ for all $i \in \alpha$. Let V be the weak space ${}^{\alpha}U^{(s)}$, and C be the full Ws_α with unit V. Recall that S_α is the set of permutations in ${}^{\alpha}\alpha^{(Id)}$. Next we define $B \subseteq A \in Ws_\alpha$, each with unit V, like we did before. The resulting subalgebras as easily checked are isomorphic to the previous ones.

Definining our new algebras

Let $u \in S_\alpha$ and $r \in R$. Then we set

$$p(u,r) = D_r \cap U_{u_0} \times U_{u_1} \times U_{u_i} \times \ldots \cap {}^{\alpha}U^{(s)}.$$

$$P(u) = \{p(u,r) : r \in R\}.$$

Let N be a fixed countable subset of R. Let

$$P_\omega(u) = \{p(u,r) : r \in R\}.$$

We let A' and B' be the (new) algebras constructed out of the new structure $\mathbf{U} = \langle U, D_r\rangle_{r\in R}$ described in Lemma 6 That is $A' \in Ws_\alpha$ and $B' \subseteq A'$ are defined as follows:

$$A' = Sg^C(\cup\{P(u) : u \in S_\alpha\});$$

and

$$B' = Sg^C((\cup\{P(u) : u \in S_\alpha \smallsetminus \{Id\}) \cup P_\omega(Id)).$$

It is easy to see that $A' \cong A$ and that $B' \cong B$. In particular B is not a neat reduct and B is an elementary subalgebra of, A. From now on we pretend that $A = A'$. The rest of the chapter is devoted to showing that A is a neat reduct. To "lift" A into higher (finite extra) dimensions, we shall need:

Some more definitions.

Let $0 \leq k < \omega$. Fix $u \in U$. Now let

$$s_k = s \cup \langle u : \alpha \leq i < \alpha + k\rangle.$$

Then $s_k \in {}^{\alpha+k}U$. Let V_k be the weak space ${}^{\alpha+k}U^{(s_k)}$ and let $C(\alpha+k)$ be the full $Ws_{\alpha+k}$ with unit V_k. For $u \in {}^{\alpha+k}\alpha^{(Id)}$, we set

$$ker(u) = \{\langle i,j\rangle \in {}^{2}(\alpha+k) : u(i) = u(j)\}$$

and

$$Eq(u) = \{x_i = x_j : x_i \neq x_j : \langle i, j\rangle \in ker(u)\}.$$

Now set

$$F_k(u) = \{D_r(x_{i_0} \cdots x_{i_j} \cdots)_{j<\alpha}, \neg D_r(x_{i_0} \cdots x_{i_j} \cdots)_{j<\alpha} : r \in R, i \in {}^{\alpha}\alpha + k^{(Id)}$$

$$\text{and } \langle u_{i_0} \cdots u_{i_j} \cdots\rangle_{j<\alpha} \text{ is one to one } \} \cup Eq(u).$$

Forming finite conjunctions of formulas in $F_k(u)$, we set

$$F_k(u)^* = \{\bigwedge J : J \subseteq_\omega F_k(u)\}$$

and forming finite disjunctions of formulas in $F_k(u)^*$, we set

$$F_k(u)^{**} = \{\bigvee J : J \subseteq_\omega F_k(u)^*\}.$$

For $u \in {}^{\alpha+k}\alpha^{(Id)}$, and $\phi \in F_k(u)^{**}$ we let $E(u, \phi)$ denote the following $\alpha+k$ ary-relation on V_k.

$$E(u, \phi) = \{s \in V_k : s_j \in U_{u_j} \text{ for all } j < \alpha + k \text{ and } \mathbf{U} \models \phi[\langle s_j\rangle]\}.$$

Here, satisfiability is defined the usual way. In particular, when $k = 0$, $u \in S_\alpha$ and $r \in R$, then $E(u, D_r(x_0, x_1 \cdots)$ is the same as $p(u, r)$.

For a formula ϕ, we let $ind(\phi)$ denote the set

$$\{i \in \alpha + k : x_i \in var(\phi)\}.$$

For $a \in G(k)$, we write $-a$ in place of $V_k \smallsetminus a$. Now we let

$$G(k) = \{C_{(\Delta)}E(u, \phi), -C_{(\Delta)}E(u, \phi) : \Delta \subseteq_\omega \alpha + k, u \in {}^{\alpha+k}\omega^{(Id)}, \phi \in F_k(u)^{**},$$

$$\Delta \cap ind(\phi) = 0\} \cup \{D_{ij}, -D_{ij} : \ i < j < \alpha + k\}.$$

Forming finite intersections of elements of $G(k)$, we let

$$G(k)^* = \{\cap J : J \subseteq_\omega G(k)\}$$

Finally, forming finite unions of elements in $G(k)^*$, we let

$$G(k)^{**} = \{\cup J : J \subseteq_\omega G(k)^*\}.$$

We will show that $G(k)^{**}$ is a $Ws_{\alpha+k}$ that $G(0)^{**} = Nr_\alpha G(k)^{**}$ for all $k \in \omega$, and finally that $G(0)^{**} = A$. This will show that $A \in \cap_{k\in\omega} Nr_\alpha Ws_{\alpha+k}$. That $A \in Nr_\alpha Ws_{\alpha+\omega}$ will be proved by a limiting construction. But first we need to recall the notion of generalized cylindrifications. Let $D \in K_\alpha$, $a \in D$ and $\Delta \subseteq_\omega \alpha$. Then $C_{(\Delta)}a = C_{i_0}\cdots C_{i_{n-1}}a$, where $\Delta = \{i_0, \cdots i_{n-1}\}$.

Claim 8 . *Let $u \in {}^{\alpha+k}\alpha^{(Id)}$, $\Delta \subseteq_\omega \alpha + k$ and $\phi \in F_k(u)^{**}$. Then the following hold:*

(i) $C_{(\Delta)}E(u, \phi) \in G(k)^{**}$.

(ii) Let $g \in G(k)^{**}$ and $i, j \in \alpha+k$ be distinct. Then $S_i^j(g) = C_j(g \cap D_{ij}) \in G(k)^{**}$.

(iii) Let $H(k) = \{g \in G(k) : g = C_{(\Delta)}E(u, \phi) : \Delta \subseteq_\omega \alpha + k, u \in {}^{\alpha+k}\alpha^{(Id)}$ and $\phi \in F_k(u)^{**}\}$. Then $H(k)$ is closed under finite intersections.

Proof.

(i) We first prove (i) when Δ consists of one element j, say. That is we will show that for u and ϕ as indicated, and $j < \alpha + k$ we have $C_jE(u, \phi) \in G(k)^{**}$. It is in this part of the proof that we use the condition of saturation. We can and will assume that $\phi \in F_k(u)^*$ since

$$C_jE(u, \phi_1 \vee \cdots \phi_n) = C_jE(u, \phi_1) \cup \cdots C_jE(u, \phi_n),$$

and $G(k)^{**}$ is closed under finite unions. Then $\phi = \bigwedge J$, where $J \subseteq_\omega F_k(u)$. Next, we break up ϕ into two parts, that is we let

$$J^+ = \{\psi \in J : x_j \in var(\psi)\}$$

and

$$J^- = \{\psi \in J : x_j \notin var(\psi)\}.$$

For the sake of brevity, we set $\phi^+ = \bigwedge J^+$ and $\phi^- = \bigwedge J^-$, where by the empty conjunction we understand the formula $\top$. Then $\phi = \phi^+ \wedge \phi^-$, $\phi^- \in F_k(u)^*$ and x_j occurs in ϕ^+ and x_j does not occur in ϕ^-.

Notation . *For $i, l \in \alpha + k$ and $\phi \in F_k(u)^{**}$, $\phi[i|l]$ stands for the formula obtained from ϕ by replacing all (free and boound) occurences of x_i in ϕ by x_l.*

We distinguish between two cases:

Case (a):

$x_j = x_m \notin J^+$ and $x_m = x_j \notin J^+$ for all $m \in \alpha + k \smallsetminus \{j\}$.

