We will prove that $c_k = \infty$.

Intuitive idea of the proof: We will see that there is a neighborhood $S(p,\varepsilon) \subseteq {}^nF$ such that k "sees" all those events which m sees in $S(p,\varepsilon)$. By $c_m = \infty$, observer m sees three photons such that they "form a triangle" inside $S(p,\varepsilon)$. See Figure 39. Since k sees all events which m sees in $S(p,\varepsilon)$, those three photons form a triangle in the world-view of k, too. This can only happen if $c_k = \infty$, and this will complete the proof.

Formally: Let $p \in Dom(\mathsf{f}_{mk})$. Such a p exists by $Rng(w_m) \cap Rng(w_k) \neq \emptyset$. Now by $\mathbf{Ax6_{01}}$ there is $\varepsilon \in {}^+F$ such that $S(p,\varepsilon) \subseteq Dom(\mathsf{f}_{mk})$. Let such an ε be fixed. Let $q,r \in S(p,\varepsilon)$ such that p,q,r are non-collinear and $ang^2(\overline{pq}) = ang^2(\overline{qr}) = ang^2(\overline{pr}) = \infty$. Such q,r exist by $n \geq 3$. By $\mathbf{Ax5^{Ph}}$ and $c_m = \infty$, there are $ph_1, ph_2, ph_3 \in Ph$ such that $tr_m(ph_1) = \overline{pq}$, $tr_m(ph_2) = \overline{qr}$, $tr_m(ph_3) = \overline{pr}$. Let such ph_1, ph_2, ph_3 be fixed. We have

$$\begin{aligned} ph_1 &\in w_m(p) \cap w_m(q), & ph_1 &\notin w_m(r),\\ ph_2 &\in w_m(q) \cap w_m(r), & ph_2 &\notin w_m(p),\\ ph_3 &\in w_m(p) \cap w_m(r), & ph_3 &\notin w_m(q). \end{aligned} \tag{35}$$

By $p,q,r \in S(p,\varepsilon) \subseteq Dom(\mathsf{f}_{mk})$, there are $p',q',r' \in {}^nF$ such that

$$w_m(p) = w_k(p'),\ \ w_m(q) = w_k(q') \ \text{ and } \ w_m(r) = w_k(r'). \tag{36}$$

By (35), (36) and $\mathbf{Ax3_0}$, we have that p',q',r' are non-collinear and $tr_k(ph_1) = \overline{p'q'}$, $tr_k(ph_2) = \overline{q'r'}$ and $tr_k(ph_3) = \overline{p'r'}$. By this and by $\mathbf{AxE_{00}}$, we have $v_k(ph_1) = v_k(ph_2) = v_k(ph_3) = \infty$. Hence $c_k = \infty$.

Proof of (ii): Item (ii) follows by item (i) because

$$\mathbf{Ax6_{00}} \models (m \overset{\odot}{\to} k \ \Rightarrow Rng(w_m) \cap Rng(w_k) \neq \emptyset).$$

Proof of (iii): Throughout the proof the reader is asked to consult Figure 40.

Let $\mathfrak{M}$ be a frame model of $Bax \setminus \{\mathbf{AxE_{01}}\}$. Let $m,k \in Obs$ with

$$Rng(w_m) \cap Rng(w_k) \neq \emptyset \ \text{ and } \ c_m \neq 0.$$

We will prove that $c_k \neq 0$.

Intuitive idea of the proof: We will see that there is a neighborhood $S(p,\varepsilon) \subseteq {}^nF$ such that k "sees" all those events which m sees in $S(p,\varepsilon)$. Now by $c_m \neq 0$, m sees two photons intersecting each other in one point which is in $S(p,\varepsilon)$. See Figure 40.

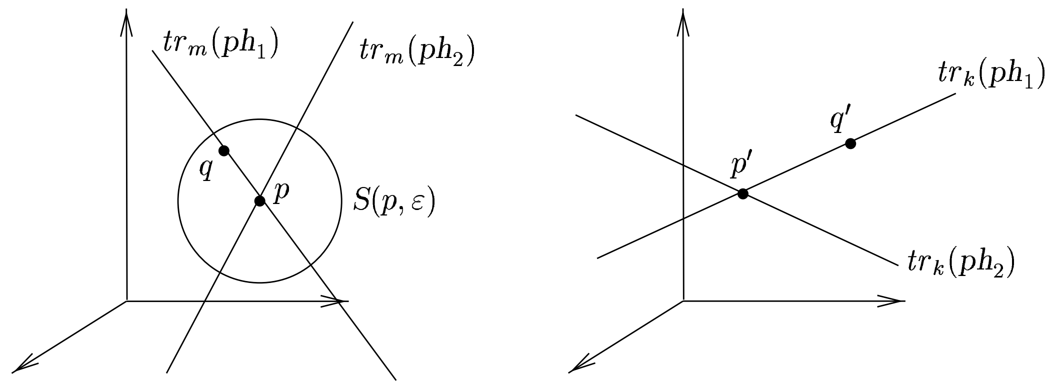

Figure 40: Illustration for the proof of Lemma 3.76(iii).

But then k sees these two photons intersecting each other in one point. See Figure 40. But this implies $c_k \neq 0$, and this will complete the proof.

Formally: Let $ph_1, ph_2 \in Ph$ such that $tr_m(ph_1) \neq tr_m(ph_2)$ and $p \in tr_m(ph_1) \cap tr_m(ph_2)$. Such ph_1, ph_2 exist because of the following. By $\mathbf{Ax5^{Obs}}$, we have $m \overset{\odot}{\to} ph$, for some $ph \in Ph$. Let such a ph be fixed. Let A be the linear transformation which takes $1_t, 1_x, 1_y, e_3, \dots, e_{n-1}$ to $1_t, -1_y, 1_x, e_3, \dots, e_{n-1}$, respectively (for $n = 3$ A is the rotation around $\bar{t}$ axis with 90 degrees). Now let $\ell_1, \ell_2 \in \mathsf{Eucl}$ such that $p \in \ell_1 \cap \ell_2$, $\ell_1 \parallel tr_m(ph)$, $\ell_2 \parallel A[tr_m(ph)]$. Obviously $\ell_1 \neq \ell_2$ and $ang^2(\ell_1) = ang^2(\ell_2) = v_m(ph)$. Hence by $\mathbf{Ax5^{Ph}}$, there are $ph_1, ph_2 \in Ph$ such that $tr_m(ph_1) = \ell_1$ and $tr_m(ph_2) = \ell_2$. For such ph_1 and ph_2, $tr_m(ph_1) \neq tr_m(ph_2)$ and $p \in tr_m(ph_1) \cap tr_m(ph_2)$ hold.

$p \in Dom(\mathsf{f}_{mk})$ and $\mathbf{Ax6_{01}}$ implies that $S(p,\varepsilon) \subseteq Dom(\mathsf{f}_{mk})$, for some $\varepsilon \in {}^+F$. Let such an ε be fixed. Let $q \in S(p,\varepsilon)$ such that

$$ph_1 \in w_m(q) \quad \text{and} \quad ph_2 \notin w_m(q). \tag{37}$$

By $p, q \in S(p,\varepsilon) \subseteq Dom(\mathsf{f}_{mk})$ there are $p', q' \in {}^nF$ such that

$$w_m(p) = w_k(p') \quad \text{and} \quad w_m(q) = w_k(q'). \tag{38}$$

$p \in tr_m(ph_1) \cap tr_m(ph_2)$, (37) and (38) implies that $p' \in tr_k(ph_1) \cap tr_k(ph_2)$ and $tr_k(ph_1) \neq tr_k(ph_2)$. This means that observer k "sees" two photons which traces are different and contain point p'. But $tr_k(ph_1), tr_k(ph_2) \in \mathsf{Eucl}$ by **Ax1**, **Ax2**, $\mathbf{Ax3_0}$. By this, we conclude that $c_k \neq 0$ because there is exactly one $\ell \in \mathsf{Eucl}$ such that $p' \in \ell$ and $ang^2(\ell) = 0$.

Proof of (iv): Item (iv) follows from item (iii) because

$$\mathbf{Ax6_{00}} \models (m \overset{\odot}{\to} k \;\Rightarrow\; Rng(w_m) \cap Rng(w_k) \neq \emptyset). \blacksquare$$

Prop.3.77 below is an analogon of Prop.2.6(iii) (§2.3).

PROPOSITION 3.77

(i) $Bax \setminus \mathbf{AxE_{01}} \models (\forall m \in Obs)\big(c_m \neq 0 \;\Rightarrow\; (w_m \textit{ is an injection})\big).$

(ii) $Bax \models (\forall m \in Obs)(w_m \textit{ is an injection}).$

Proof: It is enough to prove item (i) because item (ii) follows from item (i). Let $\mathfrak{M}$ be a frame model of $Bax \setminus \{\mathbf{AxE_{01}}\}$. Let $m \in Obs$ with $c_m \neq 0$. Let $p, q \in {}^nF$ with $p \neq q$. We will prove that $w_m(p) \neq w_m(q)$. By $c_m \neq 0$, there is $\ell \in \mathsf{Eucl}$ such that $ang^2(\ell) < c_m$, $p \in \ell$ and $q \notin \ell$. $ang^2(\ell) < c_m$, $\mathbf{Ax5^{Obs}}$ and $\mathbf{AxE_{00}}$ implies that there is $k \in Obs$ with $tr_m(k) = \ell$. For such a k, $k \in w_m(p)$ and $k \notin w_m(q)$. Thus $p \neq q$. $\blacksquare$

Lemma 3.78 below is an analogon of Lemma 3.26 (§3.3).

LEMMA 3.78

(i) $Bax \setminus \{\mathbf{AxE_{01}}\} \models (\forall m, k \in Obs)\big((m \overset{\odot}{\to} k \;\wedge\; c_m \neq 0) \;\Rightarrow\; v_m(k) \neq c_m\big).$

(ii) $Bax \models (\forall m, k \in Obs)(m \overset{\odot}{\to} k \;\Rightarrow\; v_m(k) \neq c_m).$

Proof: It is enough to prove item (i) because item (ii) follows from item (i). The proof goes by contradiction. Let $\mathfrak{M}$ be a frame model of $Bax \setminus \{\mathbf{AxE_{01}}\}$. Let $m, k \in Obs$ with $m \overset{\odot}{\to} k$, $c_m \neq 0$ and $v_m(k) = c_m$. By $\mathbf{Ax5^{Ph}}$ and $v_m(k) = c_m$, there is $ph \in Ph$ such that $tr_m(ph) = tr_m(k)$. Let such a ph be fixed. Let $p, q \in tr_m(k)$ such that $p \neq q$. Then $w_m(p) \neq w_m(q)$ by Prop.3.77(i). Now by $\mathbf{Ax6_{00}}$ we have that

(39) $$w_m(p) = w_k(p') \;\text{ and }\; w_m(q) = w_k(q'),$$

for some $p', q' \in {}^nF$. Let such p', q' be fixed. By $w_m(p) \neq w_m(q)$ and (39), we have $p' \neq q'$. Further $ph, k \in w_k(p') \cap w_k(q')$ because $ph, k \in w_m(p) \cap w_m(q)$. By $k \in w_k(p') \cap w_k(q')$ and $\mathbf{Ax4}$, we have $\overline{p'q'} = \bar{t}$. Now $\overline{p'q'} = \bar{t}$, $ph \in w_k(p') \cap w_k(q')$ and $\mathbf{Ax3_0}$ implies that $tr_k(ph) = \bar{t}$. By this and $\mathbf{AxE_{00}}$, we have that $c_k = 0$. But by Lemma 3.76(iv), it follows that $c_k \neq 0$ because $c_m \neq 0$ and $m \overset{\odot}{\to} k$. $\blacksquare$

Proof of Thm.3.40 for n = 3: It is enough to prove item (ii) because item (i) follows from item (ii).

Assume $n = 3$. Let

$$\mathfrak{M} = \langle (B, Obs, Ph, Ib), \mathfrak{F}, G; \mathrm{E}, W\rangle \models Bax \setminus \{\mathbf{AxE_{01}}\}.$$

Let $m_0, m_1 \in Obs$ with $m_0 \overset{\odot}{\to} m_1$. We have to prove the following. If $c_{m_0} \neq 0$ then $v_{m_0}(m_1) < c_{m_0}$, and if $c_{m_0} = 0$ then $v_{m_0}(m_1) = 0$.

<u>*Case 1: $c_{m_0} \neq 0$ and $c_{m_0} \neq \infty$.*</u> Throughout the proof for Case 1 the reader is asked to consult Figure 41.

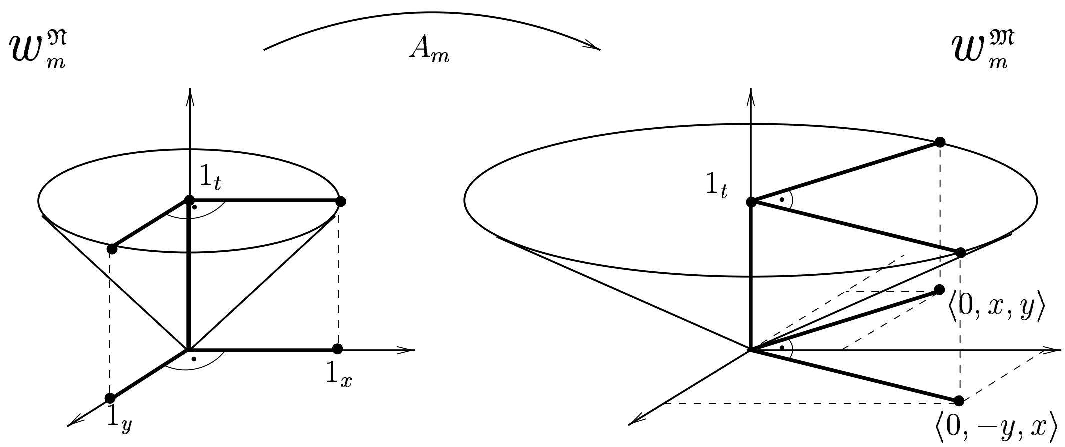

Figure 41: Illustration for the proof of Thm.3.40 for $n = 3$.

<u>*Intuitive idea of the proof:*</u> For the beginning of the intuitive idea of the proof the reader is referred to the formulation of Thm.3.40 in §3.4.2. Recall from there that from the model $\mathfrak{M}$ above we want to construct another model $\mathfrak{N} \in \mathsf{Mod}_{\mathrm{OFG}}(Newbasax)$ such that certain connections between $\mathfrak{M}$ and $\mathfrak{N}$ hold. In particular we will have to check that an observer say m is FTL in $\mathfrak{N}$ iff it is FTL in $\mathfrak{M}$. We will construct $\mathfrak{M}$ in the following way. For each observer m we will change the world-view $w_m^{\mathfrak{M}}$ of m in such a way that the speed of light (for m) becomes 1. See Figure 41. This change will be implemented by using a linear transformation A_m. We note that A_m will leave the time-axis $\bar{t}$ point-wise fixed, will take the vectors $1_x, 1_y$ to two orthogonal vectors $(\langle 0, x, y\rangle, \langle 0, -y, x\rangle)$ of the same length.

Formally: Assume $c_{m_0} \neq 0$ and $c_{m_0} \neq \infty$. Then we define a frame model

$$\mathfrak{N} = \langle (B, Obs^{\mathfrak{N}}, Ph^{\mathfrak{N}}, Ib^{\mathfrak{N}}), \mathfrak{F}, G;\ \mathrm{E}\ , W^{\mathfrak{N}} \rangle$$

as follows.

$$\begin{aligned} Obs^{\mathfrak{N}} &\stackrel{\text{def}}{=} \{m \in Obs\ :\ c_m \neq 0\ \&\ c_m \neq \infty\}, \\ Ph^{\mathfrak{N}} &\stackrel{\text{def}}{=} \{ph \in Ph\ :\ (\exists m \in Obs^{\mathfrak{N}})(m \stackrel{\odot}{\to} ph \text{ holds in } \mathfrak{M})\}, \\ Ib^{\mathfrak{N}} &\stackrel{\text{def}}{=} \{b \in Ib\ :\ (\exists m \in Obs^{\mathfrak{N}})(m \stackrel{\odot}{\to} b \text{ holds in } \mathfrak{M})\}. \end{aligned}$$

Now we are going to define $W^{\mathfrak{N}}$. For every $m \in Obs^{\mathfrak{N}}$ first we will define a linear transformation A_m of ${}^3\mathbf{F}$ and then we will define $w_m^{\mathfrak{N}}$ as follows. Let $m \in Obs$. By $\mathbf{Ax5^{Obs}}$ and $\mathbf{Ax5^{Ph}}$, $m \stackrel{\odot}{\to} ph$ for some $ph \in Ph$ with $\bar{0} \in tr_m(ph)$. Let such a ph be fixed. By $v_m(ph) = c_m \neq \infty$ there is $\langle 1, x, y\rangle \in tr_m(ph)$. Let this $\langle 1, x, y\rangle$ be fixed. Let A_m be the linear transformation of ${}^3\mathbf{F}$ which takes $1_t, 1_x, 1_y$ to $1_t, \langle 0, x, y\rangle, \langle 0, -y, x\rangle$, respectively. By $c_m \neq 0$, we have that A_m is invertible. Let

$$w_m^{\mathfrak{N}} \stackrel{\text{def}}{=} A_m \circ w_m.$$

Now

$$W^{\mathfrak{N}} \stackrel{\text{def}}{=} \{\langle m, p, b\rangle\ :\ m \in Obs^{\mathfrak{N}}\ \&\ p \in {}^3F\ \&\ b \in w_m^{\mathfrak{N}}(p)\}.$$

By the above $\mathfrak{N}$ is defined.

We will prove that (I)–(III) below hold.

(I) $(\forall m \in Obs^{\mathfrak{N}})(\forall b \in B)\left(tr_m^{\mathfrak{M}}(b) = A_m\left[tr_m^{\mathfrak{N}}(b)\right]\ \wedge\ tr_m^{\mathfrak{N}}(b) = A_m^{-1}\left[tr_m^{\mathfrak{M}}(b)\right]\right).$

(II) $(\forall m \in Obs^{\mathfrak{N}})(\forall \ell \in \mathsf{Eucl})(ang^2(\ell) = 1\ \Leftrightarrow\ ang^2(A_m[\ell]) = c_m).$

(III) $(\forall m \in Obs^{\mathfrak{N}})(\forall \ell \in \mathsf{Eucl})(ang^2(\ell) < 1\ \Leftrightarrow\ ang^2(A_m[\ell]) < c_m).$

To prove (I) let $m \in Obs^{\mathfrak{N}}$ and $b \in B$. Then

$$\begin{aligned} tr_m^{\mathfrak{N}}(b) &= \{p \in {}^3F\ :\ b \in w_m^{\mathfrak{N}}(p)\} \\ &= \{p \in {}^3F\ :\ b \in A_m \circ w_m^{\mathfrak{M}}(p)\} \\ &= \{p \in {}^3F\ :\ b \in w_m^{\mathfrak{M}}(A_m(p))\} \\ &= A_m^{-1}\left[\{p \in {}^3F\ :\ b \in w_m^{\mathfrak{M}}(p)\}\right] \\ &= A_m^{-1}[tr_m^{\mathfrak{M}}(b)]. \end{aligned}$$

Hence $tr_m^{\mathfrak{N}}(b) = A_m^{-1}[tr_m^{\mathfrak{M}}(b)]$ and $tr_m^{\mathfrak{M}}(b) = A_m[tr_m^{\mathfrak{N}}(b)]$. To prove (II) and (III) let $m \in Obs^{\mathfrak{N}}$ and let $\ell \in \mathsf{Eucl}$. We will prove that

$$(ang^2(\ell) = 1\ \Leftrightarrow\ ang^2(A_m[\ell]) = c_m) \text{ and } (ang^2(\ell) < 1\ \Leftrightarrow\ ang^2(A_m[\ell]) < c_m).$$

Without loss of generality we can assume that $\bar{0} \in \ell$ and $ang^2(\ell) \neq \infty$ because A_m takes parallel lines to parallel lines and because
$(\forall \ell \in \mathsf{Eucl})(ang^2(\ell) = \infty \;\Leftrightarrow\; ang^2(A_m[\ell]) = \infty)$. By $\bar{0} \in \ell$ and $ang^2(\ell) \neq \infty$, we have $\ell = \overline{\bar{0}\langle 1, \lambda, \mu\rangle}$, for some $\lambda, \mu \in F$. Let this λ, μ be fixed. By the definition of A_m, A_m takes $1_t, 1_x, 1_y$ to $1_t, \langle 0, x, y\rangle, \langle 0, -y, x\rangle$, for some $x, y \in F$ with $\langle 1, x, y\rangle \in tr_m(ph)$, for some $ph \in Ph$ with $m \overset{\odot}{\to} ph$ and $\bar{0} \in tr_m(ph)$. Let this x, y and ph be fixed. By $\langle 1, x, y\rangle \in tr_m(ph)$, we have $x^2 + y^2 = c_m$. Now

$$
\begin{array}{lll}
 & ang^2(A_m[\ell]) = c_m & \\
\Leftrightarrow & ang^2(\overline{\bar{0}, \langle 1, \lambda x + \mu y, \mu x - \lambda y\rangle}) = c_m & (\text{by } \ell = \overline{\bar{0}\langle 1, \lambda, \mu\rangle} \text{ and by the def. of } A_m) \\
\Leftrightarrow & (\lambda x + \mu y)^2 + (\mu x - \lambda y)^2 = c_m & (\text{by the def. of } ang^2) \\
\Leftrightarrow & (\lambda^2 + \mu^2)(x^2 + y^2) = c_m & (\text{by computation}) \\
\Leftrightarrow & \lambda^2 + \mu^2 = 1 & (\text{by } x^2 + y^2 = c_m) \\
\Leftrightarrow & ang^2(\ell) = 1 & (\text{by } \ell = \overline{\bar{0}\langle 1, \lambda, \mu\rangle}).
\end{array}
$$

(II) is proved. The proof of (III) is analogous, but for completeness we write down all the details.

$$
\begin{array}{lll}
 & ang^2(A_m[\ell]) < c_m & \\
\Leftrightarrow & ang^2(\overline{\bar{0}, \langle 1, \lambda x + \mu y, \mu x - \lambda y\rangle}) < c_m & (\text{by } \ell = \overline{\bar{0}\langle 1, \lambda, \mu\rangle} \text{ and by the def. of } A_m) \\
\Leftrightarrow & (\lambda x + \mu y)^2 + (\mu x - \lambda y)^2 < c_m & (\text{by the def. of } ang^2) \\
\Leftrightarrow & (\lambda^2 + \mu^2)(x^2 + y^2) < c_m & (\text{by computation}) \\
\Leftrightarrow & \lambda^2 + \mu^2 < 1 & (\text{by } x^2 + y^2 = c_m) \\
\Leftrightarrow & ang^2(\ell) < 1 & (\text{by } \ell = \overline{\bar{0}\langle 1, \lambda, \mu\rangle}).
\end{array}
$$

(III) is proved.

Now we will prove that $\mathfrak{N} \models Newbasax$.

$\mathfrak{N} \models$ **Ax1** by $\mathfrak{M} \models$ **Ax1**.

$\mathfrak{N} \models$ **Ax2** because of the following. Let $m \in Obs^{\mathfrak{N}}$. By $m \in Ib$ and by $m \overset{\odot}{\to} m$ holds in $\mathfrak{M}$, we have $m \in Ib^{\mathfrak{N}}$. Hence $Obs^{\mathfrak{N}} \subseteq Ib^{\mathfrak{N}}$. $Ph^{\mathfrak{N}} \subseteq Ib^{\mathfrak{N}}$ holds by $Ph \subseteq Ib$ and by the definitions of $Ph^{\mathfrak{N}}$, $Ib^{\mathfrak{N}}$. $\mathfrak{N} \models$ **Ax2** is proved.

$\mathfrak{N} \models$ **Ax3$_0$** because of the following. $(\forall b \in Ib^{\mathfrak{N}})(tr_m^{\mathfrak{N}}(b) \in G \cup \{\emptyset\})$ holds because of (I), $\mathfrak{M} \models$ **Ax3$_0$** and $Ib^{\mathfrak{N}} \subseteq Ib$. By (I), we have that

$$(\forall b \in Ib^{\mathfrak{N}})(\forall m \in Obs^{\mathfrak{N}})\Big((m \overset{\odot}{\to} b \text{ holds in } \mathfrak{M}) \Leftrightarrow (m \overset{\odot}{\to} b \text{ holds in } \mathfrak{N})\Big).$$

Hence we have

$$(\forall b \in Ib^{\mathfrak{N}})(\exists m \in Obs^{\mathfrak{N}})(m \overset{\odot}{\to} b \text{ holds in } \mathfrak{N})$$

by the definition of $Ib^{\mathfrak{N}}$. $\mathfrak{N} \models$ **Ax3$_0$** is proved.

$\mathfrak{N} \models \mathbf{Ax4}$ because $\mathfrak{M} \models \mathbf{Ax4}$, because of (I) and because $(\forall m \in Obs^{\mathfrak{N}})\ A_m^{-1}[\bar{t}] = \bar{t}$.

$\mathfrak{N} \models \mathbf{Ax5}$ because of the following. Let $m \in Obs^{\mathfrak{N}}$ and let $\ell_1, \ell_2 \in \mathsf{Eucl}$ with $ang^2(\ell_1) = 1$ and $ang^2(\ell_2) < 1$. We have to prove that there are $ph \in Ph^{\mathfrak{N}}$ and $k \in Obs^{\mathfrak{N}}$ such that $tr_m(ph) = \ell_1$ and $tr_m(k) = \ell_2$. $ang^2(A_m[\ell_1]) = c_m$ and $ang^2(A_m[\ell_2]) < c_m$ hold by (II) and (III). Thus by $\mathfrak{M} \models \{\mathbf{Ax5^{Obs}}, \mathbf{Ax5^{Ph}}, \mathbf{AxE_{00}}\}$, we have that

$$tr_m^{\mathfrak{M}}(ph) = A_m[\ell_1] \text{ and } tr_m^{\mathfrak{M}}(k) = A_m[\ell_2], \tag{40}$$

for some $ph \in Ph$ and $k \in Obs$. Let such ph and k be fixed. By $m \overset{\odot}{\rightarrow} ph$ holds in $\mathfrak{M}$, we have $ph \in Ph^{\mathfrak{N}}$. By $m \in Obs^{\mathfrak{N}}$, we have $c_m \neq 0$ and $c_m \neq \infty$. Hence by $m \overset{\odot}{\rightarrow} k$ and Lemma 3.76, we have that $c_k \neq 0$ and $c_k \neq \infty$. Therefore $k \in Obs^{\mathfrak{N}}$ by the definition of $Obs^{\mathfrak{N}}$. Now by (40) and (I), we get

$$tr_m^{\mathfrak{N}}(ph) = \ell_1 \text{ and } tr_m^{\mathfrak{N}}(k) = \ell_2.$$

Hence $\mathfrak{N} \models \mathbf{Ax5}$.