Then either $\mathbf{U} \models \phi^+ \longleftrightarrow \bot$, in which case $\mathbf{U} \models \exists x_j \phi^+ \longleftrightarrow \bot$ and so

$$C_j E(u, \phi) = C_j E(u, F) = 0 \in G(k)^{**}.$$

Else, ϕ^+ is satisfiable, in which case by the saturation condition (v) of Lemma 7, we get

$$\mathbf{U} \models (\exists x_{\alpha+k-1} \phi^+[j|\omega + k - 1]) \longleftrightarrow \top,$$

which is the same as

$$\mathbf{U} \models (\exists x_j \phi^+) \longleftrightarrow \top,$$

and so

$$C_j E(u, \phi) = C_j E(u, (\exists x_j \phi^+) \wedge \phi^{-1}) = C_j E(u, \phi^{-1}).$$

The latter is in $G(k)^{**}$, since now x_j does not occur in ϕ^-, and $\phi^- \in F_k(u)^*$. Note that, in fact we have, $\phi^- \in F_k(v)^*$ for any v that differs from u in at most the j-th place, and for any such v, we have

$$C_j E(u, \phi) = C_j E(v, \phi^-).$$

Case (b):

$x_j = x_m \in J^+$ or $x_m = x_j \in J^+$ for some $m \in \omega + k$ and $m \neq j$.

Assume without loss that $x_j = x_m \in J^+$. Then, by definition of $F_k(u)$, we have $u_j = u_m$. Also

$$\phi = (\bigwedge I) \wedge x_j = x_m \text{ for some } I \subseteq_\omega F_k(u).$$

For brevity, set $\psi = \bigwedge I$. Then, of course, $\psi \in F_k(u)^*$. Computing we get

$$C_j E(u, \phi) = C_j E(u, \psi \wedge x_j = x_m) = C_j E(u, \psi[j|m]).$$

The latter is in $G(k)^{**}$ since x_j does not occur in $\psi[j|m]$, and $\psi[j|m] \in F_k(u)^*$ because $\psi \in F_k(u)^*$ and $u_j = u_m$. In fact, it is not hard to see that $\psi[j|m] \in$

$F_k(v)^*$ for any v that differs from u in at modst the j-th place. The case when Δ is an arbitrary finite subset of $\alpha + k$ now follows by using the above reasoning eliminating the variables occuring in ϕ, with indices in Δ, one by one. Alternatively, one can use a straightforward induction on $|\Delta|$. We have, in fact, proved the following :

(+) . *Let $u \in {}^{\alpha}(\alpha + k)^{(Id)}$ and $\phi \in F_k(u)^{**}$. Let $\Delta = \{i_0, \cdots, i_{n-1}\}$ be a finite subset of $\alpha + k$. Let $v \in {}^{\alpha}(\alpha + k)^{(Id)}$ be such that $v(j) = u(j)$ for all $j \notin \Delta$. Then there exists $\psi \in F_k(v)^{**}$ such that $var(\psi) \cap \{x_{i_0}, \cdots, x_{i_{n-1}}\} = \emptyset$, i.e $ind(\psi) \cap \Delta = \emptyset$ and $\mathbf{U} \models \psi \longleftrightarrow \exists x_{i_0} \cdots \exists x_{i_{n-1}} \phi$. In particular, $C_{(\Delta)}E(u, \phi) = C_{(\Delta)}E(v, \phi)$.*

(ii) Let $g \in G(k)^{**}$ and $i, j \in \alpha + k$ be distinct.

Case (a):

$$g = C_{(\Delta)}E(u, \phi), u \in {}^{\alpha+k}\alpha^{(Id)} \text{ and } \phi \in F_k(u)^{**}.$$

Subcase (a): $j \in \Delta$.

Then by [50] 1.5.8 (i) and 1.7.3 we have

$$S_i^j(g) = S_i^j C_j C_{(\Delta \smallsetminus \{j\})} E(u, \phi) = C_j C_{(\Delta \smallsetminus \{j\})} E(u, \phi) = g.$$

Subcase (b): $j \notin \Delta$ and $i \notin \Delta$.

Then by [50] 1.5.8 (ii) we have

$$S_i^j g = S_i^j C_{(\Delta)} E(u, \phi) = C_{(\Delta)} S_i^j E(u, \phi).$$

Let $g^* = E(u, \phi) \cap D_{ij}$. Then g^* and, hence $S_i^j g = C_{(\Delta)} C_j g^*$, is equal to zero if $u_i \neq u_j$. Else, as easily checked, $g^* = E(u, \phi \wedge x_i = x_j)$. By $u_i = u_j$ and $\phi \in F_k(u)^{**}$, we have $\phi \wedge x_i = x_j \in F_k(u)^{**}$. By Claim 8 (i) we get that

$$S_i^j g = C_{(\Delta)} C_j g^* = C_{(\Delta)} C_j E(u, \phi \wedge x_i = x_j) \in G(k)^{**}.$$

Subcase (c): $j \notin \Delta$ and $i \in \Delta$.

By [50] 1.7.3 and 1.5.8 (ii), we have

$$S_i^j C_{(\Delta)} E(u,\phi) = S_i^j C_{(\Delta \smallsetminus \{i\})} C_i E(u,\phi) = C_{(\Delta \smallsetminus \{i\})} S_i^j C_i E(u,\phi).$$

Now put

$$g^* = S_i^j C_i E(u,\phi).$$

Then by [50] 1.5.1 we have

$$g^* = C_j(C_i(E(u,\phi) \cap D_{ij}).$$

Let

$$v = u^i_{u(j)} = u \circ [i|j].$$

Here, and elsewhere throughout this chapter, $[i|j]$ denotes the replacement that maps i to j and otherwise coincides with the identity. Then we have

$$v(i) = u(j) = v(i).$$

Also, it is not difficult to see that

$$C_i(E(u,\phi) \cap D_{ij}) = E(v, \phi \wedge x_i = x_j).$$

Since $i \in \Delta$, we can assume by Claim 8(i) that x_i does not occur in ϕ. Since v differs from u in at most the i th place, we have $\phi \in F_k(v)^{**}$. But $v(i) = v(j)$, and so $x_i = x_j \in F_k(v)^{**}$, thus $\phi \wedge x_i = x_j \in F_k(v)^{**}$. By Claim 8(i) we get that g^*, hence g is in $G(k)^{**}$, too.

Case (b): $g = D_{kl}$ where $k, l \in \alpha + k$ and $k \neq l$.

By [50] 1.5.4 we have $S_i^k(D_{kl}) = D_{il}$ if $k \neq l$ and $S_i^j D_{kl} = D_{kl}$ if $j \notin \{k, l\}$. In either case, we have $S_i^j g \in G(k)^{**}$.

Case (c): g is an element of $G(k)^{**}$.

Follows immediately from the two previous cases, since the S_i^j's $(i, j \in \alpha + k)$ are boolean endomorphisms of the boolean algebra (with universe) $G(k)^{**}$ (cf. [50] 1.5.3 (i).)

(iii) Let $g = C_{(\Delta_1)}E(u_1, \phi_1) \cap C_{(\Delta_2)}E(u_2, \phi_2)$ where $\Delta_i \cap ind(\phi_i) = 0$ for $i \in \{1, 2\}$. Then $g = 0$ if there exists $j \in (\alpha + k) \smallsetminus (\Delta_1 \cup \Delta_2)$ such that $u_1(j) \neq u_2(j)$. Else, it is not difficult to check that

$$g = C_{(\Delta_1 \cap \Delta_2)}E(w, \phi_1 \wedge \phi_2)$$

where w is defined as follows:

$$w(j) = u_1(j) = u_2(j) \text{ for } j \in \omega + k \smallsetminus (\Delta_1 \cup \Delta_2)$$

$$w(j) = u_1(j) \text{ if } j \in \Delta_2 \smallsetminus \Delta_1$$

$$w(j) = u_2(j) \text{ if } j \in \Delta_1 \smallsetminus \Delta_2$$

and w is defined arbitrarily on $\Delta_1 \cap \Delta_2$. Also by $ind(\phi_i) \cap \Delta_i = 0$ for $i \in \{1, 2\}$ and by $w(j) = u_1(j) = u_2(j)$ for all $j \in (\alpha + k) \smallsetminus (\Delta_1 \cup \Delta_2)$ we can well assume by Claim 8 (i) that $\phi_i \in F_k(w)^{**}$, hence it follows that $\phi_1 \wedge \phi_2 \in F_k(w)^{**}$. $\square$

$\blacksquare$

To economise on notation we recall a few notions from [50]. A *generalized diagonal element* is a finite intersection of diagonal elements. A *co-diagonal element* is the complement of a diagonal element. A *generalized co-diagonal* element is a finite intersection of co-diagonal elements. In particular the unit of an algebra can be viewed as both a generalized diagonal and a generalized co-diagonal. Our next fact says that for every $k \in \omega$, $G(k)^{**}$ is a $CA_{\alpha+k}$. More precisely:

Claim 9 . *For all* $0 \leq k < \omega$, $G(k)^{**} \in Ws_{\alpha+k}$.

Proof. Clearly $G(k)^{**}$ is a boolean field of sets with greatest element V_k. Also $G(k)^{**}$ by definition, contains all the diagonal elements. We are thus left to check that it is closed under cylindrifications. Since cylindrifications are additive (they distribute over the boolean join which is the union operation) , it suffices to show that for every $g \in G(k)^*$ and $j < \alpha + k$, we ahve $C_j g \in G(k)^{**}$.

By Claim 8 (iii) a typical element $g \in G(k)^*$ is of the form

$$a_0 \cap b \cap c \cap -a_1 \cap -a_2 \cdots \cap -a_n,$$

where $n \in \omega$, $a_0, a_1 \dots a_n \in H(k)$, b is a generalized diagonal element, and c is a generalized co-diagonal element.

Case a:

$$b = D_{jl} \cap b',$$

where b' is a generalized diagonal element and $j \neq l$. In this case, put

$$g^* = a_0 \cap b' \cap -a_1 \cap \dots \cap -a_n.$$

Then of course $g^* \in G(k)^*$ and $C_j g = S_l^j g^*$. By Claim 8 (i) the latter is in $G(k)^{**}$.

Case b:

$$a_0 = b = V_k.$$

In this case, we have

$$g = -C_{(\Delta_1)} E(u_1, \phi_1) \cap \dots \cap -C_{(\Delta_n)} E(u_n, \phi_n) \cap -D_{jl_1} \dots \cap -D_{jl_m},$$

where $n, m \in \omega$, $\Delta_k \subseteq_\omega \omega + k$ for $1 \leq k \leq n$ and since $C_j(x \cap C_j y) = C_j x \cap C_j y$ we can further assume that $j \notin \Delta_1 \cup \dots \Delta_n$. Assume that $g \neq 0$, for else there is nothing more to prove. In particular, $j, l_1 \dots l_m$ are pairwise distinct. We claim that in this case we have

$$C_j g = V_k.$$

To see this, let $t = \langle t_l \rangle_{l < \alpha + k}$ be an arbitrary element in V_k. We will show that $t \in C_j g$, by which we will be done. Choose

$$k_j \notin \{u_1(j), \cdots u_n(j)\} \cup \{i_{l_1}, \cdots i_{l_m}\}$$

and let u be an arbitrary element in U_{k_j}. Let $z = t(j|u)$. Assume that $1 \leq i \leq n$. Then $z \notin C_{(\Delta_i)} E(u_i, \phi_i)$ because

$$z(j) = t(u|j)(j) = u \notin \cup_{i<n} W_{u_i(j)}.$$

Assume now that $1 \leq i \leq m$. Then

$$z(j) \neq z(l_i),$$

since they are in disjoint copies by the choice of k_j, that is $z \notin D_{jl_i}$. We have shown that $z \in g$. Since t differs from z in at most the jth place we get that $t \in C_j g$. We have proved that $C_j g = V_k$.

Case c:

$$g = C_{(\Delta_0)}E(u_0, \phi_0) \cap -C_{(\Delta_1)}E(u_1, \phi_1) \cap \cdots \cap -C_{(\Delta_n)}E(u_n, \phi_n) \cap -D_{jl_1} \cdots \cap -D_{jl_m},$$

where $n, m \in \omega$, $\Delta_k \subseteq \alpha + k$ for $0 \leq k \leq n$ and as in the previous case, we can well assume that $j \notin \Delta_0 \cup \ldots \Delta_n$ and $j, l_1 \cdots l_m$ are pairwise distinct. We will and can further assume that $\{l_1 \cdots l_m\} \subseteq \Delta_0$, for if $l_k \notin \Delta_0$ for some $1 \leq k \leq n$, then we can simply "ignore" the co-diagonal element with indices j and l_k, i.e. the element $-D_{jl_k}$, because of the following:

$$C_{(\Delta_0)}E(u_0, \phi_0) \cap -D_{jl_k} = C_{(\Delta_0)}E(u_0, \phi_0 \wedge x_j \neq x_{l_k})$$

if $u_0(j) = u_0(l_k)$ and is equal to $C_{(\Delta_0)}E(u_0, \phi_0)$ otherwise. Note too, that $\phi_0 \wedge x_j \neq x_{l_k} \in F_k(u)^{**}$ whenever $\phi_0 \in F_k(u_0)^{**}$ and $u_0(j) = u_0(l_k)$. In either case, g can be written in the form

$$C_{(\Delta_0)}E(u_0, \phi_0) \cap -C_{(\Delta_1)}E(u_1, \phi_1) \cap \cdots \cap -C_{(\Delta_n)}E(u_n, \phi_n) \cap \bigcap \{-D_{jl} : l \in \{l_1, \cdots l_m\} - \{l_k\}\}$$

where $C_{(\Delta_0)}E(u_0, \psi) \in H(k)$. Now let

$$\Gamma = \Delta_0 \smallsetminus \{l_1 \cdots l_m\}$$

and let

$$\Gamma_1 = \{l_1, \cdots l_m\}.$$

Then

$$\Delta_0 = \Gamma \cup \Gamma_1.$$

Let

$$S = \{j\} \times \Gamma_1 = \{(j, l) : l \in \Gamma_1\}$$

and let D_S be the co-diagonal element

$$\bigcap \{-D_{jl} : (j, l) \in S\}.$$

Let

$$g^* = -C_{(\Delta_1)}E(u_1, \phi_1) \cap \cdots \cap -C_{(\Delta_n)}E(u_n, \phi_n),$$

$$g_1 = C_{(\Gamma)}E(u_0, \phi_0) \cap g^*,$$

and

$$g_2 = C_{(\Gamma_1)}E(u_0, \phi_0) \cap D_S \cap g^*.$$

Then by [50] 1.7.3, and the additivity of C_j we have $C_j g = C_j g_1 \cup C_j g_2$. We will show that $C_j g_1 \in G(k)^{**}$ and $C_j g_2 \in G(k)^{**}$, by which we will be done.

Proof of $C_j g_1 \in G(k)^{**}$.