$\mathfrak{N} \models \mathbf{Ax6_{00}}$ because

$$(\forall m, k \in Obs^{\mathfrak{N}})\Big(w_m^{\mathfrak{N}}[tr_m^{\mathfrak{N}}(k)] = w_m^{\mathfrak{M}}[tr_m^{\mathfrak{M}}(k)] \text{ and } Rng(w_k^{\mathfrak{N}}) = Rng(w_k^{\mathfrak{M}})\Big)$$

and because $\mathfrak{M} \models \mathbf{Ax6_{00}}$.

$\mathfrak{N} \models \mathbf{Ax6_{01}}$ because $\mathfrak{M} \models \mathbf{Ax6_{01}}$, because

$$(\forall m, k \in Obs^{\mathfrak{N}})\left(Dom(\mathsf{f}_{mk}^{\mathfrak{N}}) = A_m^{-1}\left[Dom(\mathsf{f}_{mk}^{\mathfrak{M}})\right]\right),$$

and because A_m is a continuous function.

$\mathfrak{N} \models \mathbf{AxE_0}$ by $\mathfrak{M} \models \mathbf{AxE_{00}}$, by the definition of c_m and by (I) and (II).

By the above, $\mathfrak{N} \models Newbasax$ is proved.

Since $m_0 \neq 0$, $m_0 \neq \infty$ and $m_0 \overset{\odot}{\rightarrow} m_1$, we have $c_{m_1} \neq 0$ and $c_{m_2} \neq \infty$ by Lemma 3.76. Hence $m_0, m_1 \in Obs^{\mathfrak{N}}$. By Thm.3.29, which says that *Newbasax* does not allow FTL observers, we have $ang^2(tr_{m_0}^{\mathfrak{N}}(m_1)) < 1$. By this and by (III), we get $ang^2(A_{m_0}[tr_{m_0}^{\mathfrak{N}}(m_1)]) < c_{m_0}$. By (I), this is equivalent with $ang^2(tr_{m_0}^{\mathfrak{M}}(m_1)) < c_{m_0}$. Hence $v_{m_0}^{\mathfrak{M}}(m_1) < c_{m_0}$.

<u>*Case 2:* $c_{m_0} = \infty$.</u> Assume $c_{m_0} = \infty$. Then $v_{m_0}(m_1) < c_{m_0}$ holds by Lemma 3.78.

Case 3: $c_{m_0} = 0$. Assume $c_{m_0} = 0$. Then we have to prove that $v_{m_0}(m_1) = 0$. Since $c_{m_0} = 0$ and $m_0 \overset{\odot}{\to} m_1$, we have $c_{m_1} = 0$ by Lemma 3.76. Then by **Ax4** and $\mathbf{Ax5^{Ph}}$, there is $ph \in Ph$ such that $tr_{m_1}(m_1) = tr_{m_1}(ph)$. Let this ph be fixed. By $\mathbf{Ax6_{00}}$, we have that $w_{m_0}[tr_{m_0}(m_1)] \subseteq Rng(w_{m_1})$. By this and $tr_{m_1}(m_1) = tr_{m_1}(ph)$, we have $(\forall p \in tr_{m_0}(m_1))\, ph \in w_{m_0}(p)$. Hence $tr_{m_0}(m_1) = tr_{m_0}(ph)$. By this, we have $v_{m_0}(m_1) = 0$ since $v_{m_0}(ph) = c_{m_0} = 0$. ∎

Now we turn to the proof of Thm.3.40 for $n = 4$. As we said, to do this we have to formulate and prove analogous counterparts of theorems and statements of §3.2 and §3.3 for *Bax*.

Claim 3.79 below is an analogon of Claim 2.10(ii) (§2.3).

Claim 3.79 $Bax \models$ (f_{mk} *is a (possibly) partial one-to one function*).

Proof: The proof follows by Prop.3.77(ii). ∎

Lemma 3.80 below is an analogon of Lemma 3.27 (§3.3).

LEMMA 3.80

$$Bax \models (\forall m, k \in Obs)(\forall p, q \in {}^nF)\Big((ang^2(\overline{pq}) = c_m \ \wedge \ p \in Dom(\mathsf{f}_{mk})) \Rightarrow \ q \in Dom(\mathsf{f}_{mk})\Big).$$

Proof: The proof is analogous to the proof of Lemma 3.27. Checking the details is left to the reader. ∎

Thm.3.81 below is an analogon of Thm.3.21 (§3.3).

THEOREM 3.81

$$Bax \models (\forall m, k \in Obs)(Rng(w_m) = Rng(w_k) \ \vee \ Rng(w_m) \cap Rng(w_k) = \emptyset).$$

Proof: The proof follows by Lemma 3.80 as follows. Let $\mathfrak{M}$ be a frame model of *Bax*. Let $m, k \in Obs$ with $Rng(w_m) \cap Rng(w_k) \neq \emptyset$. We will prove that $Rng(w_m) = Rng(w_k)$. To prove this it is enough to prove that $Dom(\mathsf{f}_{mk}) = {}^nF$ and $Dom(\mathsf{f}_{km}) = {}^nF$. We will prove that $Dom(\mathsf{f}_{mk}) = {}^nF$, the proof of $Dom(\mathsf{f}_{km}) = {}^nF$ is analogous. To prove this let $q \in {}^nF$. We will prove that $q \in Dom(\mathsf{f}_{mk})$.

Case 1: $c_m = \infty$. Assume $c_m = \infty$. Then $c_k = \infty$ by Lemma 3.76(i). First we will prove that $m \overset{\odot}{\to} k$. Let $p \in Dom(\mathsf{f}_{km})$. Such a p exists by $Rng(w_m) \cap Rng(w_k) \neq \emptyset$. Then there is $r \in \bar{t}$ such that $ang^2(\overline{pr}) = \infty = c_k$. Let this r be fixed. Since $p \in Dom(\mathsf{f}_{km})$ and $ang^2(\overline{pr}) = c_k$, we have that $r \in Dom(\mathsf{f}_{km})$ by Lemma 3.80.

By $r \in \bar{t} = tr_k(k)$, we have that $k \in w_k(r)$. This and $r \in Dom(\mathsf{f}_{km})$ implies that $k \in w_m(r')$, for some $r' \in {}^nF$. Hence $tr_m(k) \neq \emptyset$, i.e. $m \overset{\odot}{\to} k$. By Lemma 3.78, we have $v_m(k) \neq c_m = \infty$. Since $v_m(k) \neq \infty$, there is $s \in tr_m(k)$ with $ang^2(\overline{sq}) = \infty = c_m$. Now we have $s \in Dom(\mathsf{f}_{mk})$ by $s \in tr_m(k)$ and $\mathbf{Ax6_{00}}$. $s \in Dom(\mathsf{f}_{mk})$ and $ang^2(\overline{sq}) = c_m$ implies $q \in Dom(\mathsf{f}_{mk})$ by Lemma 3.80.

Case 2: $c_m \neq \infty$. Assume $c_m \neq \infty$. Let $p \in Dom(\mathsf{f}_{mk})$. We will show at the end of the proof that

$$(41) \quad (\exists r^0, r^1, \dots, r^n \in {}^nF)\Big(r^0 = p \ \wedge \ r^n = q \ \wedge \ (\forall i \in n) ang^2(\overline{r^i r^{i+1}}) = c_m\Big).$$

Now by (41) and $p \in Dom(\mathsf{f}_{mk})$, by applying Lemma 3.80 n times, we get $q \in Dom(\mathsf{f}_{mk})$. Thm.3.81 is proved modulo (41). To prove (41) we need Claim 3.82 below.

Claim 3.82 *Let $c \in {}^+F$ such that there are $a_1, a_2, \dots, a_{n-1} \in F$ with $c = a_1^2 + a_2^2 + \dots + a_{n-1}^2$. Then the vector-space ${}^n\mathbf{F}$ is generated by*

$$\{\langle 1, p_1, p_2, \dots, p_{n-1}\rangle \ : \ p_1, p_2, \dots, p_{n-1} \in F \ \& \ c = p_1^2 + p_2^2 + \dots + p_{n-1}^2\}.$$

Proof of Claim 3.82: The proof goes via straightforward induction on n. For completeness we write down all the details.

Assume $n = 2$. Let $c \in {}^+F$ such that there is a_1 with $c = a_1^2$. Let such an a_1 be fixed. Then $\langle 1, a_1\rangle, \langle 1, -a_1\rangle$ are linearly independent and

$$\langle 1, a_1\rangle, \langle 1, -a_1\rangle \in \{\langle 1, p_1\rangle \ : \ p_1 \in F \ \& \ c = p_1^2\}.$$

Hence ${}^2\mathbf{F}$ is generated by $\{\langle 1, p_1\rangle \ : \ p_1 \in F \ \& \ c = p_1^2\}$. Thus Claim 3.82 holds for $n = 2$.

Assume that Claim 3.82 holds for $n = k$, where $k \geq 2$. We will prove that Claim 3.82 holds for $n = k + 1$. To prove this let $c \in {}^+F$ such that there are $a_1, a_2, \dots, a_k \in F$ with $c = a_1^2 + a_2^2 + \dots + a_k^2$. Let such $a_1, a_2, \dots, a_k$ be fixed. We have to prove that ${}^{k+1}\mathbf{F}$ is generated by

$$A := \{\langle 1, p_1, p_2, \dots, p_k\rangle \ : \ p_1, p_2, \dots, p_k \in F \ \& \ c = p_1^2 + p_2^2 + \dots + p_k^2\}.$$

By $c = a_1^2 + a_2^2 + \dots + a_k^2$ and $k \geq 2$, there is a_i $(1 \leq i \leq k)$ such that $c > a_i^2$. Without loss of generality we can assume that $c > a_k^2$. Then we have $c - a_k^2 = a_1^2 + a_2^2 + \dots + a_{k-1}^2$ and $c - a_k^2 \in {}^+F$. Then by the assumption that Claim 3.82 holds for $n = k$, we have that ${}^k\mathbf{F}$ is generated by

$$\{\langle 1, p_1, p_2, \dots, p_{k-1}\rangle \ : \ p_1, p_2, \dots, p_{k-1} \in F \ \& \ c - a_k^2 = p_1^2 + p_2^2 + \dots + p_{k-1}^2\}.$$

Hence the sub-space of ${}^{k+1}\mathbf{F}$ generated by

$C := \{\langle 1, p_1, p_2, \ldots, p_{k-1}, a_k\rangle \,:\, p_1, p_2, \ldots, p_{k-1} \in F \;\&\; c - a_k^2 = p_1^2 + p_2^2 + \ldots + p_{k-1}^2\}$

is k-dimensional.

Case I: $a_k \neq 0$. Assume $a_k \neq 0$. Then $\langle 1, a_1, a_2, \ldots, a_{k-1}, -a_k\rangle$ is not an element of the subspace generated by C and it is an element of A. By this, by $C \subseteq A$ and by C generates a k dimensional subspace, we have that A generates a $(k+1)$-dimensional subspace, i.e. ${}^{k+1}\mathbf{F}$ is generated by A.

Case II: $a_k = 0$. Assume $a_k = 0$. Then there is a_i $(1 \leq i \leq k-1)$ such that $a_i \neq 0$. Without loss of generality we can assume $a_{k-1} \neq 0$. Then $\langle 1, a_1, a_2, \ldots, a_{k-2}, a_k, a_{k-1}\rangle$ is not an element of the subspace generated by C and it an element of A. By this as in Case I, it follows that ${}^{k+1}\mathbf{F}$ is generated by A. This completes the proof of Claim 3.82.

QED (Claim 3.82)

Proof of (41): By $\mathbf{Ax5^{Ph}}$, there is $ph \in Ph$ with $v_m(ph) = c_m$ and $tr_m(ph) \ni \bar{0}$. Let such a ph be fixed. By $v_m(ph) = c_m \neq \infty$ there is $\langle 1, a_1, \ldots, a_{n-1}\rangle \in tr_m(ph)$. Let this $\langle 1, a_1, a_2, \ldots, a_{n-1}\rangle$ be fixed. By $\bar{0} \in tr_m(ph)$ and $v_m(ph) = c_m$, we have $c_m = a_1^2 + a_2^2 + \ldots + a_{n-1}^2$. Then by Claim 3.82, ${}^{n}\mathbf{F}$ is generated by

$$A := \{\langle 1, p_1, p_2, \ldots p_{n-1}\rangle \,:\, p_1, p_2, \ldots, p_{n-1} \in F \;\&\; c_m = p_1^2 + p_2^2 + \ldots + p_{n-1}^2\}.$$

For every $u \in A$ we have $ang^2(\overline{\bar{0}u}) = c_m$. By this and because ${}^{n}\mathbf{F}$ is generated by A, there is a basis $u^1, u^2, \ldots, u^n$ of ${}^{n}\mathbf{F}$ such that

$$ang^2(\overline{\bar{0}u^1}) = ang^2(\overline{\bar{0}u^2}) = \ldots = ang^2(\overline{\bar{0}u^n}) = c_m.$$

Let such $u^1, \ldots, u^n$ be fixed. Recall that q is a fixed element of ${}^{n}F$ and p is a fixed element of $Dom(\mathsf{f}_{mk})$.

$$q - p = \lambda_1 u^1 + \lambda_2 u^2 + \ldots + \lambda_n u^n,$$

for some $\lambda_1, \lambda_2, \ldots, \lambda_n \in F$. Let such $\lambda_1, \lambda_2, \ldots, \lambda_n$ be fixed. Let

$$r^0 := p \text{ and } (\forall i \in n)\; r^{i+1} := r^i + \lambda_i u^i.$$

Now for $r^0, r^1, \ldots, r^n$ we have

$$r^0 = p,\;\; r^n = q,\;\text{ and } (\forall i \in n) ang^2(\overline{r^i r^{i+1}}) = c_m.$$

Hence (41) above holds, and this completes the proof of Thm.3.81. ∎

Thm.3.83 below is an analogon of Thm.3.22 (§3.3).

THEOREM 3.83 $Bax \models (\forall m, k \in Obs)(m \overset{\odot}{\to} k \;\Leftrightarrow\; Rng(w_m) = Rng(w_k))$.

Proof: The proof follows by Thm.3.81. ∎

Thm.3.84 below is an analogon of Thm.3.23 (§3.3).

THEOREM 3.84
$Bax \models$ "$\overset{\odot}{\to}$ is an equivalence relation when restricted to Obs."

Proof: The proof follows by Thm.3.81. ∎

Proposition 3.85 below is an analogon of Proposition 2.6(iv) (§2.3).

PROPOSITION 3.85
$Bax \models (\forall m, k \in Obs)\Big(m \overset{\odot}{\to} k \;\Rightarrow\; (\mathsf{f}_{mk} \textit{ is a bijection } \mathsf{f}_{mk} : {}^nF \to {}^nF)\Big).$

Proof: One could think that this proposition immediately follows from Thm.3.45 about Bax^-. However in Bax^- we assumed that there is a photon in every direction. This is not assumed in Bax, therefore we include the proof here. The proof follows by Claim 3.79 and Thm.3.83. ∎

Thm.3.86 below is an analogon of Thm.3.3 (§3.2) and Thm.3.45 (§3.4.2).

THEOREM 3.86 $Bax \models (\forall m, k \in Obs)(\forall \ell \in \mathsf{Eucl})(m \overset{\odot}{\to} k \;\Rightarrow \mathsf{f}_{mk}[\ell] \in \mathsf{Eucl}).$

Proof: One could think that this theorem immediately follows from Thm.3.45 about Bax^-. However in Bax^- we assumed that there is a photon in every direction. This is not assumed in Bax, therefore we include the proof here. Let $\mathfrak{M}$ be a frame model of Bax. Let $m, k \in Obs$ with $m \overset{\odot}{\to} k$. To prove that f_{mk} takes lines to lines we need Claims 3.87, 3.88 and 3.89 below which are analogons of Prop.2.6(viii) (§2.3), Lemma 3.6 (§3.2) and Lemma 3.7 (§3.2), respectively.

Claim 3.87 $(\forall \ell \in \mathsf{Eucl})(ang^2(\ell) < c_m \;\Rightarrow\; \mathsf{f}_{mk}[\ell] \in \mathsf{Eucl}).$

Proof of Claim 3.87: The proof is analogous to the proof of Prop.2.6(viii).

Claim 3.88 $(\forall \ell_1, \ell_2 \in \mathsf{Eucl})\Big((ang^2(\ell_1) < c_m \;\wedge\; \ell_1 \parallel \ell_2) \;\Rightarrow\; \mathsf{f}_{mk}[\ell_1] \parallel \mathsf{f}_{mk}[\ell_2]\Big).$

Proof of Claim 3.88: The proof is analogous to the proof of Lemma 3.6.

Claim 3.89 $(\forall p, q \in {}^nF)\left(ang^2(\overline{pq}) < c_m \;\Rightarrow\; \mathsf{f}_{mk}(\frac{p+q}{2}) = \frac{\mathsf{f}_{mk}(p)+\mathsf{f}_{mk}(q)}{2}\right).$

Proof of Claim 3.89: The proof is analogous to the proof of Lemma 3.7.

Now $(\forall \ell \in \mathsf{Eucl})\mathsf{f}_{mk}[\ell] \in \mathsf{Eucl}$ can be proved by Claims 3.87, 3.88, 3.89 as Thm.3.3 was proved by Prop.2.6(viii) and Lemmas 3.6, 3.7. ∎

Thm.3.90 below is an analogon of Thm.3.4 (§3.2).

THEOREM 3.90
$Bax \models (\forall m, k \in Obs)\Big(m \overset{\odot}{\to} k \;\Rightarrow\; (\mathsf{f}_{mk} = \widetilde{\varphi} \circ \mathsf{f}$, *for some* $\mathsf{f} \in Aftr$ *and* $\varphi \in Aut(\mathbf{F}))\Big)$.

Proof: The proof follows by Prop.3.85, Thm.3.86 and Lemma 3.5 (§3.2). ∎

Lemma 3.91 below is a generalization of Lemma 3.30 (§3.4.1).

LEMMA 3.91 *Assume* $n \geq 3$ *and* $\mathfrak{F}$ *is Euclidean. Let* $c_1, c_2 \in {}^+F$. *Assume* $\mathsf{f} : {}^nF \to {}^nF$ *is a bijection such that*

$$(\star) \qquad (\forall \ell \in \mathsf{Eucl})\Big(\mathsf{f}[\ell] \in \mathsf{Eucl} \;\wedge\; (ang^2(\ell) = c_1 \;\Leftrightarrow\; ang^2(\mathsf{f}[\ell]) = c_2)\Big).$$

Then $ang^2(\mathsf{f}[\bar{t}\,]) < c_2$.

Proof: Let $c_1, c_2 \in {}^+F$ and $\mathsf{f} : {}^nF \to {}^nF$ be a bijection such that $(\star)$ holds. Let $\mathsf{f}_1, \mathsf{f}_2$ be the linear transformations for which $\mathsf{f}_1(1_t) = 1_t$, $(\forall i \in n \setminus \{0\})\mathsf{f}_1(e_i) = \sqrt{c_1} \cdot e_i$, $\mathsf{f}_2(1_t) = 1_t$ and $(\forall i \in n \setminus \{0\})\mathsf{f}_2(e_i) = \sqrt{c_2} \cdot e_i$. Then for $\mathsf{g} := \mathsf{f}_1 \circ \mathsf{f} \circ \mathsf{f}_2^{-1}$ we have

$$(\forall \ell \in \mathsf{Eucl})\Big(\mathsf{g}[\ell] \in \mathsf{Eucl} \;\wedge\; (\mathsf{g}[\ell] \in \mathsf{PhtEucl} \;\Leftrightarrow\; \ell \in \mathsf{PhtEucl})\Big).$$

By Lemma 3.30 we have that $\mathsf{g}[\bar{t}\,] \in \mathsf{SlowEucl}$, i.e. $ang^2((\mathsf{f}_1 \circ \mathsf{f} \circ \mathsf{f}_2^{-1})[\bar{t}\,]) < 1$. But this is equvivalent with $ang^2(\mathsf{f}[\bar{t}\,]) < c_2$. ∎

For $n = 4$ Lemma 3.92 below is a generalization of Lemma 3.31 (§3.4.1).

LEMMA 3.92 *Assume* $c_1, c_2 \in {}^+F$ *such that* $c_1 = ang^2(\ell)$, *for some* $\ell \in \mathsf{Eucl}$. *Assume* $\mathsf{f} \in Aftr(\mathbf{4}, \mathbf{F})$ *satisfying* $(\star)$ *in Lemma 3.91 above. Assume* $\mathfrak{F}_* = \langle \mathbf{F}_*, \leq \rangle$ *is an ordered field such that* $\mathfrak{F} \subseteq \mathfrak{F}_*$. *Let* $\mathsf{f}_* \in Aftr(\mathbf{4}, \mathbf{F}_*)$ *for which* $\mathsf{f}_* \lceil {}^4F = \mathsf{f}$. *Then* f_* *satisfies* $(\star)$ *in Lemma 3.91 above when* f_* *and* $\mathfrak{F}_*$ *are substituted in place of* f *and* $\mathfrak{F}$, *respectively.*

We will give the **proof** of Lemma 3.92 after the proof of Thm.3.40 for $n = 4$.

Proof of Thm.3.40 for $\mathbf{n = 4}$:
Proof of (i): Assume $n = 4$. Let $\mathfrak{M}$ be frame model of Bax. Let $m, k \in Obs$ with $m \overset{\odot}{\to} k$. We have to prove that $v_m(k) < c_m$.

Intuitive idea of the proof: We want to prove that $ang^2(tr_m(k)) \leq c_m$. We will see that $\mathsf{f}_{km} = \widetilde{\varphi} \circ \mathsf{f}$ where $\varphi \in Aut(\mathbf{F})$ and f is an affine transformation satisfying $(\star)$ in 3.91 for $c_1 := \varphi(c_k)$ and $c_2 := c_m$. By 3.92 f will continue satisfying $(\star)$ in a larger field $\mathfrak{F}_*$, which, in turn will be Euclidean. Looking at it from $\mathfrak{F}_*$, $ang^2(\mathsf{f}[\bar{t}\,]) < c_m$ by

3.91. Therefore $ang^2(\mathsf{f}[\bar{t}\,]) < c_m$ in $\mathfrak{F}$, too and then $tr_m(k) = \mathsf{f}[\bar{t}\,]$ will complete the proof.

Formally: By Thm.3.90, $\mathsf{f}_{km} = \widetilde{\varphi} \circ \mathsf{f}$ for some $\mathsf{f} \in Aftr$ and $\varphi \in Aut(\mathbf{F})$. Let this f and φ be fixed. We have $\mathsf{f}_{km}[tr_k(k)] = tr_m(k)$ because f_{mk} is a bijection. By $\mathsf{f}_{km} = \widetilde{\varphi} \circ \mathsf{f}$, by $\widetilde{\varphi}[\bar{t}\,] = \bar{t}$, by $tr_k(k) = \bar{t}$ and by $\mathsf{f}_{km}[tr_k(k)] = tr_m(k)$, we have

$$\mathsf{f}[\bar{t}\,] = tr_m(k). \tag{43}$$

By $\mathbf{Ax5^{Obs}}$, $\mathbf{Ax5^{Ph}}$, $\mathbf{AxE_{00}}$ and Prop.3.85 it is easy to see that

$$(\forall \ell \in \mathsf{Eucl})(ang^2(\ell) = c_k \;\Leftrightarrow\; ang^2(\mathsf{f}_{km}[\ell]) = c_m). \tag{44}$$

By (44) and $\mathsf{f}_{km} = \widetilde{\varphi} \circ \mathsf{f}$, we have

$$(\forall \ell \in \mathsf{Eucl})\Big(\mathsf{f}[\ell] \in \mathsf{Eucl} \;\;\wedge\;\; (ang^2(\ell) = \varphi(c_k) \;\Leftrightarrow\; ang^2(\mathsf{f}[\ell]) = c_m)\Big). \tag{45}$$

$c_k = v_k(ph)$, for some $ph \in Ph$. Let such a ph be fixed. Then $\widetilde{\varphi}[tr_k(ph)] \in \mathsf{Eucl}$ and $ang^2(\widetilde{\varphi}[tr_k(ph)]) = \varphi(c_k)$. Thus

$$(\exists \ell \in \mathsf{Eucl})\, ang^2(\ell) = \varphi(c_k) \tag{46}$$

Let $\mathfrak{F}_* = \langle \mathbf{F}_*, \leq \rangle$ be such that $\mathfrak{F}_*$ is Euclidean and $\mathfrak{F} \subseteq \mathfrak{F}_*$. Such an $\mathfrak{F}_*$ exists, e.g. the real closure of $\mathfrak{F}$ is such. Let $\mathsf{f}_* \in Aftr(\mathbf{n}, \mathbf{F}_*)$ such that $\mathsf{f}_* \lceil {}^4F = \mathsf{f}$. By $\mathbf{AxE_{01}}$ and (46), we have $\varphi(c_k), c_m \in {}^+F$. By this, by (45), by (46) and by Lemma 3.92, we have that

$$(\forall \ell \in \mathsf{Eucl})\Big(\mathsf{f}_*[\ell] \in \mathsf{Eucl} \;\;\wedge\;\; (ang^2(\ell) = \varphi(c_k) \;\Leftrightarrow\; ang^2(\mathsf{f}_*[\ell]) = c_m)\Big) \tag{47}$$

Let $\bar{t}_* := F_* \times {}^{n-1}\{0\}$. Then $ang^2(\mathsf{f}_*[\bar{t}_*]) < c_m$ by Lemma 3.91. Hence $ang^2(\mathsf{f}[\bar{t}\,]) < c_m$. By this and by (43), we have $v_m(k) < c_m$.