Write

$$g_1 = C_{(\Gamma)}E(u_0, \phi_0) \cap \cdots C_{(\Delta_n)}E(u_n, \phi_n).$$

We can assume that for $1 \leq k \leq n$ and $i \notin \Gamma \cup \Delta_k$ we have $u_0(i) = u_k(i)$, for if $u_0(i) \neq u_k(i)$ for some $1 \leq k \leq n$, and some $i \notin \Gamma \cup \Delta_k$, then

$$C_{(\Gamma)}E(u_0, \psi_0) \cap -C_{(\Delta_k)}E(u_k, \phi_k) = C_{(\Gamma)}E(u_0, \phi_0).$$

For the time being, assume that

$$\phi_1 = \phi_2 = \cdots \phi_n = \top.$$

Claim

$$C_j g_1 = C_j C_{(\Gamma)}E(u_0, \phi_0) \cap -C_j C_{(\Delta_1)}E(u_1, \top) \cap \cdots \cap -C_j C_{(\Delta_n)}E(u_n, \top).$$

Proof of Claim

It is straightforward to show that the r.h.s is contained in $C_j g_1$. Now we prove the opposite slightly harder inclusion. Let $t \in C_j g_1$. Then there exists $b \in U$ such that $t(j|b) \in C_{(\Gamma)}E(u_0, \phi_0)$ and $t(j|b) \notin C_{(\Delta_k)}E(u_k, \top)$ for all $1 \leq k \leq n$. Therefore $t \in C_j C_{(\Gamma)}E(u_0, \phi_0)$. We will show that $t \notin C_j C_{(\Delta_k)}E(u_k, \top)$ for all $1 \leq k \leq n$, by which case we will be done. Fix $1 \leq k \leq n$ and assume to the contrary that $t \in C_j C_{(\Delta_k)}E(u_k, \top)$. Then there exists $c \in U$ such that $t(j|c) \in C_{(\Delta_k)}E(u_k, \top)$. Since $j \notin \Delta_k$, we get $t(j|c)(j) = c \in U_{u_k(j)}$. By $t(j|b) \in C_{(\Gamma)}E(u_0, \phi_0)$, and $j \notin \Gamma$ we get $b \in U_{u_0(j)}$. But $U_{u_0(j)} = U_{u_k(j)}$. Since $t(j|b)$ and $t(j|c)$ differ in at most the j th place, and their values b, c at j are in the same copy, we get from $t(j|c) \in C_{(\Delta_k)}E(u_k, T)$ that $t(j|b) \in C_{(\Delta_k)}E(u_k, T)$ which is a contradiction. Now the general case follows from this one by noting that

$$-C_{(\Delta)}E(u, \phi) = C_{(\Delta)}E(u, \neg\phi) \cup -C_{(\Delta)}E(u, \top);$$

and that $H(k)$ is closed under finite intersections. We now turn to showing that $C_jg_2 \in G(k)^{**}$. This will be done by showing that

$$a = C_{(\Gamma_1)}E(u_0, \phi_0) \cap D_S$$

hence $g_2 = a \cap g*$, can be written as a finite union of elements of the form

$$a_0 \cap -a_1 \cap \cdots - a_n$$

where $a_i \in H(k)$. Then by the additivity of C_j and the previous case we get the desired result. We start off when Γ_1 consists of a single element m say. We compute $a = C_mE(u_0, \phi_0) \cap -D_{jm}$. For the sake of brevity set $u = u_0$. Let

$$v = u^m_{u(j)} = u \circ [m|j].$$

Then $v(m) = v(j) = u(j)$. Moreover it is not hard to check that

$$a = [C_mE(u, \phi_0) \cap -E(v, T)] \cup E(v, \phi_0 \wedge x_j \neq x_m).$$

Since x_m does not occur in ψ_0, and v differs from u in at most the m-th place, we infer from Claim 8 (i) that $\phi_0 \in F_k(v)^{**}$. By $v(m) = v(j)$ we get that $x_j \neq x_m$ is in $F_k(v)$ hence in $F_k(v)^{**}$. It follows that $\phi_0 \wedge x_j \neq x_m \in F_k(v)^{**}$. We have shown that $E(v, \phi_0 \wedge x_j \neq x_m) \in H(k)$. Thus a is as required, since $C_m(E(u, \phi_0))$ and $E(v, \top)$ are in $H(k)$, too. Now for the general case. Assume that

$$\Gamma_1 = \{l_1, \cdots, l_m\}.$$

Having treated the case when $m = 1$, we now assume that $m > 1$. Then

$$a = C_{(\Gamma_1)}E(u, \phi) \cap -D_S = \cap\{a_k : 1 \leq k \leq m\}$$

where

$$a_k = C_{(\Gamma_1 \smallsetminus \{l_k\})}C_{l_k}E(u, \phi) \cap -D_{jl_k}.$$

Therefore a, hence $g_2 = a \cap g*$ is indeed the finite union of elements of the form $a_0 \cap -a_1 \cap -a_n$ where $a_i \in H(k)$. Thus $C_jg_2 \in G(k)^{**}$ and by this the proof of is complete.

∎

Claim 10 . *Let $0 \leq k < \omega$. Then $G(0)^{**} \cong Nr_\alpha G(k)^{**}$.*

Proof. Let $k \geq 0$. Let $i(k)$ be the following (neat embedding) function:

$$i(k) : G(0)^{**} \to C(\alpha + k)$$

$$i(k)(a) = \{t \in V_k : t \upharpoonright \alpha \in a\}, a \in G(0)^{**}.$$

Then $i(k) \in Ism(G(0)^{**}, Nr_\alpha C(\alpha + k))$, where $C(\alpha + k)$ is the full $Ws_{\alpha+k}$ with unit V_k. We will show that $i(k)G(0)^{**} \subseteq G(k)^{**}$, from which it will follow that $i(k) \in Ism(G(0)^{**}, Nr_\alpha G(k)^{**})$. For this, it suffices to show that for $u \in S_\alpha$ and $\phi \in F_k(u)$ we have $i(k)E(u, \phi) \in G(k)^{**}$. Let u and ϕ be as indicated. Then

$$i(k)E(u, \phi) = \cup\{E(v, \phi) : v \in {}^{\alpha+k}\alpha^{(Id)} : v \upharpoonright \alpha = u\}.$$

Let

$$u^* = u \cup \langle 0 : \alpha \leq i < \alpha + k\rangle.$$

Then

$$C_\alpha \cdots C_{\alpha+k-1} E(u^*, \phi) = \cup\{E(v, \phi) : v \in {}^{\alpha+k}\alpha^{(Id)} : v \upharpoonright \alpha = u\}.$$

Since $F_0(u) \subseteq F_k(v)$ whenever $v \in {}^{\alpha+k}\alpha^{(Id)}$ is such that $v \upharpoonright \alpha = u$, we get $\phi \in F_k(u^*)^{**}$, thus $i(k)E(u, \phi) \in G(k)^{**}$. We have shown that

$$i(k) \in Ism(G(0)^{**}, Nr_\alpha G(k)^{**}).$$

We will now show that $i(k)$ is actually onto $Nr_\alpha G(k)^{**}$. Since the C_i's are additive, it suffices to show that for all $g \in G(k)^{**}$, there exists $a \in G(0)^{**}$ such that

$$i(k)a = C_\alpha \cdots C_{\alpha+k-1} g.$$

Here again is where saturation comes in.[1]

Case (a):

$$g = E(v, \phi), v \in {}^{\alpha+k}\alpha^{(Id)} \text{ and } \phi \in F_k(v)^{**}.$$

Then by (+) there exists $\psi \in F_k(v)^{**}$ such that

$$var(\psi) \cap \{x_\alpha, \cdots x_{\alpha+k-1}\} = \emptyset$$

[1] The idea is that A is based on M, say and A' is based on M' which is a saturated elementary extension of M, hence $A \cong A'$.

and

$$\mathbf{U} \models \exists x_\alpha \cdots \exists x_{\alpha+k-1}\phi \longleftrightarrow \psi.$$

Put

$$u = v \upharpoonright \alpha.$$

Then $u \in {}^\alpha\alpha^{(Id)}$. Since

$$F_0(u)^{**} = \{\phi \in F_k(v)^{**} : var(\phi) \subseteq \alpha\};$$

we get that

$$\psi \in F_0(u)^{**}.$$

Moreover, we have

$$i(k)E(u,\psi) = C_\alpha \cdots C_{\alpha+k-1}E(u,\phi).$$

Case b:

$$g = a \cap c$$

where $a = -a_0 \cap \cdots - a_n$, $a_i \in H(k)$ and c is a generalized co-diagonal. If $C_\alpha \cdots C_{\alpha+k-1}g = g$, then $g \in G(0)^{**}$. Else $C_\alpha \cdots C_{\alpha+k-1}g$ is either 0 or V_k pending on whether $g = 0$ or not. The choice of $a \in G(0)^{**}$ in this case is also obvious.

Case c:

$$g = D_0 \cap -D_1 \cap -D_n \cap b \cap c,$$

$D_i \in H(k)$ b is a generalized diagonal and c a generalized co-diagonal. Assume that $1 \leq k < \omega$, for else there is nothing more to prove. Let $\Gamma = \{\alpha, \cdots, \alpha+k-1\}$. By the proof of Claim 9 we have that $C_{(\Gamma)}g$ is the finite union of elements of the form $a_0 \cap -a_1 \cap -a_n \cap b \cap c$, where $a_0 \cdots a_n \in H(k)$, b is a generalized diagonal and c is a generalized co-diagonal. Further we can assume without loss of generality that if

$$b = \cap\{D_{ij}^{V_k} : i,j \in \Delta_1\} \text{ and } c = \cap\{-D_{ij}^{V_k} : i,j \in \Delta_2, i \neq j\}$$

then $\Delta_i \subseteq_\omega \alpha$ for $i \in \{1,2\}$. By additivity of generalized cylindrification (namely of $C_{(\Gamma)}$), we can and will assume that

$$C_{(\Gamma)}g = C_{(\Gamma_0)}E(v_0,\phi_0) \cap C_{(\Gamma_1)}E(v_1,\phi_1) \cap \cdots - C_{(\Gamma_n)}E(v_n,\phi_n) \cap b \cap c,$$

where for each $i \leq n$ we have $\Gamma \subseteq \Gamma_i \subseteq_\omega \alpha + k$, $\phi_i \in F_k(v_i)^{**}$, b is a generalized diagonal and c is a generalized co-diagonal as indicated above. By invoking (+) we can assume that

$$var(\phi_i) \cap \{x_\alpha, \cdots x_{\alpha+k-1}\} = \emptyset.$$

For $i \leq n$ let $u_i = v_1 \restriction \alpha$. Then $u_i \in {}^\alpha\alpha^{(Id)}$, and by Claim 8 (i), $\phi_i \in F_0(u_i)^{**}$. Let

$$b' = \cap\{D_{ij} : i, j \in \Delta_1\}$$

and

$$c' = \cap\{D_{ij} : i, j \in \Delta_2, i \neq j\}.$$

Let

$$a = C_{(\Gamma_0 - \Gamma)} E(u_0, \phi_0) \cap \cdots - C_{(\Gamma_n - \Gamma)} E(u_n, \phi_n) \cap b' \cap c'.$$

Then $a \in G(0)^{**}$ and as easily checked

$$i(k)a = C_{(\Gamma)} g.$$

■

Claim 11 . $G(0)^{**} = A$.

Proof. Let $u \in {}^\alpha\alpha^{(Id)}$ be one to one and $r \in R$. Then by definition, we have

$$p(u, r) = E(u, D_r(x_0, x_1, \cdots x_i))$$

Hence $p(u, r) \in F_0(u)$. Since $G(0)^{**}$ is a Ws_α containing the generators of A, it follows that $A \subseteq G(0)^{**}$. For the other inclusion it is enough to show that for $u \in {}^\alpha\alpha$ and $\phi \in F_0(u)$ we have $E(u, \phi) \in A$. We start by showing that $E(u, \top) \in A$ for all such u.

Case a. u is one to one.

Let $r \in R$. Then $C_0(p(u, r)) \cap C_1(p(u, r) = E(u, \top) \in A$.

Case b. u is not one to one.

Let J be a finite subset of α such that $u(J) \subseteq J$ and $u(i) = i$ whenever $i \notin J$. Such a J exists, but of course is not unique. Fix one such J. Let $u' = u \restriction J$. Then

$u' : J \to J$. Let $n = |Rngu'|$. Then $n < |J|$ because u hence u' is not one to one. We correlate a permutation v to u' as follows: Let $y \in {}^nJ$ such that

$$y(0) < \cdots y(n-1).$$

and

$$\{u'_{y(0)}, \cdots u'_{y(n-1)}\} = Rng(u").$$

For $i < n$ put

$$m_{y(i)} = \min(u^{-1}(u'_{y(i)}).$$

Choose $v \in S_J$ (a permutation on J) such that

$$v(m_{y(i)}) = u'_{y(i)}.$$

Such a v obviously exists. Let

$$I = J \smallsetminus \{m_{y(0)}, \cdots m_{y(l-1)}\};$$

suppose that

$$I = \{j_0, \cdots j_k\}, k \in \omega \text{ and } j_0 < j_1 \cdots < j_k.$$

Let $t_i^l(x) = C_l(x) \cap D_{il}$ and let $v* = v \cup ID_{\alpha - J}$. Then v^* is a permutation on α, and it is not difficult to check that

$$t_{j(k)}^{u_j(k)} \circ \cdots t_{j(0)}^{u_j(0)} E(v^*, \top) = E(u, \top).$$

From case (a), we get that $E(u, \top) \in A$. Now let $u \in {}^\alpha\alpha^{(Id)}$ and $i, j \in ker(u)$. Then

$$E(u, x_i = x_j) = E(u, T) \cap D_{ij} \in A.$$

Also for all $u, v \in S_\alpha$, we have by symmetry condition that

$$E(u, D_r(x_0 \cdots)) = E(u, D_r(x_{v(0)}, \cdots) = p(u, r).$$

Finally by noting that

$$E(u, \neg\phi) = E(u, \top) \smallsetminus E(u, \phi)$$

we get the desired conclusion. From this it follows that $A \in \cap_{k \in \omega} Nr_\alpha Ws_{\alpha+k}$.

Finally we prove:

Claim 11 . *$A \in INr_\alpha Ws_\beta$ for all $\beta > \alpha$.*

Proof. It suffices by the previous claim to show that $A \in Nr_\alpha Ws_{\alpha+\beta}$ for infinite β. So let $\beta \geq \omega$. Let $V_0 =^\alpha U^{(s)}$ be the unit of A. Let

$$s_\beta = s \cup \langle w : \alpha \leq i < \beta \rangle.$$

Then $s_\beta \in {}^{\alpha+\beta}U^{s_\beta}$. Let V_β be the weak space ${}^{\alpha+\beta}U^{(s_\beta)}$, and let $C(\alpha+\beta)$ or simply $C(\beta)$ be the full Ws_β with unit V_β. For $k < \omega$, let $i(k,\beta)$ be the function with domain $G(k)^{**}$ such that

$$i(k,\beta) = \{s \in V_\beta : s \restriction \alpha + k \in a\}, a \in G(k)^{**}.$$

Then $i(k,\beta) : G(k)^{**} \to C(\beta)$. Let

$$G(\beta) = Sg^{C(\beta)}\{i(k,\beta)G(k)^{**} : 0 \leq k < \omega\}.$$

Then $G(\beta) \in Ws_{\alpha+\beta}$. Moreover it is not difficult to show that (*):

$$G(\beta) = \cup\{i(k,\beta)G(k)^{**} : 0 \leq k < \omega\}$$

This follows from that the left hand side is a subuniverse of of $G(\beta)$. Let $a \in G(\beta)$. We will show that $a \in Nr_\alpha G(\beta)$ iff $a \in i(\beta)A$ From (*) it follows that $a = i(k,\beta)g$ for some k and that $g \in G(k)^{**}$. Now suppose that $a \in Nr_\alpha G(\beta)$. Then $g = C_j g$ for all $\alpha \leq j < \alpha + k$ and so $g \in Nr_\alpha G(k)^{**}$. By Claim 8 we have

$$Nr_\alpha G(k)^{**} = i(k)A$$

thus it follows that $g = i(k)g'$, say, for some $g' \in A$. But

$$a = i(l,\beta)g = i(l,\beta) \circ i(n,l)g' = i(n,\beta)g'.$$

This shows that $a \in i(\beta)A$. By this the proof is complete. ■

Bibliography

[1] Adamek,J.,Herrlich,H. and Strecker,G.,*Abstract and concrete categories, or the joy of cats.* John Wiley and sons.1990.