Proof of (ii): Assume $n = 4$. Let $\mathfrak{M}$ be a a frame model of $Bax \setminus \{\mathbf{AxE_{01}}\}$. Let $m_0, m_1 \in Obs$ such that $m_0 \overset{\odot}{\to} m_1$. We have to prove the following. If $c_{m_0} \neq 0$ then $v_{m_0}(m_1) < c_{m_0}$, and if $c_{m_0} = 0$ then $v_{m_0}(m_1) = 0$. For $c_{m_0} = 0$ or $c_{m_0} = \infty$ the proof is analogous to the proof when $n = 3$. Assume $c_{m_0} \neq 0$ and $c_{m_0} \neq \infty$. We define a frame model

$$\mathfrak{N} = \langle (B, Obs^{\mathfrak{N}}, Ph^{\mathfrak{N}}, Ib^{\mathfrak{N}}), \mathfrak{F}, G;\ \mathrm{E}\ , W^{\mathfrak{N}} \rangle$$

as follows.

$$\begin{aligned}
Obs^{\mathfrak{N}} &\overset{\text{def}}{=} \{m \in Obs \;:\; c_m \neq 0 \;\&\; c_m \neq \infty\}, \\
Ph^{\mathfrak{N}} &\overset{\text{def}}{=} \{ph \in Ph \;:\; (\exists m \in Obs^{\mathfrak{N}})(m \overset{\odot}{\to} ph \text{ holds in } \mathfrak{M})\}, \\
Ib^{\mathfrak{N}} &\overset{\text{def}}{=} \{b \in Ib \;:\; (\exists m \in Obs^{\mathfrak{N}})(m \overset{\odot}{\to} b \text{ holds in } \mathfrak{M})\}, \\
W^{\mathfrak{N}} &\overset{\text{def}}{=} W\lceil(Obs^{\mathfrak{N}} \times {}^4F \times B).
\end{aligned}$$

It is easy to check that $\mathfrak{N} \models Bax$ by $\mathfrak{M} \models Bax \setminus \{\mathbf{AxE_{01}}\}$ and Lemma 3.76. Further $m_0, m_1 \in Obs^{\mathfrak{N}}$, $v^{\mathfrak{M}}_{m_0}(m_1) = v^{\mathfrak{N}}_{m_0}(m_1)$ and $c^{\mathfrak{M}}_{m_0} = c^{\mathfrak{N}}_{m_0}$. By item (i) we have $v^{\mathfrak{N}}_{m_0}(m_1) < c^{\mathfrak{N}}_{m_0}$, hence $v^{\mathfrak{M}}_{m_0}(m_1) < c^{\mathfrak{M}}_{m_0}$. ∎

Proof of Lemma 3.92:

Claim 3.93 *Let $c_1, c_2, \mathfrak{F}_*, \mathsf{f}, \mathsf{f}_*$ be as in the formulation of Lemma 3.92. Then*

$$(\star\star) \qquad (\forall \ell \in \mathsf{Eucl}(\mathbf{4}, \mathbf{F}_*))(ang^2(\ell) = c_1 \ \Rightarrow\ ang^2(\mathsf{f}_*[\ell]) = c_2).$$

We will prove Claim 3.93 very soon. Lemma 3.92 follows from Claim 3.93 because of the following. Intuitively: If f satisfies $(\star)$ in 3.91 then f^{-1} satisfies $(\star)$ when $\mathsf{f}^{-1}, c_2, c_1$ are substituted in place of f, c_1, c_2, respectively. Then applying Claim 3.93 to f, c_1, c_2 and to $\mathsf{f}^{-1}, c_2, c_1$, respectively, we obtain Lemma 3.92. More formally: Let $c_1, c_2, \mathfrak{F}_*, \mathsf{f}, \mathsf{f}_*$ as in the formulation of Lemma 3.92. Then by Claim 3.93, we have

$$(49) \qquad (\forall \ell \in \mathsf{Eucl}(\mathbf{4}, \mathbf{F}_*))(ang^2(\ell) = c_1 \ \Rightarrow\ ang^2(\mathsf{f}_*[\ell]) = c_2).$$

Let $(\mathsf{f}^{-1})_* \in Aftr(\mathbf{4}, \mathbf{F}_*)$ such that $(\mathsf{f}^{-1})_* \lceil {}^4F = \mathsf{f}^{-1}$. Obviously $(\mathsf{f}^{-1})_* = (\mathsf{f}_*)^{-1}$.

By f satisfying $(\star)$ in Lemma 3.91 we have

$$(50) \qquad (\forall \ell \in \mathsf{Eucl}(\mathbf{4}, \mathbf{F})) \qquad (ang^2(\ell) = c_2 \ \Leftrightarrow\ ang^2(\mathsf{f}^{-1}[\ell]) = c_1).$$

$$(51) \qquad (\exists \ell \in \mathsf{Eucl}(\mathbf{4}, \mathbf{F})) \qquad ang^2(\ell) = c_2$$

because there is $\ell \in \mathsf{Eucl}$ such that $ang^2(\ell) = c_1$, and $ang^2(\mathsf{f}[\ell]) = c_2$ for that ℓ by $(\star)$. By (50) and (51), Claim 3.93 can be applied to $c_2, c_1, \mathfrak{F}_*, \mathsf{f}^{-1}, (\mathsf{f}^{-1})_*$. So we have

$$(52) \qquad (\forall \ell \in \mathsf{Eucl}(\mathbf{4}, \mathbf{F}_*))(ang^2(\ell) = c_2 \ \Rightarrow\ ang^2((\mathsf{f}^{-1})_*[\ell]) = c_1).$$

By (49), (52) and $(\mathsf{f}^{-1})_* = (\mathsf{f}_*)^{-1}$, we have

$$(\forall \ell \in \mathsf{Eucl}(\mathbf{4}, \mathbf{F}_*))(ang^2(\ell) = c_1 \ \Leftrightarrow\ ang^2(\mathsf{f}_*[\ell]) = c_2).$$

Thus Lemma 3.92 follows by Claim 3.93.

Proof of Claim 3.93: Let $c_1, c_2, \mathfrak{F}_*, \mathsf{f}, \mathsf{f}_*$ be as in formulation of Lemma 3.91. We have to prove that f_* satisfies $(\star\star)$ in Claim 3.93. Without loss of generality may assume that $\mathsf{f}(\bar{0}) = \bar{0}$.

On the structure of the proof: Items (53) an (54) below are reformulations of saying that f and f_* satisfies $(\star\star)$, respectively. Items (57) and (58) below are equivalent forms of (53) and (54), respectively. Hence our task is to prove (58) from (57). This is done by the linear algebraic considerations given below.

By our assumption that f is a linear transformation, we have that

$$(\forall p \in {}^4F)\mathsf{f}(p) = \langle \sum_{i=0}^{3} p_i a_{i0}, \sum_{i=0}^{3} p_i a_{i1}, \sum_{i=0}^{3} p_i a_{i2}, \sum_{i=0}^{3} p_i a_{i3} \rangle,$$

for some $a_{ij} \in F$, where $i, j \in 4$. Let these a_{ij}'s be fixed. By the definition of f_*, we have

$$(\forall p \in {}^4F_*)\mathsf{f}_*(p) = \langle \sum_{i=0}^{3} p_i a_{i0}, \sum_{i=0}^{3} p_i a_{i1}, \sum_{i=0}^{3} p_i a_{i2}, \sum_{i=0}^{3} p_i a_{i3} \rangle.$$

By f satisfies $(\star)$, we have that (53) below holds, and to prove that f_* satisfies $(\star\star)$ we have to prove (54) below.

$$(53)\ (\forall p \in {}^4F) \qquad \left(c_1 p_0^2 = p_1^2 + p_2^2 + p_3^2 \ \Rightarrow\ c_2 \left(\sum_{i=0}^{3} p_i a_{i0} \right)^2 = \sum_{j=1}^{3} \left(\sum_{i=0}^{3} p_i a_{ij} \right)^2 \right).$$

$$(54)\ (\forall p \in {}^4F_*) \qquad \left(c_1 p_0^2 = p_1^2 + p_2^2 + p_3^2 \ \Rightarrow\ c_2 \left(\sum_{i=0}^{3} p_i a_{i0} \right)^2 = \sum_{j=1}^{3} \left(\sum_{i=0}^{3} p_i a_{ij} \right)^2 \right).$$

Let $(\forall i, j \in 4)$ $d_{ij} := c_2 a_{i0} a_{j0} - \sum_{k=1}^{3} a_{ik} a_{jk}$, and let $b_0 := d_{00}$, $b_1 := d_{11}$, $b_2 := d_{22}$, $b_3 := d_{33}$, $b_4 := d_{01} + d_{10}$, $b_5 := d_{02} + d_{20}$, $b_6 := d_{03} + d_{30}$, $b_7 := d_{12} + d_{21}$, $b_8 := d_{13} + d_{31}$, $b_9 := d_{23} + d_{32}$. Then (53) and (54) above are equivalent with (55) and (56) below, respectively.

$$(55) \qquad (\forall p \in {}^4F) \ \big(c_1 p_0^2 = p_1^2 + p_2^2 + p_3^2 \ \Rightarrow\ p_0^2 b_0 + p_1^2 b_1 + p_2^2 b_2 + p_3^2 b_3 + \\ + p_0 p_1 b_4 + p_0 p_2 b_5 + p_0 p_3 b_6 + p_1 p_2 b_7 + p_1 p_3 b_8 + p_2 p_3 b_9 = 0 \big).$$

$$(56) \qquad (\forall p \in {}^4F_*) \big(c_1 p_0^2 = p_1^2 + p_2^2 + p_3^2 \ \Rightarrow\ p_0^2 b_0 + p_1^2 b_1 + p_2^2 b_2 + p_3^2 b_3 + \\ + p_0 p_1 b_4 + p_0 p_2 b_5 + p_0 p_3 b_6 + p_1 p_2 b_7 + p_1 p_3 b_8 + p_2 p_3 b_9 = 0 \big).$$

Let E and E_* be the following set of linear equations.

$$E := \{ p_0^2 x_0 + p_1^2 x_1 + p_2^2 x_2 + p_3^2 x_3 + p_0 p_1 x_4 + p_0 p_2 x_5 + p_0 p_3 x_6 + p_1 p_2 x_7 + \\ + p_1 p_3 x_8 + p_2 p_3 x_9 = 0 \ :\ p \in {}^4F \ \&\ c_1 p_0^2 = p_1^2 + p_2^2 + p_3^2 \}.$$

$$E_* := \{ p_0^2 x_0 + p_1^2 x_1 + p_2^2 x_2 + p_3^2 x_3 + p_0 p_1 x_4 + p_0 p_2 x_5 + p_0 p_3 x_6 + p_1 p_2 x_7 + \\ + p_1 p_3 x_8 + p_2 p_3 x_9 = 0 \ :\ p \in {}^4F_* \ \&\ c_1 p_0^2 = p_1^2 + p_2^2 + p_3^2 \}.$$

Now (55) and (56) above are equivalent with (57) and (58) below.

(57) $\langle b_0, b_1, b_2, b_3, b_4, b_5, b_6, b_7, b_8, b_9 \rangle$ is a solution for the system of equations E.

(58) $\langle b_0, b_1, b_2, b_3, b_4, b_5, b_6, b_7, b_8, b_9 \rangle$ is a solution for the system of equations E_*.

Thus to prove Claim 3.93 it is enough to prove (57)$\Rightarrow$(58). To prove (57)$\Rightarrow$(58) it is enough to prove that each linear equation from E_* is a linear combination of some equations from E, i.e. that each vector from A_* is a linear combination of some vectors from A, where A and A_* are defined below.

$A := \{\langle p_0^2, p_1^2, p_2^2, p_3^2, p_0p_1, p_0p_2, p_0p_3, p_1p_2, p_1p_3, p_2p_3\rangle : p \in {}^4F \ \&\ c_1p_0^2 = p_1^2 + p_2^2 + p_3^2\}$.
$A_* := \{\langle p_0^2, p_1^2, p_2^2, p_3^2, p_0p_1, p_0p_2, p_0p_3, p_1p_2, p_1p_3, p_2p_3\rangle : p \in {}^4F_* \ \&\ c_1p_0^2 = p_1^2 + p_2^2 + p_3^2\}$.

Both A and A_* are at most 9-dimensional because of the "condition" $c_1p_0^2 = p_1^2 + p_2^2 + p_3^2$ in the definitions of A and A_*. Hence to prove that each vector from A_* is a linear combination of some vectors from A it is enough to prove that A generates a 9-dimensional sub-space of ${}^{10}\mathbf{F}$ because $A \subseteq A_*$. Now to prove that subspace generated by A is 9-dimensional it is enough to prove that the subspace $\mathbf{W}$ of ${}^9\mathbf{F}$ generated by

$$C := \{\langle p_1^2, p_2^2, p_3^2, p_1, p_2, p_3, p_1p_2, p_1p_3, p_2p_3\rangle \ :\ p_1, p_2, p_3 \in F \ \&\ c_1 = p_1^2 + p_2^2 + p_3^2\}$$

is 9-dimensional, i.e. $\mathbf{W} = {}^9\mathbf{F}$. By $(\exists \ell \in \mathsf{Eucl})ang^2(\ell) = c_1$, we have that there is ℓ with $ang^2(\ell) = c_1$ and $\bar{0} \in \ell$. Let such an ℓ be fixed. Then $\langle 1, \lambda, \mu, \nu\rangle \in \ell$, for some $\langle 1, \lambda, \mu, \nu\rangle$. Let this $\langle 1, \lambda, \mu, \nu\rangle$ be fixed. Now $c_1 = \lambda^2 + \mu^2 + \nu^2$ by $ang^2(\ell) = c_1$. We can assume that $0 \neq |\lambda| \neq |\mu| \neq 0$ (we checked that this is true, but we do not include the details here). We have

$$\begin{array}{lclcccccccccccc}
v_1 & := & \langle & \lambda^2, & \mu^2, & \nu^2, & \lambda, & \mu, & \nu, & \lambda\mu, & \lambda\nu, & \mu\nu & \rangle & \in & C,\\
v_2 & := & \langle & \lambda^2, & \mu^2, & \nu^2, & -\lambda, & -\mu, & -\nu, & \lambda\mu, & \lambda\nu, & \mu\nu & \rangle & \in & C,\\
v_3 & := & \langle & \lambda^2, & \mu^2, & \nu^2, & -\lambda, & \mu, & \nu, & -\lambda\mu, & -\lambda\nu, & \mu\nu & \rangle & \in & C,\\
v_4 & := & \langle & \lambda^2, & \mu^2, & \nu^2, & \lambda, & -\mu, & -\nu, & -\lambda\mu, & -\lambda\nu, & \mu\nu & \rangle & \in & C,\\
v_5 & := & \langle & \lambda^2, & \mu^2, & \nu^2, & \lambda, & -\mu, & \nu, & -\lambda\mu, & \lambda\nu, & -\mu\nu & \rangle & \in & C,\\
v_6 & := & \langle & \lambda^2, & \mu^2, & \nu^2, & \lambda, & \mu, & -\nu, & \lambda\mu, & -\lambda\nu, & -\mu\nu & \rangle & \in & C,\\
v_7 & := & \langle & \mu^2, & \nu^2, & \lambda^2, & \mu, & \nu, & \lambda, & \mu\nu, & \lambda\mu, & \lambda\nu & \rangle & \in & C,\\
v_8 & := & \langle & \nu^2, & \lambda^2, & \mu^2, & \nu, & \lambda, & \mu, & \lambda\nu, & \mu\nu, & \lambda\mu & \rangle & \in & C,\\
v_9 & := & \langle & \mu^2, & \lambda^2, & \nu^2, & \mu, & \lambda, & \nu, & \lambda\mu, & \mu\nu, & \lambda\nu & \rangle & \in & C.
\end{array}$$

It is easy to check that $e_3 = \frac{1}{4\lambda}(v_1 - v_2 - v_3 + v_4)$. Hence $e_3 \in W$. Similarly $e_4, e_5 \in W$. It is easy to check that $e_6 = \frac{1}{4\lambda\mu}(v_1 - v_4 - v_5 + v_6) + v$, for some v which is in the sub-space generated by $\{e_3, e_4, e_5\}$. Hence $e_6 \in W$. Similarly $e_7, e_8 \in W$. It is easy to check that $e_0 + e_1 + e_2 = \frac{1}{\lambda^2+\mu^2+\nu^2}(v_1 + v_7 + v_8) + v$, for some v which is in the sub-space generated by $\{e_3, e_4, e_5, e_6, e_7, e_8\}$. Thus $e_0 + e_1 + e_2 \in W$. It is easy to check that $e_0 - e_2 = \frac{1}{\lambda^2 - \mu^2}(v_1 - v_9) + v$, for some v which is in the sub-space generated by $\{e_3, e_4, e_5, , e_6, e_7, e_8\}$. Hence $e_0 - e_2 \in W$. Similarly $e_0 - e_3 \in W$. But $e_0 + e_1 + e_2, e_0 - e_1, e_0 - e_2 \in W$ implies $e_0 \in W$. Similarly $e_1, e_2 \in W$. We proved $(\forall i \in 9)e_i \in W$, and this completes the proof of Claim 3.93 and Lemma 3.92. ∎

3.5 Simple models for *Basax*

In this sub-section we show that *Basax* is consistent for arbitrary $n \geq 2$ by defining a class of frame models, which we call the class of simple models, in symbols SM, and showing that $\mathsf{SM} \models Basax$. Let us recall that in §2.4 a set of models for $Basax(\mathbf{2})$ was given. In this sub-section we will give models for *Basax* in the same spirit.[59] We suggest the reader to concentrate on $n = 2$ and $n = 3$, at the first reading.

To define the class of simple models, we need the definition of the set of "Newtonian transformations" which is a subset of the set of affine transformations. (Recall that we have a fixed $\mathfrak{F}$ in the background.) This comes next. In this definition we use notation $\bar{t}$, S, $|p|$, τ_p, $Aftr$ which were introduced in items 4, 5, 8, 13 of Notation 3.1 and item 1 of Def.3.2, respectively. We recall that τ_p denotes the translation by vector p.

Definition 3.94

$$Newt_0 \stackrel{\text{def}}{=} \{g \in Aftr \,:\, (\forall p \in \bar{t})\ g(p) = p \quad \& \quad (\forall p \in S)\ |g(p)| = |p|\}\,,$$

$$Newt \stackrel{\text{def}}{=} \{g \in Aftr \,:\, g = f \circ \tau_p,\ \text{for some } f \in Newt_0 \text{ and } p \in {}^nF\}\,. \qquad \triangleleft$$

Remark 3.95 $\langle Newt_0, \circ, {}^{-1}, \mathrm{Id}\rangle$ and $\langle Newt, \circ, {}^{-1}, \mathrm{Id}\rangle$ are groups. $\triangleleft$

On the intuitive meaning of Newtonian transformations we will write later. We note that elements of $Newt_0$ are invertible linear transformations, moreover $Newt_0$ is the set of all congruence (i.e. distance preserving) linear transformations which leave $\bar{t}$ point-wise fixed. We also note that $Newt_0 = Newt_0(\mathbf{n}, \mathbf{F})$ and $Newt = Newt(\mathbf{n}, \mathbf{F})$, in accordance with the conventions of the present section (§3).

Lemmas 3.96 and 3.97 below are needed for the definition of the class of simple models.

[59] Let $\mathsf{SM}(\mathbf{n})$ denote the n–dimensional version of the class SM of models to be defined in this section. The class $\mathsf{SM}(\mathbf{2})$ is almost the same as the class $\{\mathfrak{M}_0^P : P$ is an appropriate choice function (cf. §2.4 first 8 lines)$\}$ defined in §2.4. The only difference is that $\mathsf{SM}(\mathbf{2})$ is slightly bigger in the following sense: In $\mathsf{SM}(\mathbf{2})$ we have an extra parameter N, which on the other hand cannot do too much for $n = 2$. Another difference is that in $\mathsf{SM}(\mathbf{2})$ $\mathfrak{F}$ is allowed to be an arbitrary Euclidean field, while in $\mathfrak{M}_0^P$ it was fixed to be $\mathfrak{R}$.

LEMMA 3.96 *Assume $\mathfrak{F}$ is Euclidean (i.e. "positive" square-roots exist in $\mathfrak{F}$). Assume $\ell \in \mathsf{Eucl}$ and p is a point lying on ℓ. Then there is $N \in Newt$ such that $N[\ell] \subseteq \mathsf{Plane}(\bar{t}, \bar{x})$ and $N(p) = \bar{0}$.*

We will give the **proof** at the end of this sub-section.

LEMMA 3.97 *Assume $\ell \in \mathsf{Eucl}$ and $N \in Newt$. Then $ang^2(\ell) = ang^2(N[\ell])$.*

Proof: The proof is straightforward. We omit it. ∎

In Def.3.98 below the class SM of simple models will be defined in the following way. For each Euclidean field $\mathfrak{F}$ and for each function P that to each $\ell \in \mathsf{SlowEucl}$ associates two distinct points o_ℓ and t_ℓ lying on ℓ and $N_\ell \in Newt$ with $N_\ell[\ell] \subseteq \mathsf{Plane}(\bar{t}, \bar{x})$ and $N_\ell(o_\ell) = \bar{0}$, a frame model $\mathfrak{M}$ will be defined. We suggest the reader to read the following definition only for $n = 2$ and $n = 3$, at the first reading.

Definition 3.98 (**Simple Models,** SM)
Let $\mathfrak{F}$ be a Euclidean field (i.e. "positive" square roots exist in $\mathfrak{F}$). Let P be a "choice" function that to each $\ell \in \mathsf{SlowEucl}$ associates two distinct points o_ℓ and t_ℓ lying on ℓ and $N_\ell \in Newt$ with $N_\ell[\ell] \subseteq \mathsf{Plane}(\bar{t}, \bar{x})$ and $N_\ell(o_\ell) = \bar{0}$. By Lemma 3.96 above, such a P exists. We will denote $P(\ell)$ by $\langle o_\ell, t_\ell, N_\ell \rangle$. To each such $\mathfrak{F}$ and function P, we will define a frame model $\mathfrak{M}^P_{\mathfrak{F}}$.

We define $\mathfrak{M} \stackrel{\text{def}}{=} \mathfrak{M}^P_{\mathfrak{F}} \stackrel{\text{def}}{=} \langle B, F, G; Obs, Ph, Ib, +, \cdot, \leq, \mathrm{E}, W \rangle$, where

$\langle F, +, \cdot, \leq \rangle$ is $\mathfrak{F}$,

$\langle G, \mathrm{E} \rangle \stackrel{\text{def}}{=} \langle \mathsf{Eucl}, \in \rangle$,

$Obs \stackrel{\text{def}}{=} \mathsf{SlowEucl}$,

$Ph \stackrel{\text{def}}{=} \mathsf{PhtEucl}$,

$B \stackrel{\text{def}}{=} Ib \stackrel{\text{def}}{=} Obs \cup Ph = \{\ell \in \mathsf{Eucl} : ang^2(\ell) \leq 1\} = \mathsf{SlowEucl} \cup \mathsf{PhtEucl}$.

By the above, **Ax1** and **Ax2** are true in $\mathfrak{M}$. It remains to define W. Let

$$m_0 \stackrel{\text{def}}{=} \bar{t}\,.$$

First we will define $w_{m_0} : {}^nF \longrightarrow \mathcal{P}(B)$ and $\mathsf{f}_{km_0} : {}^nF \longrightarrow {}^nF$ for all $k \in \mathsf{SlowEucl}$, $k \neq m_0$. To define w_{m_0}, let $p \in {}^nF$. Then

$$w_{m_0}(p) \stackrel{\text{def}}{=} \{\ell \in B \; : \; p \in \ell\} \, .$$

By this, we have that for all $\ell \in B$,

$$tr_{m_0}(\ell) = \ell \, ,$$

in particular, $tr_{m_0}(m_0) = m_0$. Thus **Ax3**, **Ax4**, **Ax5**, **AxE** are satisfied when m is replaced in them with m_0.

Definition of f_{km_0}*:*

Let us recall that for every $k \in \mathsf{SlowEucl}$ by parameter P of the model $\mathfrak{M} \stackrel{\text{def}}{=} \mathfrak{M}^P_{\mathfrak{F}}$ a triple $\langle o_k, t_k, N_k \rangle$ was given such that $\overline{o_k t_k} = k$ $(o_k \neq t_k)$ and $N_k \in Newt$ with $N_k[k] \subseteq \mathsf{Plane}(\bar{t}, \bar{x})$ and $N_k(o_k) = \bar{0}$.

First we will define f_{km_0} for the case $n = 2$, after that for the case $n = 3$, and finally for arbitrary $n \geq 2$.

Recall that Id is the identical transformation of ${}^nF$ taking p to p.

Definition of f_{km_0} *for the case n=2:*

Assume $n = 2$. Let $k \in \mathsf{SlowEucl}$, $k \neq m_0 \stackrel{\text{def}}{=} \bar{t}$ be arbitrary and fixed. First we will define f_{km_0} for the special case when $N_k = \text{Id}$. This will be implemented in item (i) below. After that in item (ii) we will define f_{km_0} for $N_k \neq \text{Id}$.

(i) We define f_{km_0} for the special case $N_k = \text{Id}$ as follows.

$N_k = \text{Id}$ implies that $o_k = \bar{0} \in k$.

Let x_k be the mirror image of t_k w.r.t. the line $\overline{\bar{0}\langle 1, 1\rangle}$. In more detail: If $t_k = \langle t_0, t_1 \rangle$ then we define $x_k \stackrel{\text{def}}{=} \langle t_1, t_0 \rangle$.

Let f_{km_0} be the linear transformation which takes 1_t, 1_x to t_k, x_k, respectively. Clearly such an f_{km_0} exists and is unique. It is easy to check that this f_{km_0} is an invertible linear transformation.[60]

The reason why we chose x_k exactly the way we did can be explained by Prop.3.15 in §3.2.

[60]This is so because $t_k \neq o_k = \bar{0}$, and by $ang^2(k) \neq 1$ one can check that vectors t_k and x_k are linearly independent.

(ii) We define f_{km_0} for the case $N_k \neq \mathrm{Id}$ as follows.

Let $t'_k \stackrel{\text{def}}{=} N_k[t_k]$. Then $t'_k \neq \bar{0}$ by $t_k \neq o_k$ and $\bar{0} = N_k(o_k)$.

Let x'_k be chosen for t'_k exactly as we chose x_k for t_k in item (i). Let $x_k \stackrel{\text{def}}{=} N_k^{-1}(x'_k)$.

We define f_{km_0} to be the affine transformation which takes $\bar{0}$, 1_t, 1_x to o_k, t_k, x_k, respectively. Clearly such an f_{km_0} exists[61] and is unique.

<u>*Definition of* f_{km_0} *for the case n=3:*</u>

Assume $n = 3$. Let $k \in \mathsf{SlowEucl}$, $k \neq m_0 \stackrel{\text{def}}{=} \bar{t}$ be arbitrary and fixed. Let us recall $P(k) = \langle o_k, t_k, N_k \rangle$ (see at the beginning of Def.3.98). First we will define f_{km_0} for the case when $N_k = \mathrm{Id}$. This will be implemented in item (i) below. After that in item (ii) we will define f_{km_0} for $N_k \neq \mathrm{Id}$.

(i) We define f_{km_0} for the case $N_k = \mathrm{Id}$ as follows. Throughout this definition the reader is asked to consult Figure 42.

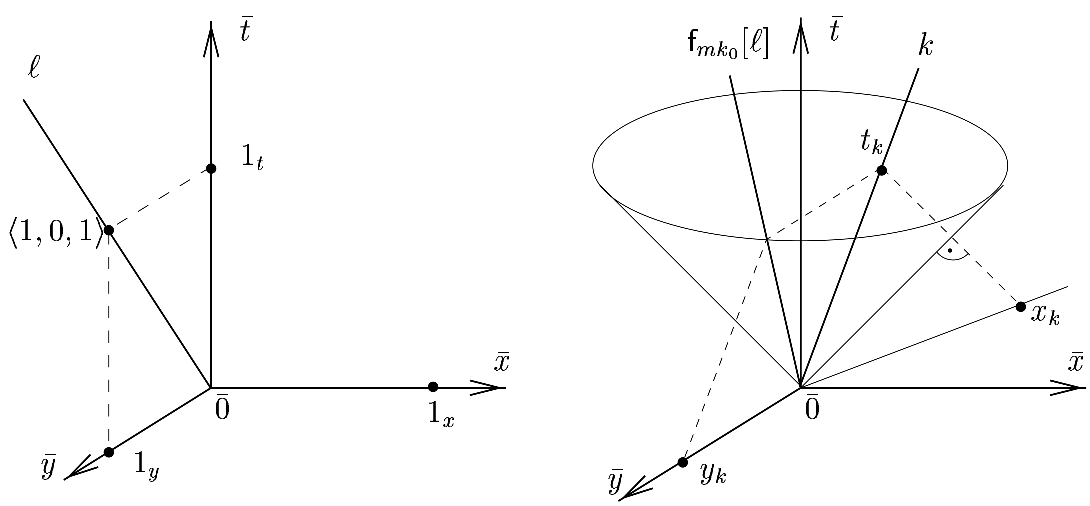

Figure 42: Illustration for item (i) of def. of f_{km_0} for the case n=3.