[2] Amer,M., *Cylindric algebras of sentences.* Abstract. Journal of Symbolic logic, volume 58, number 2, p.743, 1993.

[3] Andréka,H., *Complexity of equations valid in algebras of relations.* Annals of Pure and Applied logic, volume 89, p. 149 - 209, 1997.

[4] Andréka, H., *Atomic representable relation and cylindric algebras wth non representable completions.* Manuscript.

[5] Andréka H., Comer,J., Németi,I., *Epimorphisms in cylindric algebras.* Preprint.

[6] Andréka,H., Goldblatt, R., Németi, I., *Relativised quantification: some canonical varieties of sequence-set algebras.* The Journal of Symbolic Logic, volume 63, number 3, p. 163-184, March 1998.

[7] Andréka,H., Givant,S., Mikulas,S. Németi,I.,Simon A., *Notions of density that imply representability in algebraic logic.* Annals of pure and applied logic, volume 91, p.93-190, 1998.

[8] Andréka H., Givant,S., Németi I., *Decision problems for equational theories of relation algebras.* Memoirs, volume 126, American Mathematical Society, Providence, Rhode Island, 1997.

[9] Andréka,H., Gregely,T., and Németi,I., *On universal algebraic constructions of logics.* Studia Logica , volume 36, p.9-47, 1977.

[10] Andréka,H., Van Banthem.J., and Németi.I. *Modal languages and bounded fragments of predicate logic.* Journal of Philosophical Logic, volume 27, p 217-274, 1998.

[11] Andréka.H.,Kurucz.A.,Németi,I.,Sain.,I *Applying algebraic logic; a general methodology.* Preprint Math.Inst., Budapest, 1994, 72 pages. An overview . In Logic Colloquium 92; editors Csirmaz, L.Gabbay, D., and Rijke.M., Studies in Logic Language and Computation. CSLI Publications. 1995 p.1-60

[12] Andréka,H, Monk., J.D., Németi,I. (editors) *Algebraic Logic.* North- Holland Amsterdam, 1991.

[13] Andréka,H., Németi,I., *A simple purely algebraic proof of the completeness of some first order logics.* Algebra Universalis. vol 5, p. 8-15, 1975.

[14] Andréka,H., Németi,I., *Neat reducts of varieties* Studia. Sci. Math. Hungarica. vol 13, p.47-51, 1978.

[15] Andreka,H.,Nemeti,I.,

On Systems of varieties definable by schemes of equations. Algebra Universalis, vol 11, p.105-116, 1980.

[16] Andréka,H., Németi,I., *Definability of substitutions in Gs's.* Preprint. Mathematical Institute. Hungarian Academy of Sciences.

[17] Andréka,H., Németi,I., *On the Number of Generators in Cylindric algebras* Journal of Symbolic Logic, vol 50, number 4 , December 1985.

[18] Andréka H., Németi,I., Sain,I., *Algebraic Logic.* In handbook of Philosophical Logic, second edition. Gabbay, D., and Guenthner F., editors, Kluwer, second edition, 2001.

[19] Andréka, H.,Németi,I. and Madarász,J., *Relativity Theory.* Electronically available at http://www.math-inst.hu/pub/algebraic-logic

[20] Andréka.,H, Sain I., *Connections between initial algebra semantics of CF languages and and algebraic logic* Mathematical Logic in Computer Science.p.25-83.

[21] Andréka,H., Németi,I., and Sáyed Ahmed,T., *On neat reducts of algebras of logic.* Presented in Logic Colloquium 1996. Abstract appeared in the Bulletin of Journal of Symbolic Logic. Vol 3. Number 2. p. 249. June 1997.

[22] Andréka.,H, Németi.,I, and Sáyed Ahmed,T., *Amalgamation in algebras of logic.* Presented in the "Universal Algebra" conference held in Szeged , Hungary 1998. In preparation.

[23] Andréka,H., and Sayed Ahmed,T., *Omitting types in logics with finitely many variables.* Presented in the Logic Colloquium 1998. Abstract appeared in the Bulletin of the Journal of Symbolic Logic. Vol 5, Number 1, p. 88, March 1999.

[24] Andréka,H., and Thompson, R.J., *A stone-type representation theorem for algebras of relation of higher rank.* Transactions of the American Mathematical Society, volume 309. number 2, p. 671-682, 1988.

[25] Anellis, H., Houser N., *Nineteenth Century Roots of Algebraic Logic and Universal Algebra.* In [12] p.1-37. 1991.

[26] Azriel L., *Basic set Theory* Springer-Verlag Berlin Heidelberg. New York, 1979.

[27] Basich, P.D. *Amalgamation properties and interpolation theorems for equational theories.* Algebra Universalis, vol 5, p.45-55. 1975

[28] Blok,W.J., and Pigozzi,D. *Algebraizable logics.* Memoirs of American Mathematical Society, vol 77, Number 396, 1989.

[29] Biró, B. *Non-finite axiomatizability results in algebraic logic.* Journal of Symbolic Logic, Vol 57, Number 3, p.832-843. 1992

[30] Biro,B. Shelah,S. *Isomorphic but not lower base isomorphic cylindric set algebras* The Journal of Symbolic Logic, volume 53, Number 3, p. 846-853, Sep 1988.

[31] S.Burris and H.P. Sankappanavar. *A Course in Universal Algebra.* Graduate Texts in Mathematics. Springer-Verlag, New York, 1981.

[32] Casanovas, E., and Farre, R. *Omitting Types in Incomplete Theories.* Journal of Symbolic logic, Vol. 61, Number 1, p. 236-245. March 1996.

[33] Comer.D., *A Sheaf theoretic duality theory for cylindric algebras* Transactions American Mathemnatical Society vol 169, p. 75-87.

[34] Craig,W. *Logic in algebraic form.* North Holland, Amsterdam (1974), 204 pages.

[35] Chang.C, Keisler. J *Model Theory.* North Holland, 1994.

[36] Daigneault,A., *Freedom in polyadic algebras and two theorems of Beth and Craig.* Michigan Math.J.11 (1964) 129-135.

[37] De Morgan *On the Syllogism, no iv, and on the logic of relations*, *Transactions of the Cambridge Philosophical Society*, 10;331-358, republished in [38].

[38] De Morgan *On the Syllogism and other logical writings* Rare masterpieces of philosophy and Science. Routledge and Kegan Paul, 1966. W.Stark.

[39] Daigneault,A., and Monk,J.D., *Representation Theory for Polyadic algebras.* Fund.Math. Vol 52 p.151-176, 1963

[40] Elklof P. *Ultraproducts for Algebraists* In Handbook of Mathematical Logic. Editor. Barwise J., North-Holland, Amsterdam. 1991's edition.

[41] Ferenczi M., *On representability of cylindric algebras.* Abstracts of papers presented to the American Mathematical Society, vol 13, No 3, p. 336, 1992.

[42] Ferenczi, M., *On representability of neatly embeddable cylindric algebras.* Journal of applied non-classical logics. To appear.

[43] Goldblatt, R, *On the Role of the Baire Category Theorem and Dependent Choice in the Foundations of Logic.* The Journal of Symbolic Logic, vol 50, num 2, p.412-422, June 1985.

[44] Goldblatt R.,*Varieties of Complex algebras.* Annals of Pure and Applied Logic, vol 38 p. 173-241, 1989.

[45] Goldblatt R.,*Algebraic Polymodal Logic: A survey* Logic Journal of the IGPL, vol 8, No 4. p.393-450, 2000.

[46] Goldblatt R., *Persistance and atomic generation for atomic boolean algebras with operators.* Studia Logica vol 68, number 2, p.155-171, 2001.

[47] Halmos P., *Algebraic Logic.* Chelsea Publishing Co., New York, 1962.

[48] Halmos P., An autobiography of Polyadic Algebras. Logic Journal of IGPL, vol 8, No 4, pp. 363-392

[49] Henkin,L. Monk, J.D. *Cylindric algebras and related structures.* Proc. Tarski Symp., American Mathematical Society, vol 25, p. 105-121. 1974.

[50] Henkin,L., Monk,J.D., and Tarski, A., *Cylindric Algebras Part I.* North Holland, 1971.

[51] Henkin,L., Monk,J.D., and Tarski,A., *Cylindric Algebras Part II.* North Holland, 1985.

[52] Henkin,L., Monk,J.D., Tarski,A., Andréka,H., and Németi,I., *Cylindric Set Algebras*, lecture notes in Mathematics 883, Springer Verlag, 1981.

[53] Hirsch,R., and Hodkinson,I., *Step by step-building representations in algebraic logic.* Journal of Symbolic Logic, vol 62, number 1, p.225-279, 1997.

[54] Hirsch,R., and Hodkinson,I., *Complete representations in algebraic logic.* Journal of Symbolic Logic vol 62, number 3, p.816-847, 1997.

[55] Hirsch,R., and Hodkinson, I. *Relational algebras with n-dimensional relational basis* Annals of Pure and Applied Logic, (101) p. 227-274 , 2000

[56] Hirsch R. and Hodkinson, I. *Relation algebras from cylindric algebras,I* Annals of Pure and Applied Logic, to appear.

[57] Hirsch R. and Hodkinson, I. *Relation algebras from cylindric algebras,II* Annals of Pure and Applied Logic, to appear.

[58] Hirsch,R., and Hodkinson,I., *Strongly representable atom structures.* To appear in the Proceedings of the American Mathematical Society.

[59] Hirsch R., and Hodkinson .I. *Synthesising axioms by games.* a CD-ROM of essays dedicated to Johan Van-Banthem on occasion of his 50th birthday.

[60] Hirsch R., Hodkinson.I., *Relation algebras by games.* Draft for a new book on relation algebras.

[61] Hirsch,R., and Hodkinson,I., Maddux,R. *Relation algebra reducts of cylindric algebras and an application to proof theory.* To appear in the Journal of Symbolic Logic.

[62] Hirsch R., Hodkinson. I., , Maddux, R., *On provability with finitely many variables.* To appear in the Bulletin of the Journal of Symbolic Logic.

[63] Hirsch R., and Sayed Ahmed T.,*The class of relation algebras with n-dimensional hyperbasis and the class* $RaCA_n$ *are not elementary.* Manuscript 2001.

[64] Hirsch,R. and Sayed Ahmed T., *Neat embeddability is not sufficient for complete representations.*, Manuscript 2001.

[65] Hodkinson, I. *Atom structures of cylindric algebras and relation algebras.* Annals of pure and applied logic, vol 89 , 117-148, 1997.

[66] Hodkinson,I., Mikulás, Sz., *Non-finite axiomatizability of algebras of relations.* Algebra Universalis, vol 43, p. 127-156, 2000.

[67] Hodges,W. *Model Theory*, volume 42 of Encyclopedia of mathematics and its applications. Cambridge University Press. 1993.

[68] Hodges W. *A shorter Model Theory* Cambridge. University press. 1995

[69] Johnson and Tarski *Boolean algebras with operators, Part 1* American Journal of Mathematics, vol 73 (1951) p. 891-939.

[70] Johnson, J.S. *Non-finitizability of classes of representable polyadic algebras.* Journal of Symbolic Logic, vol 34, p. 344-354 .1969

[71] Johnson, J.S. *Amalgamation of Polyadic Algebras.* Trans. Amer. Math.Soc., vol 17, p.834. 1970

[72] Gabbay D.M. *An irreflexitivity lemma with applications to axiomatizations of conditions in linear frames*, in U.Monnich (editor) , *Aspects of Philosophical Logic* Reidel, Dordrecht, 1981.

[73] Givant. S, Venema. Y *The preservation of Sahlqvist equations in completions of Boolean algebras with operators.* Algebra Universalis, vol 41, p. 47-84, 1999.

[74] Lyndon. I, *The representation of relational algebras.* Annals of Mathematics, vol 51 no 3. p. 707-729. 1950.

[75] Lyndon. I, *Relation algebras and Projective Goemetries.* Michigan Mathematics Journal, vol 8, p. 207-210, 1961.

[76] Madarász, J *Interpolation and Amalgamation; Pushing the Limits. Part I* Studia Logica, vol 61, p. 316-345, 1998.

[77] Madarász, J *interpolation and Amalgamation; Pushing the Limits. Part II* Studia Logica, vol 62, p. 1-19, 1999.

[78] Madarász.J *Interpolation in algebraizable logics; semantics for non-normal multi-modal logic.* Journal of Applied non Classical logics. Vol 8, no 1-2 1998 p. 67-105.

[79] Madarász,J. Sayed Ahmed, T. *On amalgamation of algebras of logic.* In preparation.

[80] Madarász,J. Sayed Ahmed, T. *Amalgamation, interpolation and epimorphisms, solutions to all problems of Pigozzi's paper , and some more.* In preparation.

[81] Maddux ,R., *Relation algebras and neat embeddability of cylindric algebras.* Notices American Math. Society. vol 24. p.298, 1977.

[82] Maddux,R., *Topics in relation algebras* P.h.D dissertation. University of California, Berkley. 1978.

[83] Maddux,R., *A sequent calculus for relation algebras* Annals of pure and applied logic, vol 25, p. 73-101, 1983.

[84] Maddux R., *Non-finite axiomatizability results for cylindric and relational algebras.* The Journal of Symbolic Logic, vol 54, number 3, p. 951-974.

[85] Maddux,R., *The neat embedding problem and the number of variables in proofs.* Proceedings of American Mathematical Society, vol 112, p.195-202.

[86] Maddux,R., Tarski,A. *A sufficient condition for the representability of relation algebras.* Notices American Math.Society, 23:A-477, 1976.

[87] Maksimova .L. *Amalgamation and interpolation in normal modal logics.* Studia Logica, vol 50, p.451-471, 1991.

[88] Marx. M. *Algebraic relativization and arrow logic.* PhD thesis, University of Amsterdam, 1995.

[89] Marx M. *Amalgamation in relation algebras.* Journal of Symbolic Logic, vol 63, number 2 June 1998. p. 479-484.

[90] Marx.M, *Relativized relation algebras* Algebra Universalis, vol 41 23-45, 1999.

[91] Marx.M. *Amalgamation in finite dimensional cylindric algebras.* Algebra Universalis, vol 43, p. 41-49. 2000

[92] Marx M. *Algebraic relativization and arrow logic.* PhD thesis, Institute for Logic, Language and Computation, University of Amsterdam, ILLC Dissertation Series 1995.

[93] Marx M, Polos L, Masuch M, editors *Arrow logic and Multi-modal logic* 1996 Studies in Logic Language and Information. CSLI Publications, Centre for the Study of Language and Information.

[94] Mikulús, Sz., Németi, I. *Taming logic.* Journal of Logic, Language and information, p. 207-226, 1995.

[95] Monk., J.D., *Non-finitizability of classes of representable cylindric algebras.* Journal of Symbolic Logic. vol 34, p.331-343. 1969.

[96] Monk,.J.D., *Provability with finitely many variables* Proceedings of the American Mathematical Society, vol 27, p.353-352 1971.

[97] Monk, J.D., *Omitting types algebraically* Ann. Sci. Univ. Clermont Math., No 16, p. 101-105.

[98] Monk,.J.D.,*Connections between combinatorial theory and algebraic logic* In Studies in Math, vol 9, Math Assoc Amer p. 58-91, 1974.

[99] Monk.,J.D., *Remarks on the problems in the books Cylindric Algebras, Part I and Part II and Cylindric set algebras.* In [12] p.723-726

[100] Monk.,J,D *Open problems* In [12]p 727-746

[101] *Handbook of boolean algebras.* Edited by J.D.Monk.

[102] Monk.J.D, *An introduction to cylindric set algebras.* Logic Journal of IGPL, vol 8, no 4, pp 346-392.

[103] Miller, A. *Covering* ${}^{\omega}2$ *with* ω_1 *disjoint closed sets.* The Kleene Symposuim (proceedings, Madison, Wisconsin, 1978). Studies in Logic and the Foundation of Mathematics, vol 101, North-Holland, Amsterdam, 1980 pp. 415-421.