$N_k = \mathrm{Id}$ implies that $o_k = \bar{0}$ and $\overline{o_k t_k} = k \subseteq \mathsf{Plane}(\bar{t}, \bar{x})$.

In what follows we will define points x_k and y_k, and f_{km_0} will be the linear transformation which will take 1_t, 1_x, 1_y to t_k, x_k, y_k, respectively. The reason

[61] f_{km_0} exists because vectors $t_k - o_k$, $x_k - o_k$ are linearly independent (this is so because vectors t'_k, x'_k are linearly independent and N_k^{-1} is the affine transformation taking $\bar{0}$, t'_k, k'_x to o_k, t_k, x_k, respectively).

why we will choose x_k and y_k exactly the way we will do, can be explained (and motivated) in a completely similar style as in the proof of Prop.3.15 in §3.2. Such an explanation will be included in the present work at a later stage of its development.

First we define the point x_k as the mirror image of t_k w.r.t. the line $\overline{\bar{0}\langle 1,1,0\rangle}$. In more detail: Let $t_k = \langle t_0, t_1, 0\rangle$ then we define $x_k \stackrel{\text{def}}{=} \langle t_1, t_0, 0\rangle$.

Let $r \in {}^+F$ and $y_r = \langle 0, 0, r\rangle$. We will use r as a parameter. We define $\mathsf{f}_r : {}^3F \longrightarrow {}^3F$ to be the linear transformation for which $\mathsf{f}_r(1_t) = t_k$, $\mathsf{f}_r(1_x) = x_k$, and $\mathsf{f}_r(1_y) = y_r$. Clearly, such an f_r exists and is unique. It is easy to check that this f_r is an invertible linear transformation.[62]

Let $\ell \stackrel{\text{def}}{=} \overline{\bar{0}\langle 1,0,1\rangle}$. Let us notice that $\ell \in \mathsf{PhtEucl}$. We claim that $ang^2(\mathsf{f}_r[\ell])$ depends on the choice of r. If r is very big (e.g. $r > 100 \cdot |t_k|$), then $ang^2(\mathsf{f}_r[\ell]) > 1$, while for small r (e.g. $r < |t_k|/100$) $ang^2(\mathsf{f}_r([\ell]) < 1$, the latter is so because $\overline{\bar{0}t_k} = k \in \mathsf{SlowEucl}$. Cheking this claim is left to the reader.

Next we use our assumption that $\mathfrak{F}$ is Euclidean, i.e. that "positive" square-roots exist in $\mathfrak{F}$. Namely, we claim that between the two extremes (big and small choices of r) there exists $r \in {}^+F$ such that

$$ang^2(\mathsf{f}_r[\ell]) = 1\,, \tag{59}$$

because $r \mapsto ang^2(\mathsf{f}_r[\ell])$ is a quadratic polynomial function (and "positive" square-roots exist).[63]

Let this r be fixed. Now, we define

$$\begin{aligned} y_k &\stackrel{\text{def}}{=} y_r\,,\\ \mathsf{f}_{km_0} &\stackrel{\text{def}}{=} \mathsf{f}_r\,, \end{aligned}$$

for the above choice of r. Let us notice that condition (59) above was needed because $\ell \in \mathsf{PhtEucl}$ and because of Prop.3.13 in §3.2.

By this, f_{km_0} is defined for the case $N_k = \mathrm{Id}$, i.e. f_{km_0} is the invertible linear transformation which takes 1_t, 1_x, 1_y to t_k, x_k, y_k, respectively.

(ii) We define f_{km_0} for $N_k \neq \mathrm{Id}$ as follows.

Let $t'_k \stackrel{\text{def}}{=} N_k[t_k]$. Then $\overline{\bar{0}t'_k} \in \left(\mathsf{SlowEucl} \cap \mathsf{Plane}(\bar{t}, \bar{x})\right)$ by $N_k(o_k) = \bar{0}$, by $\overline{o_k t_k} = k \in \mathsf{SlowEucl}$, by Lemma 3.97, and by $N_k[k] \subseteq \mathsf{Plane}(\bar{t}, \bar{x})$.

[62]This is so because vectors t_k, x_k, y_r are linearly independent.

[63]As a curiosity we mention that $r = \sqrt{t_0^2 - t_1^2}$ is such, where $t_k = \langle t_0, t_1, 0\rangle$. We will see this in Claim 3.100, but it is irrelevant at the present point.

Let x'_k and y'_k be chosen for t'_k exactly the way as we chose x_k and y_k for t_k in item (i). It can be done because $\overline{\bar{0}t'_k} \in \big(\mathsf{SlowEucl} \cap \mathsf{Plane}(\bar{t}, \bar{x})\big)$.

Let $x_k \stackrel{\text{def}}{=} N_k^{-1}(x'_k)$, and $y_k \stackrel{\text{def}}{=} N_k^{-1}(y'_k)$.

We define f_{km_0} to be the affine transformation which takes $\bar{0}$, 1_t, 1_x, 1_y to o_k, t_k, x_k, y_k, respectively. Clearly such an f_{km_0} exists[64] and is unique.

Definition of f_{km_0} *for the case of arbitrary n:*

The definition of f_{km_0} for arbitrary n is analogous to the definition of f_{km_0} for $n = 3$. (We recommend the reader to consult Figure 42 there.) Let $n \geq 2$ be arbitrary. Let $k \in \mathsf{SlowEucl}$, $k \neq m_0$ be arbitrary and fixed. Recall again that $P(k) = \langle o_k, t_k, N_k \rangle$. First we will define f_{km_0} for the case when $N_k = \text{Id}$. This will be implemented in item (i) below. After that in item (ii) we will define f_{km_0} for $N_k \neq \text{Id}$.

(i) We define f_{km_0} for the case $N_k = \text{Id}$ as follows.

$N_k = \text{Id}$ implies that $o_k = \bar{0}$ and $\overline{o_k t_k} = k \subseteq \mathsf{Plane}(\bar{t}, \bar{x})$.

First we define the point x_k as the mirror image of t_k w.r.t. the line $\overline{\bar{0}\langle 1, 1, 0, \ldots, 0\rangle}$. In more detail: Let $t_k = \langle t_0, t_1, 0, \ldots, 0\rangle$ then we define $x_k \stackrel{\text{def}}{=} \langle t_1, t_0, 0, \ldots, 0\rangle$.

Let $r \in {}^+F$. We will use r as a parameter. The notation e_i was introduced in item 6 of Notation 3.1. We define $\mathsf{f}_r : {}^nF \longrightarrow {}^nF$ to be the linear transformation for which $\mathsf{f}_r(1_t) = t_k$, $\mathsf{f}_r(1_x) = x_k$, and $\mathsf{f}_r(e_i) = r \cdot e_i$, for all $i \in n \setminus 2$.[65] Clearly, such an f_r exists and is unique. It is easy to check that f_r is an invertible linear transformation.

Let $\ell_i \stackrel{\text{def}}{=} \overline{\bar{0}(1_t + e_i)}$, for all $i \in n \setminus 2$. Let us notice that $\ell_i \in \mathsf{PhtEucl}$, for all $i \in n \setminus 2$.

It is easy to check that $ang^2(\mathsf{f}_r[\ell_i]) = ang^2(\mathsf{f}_r[\ell_j])$, for all $i, j \in n \setminus 2$.

We claim that $ang^2(\mathsf{f}_r[\ell_i])$ depends on the choice of r. Now if r is very big, then $ang^2(\mathsf{f}_r[\ell_i]) > 1$, while for small r $ang^2(\mathsf{f}_r[\ell_i]) < 1$, the latter is so because $\overline{\bar{0}t_k} = k \in \mathsf{SlowEucl}$. Checking this claim is left to the reader.

Next we use our assumption that $\mathfrak{F}$ is Euclidean, i.e. that "positive" square-roots exist in $\mathfrak{F}$. Namely, we claim that between the two extremes (big and

[64] f_{km_0} exists because vectors $t_k - o_k$, $x_k - o_k$, $y_k - o_k$ are linearly independent (this is so because vectors t'_k, x'_k, y'_k are linearly independent and N_k^{-1} is the affine transformation taking $\bar{0}$, t'_k, x'_k, y'_k to o_k, t_k, x_k, y_k, respectively).

[65] Recall from set theory that $n \setminus 2 = \{2, \ldots, n-1\}$.

small choices of r) there exists $r \in {}^+F$ such that

$$ang^2(\mathsf{f}_r[\ell_i]) = 1\,, \text{ for all } i \in n \setminus 2\,,$$

because $r \mapsto ang^2(\mathsf{f}_r[\ell])$ is a quadratic polynomial function (and "positive" square-roots exist). Let this r be fixed.

Now, we define

$$\begin{aligned} e_{k,i} &\stackrel{\text{def}}{=} r \cdot e_i\,, \text{ and} \\ \mathsf{f}_{km_0} &\stackrel{\text{def}}{=} \mathsf{f}_r\,, \end{aligned}$$

for the above choice of r, and for all $i \in n \setminus 2$.

By this, f_{km_0} is defined for the case $N_k = \mathrm{Id}$, i.e. f_{km_0} is the invertible linear transformation which takes 1_t, 1_x, $e_2, \ldots, e_{n-1}$ to t_k, x_k, $e_{k,2}, \ldots, e_{k,n-1}$, respectively.

(ii) The definition of f_{km_0} for the case $N_k \neq \mathrm{Id}$ is obtained from item (i) in a completely analogous way as we did this for the case $n = 3$. In more detail:

Let $t'_k \stackrel{\text{def}}{=} N_k[t_k]$. Then $\overline{\bar{0}t'_k} \in \big(\mathsf{SlowEucl} \cap \mathsf{Plane}(\bar{t}, \bar{x})\big)$ by $N_k(o_k) = \bar{0}$, by $\overline{o_k t_k} = k \in \mathsf{SlowEucl}$, by Lemma 3.97, and by $N_k[k] \subseteq \mathsf{Plane}(\bar{t}, \bar{x})$.

Let x'_k, $e'_{k,i}$ ($i \in n \setminus 2$) be obtained from t'_k exactly as x_k, $e_{k,i}$ ($i \in n \setminus 2$) were obtained from t_k in item (i). This can be done because $\overline{\bar{0}t'_k} \in \big(\mathsf{SlowEucl} \cap \mathsf{Plane}(\bar{t}, \bar{x})\big)$. Then let $x_k \stackrel{\text{def}}{=} N_k^{-1}(x'_k)$ and $e_{k,i} \stackrel{\text{def}}{=} N_k^{-1}(e'_{k,i})$. Then f_{km_0} is defined to be the affine transformation wich takes $\bar{0}$, 1_t, 1_x, $e_2, \ldots, e_{n-1}$ to o_k, t_k, x_k, $e_{k,2}, \ldots, e_{k,n-1}$, respectively.

Definition of W:

By the above, f_{km_0} is defined for all $k \in \mathsf{SlowEucl}$, $k \neq m_0$. Recall that w_{m_0} was defined below the definition of m_0 at the beginning of Def.3.98. We now define

$$w_k \stackrel{\text{def}}{=} \mathsf{f}_{km_0} \circ w_{m_0}, \text{ for all } k \in Obs \setminus \{m_0\}, \text{ and}$$

$$W \stackrel{\text{def}}{=} \{\langle m, p, h\rangle \,:\, m \in Obs, h \in w_m(p)\}.$$

By this, the model $\mathfrak{M} \stackrel{\text{def}}{=} \mathfrak{M}^P_{\mathfrak{F}} \stackrel{\text{def}}{=} \langle B, \ldots, W\rangle$ has been defined.

For fixed $n \geq 2$, the class of the above defined models is called the class of *simple models*, and we denote this class by SM.

END OF DEF. OF SM. $\triangleleft$

THEOREM 3.99 $\mathsf{SM} \models Basax$.

Proof: We will give the proof for the case $n = 3$. The proof for arbitrary n is similar and we omit it. (The proof for $n = 2$ is obtainable from that for $n = 3$ in the obvious way.) The organization of the proof will be analogous with that of a similar proof given for $n = 2$ for Thm.2.12 in §2.4 ("Models for *Basax* in dimension 2").

Let $\mathfrak{M} \stackrel{\text{def}}{=} \mathfrak{M}^P_{\mathfrak{F}}$ be such that $\mathfrak{F}$ is Euclidean. Recall that $P(k) = \langle o_k, t_k, N_k \rangle$ (cf. the beginning of Def.3.98). We have already observed that $\mathfrak{M} \models$ **Ax1**, **Ax2**, and that **Ax3**, **Ax4**, **Ax5**, **AxE** hold for the fixed observer $m_0 \in Obs$ (cf. the first two pages of Def.3.98, above "Definition of f_{km_0}"). Let $k \in Obs \setminus \{m_0\}$ be arbitrary but fixed.

We will prove that (I)-(V) hold for f_{km_0}:

(I) $\mathsf{f}_{km_0} : {}^3F \longrightarrow {}^3F$ is a bijection.

(II) $\mathsf{f}_{km_0}[\ell] \in \mathsf{Eucl}$, for all $\ell \in \mathsf{Eucl}$.

(III) $\mathsf{f}_{km_0}[\bar{t}] = k$.

(IV) $\mathsf{f}_{km_0}[\ell] \in Ph$ iff $\ell \in Ph$, for all $\ell \in \mathsf{Eucl}$.

(V) $\mathsf{f}_{km_0}[\ell] \in Obs$, for all $\ell \in Obs$.

Indeed, (I)-(II) hold because f_{km_0} is defined to be an affine transformation. (III) holds because of (II) and because we defined f_{km_0} to take $\bar{0}$, 1_t, respectively, to o_k, t_k, and $k = \overline{o_k t_k}$. We will prove (IV) and (V) at the end of the proof.

Now, in $\mathfrak{M}$ we have that for all $\ell \in B$ that

$$\ell = tr_k(\mathsf{f}_{km_0}[\ell]) . \tag{60}$$

(The proof of (60) is exactly like in §2.4, in the proof of Thm.2.12.) Since f_{km_0} is a bijection, by (II) we have that both f_{km_0} and $\mathsf{f}^{-1}_{km_0}$ preserve Eucl. Using this, together with (II)-(V), (60), and the fact that **Ax3**, **Ax4**, **Ax5**, **AxE** hold for $m_0 \stackrel{\text{def}}{=} \bar{t}$, we get that **Ax3**, **Ax4**, **Ax5**, **AxE** hold for k, too. From (I) and from the definition[66] of w_k we get that $Rng(w_k) = Rng(w_{m_0})$. Since k was arbitrary, this proves $\mathfrak{M} \models Basax$.

Thm.3.99 is proved modulo (IV) and (V) above. Now we turn to prove these.

[66] Recall that $w_k = \mathsf{f}_{km_0} \circ w_{m_0}$.

Proof of (IV):
To prove (IV), by $Ph = \mathsf{PhtEucl}$ it is sufficient to prove (61) below.

$$(61) \qquad (\forall \ell \in \mathsf{Eucl})\Big(\ell \in \mathsf{PhtEucl} \;\Leftrightarrow\; \mathsf{f}_{km_0}[\ell] \in \mathsf{PhtEucl}\Big).$$

Next we turn to prove (61). The proof of (61) will consist of two cases: (i) $N_k = \mathrm{Id}$, (ii) $N_k \neq \mathrm{Id}$.

Proof of (61) for case $N_k = \mathrm{Id}$:
Let us recall that for this case f_{km_0} was defined in item (i) of the "definition of f_{km_0} for the case $n = 3$". Let us recall that in that definition we have that

$$(62) \qquad o_k = \bar{0} \quad \text{and} \quad \overline{o_k t_k} = k \subseteq \mathsf{Plane}(\bar{t}, \bar{x})\,,$$

and f_{km_0} is the linear transformation which takes 1_t, 1_x, 1_y, respectively, to $t_k = \langle t_0, t_1, 0\rangle$, $x_k = \langle t_1, t_0, 0\rangle$, $y_k = \langle 0, 0, r\rangle$, for fixed $r \in {}^+F$, where r was fixed in such a way that

$$ang^2(\mathsf{f}_{km_0}[\ell]) = 1, \text{ where } \ell \stackrel{\text{def}}{=} \overline{\bar{0}\langle 1, 0, 1\rangle}\,.$$

Claim 3.100 *$t_0^2 - t_1^2 > 0$ and $r = \sqrt{t_0^2 - t_1^2}$.*

Proof of Claim 3.100: We have $t_0^2 - t_1^2 > 0$ because $t_k = \langle t_0, t_1, 0\rangle$, because $k \in Obs \stackrel{\text{def}}{=} \mathsf{SlowEucl}$ and because by (62), $\overline{\bar{0}t_k} = k$.

By f_{km_0} being a linear transformation taking 1_t, 1_x, 1_y to t_k, x_k, y_k, respectively, it is easy to see that $\mathsf{f}_{km_0}[\ell] = \overline{\bar{0}\langle t_0, t_1, r\rangle}$. Now by this

$$ang^2(\mathsf{f}_{km_0}[\ell]) = 1 \quad \Leftrightarrow \quad \frac{t_1^2 + r^2}{t_0^2} = 1\,.$$

By this and by $r \in {}^+F$, we have

$$ang^2(\mathsf{f}_{km_0}[\ell]) = 1 \quad \Leftrightarrow \quad r = \sqrt{t_0^2 - t_1^2}\,;$$

where $\sqrt{t_0^2 - t_1^2}$ exists because $t_0^2 - t_1^2 > 0$ and $\mathfrak{F}$ is Euclidean. This proves Claim 3.100.

QED (Claim 3.100)

We have that f_{km_0} takes parallel lines to parallel lines because f_{km_0} is a linear transformation. Hence to show (61) it is enough to show (63) below.

$$(63) \qquad (\forall \ell \in \mathsf{Eucl})\Big(\bar{0} \in \ell \;\Rightarrow\; (\ell \in \mathsf{PhtEucl} \;\Leftrightarrow\; \mathsf{f}_{km_0}[\ell] \in \mathsf{PhtEucl}\Big)\,.$$

To show (63) let $\ell \in \mathsf{Eucl}$ with $\bar{0} \in \ell$. Without loss of generality we may assume that $\ell = \overline{\bar{0}\langle 1, a, b\rangle}$, for some $a, b \in F$. Let these a, b be fixed. Now by using Claim 3.100, we have that f_{km_0} is the linear transformation taking 1_t, 1_x, 1_y to $t_k = \langle t_0, t_1, 0\rangle$, $x_k = \langle t_1, t_0, 0\rangle$, $y_k = \langle 0, 0, \sqrt{t_0^2 - t_1^2}\rangle$, respectively ($t_0^2 - t_1^2 > 0$). By this, it is easy to see that

$$\mathsf{f}_{km_0}(1, a, b) = \langle t_0 + at_1,\ t_1 + at_0,\ b\sqrt{t_0^2 - t_1^2}\ \rangle\,, \text{ hence}$$

$$\mathsf{f}_{km_0}[\ell] = \overline{\bar{0}\ \langle t_0 + at_1,\ t_1 + at_0,\ b\sqrt{t_0^2 - t_1^2}\ \rangle}\,. \tag{64}$$

Now

$$ang^2(\mathsf{f}_{km_0}[\ell]) \;=\; \frac{(t_1 + at_0)^2 + \left(b\sqrt{t_0^2 - t_1^2}\right)^2}{(t_0 + at_1)^2} \qquad \text{(by (64))}$$

$$=\; \frac{(t_0 + at_1)^2 + (a^2 + b^2 - 1)(t_0^2 - t_1^2)}{(t_0 + at_1)^2} \qquad \text{(by some computation).}$$

Now

$$\begin{aligned}
ang^2(\mathsf{f}_{km_0}[\ell])) = 1 \quad &\Leftrightarrow \\
(a^2 + b^2 - 1)(t_0^2 - t_1^2) = 0 \quad &\Leftrightarrow \quad (\text{by } t_0^2 - t_1^2 > 0) \\
a^2 + b^2 = 1 \quad &\Leftrightarrow \quad (\text{by def. of } a, b) \\
ang^2(\ell) = 1. &
\end{aligned}$$

By the above computation, we have $\mathsf{f}_{km_0}[\ell] \in \mathsf{PhtEucl}$ iff $\ell \in \mathsf{PhtEucl}$. By this, (61) is proved for case $N_k = \mathrm{Id}$.

Proof of (61) for case $N_k \neq \mathrm{Id}$*:*
Let us recall that for this case f_{km_0} was defined in item (ii) of the "Definition of f_{km_0} for the case $n = 3$". Let t'_k, x'_k, y'_k be exactly those points which were defined there and let f_{km_0} be the affine transformation which was defined there (in item (ii) of the "Definition of f_{km_0} for the case $n = 3$"). Now we define f'_{km_0} to be the linear transformation which takes 1_t, 1_x, 1_y to t'_k, x'_k, y'_k, respectively.

Now we can prove that f'_{km_0} satisfies condition (61) above in a completely analogous way as we did for case $N_k = \mathrm{Id}$ for f_{km_0} there (to see the analogy, consider the definition of x'_k, y'_k in line 4 of item (ii)).

It is easy to check that $\mathsf{f}_{km_0} = \mathsf{f}'_{km_0} \circ N_k^{-1}$. Now by f'_{km_0} satisfying condition (61) above, by $\mathsf{f}_{km_0} = \mathsf{f}'_{km_0} \circ N_k^{-1}$ and by Lemma 3.97, we have that f_{km_0} satisfies (61). We proved (61) for case $N_k \neq \mathrm{Id}$, too. This completes the proof of (IV).

Proof of (V): To prove (V), by $Obs = \mathsf{SlowEucl}$ it is sufficient to prove (65) below.

$$(65) \qquad (\forall \ell \in \mathsf{SlowEucl})\ \mathsf{f}_{km_0}[\ell] \in \mathsf{SlowEucl}\,.$$

Next we turn to prove (65). The proof of (65) will be similar to that of (61) in the proof of (IV). The proof again will consist of two cases.

Proof of (65) for case $N_k = \mathrm{Id}$*:*
As we have shown at the beginning of the proof of (IV) (cf. Claim 3.100), for this case we have that f_{km_0} is a linear transformation which takes 1_t, 1_x, 1_y to $t_k = \langle t_0, t_1, 0\rangle$, $x_k = \langle t_1, t_0, 0\rangle$, $y_k = \langle 0, 0, \sqrt{t_0^2 - t_1^2}\rangle$, respectively ($t_0^2 - t_1^2 > 0$).

By f_{km_0} being a linear transformation, we have that f_{km_0} takes parallel lines to parallel lines. Thus to prove (65) above it is enough to prove

$$(\forall \ell \in \mathsf{SlowEucl})\Big(\bar{0} \in \ell \ \Rightarrow\ \mathsf{f}_{km_0}[\ell] \in \mathsf{SlowEucl}\Big)\,.$$

To see this, let $\ell \in \mathsf{SlowEucl}$ with $\bar{0} \in \ell$. Then it is easy to see that $\ell = \overline{\bar{0}\langle 1, a, b\rangle}$, for some $a, b \in F$ with $a^2 + b^2 < 1$. Let these a, b be fixed. Now it is easy to see that

$$(66) \qquad \mathsf{f}_{km_0}[\ell] = \overline{\bar{0}\ \langle t_0 + at_1,\ t_1 + at_0,\ b\sqrt{t_0^2 - t_1^2}\ \rangle}\,.$$

Now

$$\begin{aligned}
ang^2(\mathsf{f}_{km_0}[\ell]) &= \frac{(t_1 + at_0)^2 + \left(b\sqrt{t_0^2 - t_1^2}\right)^2}{(t_0 + at_1)^2} && \text{(by(66))}\\
&= \frac{(t_0 + at_1)^2 + (a^2 + b^2 - 1)(t_0^2 - t_1^2)}{(t_0 + at_1)^2} && \text{(by some computation)}\\
&< \frac{(t_0 + at_1)^2}{(t_0 + at_1)^2} && \text{(by } a^2 + b^2 < 1 \text{ and } t_0^2 - t_1^2 > 0)\\
&= 1\,.
\end{aligned}$$

If we summarize the above computation we get $ang^2(\mathsf{f}_{km_0}[\ell]) < 1$, hence $\mathsf{f}_{km_0}[\ell] \in \mathsf{SlowEucl}$. By this, (65) is proved for case $N_k = \mathrm{Id}$.

Proof of (65) for case $N_k \neq \mathrm{Id}$*:*
The proof of this will be analogous with that of (61) in the proof of (IV). Let us recall that for this case f_{km_0} was defined in item (ii) of the "Definition of f_{km_0} for the case $n = 3$". Let t_k', x_k', y_k' be exactly those points which were defined there

and let f_{km_0} be the affine transformation which was defined there (in item (ii) of the "Definition of f_{km_0} for the case $n = 3$"). Now we define f'_{km_0} to be the linear transformation which takes 1_t, 1_x, 1_y to t'_k, x'_k, y'_k, respectively.

Now we can prove that f'_{km_0} satisfies condition (65) above in a completely analogous way as we did for case $N_k = \mathrm{Id}$ for f_{km_0} there (to see the analogy, consider the definition of x'_k, y'_k in line 4 of item (ii)).

It is easy to check that $\mathsf{f}_{km_0} = \mathsf{f}'_{km_0} \circ N_k^{-1}$. Now by f'_{km_0} satisfying condition (65) above, by $\mathsf{f}_{km_0} = \mathsf{f}'_{km_0} \circ N_k^{-1}$ and by Lemma 3.97 we have that f_{km_0} satisfies (65). We proved (65) for case $N_k \neq \mathrm{Id}$, too. This completes the proof of (V). ∎

Proof of Lemma 3.96:

Definition: By a *congruence* transformation $h : {}^nF \longrightarrow {}^nF$ we understand an affine transformation which preserves Euclidean distances, i.e. $(\forall p, q \in {}^nF)$ $|h(p) - h(q)| = |p - q|$. We note that in the present proof we will use such transformations which preserve $\bar{0}$.

Let us turn to proving Lemma 3.96.

(1) Without loss of generality we may assume $p = \bar{0}$.

(2) The assumption that $\mathfrak{F}$ is Euclidean (i.e. "positive" square-roots exist) is essential in proving this lemma (it is not true without this assumption).

(3) We need a proof only for $n \leq 4$ because, we mentioned earlier in this work that we treat $n > 4$ only if no extra effort is needed for that. (All the same, the present lemma is true for arbitrary n).

Sub-lemma: Let $q \in {}^jF$, $j \in \omega$ arbitrary, and let $\mathfrak{F}$ be Euclidean. Then there is a *congruence* transformation $f : {}^jF \longrightarrow {}^jF$ with $f(\bar{0}) = \bar{0}$ and $f(q) \in \bar{x}$.

Proof of Sub-lemma:

We prove this only for $j \leq 3$ because that will be sufficient for $n \leq 4$, cf. item (3) above. Let $j = 3$. Throughout the proof of Sub-lemma the reader is asked to consult Figure 43.

Let $q' \stackrel{\text{def}}{=} \langle 0, q_1, q_2 \rangle$ and $\lambda \in F$ be such that $1'_x \stackrel{\text{def}}{=} \lambda \cdot q'$ is of length 1, i.e. $|1'_x| = 1$. Such a λ exists because $\mathfrak{F}$ is Euclidean.

Let $1'_y = \langle 0, a, b \rangle$ be arbbitrary but orthogonal (in the Euclidean sense)[67] to $1'_x$ with $|1'_y| = 1$. This obviously exists.

Let h be the linar trasformation defined by $1_t, 1'_x, 1'_y \mapsto 1_t, 1_x, 1_y$. By the choice of $1'_x, 1'_y$, h exists and is invertible. Further, $h(q) \in \mathsf{Plane}(\bar{t}, \bar{x})$.

[67]For the definition of orthogonality see item 4 of Def.3.2.

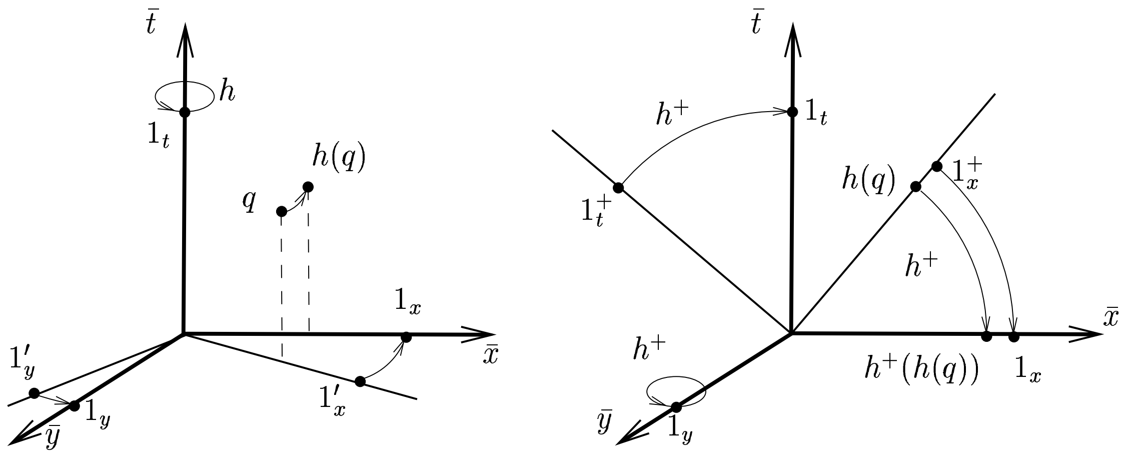

Figure 43: Illustration for the proof of Sub-lemma.

By a completely similar argument, there is another invertible linear transformation h^+ with $h^+(h(q)) \in \bar{x}$. But then $f = h \circ h^+$ has the desired properties. Further f is a *congruence* transformation because $1'_x, 1'_y$ had lenght 1, $1'_x \perp 1'_y$ etc.

The proof for $j < 3$ is obtained from the above one the obvious way.

END of Proof of Sub-lemma.

Let us turn to proving Lemma 3.96. Throughout the proof the reader is asked to consult Figure 44.

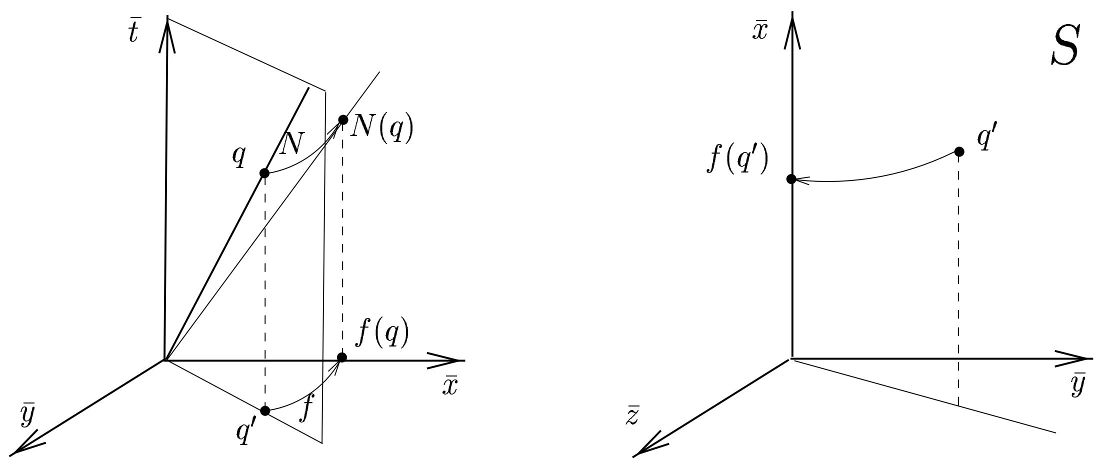

Figure 44: Illustration for the proof of Lemma 3.96.

Let $\ell \in \mathsf{Eucl}$, $\bar{0} \in \ell$ and $\bar{0} \neq q \in \ell$. Let $q' = \langle 0, q_1, \ldots, q_{n-1} \rangle \in S$.

Clearly, any congruence transformation $f : S \longrightarrow S$ of S preserving $\bar{0}$ induces an $N \in Newt_0$ as follows.

$$N(p_0, \ldots, p_{n-1}) \stackrel{\text{def}}{=} \langle p_0, f(p_1, \ldots, p_{n-1})\rangle \, .$$

By Sub-lemma (applied to q', S in place of $q, {}^jF$), there is $f : S \longrightarrow S$ with $\langle 0, f(q_1, \ldots, q_{n-1})\rangle \in \bar{x}$.

The "N" induced by this f has the desired properties.

This completes the proof of Lemma 3.96 for $n \leq 4$, because we proved Sub-lemma only for $j \leq 3$.

The generalization for $n > 4$ goes by proving Sub-lemma for arbitrary j. This can be done via a straightforward induction. We omit it for the already indicated reasons. ∎

3.6 Models of *Basax*

In this sub-section we will characterize the models of $Basax(\mathbf{3})$, $Basax(\mathbf{n}) \cup \{\mathbf{Ax}(\sqrt{\ })\}$, and $Basax(\mathbf{n}) \cup \{\mathbf{Ax7}\}$, for $n \geq 3$, where **Ax7** will be introduced very soon. We will construct a class of frame models which we call the class of general models, in symbols GM, and we will show that $\mathsf{GM} \models Basax$. Let $\mathsf{GM}(\mathbf{n})$ denote the n–dimensional version of the class GM of models to be defined in this sub-section. We will show that every model of $Basax(\mathbf{3})$ is isomorphic to a model in $\mathsf{GM}(\mathbf{3})$, and every model of $Basax(\mathbf{n}) \cup \{\mathbf{Ax7}\}$ or $Basax(\mathbf{n}) \cup \{\mathbf{Ax}(\sqrt{\ })\}$, for $n \geq 3$, is isomorphic to a model in $\mathsf{GM}(\mathbf{n})$. Further in this sub-section we will show that both $Basax(\mathbf{3})$ and $Basax(\mathbf{n}) \cup \{\mathbf{Ax7}\}$ imply $\mathbf{Ax}(\sqrt{\ })$ for $n \geq 3$, where **Ax7** is a more natural (or more "physical" in some sense) axiom than $\mathbf{Ax}(\sqrt{\ })$.

Below we postulate axiom **Ax7**.

Ax7 $(\forall m \in Obs)(\forall \ell \in \mathsf{SlowEucl} \cap \mathsf{Plane}(\bar{t}, \bar{x}))(\exists k \in Obs)$
$$(tr_m(k) = \ell \ \wedge\ \mathsf{f}_{km}[\bar{y}] \parallel \bar{y})\,.$$

See Figure 45.

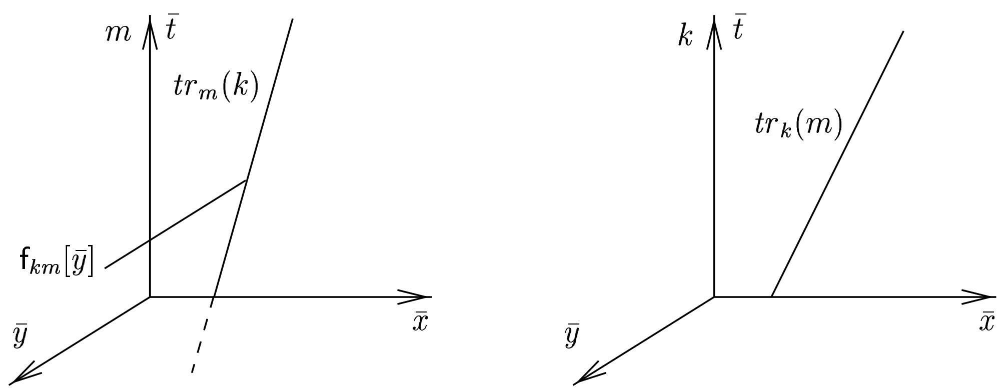

Figure 45: Illustration for **Ax7**.

Intuitively, **Ax7** says that there are observers moving in direction $\bar{x}$ whose $\bar{y}$ axis remains parallel with the "original" $\bar{y}$ axis. This assumption is almost always taken for granted in physics books (cf. e.g. Rindler [35]). Indeed, it sounds contrary to experience to assume that for some velocity $v < 1$, if an observer k is moving with velocity v in direction $\bar{x}$ then some "magical force" would force k to point his $\bar{y}$ axis in some direction different from the original

$\bar{y}$ axis.

Summing up, we consider **Ax7** as a relatively weak and natural ("physically convincing") assumption. Actually we tend a feel that **Ax7** is more natural (in some sense) than e.g. $\mathbf{Ax}(\sqrt{\ })$.

Remark 3.101 We note that $\mathsf{SM} \not\models$ **Ax7**, where SM is defined in Def.3.98 in §3.5. However, we will see that the models of SM can be extended to richer models validating **Ax7** (cf. Prop.3.115). $\triangleleft$

In what follows we will introduce the set PT of so called *photon preserving transformations* which will be a subset of the set of all affine transformations. To motivate this definition we recall Thm.3.3 and Prop.3.13 from §3.2.

Thm.3.3 $\quad Basax \models (\mathsf{f}_{mk} = \tilde{\varphi} \circ g$, for some $g \in Aftr$ and $\varphi \in Aut(\mathbf{F}))$.

Prop.3.13 $\quad Basax \models (\forall \ell \in \mathsf{Eucl})(\ell \in \mathsf{PhtEucl} \ \Leftrightarrow\ \mathsf{f}_{mk}[\ell] \in \mathsf{PhtEucl})$.

By these two, we also have that the "g" occurring in Thm.3.3 preserves $\mathsf{PhtEucl}$. Next, we will collect these g's into something called PT.

Definition 3.102 The set of *photon preserving transformations*, in symbols $PT = PT(\mathbf{n}, \mathbf{F})$, is defined as follows.

$$PT \stackrel{\text{def}}{=} \{g \in Aftr : (\forall \ell \in \mathsf{Eucl})(\ell \in \mathsf{PhtEucl} \ \Leftrightarrow\ g[\ell] \in \mathsf{PhtEucl})\}. \qquad \triangleleft$$

Remark 3.103 $\langle PT, \circ, {}^{-1}, \mathrm{Id}\rangle$ is a group. $\triangleleft$

The next two propositions belong to the motivation of Def.3.102 above. Thm.3.108 below them describes how the elements of PT look like. Prop.3.104(ii) below also serves as a motivation for the definition of GM (Def.3.109 way below).

PROPOSITION 3.104

(i) *Basax* $\models (\forall m, k \in Obs)\big(\mathsf{f}_{mk} = \tilde{\varphi} \circ g$, *for some* $g \in PT$ *with* $g[\bar{t}] = tr_k(m)$ *and* $\varphi \in Aut(\mathbf{F})\big)$.

(ii) $Basax \cup \{\mathbf{Ax}(\sqrt{\ })\} \models (\forall m, k \in Obs)\Big(\mathsf{f}_{mk} = \tilde{\varphi} \circ g,$ *for some* $g \in PT$ *with* $g[\bar{t}] = tr_k(m)$ *and* $\varphi \in Aut(\mathfrak{F})\Big)$.

Proof: Item (i) follows by Thm.3.4, by Prop.3.13, by **Ax4**, by Prop.2.6(vii) in §2.3, and by the following property (which follows by Lemma 3.119(iii) way below):

$$(\forall \varphi \in Aut(\mathbf{F}))(\forall \ell \in \mathsf{Eucl})(\ell \in \mathsf{PhtEucl} \;\Leftrightarrow\; \tilde{\varphi}[\ell] \in \mathsf{PhtEucl}) .$$

Item (ii) is a corollary of item (i) and Remark 3.106 below. ∎

The emphasis in Prop.3.104(ii) above and Prop.3.105 below is on φ being order preserving, i.e. on writing $\mathfrak{F}$ in place of $\mathbf{F}$.

PROPOSITION 3.105 *Let* $n \geq 3$. *Then (i), (ii) below hold.*

(i) $Basax(\mathbf{n}) \cup \{\mathbf{Ax7}\} \models (\forall m, k \in Obs)\Big(\mathsf{f}_{mk} = \tilde{\varphi} \circ g,$ *for some* $g \in PT$ *with* $g[\bar{t}] = tr_k(m)$ *and* $\varphi \in Aut(\mathfrak{F})\Big)$.

(ii) $Basax(\mathbf{3}) \qquad \models (\forall m, k \in Obs)\Big(\mathsf{f}_{mk} = \tilde{\varphi} \circ g,$ *for some* $g \in PT$ *with* $g[\bar{t}] = tr_k(m)$ *and* $\varphi \in Aut(\mathfrak{F})\Big)$.

Proof: This proposition is a corollary of Prop.3.104(ii) above and Thm.3.114 way below which says that both $Basax(\mathbf{3})$ and $Basax(\mathbf{n}) \cup \{\mathbf{Ax7}\}$ for $n \geq 3$, implies $\mathbf{Ax}(\sqrt{\ })$. ∎

Remark 3.106 Assume $\mathfrak{F}$ is Euclidean, i.e. $\mathfrak{F} \models \mathbf{Ax}(\sqrt{\ })$. Then $Aut(\mathfrak{F}) = Aut(\mathbf{F})$ because of the following. $Aut(\mathfrak{F}) \subseteq Aut(\mathbf{F})$ is obvious. To prove the other inclusion let $\varphi \in Aut(\mathbf{F})$ and let $0 \leq x \in F$. By $\mathbf{Ax}(\sqrt{\ })$, we have $x = y^2$, for some $y \in F$. Let this y be fixed. Now $\varphi(x) = \varphi(y^2) = \varphi(y)^2 \geq 0$. Hence φ is order preserving, i.e. $\varphi \in Aut(\mathfrak{F})$. $\lhd$

Definition 3.107 (T_{pq})
Assume $\mathfrak{F}$ is Euclidean.

Let $N : {}^nF \times {}^nF \longrightarrow Newt_0$ be a fixed function with the property ($\star$) below. $N_{pq} \stackrel{\text{def}}{=} N(p,q)$.

$$(\star) \qquad (\forall p, q \in {}^nF)\Big(p \neq q \;\Rightarrow\; (N_{pq}[\overline{pq}] \subseteq \mathsf{Plane}(\bar{t}, \bar{x}) \;\wedge\; N_{pq}(p) = \bar{0})\Big).$$

Such a function exists by Lemma 3.96 in §3.5. Throughout, this function N is fixed.[68]

For every distinct $p, q \in {}^nF$ with $\overline{pq} \in \mathsf{SlowEucl}$ we define T_{pq} exacly as f_{km_0} was defined in the definition of SM (Def.3.98 in §3.5) but with $\langle p, q, N_{pq} \rangle$ in place of $\langle o_k, t_k, N_k \rangle$.

$\triangleleft$

The following theorem describes how the elements of PT look like.

THEOREM 3.108 *Assume $\mathfrak{F}$ is Euclidean. Assume $p, q \in {}^nF$ with $p \neq q$ and $\overline{pq} \in \mathsf{SlowEucl}$. For (i), (ii) below, we claim (i)$\Leftrightarrow$(ii).*

(i) *$A \in PT$ with $A(\bar{0}) = p$, $A(1_t) = q$.*

(ii) *$A = M \circ T_{pq}$, for some $M \in Newt_0$.*

We will give the **proof** later, after the proof of Lemma 3.118.

In the following definition we will define a class of frame models GM, so called *general models*. This definition is motivated by Prop.3.104(ii) and Thm.3.108.

Definition 3.109 (General Models, GM)
Let $\mathfrak{F}$ be Euclidean. Let B, Obs ,Ph, Ib be sets and let
$\alpha \; : \; Obs \longrightarrow {}^nF \times {}^nF \times Aut(\mathfrak{F}) \times Newt_0$ and
$\beta \; : \; B \longrightarrow \mathcal{P}({}^nF)$
be functions with properties 1-6 below. For all $k \in Obs$ we denote $\alpha(k)$ by $\langle o_k, t_k, \varphi_k, M_k \rangle$.

1. $Obs \cup Ph \subseteq Ib \subseteq B$.
2. $(\forall k \in Obs)\; o_k \neq t_k$.
3. $(\forall k \in Obs)\; \beta(k) = \overline{o_k t_k}$.
4. $\beta[Obs] = \mathsf{SlowEucl}$.
5. $\beta[Ph] = \mathsf{PhtEucl}$.
6. $\beta[Ib] \subseteq \mathsf{Eucl}$.

[68] In some sense, N is also a "choice function" similarly to P in the definition of SM (Def.3.98 in §3.5).

For every such $\mathfrak{F}$, and for every such sets B, Obs, Ph, Ib and functions α, β satisfying 1-6 we define a frame model $\mathfrak{M}$ as follows.
$\mathfrak{M} \stackrel{\text{def}}{=} \langle (B, Obs, Ph, Ib), \mathfrak{F}, G; \ \mathrm{E}\ , W \rangle$, where $\langle G, \ \mathrm{E}\ \rangle \stackrel{\text{def}}{=} \langle \mathsf{Eucl}, \ \in \ \rangle$.
It remains to define W. First we define a function $w_0 : {}^nF \longrightarrow \mathcal{P}(B)$ as follows. Let $p \in {}^nF$. Then

$$w_0(p) \stackrel{\text{def}}{=} \{b \in B \ : \ p \in \beta(b)\} .$$

Let $k \in Obs$ be arbitrary. We define

$$w_k \stackrel{\text{def}}{=} \widetilde{\varphi_k} \circ M_k \circ T_{o_k t_k} \circ w_0 ;$$

where $T_{o_k t_k}$ was defined in Def.3.107. (Let us notice that $o_k \neq t_k$ (by 2) and $\overline{o_k t_k} \in \mathsf{SlowEucl}$ (by 3,4). Therefore $T_{o_k t_k}$ is defined.)

$$W \stackrel{\text{def}}{=} \{\langle m, p, h\rangle \ : \ m \in Obs, h \in w_m(p)\} .$$

By this $\mathfrak{M}$ is defined.
For a fixed $n \geq 2$, the class of the above defined models is called the class of *general models*, and we denote this class by $\mathsf{GM} = \mathsf{GM}(\mathbf{n})$. $\lhd$

THEOREM 3.110 $\mathsf{GM} \models$ *Basax.*

We will give the **proof** later, after the proof of Lemma 3.120.

Notation 3.111 Let K be a class of models. Then IK denotes the class of isomorphic copies of members of K, that is,

$$\mathsf{IK} \stackrel{\text{def}}{=} \{\mathfrak{A} : (\exists \mathfrak{B} \in \mathsf{K})\mathfrak{A} \cong \mathfrak{B}\}. \qquad \lhd$$

In the following theorem we give "structural" characterizations for the models of the theories $Basax(\mathbf{3})$, $Basax(\mathbf{n}) \cup \{\mathbf{Ax}(\sqrt{\ })\}$, and $Basax(\mathbf{n}) \cup \{\mathbf{Ax7}\}$, for $n \geq 3$.

THEOREM 3.112 *Let $n \geq 3$. Then (i)-(iii) below hold.*

(i)	$\mathsf{Mod}_{\mathrm{OFG}}(Basax(\mathbf{3}))$	$=$	$\mathsf{IGM}(\mathbf{3})$.
(ii)	$\mathsf{Mod}_{\mathrm{OFG}}(Basax(\mathbf{n}) \cup \{\mathbf{Ax}(\sqrt{\ })\})$	$=$	$\mathsf{IGM}(\mathbf{n})$.
(iii)	$\mathsf{Mod}_{\mathrm{OFG}}(Basax(\mathbf{n}) \cup \{\mathbf{Ax7}\})$	$=$	$\mathsf{IGM}(\mathbf{n}) \cap \mathsf{Mod}(\mathbf{Ax7})$.

On the **proof**: We will give the proof of item (ii) later, after the proof of Thm.3.110. Items (i) and (iii) directly follow by item (ii) and Thm.3.114 below.

QUESTION 3.113 *Is* $\mathsf{Mod}_{\text{OFG}}(Basax(\mathbf{4})) = \mathbf{I}\mathsf{GM}(\mathbf{4})$ *true?* $\lhd$

Theorem 3.114 below says that if $n \geq 3$, then both $Basax(\mathbf{3})$ and $Basax(\mathbf{n}) \cup \{\mathbf{Ax7}\}$ imply $\mathbf{Ax}(\sqrt{\ })$. We note that these implications do not hold backward, i.e. $Basax(\mathbf{n}) \cup \{\mathbf{Ax}(\sqrt{\ })\} \not\models \mathbf{Ax7}$, for every $n \geq 3$. However, Proposition 3.115 below Thm.3.114 says that every model of $Basax(\mathbf{n}) \cup \{\mathbf{Ax}(\sqrt{\ })\}$ can be extended to a model of $Basax(\mathbf{n}) \cup \{\mathbf{Ax7}\}$, for every $n \geq 3$.

THEOREM 3.114 *Assume* $n \geq 3$. *Then (i), (ii) below hold.*

(i) $Basax(\mathbf{3}) \models \mathbf{Ax}(\sqrt{\ })$.

(ii) $Basax(\mathbf{n}) \cup \{\mathbf{Ax7}\} \models \mathbf{Ax}(\sqrt{\ })$.

The **proof** is available from the authors.

In the following proposition we will use the notation $\mathfrak{N} \subseteq \mathfrak{M}$ introduced in item 15 of Notation 3.1.

PROPOSITION 3.115 *Assume* $n \geq 3$. *Assume* $\mathfrak{M}$ *is a frame model such that* $\mathfrak{M} \models Basax(\mathbf{n}) \cup \{\mathbf{Ax}(\sqrt{\ })\}$. *Then there is a frame model* $\mathfrak{M}^+$ *such that* $\mathfrak{M} \subseteq \mathfrak{M}^+$ *and* $\mathfrak{M}^+ \models Basax(\mathbf{n}) \cup \{\mathbf{Ax7}\}$.

Proof: Let $n \geq 3$. Let
$\mathfrak{M} = \langle (B, Obs, Ph, Ib), \mathfrak{F}, G;\ \text{E}\ , W\rangle \models Basax(\mathbf{n}) \cup \{\mathbf{Ax}(\sqrt{\ })\}$.
Let $\mathfrak{M}^+ \stackrel{\text{def}}{=} \langle (B^+, Obs^+, Ph^+, Ib^+), \mathfrak{F}, G;\ \text{E}\ , W^+\rangle$ be defined as follows.
$B^+ \stackrel{\text{def}}{=} B \times Newt_0$, $Obs^+ \stackrel{\text{def}}{=} Obs \times Newt_0$, $Ph^+ \stackrel{\text{def}}{=} Ph \times Newt_0$, $Ib^+ \stackrel{\text{def}}{=} Ib \times Newt_0$, and for all $\langle m, N\rangle \in Obs^+$ and $p \in {}^nF$ $\quad w^+_{\langle m,N\rangle}(p) \stackrel{\text{def}}{=} (N \circ w_m(p)) \times Newt_0$. By this $\mathfrak{M}^+$ is defined. Now $\mathfrak{M} \subseteq \mathfrak{M}^+$ assuming that we treat $\langle b, \text{Id}\rangle \in B^+$ as identical with $b \in B$, for all $b \in B$. We claim that $\mathfrak{M}^+ \models Basax(\mathbf{n}) \cup \{\mathbf{Ax7}\}$. Checking this claim is left to the reader. ∎

QUESTION 3.116 *Is* $Basax(\mathbf{4}) \models \mathbf{Ax}(\sqrt{\ })$ *true?* $\lhd$

We note that Questions 3.113 and 3.116 are equivalent in the sense that the answer to Question 3.116 is “YES” iff the answer to Question 3.113 is “YES”.

LEMMA 3.117 $\{g \in PT \,:\, g(\bar{0}) = \bar{0} \,\wedge\, g(1_t) = 1_t\} = Newt_0$

The **proof** is available from the authors.

LEMMA 3.118 *Assume $\mathfrak{F}$ is Euclidean. Let $p, q \in {}^nF$ with $p \neq q$ and $\overline{pq} \in \mathsf{SlowEucl}$. Then for T_{pq} defined in Def.3.107, (i)-(iii) below hold.*

(i) $T_{pq} \in PT$.

(ii) $T_{pq}(\bar{0}) = p$ *and* $T_{pq}(1_t) = q$.

(iii) $(\forall \ell \in \mathsf{SlowEucl})\ T_{pq}[\ell] \in \mathsf{SlowEucl}$.

Proof: The proof of the lemma follows by the definition of T_{pq} (Def.3.107). This definition says that T_{pq} is defined exactly as f_{km_0} was defined in the definition of SM (Def.3.98 in §3.5) but with $\langle p, q, N_{pq}\rangle$ in place of $\langle o_k, t_k, N_k\rangle$. Now in the definition of SM, f_{km_0} was defined to be an affine transformation which takes $\bar{0}$, 1_t to o_k, t_k, respectively. Further, in the proof of Thm.3.99 in §3.5, which says $\mathsf{SM} \models Basax$, we have the following propositions. In the "Proof of (IV)" (61) says that

$$(\forall \ell \in \mathsf{Eucl})\Big(\ell \in \mathsf{PhtEucl} \;\Leftrightarrow\; \mathsf{f}_{km_0}[\ell] \in \mathsf{PhtEucl}\Big).$$

And in the "Proof of (V)" (65) says that

$$(\forall \ell \in \mathsf{SlowEucl})\ \mathsf{f}_{km_0}[\ell] \in \mathsf{SlowEucl}.$$

By the above, we conclude that (i)-(iii) hold for T_{pq}. ∎

Proof of Thm.3.108: Assume $\mathfrak{F}$ is Euclidean. Let $p, q \in {}^nF$ with $p \neq q$ and $\overline{pq} \in \mathsf{SlowEucl}$. By Lemma 3.118(i),(ii) we have that (67) and (68) below hold.