[104] Németi,I., *The Class of Neat Reducts of Cylindric Algebras is Not a Variety But is closed w.r.t. HP.* Notre Dame Journal of Formal logic, Volume 24, Number 3, July 1983, pp 399-409.

[105] Németi , I. *Cylindric relativized set algebras have the strong amalgamation property.* Journal of Symbolic Logic, vol 50, number 3 p.689-700 1985.

[106] Németi,I., *Strong representability of Fork Algebras, a set theoretic foundation.* Journal of IGPL, vol 5. number 1, p. 2-23 1997.

[107] Nemeti,I. *Free algebras and Decidability in Algebraic Logic.* Third Doctoral Dissertation . Mathematical Institute. Hung. Acad.Sciences. Budapest Hungary In Hungarian. The english version is [108].

[108] Németi,I. *Decidability of weakened versions of first order logic* In [93].

[109] Németi,I., *Algebraisation of quantifier logics, an introductory overview.* Math.Inst.Budapest, Preprint, No 13-1996. A shortened version appeared in Studia Logica vol 50.4 p.465-569. 1991.

[110] Németi,G.Sági *On the equational theory of representable polyadic algebras.* Journal of Symbolic logic, 2000 vol 65 p. 1143-1167.

[111] Németi. I, Simon A. *Relation algebras from cylindric and polyadic algebras.* Logic Journal of IGPL, vol 5 1997 p. 575-588

[112] Newelski, L. *Omitting types and the real line.* JSL vol 52 1987 p.1020-1026.

[113] Pigozzi,D., *Amalgamation, congruence extension, and interpolation properties in algebras.* Algebra Universalis. Vol 1. p.269-349. 1971.

[114] Sain I. *Beth and Craig's properties via epimorphism in Algebraic Logic.* Algebraic Logic and universal algebra in Computer Science , volume 425 of Lecture notes in Computer Science, pages 209-226. Springer-Verlag, Berlin 1990.

[115] Sain, I. Gyuris, V., *Finite Schematizable Algebraic Logic.* Preprint, Math.Inst. Hng.Acad.Sci., Budapest, 1996. IGPL

[116] Sain,I. *Searching for a finitizable algebraization of first order logic.* Logic Journal of IGPL. Oxford University Press. vol 8., no 4, July 2000. p.495-589.

[117] Sain,I. Németi. I. *Fork algebras in usual and in non-well founded set theories (An overview)* Preprint

[118] Sain,I. Thompson,R, *Strictly finite schema axiomatization of quasi-polyadic algebras*. In [12] p.539-571.

[119] Sagi.G. *Non computability of the axioms of the ω dimensional polyadic algebras.* Preprint . Submitted.

[120] Sági, G. *On the Finitization Problem of Algebraic Logic.* Ph.D dissertation. Budapest 1999.

[121] Sagi, G., A completeness theorem for higher order logics JSL volume 65 no 3, June 2000. p.857-884.

[122] Sagi, G, Sayed Ahmed T. *Németi's directed cylindric algebras have the strong amalgamation property*. Manuscript. 1999

[123] Sagi,G, Sayed Ahmed T., Sain I. *The Finitization Problem in algebraic logic, a survey* Manuscript 2000.

[124] Sagi, G. *On the Finitization Problem of Algebraic Logic.* Ph.D dissertation. Budapest 1999.

[125] Sereny, G. *Neatly atomic cylindric algebras and representable isomorphisms.* In Algebraic Logic p.637-645, edited by Andreka, H., Nemeti.I., and Monk.J.D., North Holland, Amsterdam. 1991.

[126] Shelah. S *Classification Theory* Second Edition North Holland P.C., Amsterdam 1990.

[127] Shelah.S *On a Problem in cylindric algebra* In Algebraic Logic p. 645-664, edited by Andreka, H., Nemeti.I., and Monk.J.D., North Holland, Amsterdam. 1991.

[128] Sayed Ahmed, T. *Algebras of sentences of Logic*. Masters Thesis, Cairo University. 1998.

[129] Sayed Ahmed, T. *The class of* 2*-dimensional neat reducts of polyadic algebras is not elementary.* Fundementa Mathematicea, to appear.

[130] Sayed Ahmed, T. *On amalgamation of reducts of polyadic algebras.* Algebra Universalis, to appear. Electronically available at http://www.math-inst.hu/pub/algebraic-logic

[131] Sayed Ahmed, T. *On a stronger version of the finitization problem.* Presented in Conference in Algebra (in honour of the 70th birthday of Ervin Fried). August 17-21, 1999. Alfred Rényi Institute of Mathematics, Budapest, Hungary. Electronically available at http:// www.renyi.hu/ rabbit/.

[132] Sayed Ahmed T. *On amalgamation of algebras of logics, a functorial formulation* In preparation.

[133] Sayed Ahmed T. *Omitting types, complete representability and atom-canonicity.* Submitted

[134] Sayed Ahmed T. *The class of neat reducts is not elementary.* Abstract. Bulletin of the Journal of Symbolic Logic, vol 5, no 3, September 1999. p.407-408.

[135] Sayed Ahmed T. *The class of neat reducts is not elementary.* Logic Journal of IGPL, volume 9, 2001 p. 31-65 electronically available at http://www.math-inst.hu/pub/algebraic-logic

[136] Sayed Ahmed T. *Martin's axiom, omitting types and complete representations in algebraic logic* . To appear in a special echoe of Studia Logica dedicated to many dimensional logical systems.

[137] Sayed Ahmed T. *The class of strongly representable atom structures of* RCA_3 *is not elementary. In particular,* RCA_3 *is not single-persistent.* Submitted.

[138] Sayed Ahmed T. *On neat reducts and amalgamation* Presented in Logic colloquium . Paris 2000. Abstract in the Bulletin of Symbolic Logic, volume 7, number 1 March 2001.

[139] Sayed Ahmed T.*On the completeness and incompleteness of finite variable fragments of first order logic.* Manuscript.

[140] Sayed Ahmed T. *On the closure of the class of neat reducts under various operators* Preprint of the mathematical institute of the Hungarian academy of Sciences.

[141] Sayed Ahmed T. *Which reducts of polyadic algebras are atom-canonical?* Preprint.

[142] Sayed Ahmed T.*A Model-theoretic Solution to a problem of Tarski* Mathematical Logic Quaterly, to appear.

[143] Sayed Ahmed T.RCA_ω *is not Sahlqvist, and is not closed under minimal completions.* Manuscript.

[144] Sayed Ahmed T.*Reducts of* L_3 *has Godel's Incompleteness, therefore the free quasipolyadic algebras of dimension* 3 *are not atomic.* Manuscript.

[145] Sayed Ahmed, T. Németi, I. *On neat reducts of algebras of logic.* Studia Logica, vol 62, number 2, 2001 p.229-262

[146] Simon,A. *What the finitization problem is not.* Algebraic methods in Logic and Computer Science, Banach Centre Publications, Vol 28, 95-116.

[147] Simon,A. *Non representable algebras of relations.* Ph.D Dissertation. Budapest 1997.

[148] Simon A., *Finite schema completeness for typeless logics and representable cylindric algebras* In [?] p. 665-670.

[149] Tarski,A., Givant,S. *A formalization of set theory without variables.* AMS Colloquium Publications vol. 41, 1987.

[150] Venema Yde *Cylindric modal logic* Journal of Symbolic Logic. Volume 60, no 2, June 1995, p.591-623.

[151] Venema Yde *Rectangular games* Journal of symbolic Logic vol 63, Number 4, Dec 1998, p.1549-1564

[152] Venema Yde Atom structures and Sahlqvist equations Algebra Universalis. 38:185-199, 1998.

[153] Weaver, G., Welaish J., *Back and forth constructions in modal logic; an interpolation Theorem for a family of modal logics.* The Journal of Symbolic Logic, vo. 51, Number 4 p. 969-980, December 1986.