(67) $$T_{pq} \in PT.$$

(68) $$T_{pq}(\bar{0}) = p \quad \text{and} \quad T_{pq}(1_t) = q.$$

Proof of (i)⇒(ii): Let $A \in PT$ with $A(\bar{0}) = p$ and $A(1_t) = q$. Then we have

(69) $$A \circ T_{pq}^{-1} \in PT \quad \& \quad A \circ T_{pq}^{-1}(\bar{0}) = \bar{0} \quad \& \quad A \circ T_{pq}^{-1}(1_t) = 1_t,$$

by (67),(68), and by Remark 3.103. Now we conclude that $A \circ T_{pq}^{-1} \in Newt_0$ by Lemma 3.117 and by (69). Therefore $A = M \circ T_{pq}$, for some $M \in Newt_0$.

Proof of (ii)⇒(i): Let $M \in Newt_0$ and $A = M \circ T_{pq}$. Then $A \in PT$ by $Newt_0 \subseteq PT$, by (67), and by Remark 3.103. By $M \in Newt_0$, we have $M(\bar{0}) = \bar{0}$ and $M(1_t) = 1_t$. Therefore $A(\bar{0}) = p$ and $A(1_t) = q$ by (68). ∎

LEMMA 3.119 *Let $\varphi \in Aut(\mathbf{F})$. Then (i)-(iii) below hold.*

(i) $\tilde{\varphi} : {}^nF \longrightarrow {}^nF$ *is a bijection.*

(ii) $(\forall \ell \in \mathsf{Eucl})\ \tilde{\varphi}[\ell] \in \mathsf{Eucl}$.

(iii) $(\forall \ell \in \mathsf{Eucl})\ ang^2(\tilde{\varphi}[\ell]) = \varphi(ang^2(\ell))$.

Proof: The proof is straightforward, we omit it. We note that (i) and (ii) follow by Lemma 3.5 in §3.2. ∎

LEMMA 3.120 *Let $\varphi \in Aut(\mathfrak{F})$. Then*

$$\ell \in \mathsf{SlowEucl} \ \Leftrightarrow\ \tilde{\varphi}[\ell] \in \mathsf{SlowEucl}\,.$$

Proof:

Proof of $\Rightarrow$*:* Let $\ell \in \mathsf{SlowEucl}$. Then $ang^2(\ell) < 1$. By Lemma 3.119(iii), we have $ang^2(\tilde{\varphi}[\ell]) = \varphi(ang^2(\ell))$. Therefore we have $\varphi(ang^2(\ell)) < 1$ by $ang^2(\ell) < 1$ and by φ being order preserving. Hence $\tilde{\varphi}[\ell] \in \mathsf{SlowEucl}$.

Proof of $\Leftarrow$*:* The proof is analogous with the proof of direction $\Rightarrow$, because $\varphi^{-1} \in Aut(\mathfrak{F})$. ∎

Proof of Thm.3.110: The proof will be analogous with the proofs of $\mathfrak{M}_0^P \models Basax(\mathbf{2})$ in §2.4 (Thm.2.12) and of $\mathsf{SM} \models Basax(\mathbf{n})$ in §3.5 (Thm.3.99). The essential novelty in the present proof is that we have to handle the field automorphisms φ_k and the new Newtonian transformations M_k too, for each observer k, (because now they too belong to an observer k).[69]

Let $\mathfrak{M} \in \mathsf{GM}$. We will show that $\mathfrak{M} \models Basax$.

For every $k \in Obs$, by Lemma 3.118, we have that (70)-(72) below hold.

(70) $\qquad T_{o_k t_k} \in PT$

(71) $\qquad T_{o_k t_k}[\bar{t}] = \overline{o_k t_k} \quad$ (by $T_{o_k t_k}(\bar{0}) = o_k$ and $T_{o_k t_k}(1_t) = t_k$).

(72) $\qquad (\forall \ell \in \mathsf{SlowEucl})\ T_{o_k t_k}[\ell] \in \mathsf{SlowEucl}$.

For every $k \in Obs$ let

$$\mathsf{f}_k \stackrel{\text{def}}{=} \widetilde{\varphi_k} \circ M_k \circ T_{o_k t_k}\,.$$

[69] In the definition of SM, an observer k (more precisely k's world-view) was determined by data $\langle o_k, t_k, N_k \rangle$, while in GM it is determined by more data like $\langle o_k, t_k, N_k, \varphi_k, M_k \rangle$.

We will prove that (73)-(77) below hold for f_k, for every $k \in Obs$.

(73) $\qquad \mathsf{f}_k : {}^nF \longrightarrow {}^nF$ is a bijection.

(74) $\qquad (\forall \ell \in \mathsf{Eucl})\ \mathsf{f}_k[\ell],\ \mathsf{f}_k^{-1}[\ell] \in \mathsf{Eucl}\,.$

(75) $\qquad (\forall \ell \in \mathsf{PhtEucl})\ \mathsf{f}_k[\ell],\ \mathsf{f}_k^{-1}[\ell] \in \mathsf{PhtEucl}\,.$

(76) $\qquad (\forall \ell \in \mathsf{SlowEucl})\ \mathsf{f}_k[\ell] \in \mathsf{SlowEucl}\,.$

(77) $\qquad \mathsf{f}_k[\bar{t}] = \overline{o_k t_k}\,.$

(73) holds by $M_k \circ T_{o_k t_k} \in Aftr$ and by Lemma 3.119(i).
(74) holds by $M_k \circ T_{o_k t_k} \in Aftr$, by Lemma 3.119(ii), and by (73).
(75) holds by $M_k \in Newt_0 \subseteq PT$, by (70), and by Lemma 3.119(iii).
(76) holds by (72), by Lemma 3.97 in §3.5, and by Lemma 3.120.
(77) holds by $\widetilde{\varphi_k} \circ M_k[\bar{t}] = \bar{t}$ and by (71).

It is easy to see that for every $k \in Obs$ and $b \in B$

(78) $\qquad w_k = \mathsf{f}_k \circ w_0\,,$

(79) $\qquad \beta(b) = \mathsf{f}_k[tr_k(b)]\,,$

(80) $\qquad tr_k(b) = \mathsf{f}_k^{-1}[\beta(b)]\,,$

by the definitions of w_k and f_k.
Now

$\mathfrak{M} \models$ **Ax1**, **Ax2** by $\mathfrak{M} \in \mathsf{GM}$.

$\mathfrak{M} \models$ **Ax3** because of the following. Let $h \in Ib$ and $m \in Obs$. By (80), we have $tr_m(h) = \mathsf{f}_k^{-1}[\beta(h)]$. By $h \in Ib$ and by item 6 in the definition of GM (Def.3.109), we have $\beta(h) \in \mathsf{Eucl}$. Therefore $tr_m(h) = \mathsf{f}_k^{-1}[\beta(h)]$ and $\beta(h) \in \mathsf{Eucl}$ imply $tr_m(h) \in \mathsf{Eucl}$ by (74).

$\mathfrak{M} \models$ **Ax4** because of the following. Let $m \in Obs$. Then $\mathsf{f}_m[\bar{t}] = \overline{o_m t_m}$ by (77). We have $\overline{o_m t_m} = \beta(m)$ by item 3 in the definition of GM (Def.3.109). By these two, we have $\mathsf{f}_m[\bar{t}] = \beta(m)$. By (79), we have $\beta(m) = \mathsf{f}_m[tr_m(m)]$. By this, by $\mathsf{f}_m[\bar{t}] = \beta(m)$, and by (73), we have $tr_m(m) = \bar{t}$.

$\mathfrak{M} \models$ **Ax5** because of the following. Let $m \in Obs$, $\ell_1 \in \mathsf{SlowEucl}$, $\ell_2 \in \mathsf{PhtEucl}$. By (76), we have $\mathsf{f}_m[\ell_1] \in \mathsf{SlowEucl}$, and by (75), we have $\mathsf{f}_m[\ell_2] \in \mathsf{PhtEucl}$. Therefore by items 4,5 in the definition of GM (Def.3.109), we have $\mathsf{f}_m[\ell_1] = \beta(k)$ and $\mathsf{f}_m[\ell_2] = \beta(ph)$, for some $k \in Obs$ and $ph \in Ph$. Let such k and ph be fixed. Now we conclude $tr_m(k) = \ell_1$ and $tr_m(ph) = \ell_2$ by $\mathsf{f}_m[\ell_1] = \beta(k)$, $\mathsf{f}_m[\ell_2] = \beta(ph)$, by (79), and by (73).

$\mathfrak{M} \models$ **Ax6** because (73) and (78) imply that $(\forall m \in Obs)$ $Rng(w_m) = Rng(w_0)$.

$\mathfrak{M} \models$ **AxE** because of the following. Let $m \in Obs$ and $ph \in Ph$. By (80), we have $tr_m(ph) = \mathsf{f}_m^{-1}[\beta(ph)]$. By item 5 in the definition of GM (Def.3.109), we have $\beta(ph) \in \mathsf{PhtEucl}$. By this, by $tr_m(ph) = \mathsf{f}_m^{-1}[\beta(ph)]$ and by (75), we have $tr_m(ph) \in \mathsf{PhtEucl}$, hence $v_m(ph) = 1$.

By the above, $\mathfrak{M} \models Basax$. ∎

Proof of Thm.3.112(ii): Let $n \geq 3$. By Thm.3.110 and $\mathsf{GM} \models \mathbf{Ax}(\sqrt{\ })$, we have that $\mathsf{IGM} \subseteq \mathsf{Mod}_{\text{OFG}}(Basax(\mathbf{n}) \cup \{\mathbf{Ax}(\sqrt{\ })\})$. To prove the other inclusion assume

$$\mathfrak{N} = \langle (B, Obs, Ph, Ib), \mathfrak{F}, G^{\mathfrak{N}};\ \mathrm{E}^{\mathfrak{N}}, W^{\mathfrak{N}} \rangle \in \mathsf{Mod}_{\text{OFG}}(Basax(\mathbf{n}) \cup \{\mathbf{Ax}(\sqrt{\ })\}).$$

Without loss of generality we may assume that

$$\langle G^{\mathfrak{N}},\ \mathrm{E}^{\mathfrak{N}} \rangle = \langle \mathsf{Eucl}(\mathbf{n}, \mathfrak{F}),\ \in \rangle.$$

Now we will prove that $\mathfrak{N} \in \mathsf{GM}$. By $\mathfrak{N} \models \mathbf{Ax}(\sqrt{\ })$, we have that $\mathfrak{F}$ is Euclidean. Let $m_0 \in Obs$ be arbitrary and fixed. We will define functions

$$\alpha : Obs \longrightarrow {}^nF \times {}^nF \times Aut(\mathfrak{F}) \times Newt_0 \quad \text{and}$$
$$\beta : B \longrightarrow \mathcal{P}({}^nF).$$

Definition of α:

Let $k \in Obs$. In what follows we will define $\alpha(k)$. By Prop.3.104(ii), we have

$$\mathsf{f}_{km_0} = \widetilde{\varphi_k} \circ A, \quad \text{for some } A \in PT \text{ with } A[\bar{t}] = tr_{m_0}(k) \text{ and } \varphi_k \in Aut(\mathfrak{F}).$$

Let these φ_k and A be fixed. Let

$$o_k \stackrel{\text{def}}{=} A(\bar{0}) \quad \text{and} \quad t_k \stackrel{\text{def}}{=} A(1_t).$$

By this and $A[\bar{t}] = tr_{m_0}(k)$, we have $o_k \neq t_k$ and $\overline{o_k t_k} = tr_{m_0}(k)$. There is no FTL observer in $\mathfrak{N}$ by Thm.3.28(i) in §3.4, hence $\overline{o_k t_k} = tr_{m_0}(k) \in \mathsf{SlowEucl}$. Now Thm.3.108 implies that

$$A = M_k \circ T_{o_k t_k}, \quad \text{for some } M_k \in Newt_0,$$

because $o_k \neq t_k$, $\overline{o_k t_k} \in \mathsf{SlowEucl}$, $A \in PT$, $A(\bar{0}) = o_k$, and $A(1_t) = t_k$. Let this M_k be fixed. Now

$$\alpha(k) \stackrel{\text{def}}{=} \langle o_k, t_k, \varphi_k, M_k \rangle.$$

By this function α is defined. We note that

(81) $$\mathsf{f}_{km_0} = \widetilde{\varphi_k} \circ M_k \circ T_{o_k t_k}\,,$$

by $\mathsf{f}_{km_0} = \widetilde{\varphi_k} \circ A$ and $A = M_k \circ T_{o_k t_k}$.

<u>*Definition of β:*</u>
Let $b \in B$. Then

$$\beta(b) \stackrel{\text{def}}{=} tr_{m_0}(b)\,.$$

By this function β is defined.

Now we will check that 1-6 in the definition of GM (Def.3.109) hold for B, Obs, Ph, Ib, α, and β.

1 holds by $\mathfrak{N} \in \mathsf{Mod}_{\text{OFG}}(\mathbf{Ax2})$.

In the definition of α we saw that 2 holds, i.e. for all $k \in Obs \quad o_k \neq t_k$.

3 holds because of the following. Let $k \in Obs$. In the definition of α we saw that $\overline{o_k t_k} = tr_{m_0}(k)$, and by def. of β we have $\beta(k) = tr_{m_0}(k)$. Hence $\beta(k) = \overline{o_k t_k}$.

4 holds because in $\mathfrak{N}$ there is no FTL observer by Thm.3.28(i), because of **Ax5**, and by the definition of β.

5 holds by def. of β and by **Ax5**,**AxE**.

6 holds by def. of β and by **Ax2**,**Ax3**.

It remains to show that W defined in the definition of GM coincides with $W^{\mathfrak{N}}$. To show this it is enough to prove $w_k^{\mathfrak{N}} = w_k$, for all $k \in Obs$. To see this let $k \in Obs$. By (81) above, we have

$$w_k^{\mathfrak{N}} \circ (w_{m_0}^{\mathfrak{N}})^{-1} = \widetilde{\varphi_k} \circ M_k \circ T_{o_k t_k}\,.$$

Therefore

(82) $$w_k^{\mathfrak{N}} = \widetilde{\varphi_k} \circ M_k \circ T_{o_k t_k} \circ w_{m_0}^{\mathfrak{N}}\,.$$

By the definition of β, it is easy to see that

$$w_{m_0}^{\mathfrak{N}} = w_0\,.$$

By this, by (82), and by the definition of w_k, we have

$$w_k^{\mathfrak{N}} = w_k\,.$$

This completes the proof of Thm.3.112(ii). ∎

Warning!

From this point on, read this material **only** at your own risk. The rest (except for section 5 Appendix) is not even a draft; it is only raw material to be used later for creating (first) a draft version.

3.7 Symmetry axioms

In this section, we will study the possibility of adding an extra symmetry axiom to *Basax*. The reason for this is that *Basax* has many different models which are not elementarily equivalent. Actually *Basax* has 2^ω many consistent theories as its extensions[70] (hence it has continuum many non-elementarily-equivalent models). While we consider this as a virtue[71], there exists a natural symmetry principle (cf. $\mathbf{Ax\triangle 1}$ way below) which could be added to *Basax*. Moreover this extra principle makes *Basax* much stronger. Roughly speaking the new principle (e.g. formalized as $\mathbf{Ax\triangle 1}$) says that if two inertial observers m and k are looking at each other then m "sees" k more or less the same way as k "sees" m. The point is the following. Of course, m thinks that k is moving fast while k thinks that m is moving fast. So far the picture is symmetric in every model of *Basax*. However, in some models of *Basax* m may think that the effect of moving fast slows down the clocks of k while k may think that moving fast speeds up the clocks of m. Now, we consider this as an *assymetric* situation, since one might even interpret this such that the "laws of the nature" are different for m and k. The principle $\mathbf{Ax\triangle 1}$ excludes such possibilities. (Actually, $\mathbf{Ax\triangle 1}$ is stronger than this, it would be interesting to see how far could one develop the theory by a weaker axiom saying only that if k's clocks slow down in m's world-view then m's clocks must also slow down in k's world-view.)

For completeness, we note that there is a form of the twin paradox which is provable in *Basax* + $\mathbf{Ax\triangle 1}$ but not in pure *Basax*. On the other hand one should also note that a large majority of the theorems of Special Relativity are already provable in *Basax* (moreover many of these already follow from *Bax*, but we did not explore this "*Bax*-direction" here).

Before studying the new principle $\mathbf{Ax\triangle 1}$ (and its equvivalent or almost equvivalent forms) we will introduce some auxiliary definitions and axioms which we need for technical reasons only.

Notation 3.121

1. Throughout, $GL = GL(\mathbf{n}, \mathbf{F})$ denotes the set of invertible linear transformations of the vector space ${}^n\mathbf{F}$.

[70] This is as much as possible for "administrative" reasons.

[71] For various reasons, some of which are methodological, we find that it is a virtue if a theory admits many extensions.

2. The set of similarity linear transformations[72] is denoted as follows:

$$ST \stackrel{\text{def}}{=} ST(\mathbf{n},\mathbf{F}) \stackrel{\text{def}}{=} \{h \in GL \,:\, (\exists r \in F \setminus \{0\})\ (\forall p \in {}^nF)\ h(p) = r \cdot p\}\,.$$

3. Let $\sigma_{\bar{t}}, \sigma_S \in GL$ denote the reflections about $\bar{t}$ and S respectively, in more detail:

$$(\forall i \in n \setminus \{0\})\ \sigma_{\bar{t}}(e_i) = -e_i \quad \& \quad \sigma_{\bar{t}}(1_t) = 1_t, \text{ and}$$
$$(\forall i \in n \setminus \{0\})\ \sigma_S(e_i) = e_i \quad \& \quad \sigma_S(1_t) = -1_t.$$

4. Let $p \in {}^nF$. Then $p_t \stackrel{\text{def}}{=} p_0$, i.e. p_t is the time-component of the vector p. ◁

$Newt_0 = Newt_0(\mathbf{n},\mathbf{F})$ and $Newt = Newt(\mathbf{n},\mathbf{F})$ were defined in §3.5, Def.3.94.

Definition 3.122

$$\begin{aligned} Newt_t \stackrel{\text{def}}{=} Newt_t(\mathbf{n},\mathbf{F}) &\stackrel{\text{def}}{=} \{g \in Newt \,:\, g[\bar{t}] = \bar{t}\},\\ Newt^\star \stackrel{\text{def}}{=} Newt^\star(\mathbf{n},\mathbf{F}) &\stackrel{\text{def}}{=} Newt \cup \{g \circ \sigma_S \,:\, g \in Newt\}\,. \end{aligned} \quad ◁$$

Remark 3.123 We note that $Newt_0 \subseteq Newt_t \subseteq Newt \subseteq Newt^\star \subseteq PT \subseteq Aftr$, $\langle Newt_0, \circ, {}^{-1}, \mathrm{Id}\rangle$, $\langle Newt_t, \circ, {}^{-1}, \mathrm{Id}\rangle$, $\langle Newt, \circ, {}^{-1}, \mathrm{Id}\rangle$, $\langle Newt^\star, \circ, {}^{-1}, \mathrm{Id}\rangle$, $\langle PT, \circ, {}^{-1}, \mathrm{Id}\rangle$, $\langle Aftr, \circ, {}^{-1}, \mathrm{Id}\rangle$ are groups, and $Newt \cap GL = Newt_t \cap GL = Newt_0$. ◁

Next we list variants of the symmetry principle which is the subject matter of this section.

Ax□1 $(\forall m,k,m' \in Obs)(\exists k' \in Obs)\ \mathsf{f}_{mk} = \mathsf{f}_{m'k'}$.

Ax□2 $(\forall m,k,m',k' \in Obs)(tr_m(k) = tr_{m'}(k') \Rightarrow (\exists N \in Newt^\star)\ \mathsf{f}_{mk} = \mathsf{f}_{m'k'} \circ N)$.

Ax△1 $(\forall m,k \in Obs)(\exists k' \in Obs)(tr_m(k) = tr_m(k') \ \wedge\ \mathsf{f}_{mk'} = \mathsf{f}_{k'm})$.

Ax△2 $(\forall m,k \in Obs)(\exists N \in Newt^\star)\ \mathsf{f}_{mk} = N \circ \mathsf{f}_{km} \circ N$.

[72] The similarity linear transformation is the same what is called expansion on p.14 of Burke [6], and sometimes it is called homothetical transformations.

Next we list some (rather natural and harmless) auxiliary axioms needed e.g. to prove the equivalence of the above symmetry principles.

Ax8 $(\forall m \in Obs)(\forall N \in Newt_t)(\exists k \in Obs)$
$$(\forall p \in \{e_i \,:\, i \in n\} \cup \{\bar{0}\})\ w_k(p) = w_m(Np).$$

Ax9$_0$ $(\forall m, k \in Obs)(tr_m(k) = \bar{t} \quad \Rightarrow \quad \mathsf{f}_{mk} \in Newt^\star).$

Ax9 $(\forall m, k \in Obs)(tr_m(k) \parallel \bar{t} \quad \Rightarrow \quad \mathsf{f}_{mk} \in Newt^\star).$

Ax($\sigma_{\bar{t}}$) $(\forall m \in Obs)(\exists k \in Obs)\ \mathsf{f}_{mk} = \sigma_{\bar{t}}.$

Remark 3.124 We note that **Ax9** $\models$ **Ax9**$_0$ and $Basax \cup \{$**Ax9**$_0\} \not\models$ **Ax9**. $\triangleleft$

PROPOSITION 3.125 $Basax \cup \{$**Ax8**, **Ax9**$_0\} \models$ **Ax($\sigma_{\bar{t}}$)**.

Proof: Assume $Basax \cup \{$**Ax8**, **Ax9**$_0\}$. Let $m \in Obs$. By **Ax8** there is $k \in Obs$ such that

$$\Big(w_k(1_t) = w_m(1_t) \quad \& \quad w_k(\bar{0}) = w_m(\bar{0}) \quad \& \quad (\forall i \in n \setminus \{0\}) w_k(e_i) = w_m(-e_i)\Big).$$

Hence $(\forall i \in n)$ $\mathsf{f}_{mk}(e_i) = \sigma_{\bar{t}}(e_i)$ and $\mathsf{f}_{mk}(\bar{0}) = \bar{0}$. By this and by **Ax9**$_0$, we get $\mathsf{f}_{mk} = \sigma_{\bar{t}}$.

CONVENTION 3.126 As in usual in the literature of partial algebras cf. [3] we will use the following convention. Let ϱ and θ be two terms involving partial symbols. Then
$$\varrho = \theta \overset{\text{def}}{\iff} (\varrho = \theta \text{ are both defined and } \varrho = \theta)\,.$$
Similarly for relation symbols in place of equality. $\triangleleft$

Definition 3.127 We define binary relations $\uparrow, \downarrow \subseteq Obs \times Obs$ as follows.
$(\forall m, k \in Obs)$

$$m \uparrow k \quad \overset{\text{def}}{\iff} \quad \Big(\mathsf{f}_{km}(1_t) - \mathsf{f}_{km}(\bar{0})\Big)_t > 0, \text{ and}$$
$$m \downarrow k \quad \overset{\text{def}}{\iff} \quad \Big(\mathsf{f}_{km}(1_t) - \mathsf{f}_{km}(\bar{0})\Big)_t < 0. \qquad \triangleleft$$

We need two further auxiliary axioms.

Ax5$^+$ $(\forall m,k,m' \in Obs)(\exists k' \in Obs)(tr_m(k) = tr_{m'}(k') \;\wedge\; (m \uparrow k \;\Leftrightarrow\; m' \uparrow k'))$.

Ax5^{++} $(\forall m \in Obs)(\forall \ell \in \mathsf{SlowEucl})(\exists k \in Obs)(tr_m(k) = \ell \;\;\wedge\;\; m \uparrow k)$.

PROPOSITION 3.128 *Assume $n \geq 3$. Then*

$$Basax(\mathbf{n}) \cup \{\mathbf{Ax5^{++}}, \mathbf{Ax}(\sqrt{\;})\} \models \mathbf{Ax5^+}.$$

We will give the **proof** after the proof of Lemma 3.161.

PROPOSITION 3.129

$$Basax \models (\mathbf{Ax\square 1} \Leftrightarrow \mathbf{Ax\square 1^*}),\ \textit{where}$$

Ax□1* *is defined below.*

$$\mathbf{Ax\square 1^*} \quad (\forall m,k,m' \in Obs)(\exists k' \in Obs)(tr_m(k) = tr_{m'}(k') \;\;\wedge\;\; \mathsf{f}_{mk} = \mathsf{f}_{m'k'})\,.$$

Proof: It is easy to see that it is enough to prove that

$$Basax \models (\forall m,k,m',k' \in Obs)(\mathsf{f}_{mk} = \mathsf{f}_{m'k'} \;\Rightarrow\; tr_m(k) = tr_{m'}(k')).$$

To see this, let $m,k \in Obs$ with $\mathsf{f}_{mk} = \mathsf{f}_{m'k'}$. By $Basax$, we have $\mathsf{f}_{mk}[tr_m(k)] = \bar{t}$ and $\mathsf{f}_{m'k'}[tr_{m'}(k')] = \bar{t}$. By this and by $\mathsf{f}_{mk} = \mathsf{f}_{m'k'}$, we have $tr_m(k) = tr_{m'}(k')$. ∎

PROPOSITION 3.130 *Assume $n \geq 3$. Then*

$$Basax(\mathbf{n}) \cup \{\mathbf{Ax}(\sqrt{\;})\} \models (\mathbf{Ax\triangle 1} \Leftrightarrow \mathbf{Ax\triangle 1^*})\,,\quad \textit{where}$$

Ax△1* *is defined below.*

$$\mathbf{Ax\triangle 1^*} \quad (\forall m \in Obs)(\forall \ell \in \mathsf{SlowEucl})(\exists k \in Obs)(tr_m(k) = \ell \;\;\wedge\;\; \mathsf{f}_{mk} = \mathsf{f}_{km})\,.$$

Proof: The proof easily follows by Thm.3.28(i) in §3.4 which says that $Basax(\mathbf{n}) \cup \{\mathbf{Ax}(\sqrt{\;})\}$ implies that there is no FTL observer, if $n \geq 3$. ∎

PROPOSITION 3.131 *Assume $n \geq 3$. Then*

$$Basax(\mathbf{n}) \cup \{\mathbf{Ax}(\sqrt{\;})\} \models (\mathbf{Ax\triangle 2} \Leftrightarrow \mathbf{Ax\triangle 2^*})\,,\ \textit{where}$$

Ax△2* *is defined below.*

$$\mathbf{Ax\triangle 2^*} \quad (\forall m,k \in Obs)(\exists N \in Newt_t)\ \mathsf{f}_{mk} = N \circ \mathsf{f}_{km} \circ N.$$

We will give the **proof** after Lemma 3.160 way below.

The following theorem says that $\mathbf{Ax\Box 2}$ almost follows from *Basax*.

THEOREM 3.132
$Basax \models (\forall m,k \in Obs)\Big(tr_m(k) = tr_{m'}(k') \;\Rightarrow\; (\mathsf{f}_{mk} = \mathsf{f}_{m'k'} \circ h \circ N \circ \widetilde{\varphi}\,,$ *for some* $h \in ST,\ N \in Newt_t$ *and* $\varphi \in Aut(\mathbf{F}))\Big).$

We will give the **proof** after Lemma 3.146.

Thm.3.133 below says that under some assumptions the world-view transformations are affine transformations.

THEOREM 3.133

$$Basax \cup \{\mathbf{Ax\triangle 2}, \mathbf{Ax}(\sqrt{\ })\} \models (\forall m,k \in Obs)\ \mathsf{f}_{mk} \in PT\,.$$

We will give the **proof** after the proof of Thm.3.132.

The following theorem says that two relatively moving observers see each other with the same velocity, under some mild assumptions.

THEOREM 3.134

$$Basax \cup \{(\forall m,k \in Obs)\ \mathsf{f}_{mk} \in Aftr\} \models (\forall m,k \in Obs)\ v_m(k) = v_k(m)\,.$$

The **proof** is available from the authors.

COROLLARY 3.135

$$Basax \cup \{\mathbf{Ax\triangle 2}, \mathbf{Ax}(\sqrt{\ })\} \models (\forall m,k \in Obs)\ v_m(k) = v_k(m)\,.$$

Proof: This is a corollary of Thm.3.133 and Thm.3.134. ∎

In the next three theorems we discuss the conditions under which our four variants of our symmetry principles are equivalent.

THEOREM 3.136 *Assume* $n \geq 3$. *Then (i)-(iv) below hold.*

(i) $Basax(\mathbf{n}) \cup \{\mathbf{Ax}(\sigma_{\bar{t}}), \mathbf{Ax}(\sqrt{\ })\} \models (\mathbf{Ax\triangle 2} \Leftrightarrow \mathbf{Ax\Box 2})$.

(ii) $Basax(\mathbf{n}) \cup \{\mathbf{Ax}(\sqrt{\ })\} \models (\mathbf{Ax\triangle 2} \Rightarrow \mathbf{Ax\Box 2})$.

(iii) $Basax(\mathbf{n}) \cup \{\mathbf{Ax9_0}, \mathbf{Ax}(\sqrt{\ })\} \models (\mathbf{Ax\triangle 2} \Leftrightarrow \mathbf{Ax\Box 2})$.

(iv) $Basax(\mathbf{n}) \cup \{\mathbf{Ax9_0}, \mathbf{Ax8}, \mathbf{Ax}(\sqrt{\ })\} \models (\mathbf{Ax\triangle 2} \Leftrightarrow \mathbf{Ax\Box 2})$.

We will give the **proof** after the proof of Lemma 3.158.

THEOREM 3.137 *(i)-(iii) below hold.*

(i) $Basax(\mathbf{n}) \cup \{\mathbf{Ax8}, \mathbf{Ax9}_0, \mathbf{Ax5}^+\} \models (\mathbf{Ax\Box1} \Leftrightarrow \mathbf{Ax\Box2})$.

(ii) $Basax(\mathbf{n}) \cup \{\mathbf{Ax9}_0\} \models (\mathbf{Ax\Box1} \Rightarrow \mathbf{Ax\Box2})$.

(iii) *If* $n \geq 3$ *then*
$Basax(\mathbf{n}) \cup \{\mathbf{Ax8}, \mathbf{Ax9}_0, \mathbf{Ax}(\sqrt{\ }), \mathbf{Ax5}^{++}\} \models (\mathbf{Ax\Box1} \Leftrightarrow \mathbf{Ax\Box2})$.

We will give the **proof** after Lemma 3.163 way below.

THEOREM 3.138 *(i) and (ii) below hold.*

(i) *If* $n \geq 3$ *then*
$Basax(\mathbf{n}) \cup \{\mathbf{Ax8}, \mathbf{Ax9}_0, \mathbf{Ax}(\sqrt{\ })\} \models (\mathbf{Ax\triangle1} \Leftrightarrow \mathbf{Ax\triangle2})$.

(ii) $Basax(\mathbf{n}) \cup \{\mathbf{Ax9}_0\} \models (\mathbf{Ax\triangle1} \Rightarrow \mathbf{Ax\triangle2})$.

We will give the **proof** after the proof of Thm.3.137 way below.

The following auxiliary axioms will help us in making our theory *Basax* complete. Their status is different from the status of the previous axioms like $\mathbf{Ax\triangle1}$, in that we do not believe that they would be true in the "real world".

Ax♣ $(\forall m, k \in Obs)(w_m = w_k \;\Rightarrow\; m = k)$.

Ax♢ $(\forall ph_1, ph_2 \in Ph)(\exists k \in Obs)(tr_k(ph_1) = tr_k(ph_2) \;\Rightarrow\; ph_1 = ph_2)$.

Ax♡ $B = Obs \cup Ph$.

Ax(↑) $(\forall m, k \in Obs)(tr_m(k) = \bar{t} \;\Rightarrow\; m \uparrow k)$.

The following theorem says that our symmetry principle makes *Basax* almost (not really) complete in some sense, and it also shows how to make it really complete. The price of this completion

THEOREM 3.139 *Assume $n \geq 3$. Assume $\mathfrak{F}$ is Euclidean.*
Let $\mathbf{Ax} \in \{\mathbf{Ax}\square\mathbf{1}, \mathbf{Ax}\square\mathbf{2}, \mathbf{Ax}\triangle\mathbf{1}, \mathbf{Ax}\triangle\mathbf{2}\}$. *Then (i) and (ii) below hold.*

(i)
$\mathsf{Mod}_{\mathfrak{F}}(Basax(\mathbf{n}) \cup \{\mathbf{Ax5}^{++}, \mathbf{Ax8}, \mathbf{Ax}(\surd), \mathbf{Ax9}_0, \mathbf{Ax}\clubsuit, \mathbf{Ax}\diamondsuit, \mathbf{Ax}\heartsuit, \mathbf{Ax}\})$
has exactly two elements up to isomorphisms.

(ii)
$\mathsf{Mod}_{\mathfrak{F}}(Basax(\mathbf{n}) \cup \{\mathbf{Ax5}^{++}, \mathbf{Ax8}, \mathbf{Ax}(\surd), \mathbf{Ax9}_0, \mathbf{Ax}\clubsuit, \mathbf{Ax}\diamondsuit, \mathbf{Ax}\heartsuit, \mathbf{Ax}(\uparrow), \mathbf{Ax}\})$
has exactly one element up to isomorphisms.

The **proof** is available from the authors

Remark 3.140 We note that in the above theorem $\mathbf{Ax}\heartsuit$ is responsible for existing only finitely many models. Without $\mathbf{Ax}\heartsuit$ the Gödel's incompleteness theorem goes through for the above axiom system. $\lhd$

Next, we turn to preparations for proving the above theorems.

LEMMA 3.141 *Let $A \in PT$. Then (i) and (ii) below hold.*

(i) $A[\bar{t}\,] = \bar{t} \quad \Rightarrow \quad (A = h \circ N\,, \quad \textit{for some } h \in ST \textit{ and } N \in Newt_t).$

(ii) $A[\bar{t}\,] \parallel \bar{t} \quad \Rightarrow \quad (A = h \circ N\,, \quad \textit{for some } h \in ST \textit{ and } N \in Newt^\star).$

Proof: The proof follows by Lemma 3.117 in §3.6. ∎

LEMMA 3.142 *Let $\ell \in \mathsf{Eucl}$ and $g_1, g_2 \in PT$ with $g_1[\ell] = g_2[\ell] = \bar{t}$. Then*

$$g_1 = g_2 \circ h \circ N\,, \quad \textit{for some } h \in ST \textit{ and } N \in Newt_t\,.$$

Proof: $g_2^{-1} \circ g_1[\bar{t}\,] = \bar{t}$. Applying Lemma 3.141(i) completes the proof. ∎

CONVENTION 3.143 For every matrix M $\;m_{ij}$ denotes the element in the i'th row and j'th column of M.

If $p \in {}^nF$ and A is an n by n matrix then pA denotes the vector obtained by matrix multiplying p with A, in more detail here we consider p as 1 by n matrix. For completeness we note that if A is an n by m matrix and B is an m by k matrix and $C = AB$, then C is an n by k matrix, and $c_{ij} = \sum_{r=1}^{m} a_{ir}b_{rj}$. $\lhd$

Definition 3.144 For every $g \in Aftr(\mathbf{n},\mathbf{F})$ and $\varphi \in Aut(\mathbf{F})$ we define $g^\varphi \in Aftr(\mathbf{n},\mathbf{F})$ as follows. Let $g \in Aftr(\mathbf{n},\mathbf{F})$. Then there is an invertible n by n matrix A and $a \in {}^nF$ such that

$$(\forall p \in {}^nF)\ g(p) = pA + a.$$

Let B be an n by n matrix such that $b_{ij} = \varphi(a_{ij})$ and let $b \in {}^nF$ be such that $b_i = \varphi(a_i)$. Now let g^φ be defined as follows.

$$(\forall p \in {}^nF)\ g^\varphi(p) \stackrel{\text{def}}{=} pB + b\,. \qquad \lhd$$

LEMMA 3.145 *For every $g \in Aftr$ and $\varphi \in Aut(\mathbf{F})$ we have*

$$(g \circ \widetilde{\varphi} = \widetilde{\varphi} \circ g^\varphi \quad \& \quad \widetilde{\varphi} \circ g = g^{\varphi^{-1}} \circ \widetilde{\varphi}).$$

Proof: The proof is straightforward. We omit it. ∎

LEMMA 3.146 *Let $\varphi \in Aut(\mathbf{F})$. Then (i)-(iii) below hold.*

(i) $g \in PT \quad \Rightarrow \quad g^\varphi \in PT.$
(ii) $g \in ST \quad \Rightarrow \quad g^\varphi \in ST.$
(iii) $g \in Newt_t \quad \Rightarrow \quad g^\varphi \in Newt_t.$

Proof: The proof is straightforward. We omit it. ∎

Proof of Thm.3.132: Assume *Basax*. Let $m, k, m', k' \in Obs$ with $tr_m(k) = tr_{m'}(k') := \ell$. Then by Prop.3.104(i) in §3.6, we have

(83) $\qquad \mathsf{f}_{mk} = g \circ \widetilde{\psi}\,,\quad$ for some $g \in PT$ and $\psi \in Aut(\mathbf{F})$ with $g[\ell] = \bar{t}\,.$

Similarly

(84) $\qquad \mathsf{f}_{m'k'} = g' \circ \widetilde{\psi'}\,,\quad$ for some $g' \in PT$ and $\psi' \in Aut(\mathbf{F})$ with $g'[\ell] = \bar{t}.$

Now by Lemma 3.142, we have

(85) $\qquad g = g' \circ h \circ N\,,\quad$ for some $h \in ST$ and $N \in Newt_t\,.$

Now

$$\begin{aligned}
\mathsf{f}_{mk} &= g \circ \widetilde{\psi} && \text{(by (83))}\\
&= g' \circ h \circ N \circ \widetilde{\psi} && \text{(by (85))}\\
&= g' \circ \widetilde{\psi'} \circ \widetilde{\psi'^{-1}} \circ h \circ N \circ \widetilde{\psi} && \\
&= \mathsf{f}_{m'k'} \circ \widetilde{\psi'^{-1}} \circ h \circ N \circ \widetilde{\psi} && \text{(by (84))}\\
&= \mathsf{f}_{m'k'} \circ h^{\psi'} \circ N^{\psi'} \circ \widetilde{\psi'^{-1} \circ \psi} && \text{(by Lemma 3.145)}
\end{aligned}$$

Now by Lemma 3.146, we have $h^{\psi'} \in ST$, $N^{\psi'} \in Newt_t$, and this completes the proof. ∎

Proof of Thm.3.133: Assume $Basax \cup \{\mathbf{Ax\triangle 2}, \mathbf{Ax}(\sqrt{\ })\}$. Let $m, k \in Obs$. Then by Prop.3.104(ii) in §3.6, we have

$$\mathsf{f}_{mk} = g \circ \widetilde{\varphi}\,, \quad \text{for some } g \in PT \text{ and } \varphi \in Aut(\mathfrak{F}). \tag{86}$$

By $\mathbf{Ax\triangle 2}$, we have

$$\mathsf{f}_{mk} = N \circ \mathsf{f}_{km} \circ N\,, \quad \text{for some } N \in Newt^{\star}\,. \tag{87}$$

Now

$$\begin{aligned} \mathrm{Id} &= \mathsf{f}_{km} \circ N \circ \mathsf{f}_{km} \circ N && \text{(by (87))} \\ &= g \circ \widetilde{\varphi} \circ N \circ g \circ \widetilde{\varphi} \circ N && \text{(by (86))} \\ &= \widetilde{\varphi^2} \circ g^{\varphi^2} \circ N^{\varphi} \circ g^{\varphi} \circ N && \text{(by Lemma 3.145).} \end{aligned}$$

Let $A := g^{\varphi^2} \circ N^{\varphi} \circ g^{\varphi} \circ N \in Aftr$. By the above computation, we got $\mathrm{Id} = \widetilde{\varphi^2} \circ A$. Hence $\widetilde{\varphi^2} = A^{-1} \in Aftr$. This implies $\varphi^2 = \mathrm{Id}$. Hence $\varphi = \mathrm{Id}$, because the members of $Aut(\mathfrak{F})$ have infinite order. Now by (86), $\mathsf{f}_{mk} = g \in PT$. ∎

Remark 3.147 We note that the above proof gives the following result.
$Basax \cup \{\mathbf{Ax\triangle 2}\} \models (\mathsf{f}_{mk} = g \circ \widetilde{\varphi}\,,\ \text{ for some } g \in PT\,,\ \varphi \in Aut(\mathbf{F}) \text{ with } \varphi^2 = \mathrm{Id})$.
◁

LEMMA 3.148 *(i) and (ii) below hold.*

(i) $Basax \models (\forall m, k \in Obs)\Big(tr_m(k) = \bar{t} \ \Rightarrow\ (\mathsf{f}_{mk} = h \circ N \circ \widetilde{\varphi},$ *for some* $h \in ST,$ $N \in Newt_t$ *and* $\varphi \in Aut(\mathbf{F}))\Big)$.

(ii) $Basax \models (\forall m, k \in Obs)\Big(tr_m(k) \parallel \bar{t} \ \Rightarrow\ (\mathsf{f}_{mk} = h \circ N \circ \widetilde{\varphi}\,,$ *for some* $h \in ST,$ $N \in Newt$ *and* $\varphi \in Aut(\mathbf{F}))\Big)$.

Proof: The proof follows by Lemma 3.141 and Prop.3.104(i) in §3.6. ∎

LEMMA 3.149 *Assume* $n \geq 3$. *Then*

$$Basax(\mathbf{n}) \cup \{\mathbf{Ax}(\sqrt{\ })\} \models (\forall m_1, m_2 \in Obs)(\exists m \in Obs)\ \sigma_{\bar{t}}[tr_m(m_1)] = tr_m(m_2)\,.$$

CONVENTION 3.150 Let $\ell \in \mathsf{Eucl}$. Then
σ_ℓ denotes the reflection about ℓ. $\lhd$

We need Lemmas 3.151, 3.152, 3.153 below for the proof of Lemma 3.149 above.

LEMMA 3.151 *Let $\ell \in \mathsf{Eucl}$ with $\ell \parallel \bar{t}$. Let $\ell_1, \ell_2 \in \mathsf{Eucl}$ with $\sigma_\ell[\ell_1] = \ell_2$. Then (i)-(iii) below hold.*

(i) *If $h \in ST$ then* $\sigma_{h[\ell]}\Big[h[\ell_1]\Big] = h[\ell_2]$.

(ii) *If $N \in Newt^\star$ then* $\sigma_{N[\ell]}\Big[N[\ell_1]\Big] = N[\ell_2]$.

(iii) *If $\varphi \in Aut(\mathbf{F})$ then* $\sigma_{\widetilde{\varphi}[\ell]}\Big[\widetilde{\varphi}[\ell_1]\Big] = \widetilde{\varphi}[\ell_2]$.

Proof: The proof is straightforward. We omit it. ∎

LEMMA 3.152 *Assume $\mathfrak{F}$ is Euclidean. Assume $n = 2$. Let $\ell \in \mathsf{SlowEucl}$ with $\bar{0} \in \ell$. Then there is $\ell' \in \mathsf{SlowEucl}$ such that*

$$(\forall g \in PT)(g[\ell'] = \bar{t} \quad \Rightarrow \quad \sigma_{\bar{t}}\Big[g[\bar{t}]\Big] = g[\ell]).$$

Proof: We omit the proof. ∎

LEMMA 3.153 *Assume $\mathfrak{F}$ is Euclidean. Let $\ell \in \mathsf{SlowEucl}$. Then there is $\ell' \in \mathsf{SlowEucl}$ such that*

$$(\forall g \in PT)(g[\ell'] = \bar{t} \quad \Rightarrow \quad \sigma_{\bar{t}}\Big[g[\bar{t}]\Big] = g[\ell])\,.$$

Proof: Let us recall from item 13 of Notation 3.1 that τ_p denotes the translation by vector p, for every $p \in {}^nF$.

Let $\ell \in \mathsf{SlowEucl}$. Then there is $\ell_1 \in \mathsf{SlowEucl}$ such that $\ell_1 \parallel \ell$ and $\bar{0} \in \ell_1$. Then by Lemma 3.152, one can prove that there is $\ell'' \in \mathsf{Plane}(\bar{t}, \ell_1)$ such that

(88) $$(\forall g \in PT)(g[\ell''] = \bar{t} \quad \Rightarrow \quad \sigma_{\bar{t}}\Big[g[\bar{t}]\Big] = g[\ell_1]).$$

(Here we missed some details.) Let $A \in PT$ be fixed such that

(89) $$A[\ell''] = \bar{t}\,.$$

By (88) and (89), we have

(90) $$\sigma_{\bar{t}}\Big[A[\bar{t}\,]\Big] = A[\ell_1].$$

By $\ell_1, \ell \in \mathsf{SlowEucl}$ and $\ell_1 \parallel \ell$, we have $A[\ell_1], A[\ell] \in \mathsf{SlowEucl}$ and $A[\ell_1] \parallel A[\ell]$. Hence there is $c \in S$ such that

(91) $$\tau_c\Big[A[\ell_1]\Big] = A[\ell].$$

Let

(92) $$\ell' \overset{\text{def}}{=} A^{-1}\Big[\tau_{\frac{c}{2}}[\bar{t}\,]\Big].$$

We will prove that

(93) $$(\forall g \in PT)(g[\ell'] = \bar{t} \quad \Rightarrow \quad \sigma_{\bar{t}}\Big[g[\bar{t}\,]\Big] = g[\ell]).$$

To prove this we need Claim 3.154 below.

Claim 3.154 $\sigma_{A[\ell']}\Big[A[\bar{t}\,]\Big] = A[\ell]$.

Proof of Claim 3.154: By (92), it is enough to prove that $\sigma_{\tau_{\frac{c}{2}}[\bar{t}]}\Big[A[\bar{t}\,]\Big] = A[\ell]$. It is easy to see that $\sigma_{\tau_{\frac{c}{2}}[\bar{t}]}\Big[A[\bar{t}\,]\Big] = A[\ell]$ is equivalent with

(94) $$\langle p_0, p_1, \ldots, p_{n-1}\rangle \in A[\bar{t}\,] \Leftrightarrow \langle p_0, -p_1 + c_1, \ldots, -p_{n-1} + c_{n-1}\rangle \in A[\ell].$$

In the following we will prove (94).

$$\begin{aligned} \langle p_0, p_1, \ldots, p_{n-1}\rangle \in A[\bar{t}\,] &\Leftrightarrow \langle p_0, -p_1, \ldots, -p_{n-1}\rangle \in A[\ell_1] && \text{(by (90))}\\ &\Leftrightarrow \langle p_0, -p_1 + c_1, \ldots, -p_{n-1} + c_{n-1}\rangle \in A[\ell] && \text{(by (91))}. \end{aligned}$$

QED (Claim 3.154)

Proof of (93): Let $g \in PT$ with $g[\ell'] = \bar{t}$. We have to prove $\sigma_{\bar{t}}\Big[g[\bar{t}\,]\Big] = g[\ell]$. By $\ell' \parallel \ell''$, by (89), and by $g[\ell'] = \bar{t}$, it is easy to see that $A^{-1} \circ g[\bar{t}\,] \parallel \bar{t}$. Hence by Lemma 3.141, we have $A^{-1} \circ g = h \circ N$, for some $h \in ST$ and $N \in Newt^{\star}$. By this, we have $g = A \circ h \circ N$. By Claim 3.154 and Lemma 3.151(i),(ii), we have

$$\sigma_{h\circ N[A[\ell']]}\Big[h \circ N\Big[A[\bar{t}\,]\Big]\Big] = h \circ N\Big[A[\ell]\Big], \text{ i.e.}$$

$$\sigma_{A\circ h\circ N[\ell']}\Big[A \circ h \circ N[\bar{t}\,]\Big] = A \circ h \circ N[\ell].$$

But by $g = A \circ h \circ N$ and $g[\ell'] = \bar{t}$, this is equivalent with

$$\sigma_{\bar{t}}\Big[g[\bar{t}\,]\Big] = g[\ell]. \ \blacksquare$$

Proof of Lemma 3.149: Assume $n \geq 3$. Assume $Basax(\mathbf{n}) \cup \{\mathbf{Ax}(\sqrt{\ })\}$. Let $m_1, m_2 \in Obs$. By Thm.3.28, we have that $tr_{m_1}(m_2) \in \mathsf{SlowEucl}$. Now by applying Lemma 3.153, we have that there is $\ell' \in \mathsf{SlowEucl}$ such that

$$(95) \qquad (\forall g \in PT)(g[\ell'] = \bar{t} \quad \Rightarrow \quad \sigma_{\bar{t}}\Big[g[\bar{t}]\Big] = g[tr_{m_1}(m_2)])\,.$$

By **Ax5**, there is $m \in Obs$ such that $tr_{m_1}(m) = \ell'$. We will prove that $\sigma_{\bar{t}}[tr_m(m_1)] = tr_m(m_2)$. By Prop.3.104(i), we have that $\mathsf{f}_{m_1 m} = A \circ \widetilde{\varphi}$, for some $A \in PT$ with $A[tr_{m_1}(m)] = \bar{t}$ and $\varphi \in Aut(\mathbf{F})$. By this, by (95), and by $tr_{m_1}(m) = \ell'$, we have

$$\sigma_{\bar{t}}\Big[A[tr_{m_1}(m_1)]\Big] = A[tr_{m_1}(m_2)]\,.$$

By this and by Lemma 3.151(iii), we have

$$\sigma_{\bar{t}}\Big[A \circ \widetilde{\varphi}[tr_{m_1}(m_1)]\Big] = A \circ \widetilde{\varphi}\Big[tr_{m_1}(m_2)\Big], \text{ i.e.}$$

$$\sigma_{\bar{t}}\Big[\mathsf{f}_{m_1 m}[tr_{m_1}(m_1)]\Big] = \mathsf{f}_{m_1 m}\Big[tr_{m_1}(m_2)\Big].$$

Hence

$$\sigma_{\bar{t}}[tr_m(m_1)] = tr_m(m_2)\,. \qquad \blacksquare$$

CONVENTION 3.155 For every $g \in Aftr(\mathbf{n}, \mathbf{F})$ by $det(g)$ we denote the following. Let $g \in Aftr(\mathbf{n}, \mathbf{F})$. Then $(\forall p \in {}^nF) g(p) = pA + a$ for some invertible n by n matrix A and vector a. Now by $det(g)$ we denote the determinant of matrix A.

$\triangleleft$

LEMMA 3.156 *Assume $N \in Newt^\star$. Then $|det(N)| = 1$.*

Proof: The proof is straightforward. We omit it. $\blacksquare$

LEMMA 3.157

$$(\forall g_1, g_2 \in Aftr)\ det(g_1 \circ g_2) = det(g_1) \cdot det(g_2).$$

Proof: The proof is known from the literature. We omit it. $\blacksquare$

LEMMA 3.158

$$Basax \cup \{\mathbf{Ax}(\sqrt{\ }), \mathbf{Ax}\triangle\mathbf{2}\} \models (\forall m, k \in Obs)(\mathsf{f}_{mk} \in PT \quad \wedge \quad |det(\mathsf{f}_{mk})| = 1)\,.$$

Proof: Assume $Basax \cup \{\mathbf{Ax}(\sqrt{\ }), \mathbf{Ax}\triangle\mathbf{2}\}$. Let $m, k \in Obs$. Then by Thm.3.133, we have $\mathsf{f}_{mk} \in PT$. By $\mathbf{Ax}\triangle\mathbf{2}$, we have

$$\mathsf{f}_{km} = N \circ \mathsf{f}_{mk} \circ N\,, \quad \text{for some } N \in Newt^\star\,.$$

By this, we have

$$\mathsf{f}_{mk} \circ N \circ \mathsf{f}_{mk} \circ N = \mathrm{Id}\,.$$

By this and by Lemmas 3.156,3.157, we have $|det(\mathsf{f}_{mk})| = 1$. ∎

Proof of Thm.3.136:
Proof of (ii): Assume $n \geq 3$. Assume $Basax(\mathbf{n}) \cup \{\mathbf{Ax}(\sqrt{\ }), \mathbf{Ax}\triangle\mathbf{2}\}$. We will prove $\mathbf{Ax}\square\mathbf{2}$. To prove this, let $m, k, m', k' \in Obs$ with $tr_m(k) = tr_{m'}(k') := \ell$.

By Thm.3.133, we have $\mathsf{f}_{mk}, \mathsf{f}_{m'k'} \in PT$. Let $p, q \in \ell$ with $|p - q| = 1$. Such p, q exist by $\mathbf{Ax}(\sqrt{\ })$. By Thm.3.28 we have that $\ell \in \mathsf{SlowEucl}$. Now let T_{pq} be as defined in Def.3.107 in §3.6. Now by the definition of T_{pq}, we have $T_{pq}[\bar{t}\,] = \ell$. We claim that $|det(T_{pq})| = 1$ because $|p - q| = 1$. Checking this claim is left to the reader. Now let $A \stackrel{\text{def}}{=} T_{pq}^{-1}$. Now by $T_{pq}[\bar{t}\,] = \ell$ and $|det(T_{pq})| = 1$, we have

$$A[\ell] = \bar{t}\,, \tag{96}$$

$$|det(A)| = 1\,. \tag{97}$$

Claim 3.159 *Let $m_1, m_2 \in Obs$ with $\mathsf{f}_{m_1m_2}[\ell] = \bar{t}$. Then $\mathsf{f}_{m_1m_2} = A \circ N$, for some $N \in Newt^\star$.*

Proof of Claim 3.159: By Lemma 3.142, we have

$$\mathsf{f}_{m_1m_2} = A \circ h \circ N_1\,, \quad \text{for some } h \in ST \text{ and } N_1 \in Newt_t\,. \tag{98}$$

By Lemma 3.158, by Lemma 3.156, by (97), respectively, we have $|det(\mathsf{f}_{m_1m_2})| = 1$, $|det(N_1)| = 1$, $|det(A)| = 1$. By this, by (98), and by Lemma 3.157, we get that $|det(h)| = 1$. By $|det(h)| = 1$ and (98), we have $\mathsf{f}_{m_1m_2} = A \circ N$, for some $N \in Newt^\star$.
QED (Claim 3.159)

Now by (97), by $\mathsf{f}_{mk}[\ell] = \mathsf{f}_{m'k'}[\ell] = \bar{t}$, and by Claim 3.159, we have

$$\begin{aligned} \mathsf{f}_{mk} &= A \circ g_1\,, \quad \text{for some } g_1 \in Newt^\star, \\ \mathsf{f}_{m'k'} &= A \circ g_2\,, \quad \text{for some } g_2 \in Newt^\star\,. \end{aligned}$$

Hence

$$\mathsf{f}_{mk} = A \circ g_2 \circ g_2^{-1} \circ g_1 = \mathsf{f}_{m'k'} \circ (g_2^{-1} \circ g_1)\,.$$

But $g_2^{-1} \circ g_1 \in Newt^\star$ and this completes the proof of Thm.3.136(ii).

Proof of Thm.3.136(i): Assume $n \geq 3$. By Thm.3.136(ii) it is enough to prove

$$Basax(\mathbf{n}) \cup \{\mathbf{Ax(\sigma_{\bar{t}})}, \mathbf{Ax}(\sqrt{\ })\} \models (\mathbf{Ax\Box 2} \Rightarrow \mathbf{Ax\triangle 2}).$$

To see this assume $Basax(\mathbf{n}) \cup \{\mathbf{Ax(\sigma_{\bar{t}})}, \mathbf{Ax}(\sqrt{\ }), \mathbf{Ax\Box 2}\}$. To prove $\mathbf{Ax\triangle 2}$, let $m_1, m_2 \in Obs$. We have to prove that there is $N \in Newt^\star$ such that $\mathsf{f}_{m_1 m_2} = N \circ \mathsf{f}_{m_1 m_2} \circ N$. By Lemma 3.149, we have

(99) $$\sigma_{\bar{t}}[tr_m(m_1)] = tr_m(m_2)\,, \quad \text{for some } m \in Obs\,.$$

By $\mathbf{Ax(\sigma_{\bar{t}})}$, we have that there is $m' \in Obs$ with

(100) $$\mathsf{f}_{mm'} = \mathsf{f}_{m'm} = \sigma_{\bar{t}}\,.$$

Now by (100), we have

$$tr_{m'}(m_1) = \mathsf{f}_{m'm}[tr_m(m_1)] = \sigma_{\bar{t}}[tr_m(m_1)] = tr_m(m_2)\,.$$

By this and $\mathbf{Ax\Box 2}$, we have

(101) $$\mathsf{f}_{mm_2} = \mathsf{f}_{m'm_1} \circ N\,, \quad \text{for some } N \in Newt^\star\,.$$

Now

$$\begin{aligned}
\mathsf{f}_{m'm_2} &= \mathsf{f}_{m'm} \circ \mathsf{f}_{mm_2} \\
&= \sigma_{\bar{t}} \circ \mathsf{f}_{mm_2} && \text{(by (100))} \\
&= \sigma_{\bar{t}} \circ \mathsf{f}_{m'm_1} \circ N && \text{(by (101))} \\
&= \mathsf{f}_{mm'} \circ \mathsf{f}_{m'm_1} \circ N && \text{(by (100))} \\
&= \mathsf{f}_{mm_1} \circ N.
\end{aligned}$$

If we summarize the above computation we get

$$\mathsf{f}_{m'm_2} = \mathsf{f}_{mm_1} \circ N.$$

This is equivalent to

(102) $$N^{-1} \circ \mathsf{f}_{m_1 m} = \mathsf{f}_{m_2 m'}.$$

By (101) and (102), we have

$$N^{-1} \circ \mathsf{f}_{m_1 m} \circ \mathsf{f}_{mm_2} = \mathsf{f}_{m_2 m'} \circ \mathsf{f}_{m'm_1} \circ N.$$

Thus

$$\mathsf{f}_{m_1 m_2} = N \circ \mathsf{f}_{m_2 m_1} \circ N.$$

This completes the proof of Thm.3.136(i).

Proof of Thm.3.136(iii),(iv): The proof follows by Thm.3.136(i) and Proposition 3.125. ∎

We need Lemma 3.160 below for the proof of Proposition 3.131.

LEMMA 3.160 *Assume $\mathfrak{F}$ is Euclidean. Let $\ell_1, \ell_2 \in \mathsf{SlowEucl}$ such that $\sigma_{\bar{t}}[\ell_1] = \ell_2$. Then there is $\ell \in \mathsf{Eucl}$ such that $\ell \perp \bar{t}$ and $\ell \cap \bar{t} \neq \emptyset$ and $\sigma_\ell[\ell_1] = \ell_2$.*

Outline of proof: Let $\ell' \in \mathsf{Eucl}$ such that $\ell' \perp \ell_1$, $\ell' \perp \ell_2$, $\ell' \perp \bar{t}$, $\ell' \cap \ell_1 \neq \emptyset$, $\ell' \cap \ell_2 \neq \emptyset$, and $\ell' \cap \bar{t} \neq \emptyset$. Now let $\ell \perp \ell'$ and $\ell \perp \bar{t}$ such that $\ell' \cap \bar{t} = \ell \cap \bar{t}$. For this ℓ we have $\sigma_\ell[\ell_1] = \ell_2$. ∎

Proof of Proposition 3.131: Assume $n \geq 3$.
$Basax(\mathbf{n}) \cup \{\mathbf{Ax}(\sqrt{\ })\} \models (\mathbf{Ax\triangle 2}^* \Rightarrow \mathbf{Ax\triangle 2})$ is obvious hence we will prove

$$Basax(\mathbf{n}) \cup \{\mathbf{Ax}(\sqrt{\ })\} \models (\mathbf{Ax\triangle 2} \Rightarrow \mathbf{Ax\triangle 2}^*)\,.$$

To prove this assume $Basax(\mathbf{n}) \cup \{\mathbf{Ax}(\sqrt{\ }), \mathbf{Ax\triangle 2}\}$. Let $m_1, m_2 \in Obs$. Then by Lemma 3.149 there is $m \in Obs$ such that

$$\sigma_{\bar{t}}[tr_m(m_1)] = tr_m(m_2)\,. \tag{103}$$

Now (104) or (105) below hold.

$$(m_1 \uparrow m \;\&\; m_2 \uparrow m) \quad \text{or} \quad (m_1 \downarrow m \;\&\; m_2 \downarrow m). \tag{104}$$
$$(m_1 \uparrow m \;\&\; m_2 \downarrow m) \quad \text{or} \quad (m_1 \downarrow m \;\&\; m_2 \uparrow m). \tag{105}$$

By Thm.3.133, we have

$$\mathsf{f}_{mm_1}, \mathsf{f}_{mm_2} \in PT.$$

Proof in case (104):

$$\begin{array}{rcll}
\sigma_{\bar{t}} \circ \mathsf{f}_{mm_2}[tr_m(m_1)] & = & \mathsf{f}_{mm_2}\Big[\sigma_{\bar{t}}[tr_m(m_1)]\Big] & \\
& = & \mathsf{f}_{mm_2}[tr_m(m_2)] & \text{(by (103))} \\
& = & tr_{m_2}(m_2) & \text{(by } Basax(\mathbf{n})) \\
& = & \bar{t} & \text{(by } \mathbf{Ax4}) \\
& = & tr_{m_1}(m_1) & \text{(by } \mathbf{Ax4}) \\
& = & \mathsf{f}_{mm_1}[tr_m(m_1)] & \text{(by } Basax(\mathbf{n})).
\end{array}$$

If we summarize the above computation we get

$$\sigma_{\bar{t}} \circ \mathsf{f}_{mm_2}[tr_m(m_1)] = \mathsf{f}_{mm_1}[tr_m(m_1)] = \bar{t}\,. \tag{106}$$

By (106) and Lemma 3.142, we have

$$\sigma_{\bar{t}} \circ \mathsf{f}_{mm_2} = \mathsf{f}_{mm_1} \circ h \circ N\,, \quad \text{for some } h \in ST, N \in Newt_t\,. \tag{107}$$

Then

$$(\forall p \in {}^nF)\ h(p) = r \cdot p\,, \quad \text{for some } r \in F \setminus \{0\}, \text{ and} \tag{108}$$

$$(109) \qquad (\forall p \in {}^nF)\ N(p) = N_0(p) + c\,, \quad \text{for some } N_0 \in Newt_t \cap GL \text{ and } c \in {}^nF\,.$$

By $N_0 \in Newt_t \cap GL$, we have

$$(110) \qquad (\forall p \in {}^nF)\ (p)_t = (N_0(p))_t.$$

$$\begin{array}{rll}
\text{Now} & (\mathsf{f}_{mm_1} \circ h \circ N(1_t) - \mathsf{f}_{mm_1} \circ h \circ N(\bar{0}))_t = & \\
= & (N(h(\mathsf{f}_{mm_1}(1_t)) - N(h(\mathsf{f}_{mm_1}(\bar{0}))))_t & \\
= & (N(r \cdot \mathsf{f}_{mm_1}(1_t)) - N(r \cdot \mathsf{f}_{mm_1}(\bar{0})))_t & \text{(by (108))} \\
= & (N_0(r \cdot \mathsf{f}_{mm_1}(1_t)) - N_0(r \cdot \mathsf{f}_{mm_1}(\bar{0})))_t & \text{(by (109))} \\
= & (r \cdot N_0(\mathsf{f}_{mm_1}(1_t) - \mathsf{f}_{mm_1}(\bar{0})))_t & \text{(by } N_0 \in GL) \\
= & r \cdot (N_0(\mathsf{f}_{mm_1}(1_t) - \mathsf{f}_{mm_1}(\bar{0})))_t & \\
= & r \cdot (\mathsf{f}_{mm_1}(1_t) - \mathsf{f}_{mm_1}(\bar{0}))_t & \text{(by (110))}
\end{array}$$

If we summarize the above computation we get

$$(111) \qquad (\mathsf{f}_{mm_1} \circ h \circ N(1_t) - \mathsf{f}_{mm_1} \circ h \circ N(\bar{0}))_t = r \cdot (\mathsf{f}_{mm_1}(1_t) - \mathsf{f}_{mm_1}(\bar{0}))_t.$$

Obviously

$$(112) \qquad \sigma_{\bar{t}}(1_t) = 1_t \quad \text{and} \quad \sigma_{\bar{t}}(\bar{0}) = \bar{0}\,.$$

Now

$$\begin{array}{rcll}
(\mathsf{f}_{mm_2}(1_t) - \mathsf{f}_{mm_2}(\bar{0}))_t & = & (\sigma_{\bar{t}} \circ \mathsf{f}_{mm_2}(1_t) - \sigma_{\bar{t}} \circ \mathsf{f}_{mm_2}(\bar{0}))_t & \text{(by (112))} \\
& = & (\mathsf{f}_{mm_1} \circ h \circ N(1_t) - \mathsf{f}_{mm_1} \circ h \circ N(\bar{0}))_t & \text{(by (107))} \\
& = & r \cdot (\mathsf{f}_{mm_1}(1_t) - \mathsf{f}_{mm_1}(\bar{0}))_t & \text{(by (111)).}
\end{array}$$

We got

$$(113) \qquad (\mathsf{f}_{mm_2}(1_t) - \mathsf{f}_{mm_2}(\bar{0}))_t = r \cdot (\mathsf{f}_{mm_1}(1_t) - \mathsf{f}_{mm_1}(\bar{0}))_t.$$

By the definition of $\uparrow, \downarrow$ and (105), we have

$$(114) \qquad (\mathsf{f}_{mm_1}(1_t) - \mathsf{f}_{mm_1}(\bar{0}))_t > 0 \quad \Leftrightarrow \quad (\mathsf{f}_{mm_1}(1_t) - \mathsf{f}_{mm_1}(\bar{0}))_t > 0.$$

By (113) and (114), we have

$$(115) \qquad r > 0.$$

By Lemmas 3.158, 3.156, 3.157, and by (107), it follows that $|det(h)| = 1$. $|det(h)| = 1$, (108) and (115) implies $h = \mathrm{Id}$. By this and by (107), we get

$$(116) \qquad \sigma_{\bar{t}} \circ \mathsf{f}_{mm_2} = \mathsf{f}_{mm_1} \circ N\,.$$

By (116), we get (117), (118) below.

$$(117) \qquad \mathsf{f}_{m_1m} = N \circ \mathsf{f}_{m_2m} \circ \sigma_{\bar{t}}\,.$$

$$(118) \qquad \mathsf{f}_{mm_2} = \sigma_{\bar{t}} \circ \mathsf{f}_{mm_1} \circ N\,.$$

Now

$$\begin{array}{lcll} \mathsf{f}_{m_1m_2} & = & \mathsf{f}_{m_1m} \circ \mathsf{f}_{mm_2} & \\ & = & (N \circ \mathsf{f}_{m_2m} \circ \sigma_{\bar{t}}) \circ (\sigma_{\bar{t}} \circ \mathsf{f}_{mm_1} \circ N) & \text{(by (117), (118))} \\ & = & N \circ \mathsf{f}_{m_2m} \circ \mathsf{f}_{mm_1} \circ N & \text{(by } \sigma_{\bar{t}} \circ \sigma_{\bar{t}} = \mathrm{Id}) \\ & = & N \circ \mathsf{f}_{m_2m_1} \circ N. & \end{array}$$

By the above computation, we got

$$\mathsf{f}_{m_1m_2} = N \circ \mathsf{f}_{m_2m_1} \circ N. \tag{119}$$

By (107), we have $N \in Newt_t$, and this completes the proof of Proposition 3.131 in case (104).

Proof in case (105): By (103) and Lemma 3.160, there is $\ell \in \mathsf{Eucl}$ such that $\ell \perp \bar{t}$, $\ell \cap \bar{t} \neq \emptyset$ and $\sigma_\ell[tr_m(m_1)] = tr_m(m_2)$. By this, analogously to (106) we get

$$\sigma_\ell \circ \mathsf{f}_{mm_2}[tr_m(m_1)] = \mathsf{f}_{mm_1}[tr_m(m_1)] = \bar{t}\,. \tag{120}$$

By Lemma 3.142, we have

$$\sigma_\ell \circ \mathsf{f}_{mm_2} = \mathsf{f}_{mm_1} \circ h \circ N\,, \quad \text{for some } h \in ST, N \in Newt_t\,. \tag{121}$$

Then

$$(\forall p \in {}^nF)\ h(p) = r \cdot p\,, \quad \text{for some } r \in F \setminus \{0\}\,. \tag{122}$$

Analogously to (111) we get (123) below.

$$(\mathsf{f}_{mm_1} \circ h \circ N(1_t) - \mathsf{f}_{mm_1} \circ h \circ N(\bar{0}))_t = r \cdot (\mathsf{f}_{mm_1}(1_t) - \mathsf{f}_{mm_1}(\bar{0}))_t. \tag{123}$$

It is easy to see that

$$(\forall A \in Aftr) \qquad \big(A(1_t) - A(\bar{0}) = -(A(-1_t) - A(\bar{0}))\big), \tag{124}$$

$$(\forall A \in Aftr)(\forall c \in {}^nF) \qquad \big(A(1_t) - A(\bar{0}) = A(1_t + c) - A(c)\big). \tag{125}$$

It is easy to see that

$$\sigma_\ell(1_t) = -1_t + \sigma_\ell(\bar{0}). \tag{126}$$

Now

$$\begin{array}{lcll} r \cdot (\mathsf{f}_{mm_1}(1_t) - \mathsf{f}_{mm_1}(\bar{0}))_t & = & (\mathsf{f}_{mm_1} \circ h \circ N(1_t) - \mathsf{f}_{mm_1} \circ h \circ N(\bar{0}))_t & \text{(by (123))} \\ & = & (\sigma_\ell \circ \mathsf{f}_{mm_2}(1_t) - \sigma_\ell \circ \mathsf{f}_{mm_2}(\bar{0}))_t & \text{(by (121))} \\ & = & (\mathsf{f}_{mm_2}(-1_t) + \sigma_\ell(\bar{0}) - \mathsf{f}_{mm_2}(\sigma_\ell(\bar{0})))_t & \text{(by (126))} \\ & = & (\mathsf{f}_{mm_2}(-1_t) - \mathsf{f}_{mm_2}(\bar{0}))_t & \text{(by (125))} \\ & = & -(\mathsf{f}_{mm_2}(1_t) - \mathsf{f}_{mm_2}(\bar{0}))_t & \text{(by (124)).} \end{array}$$

By the above computation, we got

$$r \cdot (\mathsf{f}_{mm_1}(1_t) - \mathsf{f}_{mm_1}(\bar{0}))_t = -(\mathsf{f}_{mm_2}(1_t) - \mathsf{f}_{mm_2}(\bar{0}))_t. \tag{127}$$

Now by (105) and the definition of $\uparrow, \downarrow$, we have

(128) $$(\mathsf{f}_{mm_1}(1_t) - \mathsf{f}_{mm_1}(\bar{0}))_t > 0 \quad \Leftrightarrow \quad (\mathsf{f}_{mm_2}(1_t) - \mathsf{f}_{mm_2}(\bar{0}))_t < 0.$$

By (127) and (128), we have $r > 0$. Now as in case (104) we can conclude that $h = \mathrm{Id}$. Hence by (121), we have

(129) $$\sigma_\ell \circ \mathsf{f}_{mm_2} = \mathsf{f}_{mm_1} \circ N.$$

Now by (129), we get (130) below, and the proof of (129)⇒(130) is analogous with the proof of (116)⇒(119) in case (104), the only change is that we use σ_ℓ instead of $\sigma_{\bar{t}}$.

(130) $$\mathsf{f}_{m_1m_2} = N \circ \mathsf{f}_{m_2m_1} \circ N.$$

By (121), we have $N \in Newt_t$, and this completes the proof. ∎

LEMMA 3.161 *Assume $n \geq 3$. Then (i), (ii) below hold.*

(i) $Basax(\mathbf{n}) \cup \{\mathbf{Ax}(\sqrt{\ })\} \models$ "$\uparrow$ *is an equivalence relation*".

(ii) $Basax(\mathbf{n}) \cup \{\mathbf{Ax}(\sqrt{\ })\} \models ((m_1 \downarrow m_2 \ \&\ m_1 \downarrow m_3) \Rightarrow m_2 \downarrow m_3)$.

We omit the **proof**.

LEMMA 3.162 *Assume $n \geq 3$. Then*

$$Basax(\mathbf{n}) \cup \{\mathbf{Ax}(\sqrt{\ }), \mathbf{Ax5}^{++}\} \models (\exists m_1, m_2 \in Obs)\Big(m_1 \downarrow m_2 \quad \Rightarrow$$

$$(\forall k_1 \in Obs)(\exists k_2 \in Obs)(tr_{k_1}(k_2) = \bar{t} \ \ \wedge \ \ k_1 \downarrow k_2)\Big).$$

Proof: Assume $n \geq 3$. Assume $Basax(\mathbf{n}) \cup \{\mathbf{Ax}(\sqrt{\ }), \mathbf{Ax5}^{++}\}$ and assume that $(\exists m_1, m_2 \in Obs)$ with $m_1 \downarrow m_2$. From now on let such m_1 and m_2 be fixed. We have to show that

$$(\forall k_1 \in Obs)(\exists k_2 \in Obs)(tr_{k_1}(k_2) = \bar{t} \ \ \wedge \ \ k_1 \downarrow k_2).$$

To see this let $k_1 \in Obs$. Then (131) or (132) below hold.

(131) $$m_1 \ \uparrow \ k_1\,.$$

(132) $$m_1 \ \downarrow \ k_1\,.$$

Proof in case (131): By $m_1 \downarrow m_2$, by $m_1 \uparrow k_1$, and by Lemma 3.161, we have $m_2 \downarrow k_1$. By Thm.3.28, we have $\ell := tr_{m_2}(k_1) \in \mathsf{SlowEucl}$. Now by $\mathbf{Ax5}^{++}$, we have that there is $k_2 \in Obs$ with $tr_{m_2}(k_2) = \ell = tr_{m_2}(k_1)$ and $m_2 \uparrow k_2$. By $tr_{m_2}(k_2) = tr_{m_1}(k_1)$, we have $tr_{k_1}(k_2) = \bar{t}$, and by $m_2 \downarrow k_1$ and $m_2 \uparrow k_2$ and Lemma 3.161, we have $k_1 \downarrow k_2$.

Proof in case (132): By Thm.3.28, we have $\ell := tr_{m_1}(k_1) \in \mathsf{SlowEucl}$. By $\mathbf{Ax5^{++}}$, we have that there is $k_2 \in Obs$ with $tr_{m_1}(k_2) = \ell = tr_{m_1}(k_1)$ and $m_1 \uparrow k_2$. Now $tr_{m_1}(k_2) = tr_{m_1}(k_1)$ implies $tr_{k_1}(k_2) = \bar{t}$, and $m_1 \downarrow k_1$ and $m_1 \uparrow k_2$ by Lemma 3.161 implies $k_1 \downarrow k_2$. ∎

Proof of Proposition 3.128: Assume $n \geq 3$. Assume $Basax(\mathbf{n}) \cup \{\mathbf{Ax}(\sqrt{\ }), \mathbf{Ax5^{++}}\}$. To prove $\mathbf{Ax5^{+}}$, let $m, k, m' \in Obs$. Then by Thm.3.28, we have

$$\ell \stackrel{\text{def}}{=} tr_m(k) \in \mathsf{SlowEucl}\,.$$

By $\mathbf{Ax5^{++}}$, we have that there is $k' \in Obs$ with $tr_{m'}(k') = \ell = tr_m(k)$ and $m' \uparrow k'$. If $m \uparrow k$ then we are done. If $m \downarrow k$ then by Lemma 3.162, we have that there is $k'' \in Obs$ with $tr_{m'}(k'') = tr_{m'}(k')$ and $k' \downarrow k''$. By $m' \uparrow k'$ and $k' \downarrow k''$, we have $m' \downarrow k''$ and $tr_{m'}(k'') = tr_{m'}(k') = tr_m(k)$. ∎

LEMMA 3.163

$$Basax \cup \{\mathbf{Ax8}, \mathbf{Ax9}_0\} \models (\forall k \in Obs)(\forall N \in Newt_t)$$

$$(\exists k' \in Obs)(\mathsf{f}_{kk'} = N \quad \wedge \quad tr_k(k') = \bar{t}\,).$$

We omit the **proof.**

Proof of Thm.3.137:
Proof of Thm.3.137(ii): Assume $Basax(\mathbf{n}) \cup \{\mathbf{Ax9}_0\}$. By Prop.3.129 it is enough to prove $\mathbf{Ax\Box1^*} \Rightarrow \mathbf{Ax\Box2}$. Assume $\mathbf{Ax\Box1^*}$. To prove $\mathbf{Ax\Box2}$, let $m, k, m', k' \in Obs$ with

(133) $$tr_m(k) = tr_{m'}(k').$$

By $\mathbf{Ax\Box1^*}$, there is $k'' \in Obs$ with

(134) $$tr_m(k) \;=\; tr_{m'}(k'')\,,$$
(135) $$\mathsf{f}_{mk} \;=\; \mathsf{f}_{m'k''}\,.$$

But $tr_{m'}(k') = tr_{m'}(k'')$ by (133) and (134). Hence $tr_{k'}(k'') = \bar{t}$. By this and by $\mathbf{Ax9}_0$, we have

(136) $$\mathsf{f}_{m'k''} \in Newt^{\star}\,.$$

By (135) we have

$$\mathsf{f}_{mk} = \mathsf{f}_{m'k'} \circ \mathsf{f}_{k'k''}\,.$$

This and (136) completes the proof of (ii) of Thm.3.137.

Proof of Thm.3.137(i): By Thm.3.137(ii) it is enough to prove

$$Basax(\mathbf{n}) \cup \{\mathbf{Ax8}, \mathbf{Ax9}_0, \mathbf{Ax5}^+\} \models (\mathbf{Ax\Box 2} \Rightarrow \mathbf{Ax\Box 1}) .$$

To see this, assume $Basax(\mathbf{n}) \cup \{\mathbf{Ax8}, \mathbf{Ax9}_0, \mathbf{Ax5}^+, \mathbf{Ax\Box 2}\}$. To prove $\mathbf{Ax\Box 1}$, let $m, k, m' \in Obs$. By $\mathbf{Ax5}^+$, there is $k'' \in Obs$ with

$$(137) \qquad tr_m(k) = tr_{m'}(k') \quad \text{and} \quad (m \uparrow k \Leftrightarrow m' \uparrow k'') .$$

By $\mathbf{Ax\Box 2}$, we have

$$(138) \qquad \mathsf{f}_{mk} = \mathsf{f}_{m'k'} \circ N , \quad \text{for some } N \in Newt^{\star} .$$

By (137), one can see that $N \in Newt_t$. By this and by Lemma 3.163, we have

$$(139) \qquad \mathsf{f}_{k''k'} = N , \quad \text{for some } k' \in Obs \text{ with } tr_{k''}(k') = \bar{t} .$$

Now by (138) and (139), we have

$$\mathsf{f}_{mk} = \mathsf{f}_{m'k'} \circ N = \mathsf{f}_{m'k''} \circ \mathsf{f}_{k''k'} = \mathsf{f}_{m'k'} \text{ and } tr_m(k) = tr_{m'}(k') = tr_{m'}(k'') .$$

This completes the proof of (i) of Thm.3.137.

Proof of Thm.3.137(iii): The proof follows by Prop.3.128 and (i) of Thm.3.137. ∎

Proof of Thm.3.138:

Proof of Thm.3.138(ii): Assume $Basax(\mathbf{n}) \cup \{\mathbf{Ax9}_0, \mathbf{Ax\triangle 1}\}$. To prove $\mathbf{Ax\triangle 2}$, let $m, k \in Obs$. We have to prove that there is $N \in Newt^{\star}$ such that $\mathsf{f}_{mk} = N \circ \mathsf{f}_{km} \circ N$. By $\mathbf{Ax\triangle 1}$, there is $k' \in Obs$ such that

$$(140) \qquad tr_m(k) = tr_m(k') \quad \text{and} \quad \mathsf{f}_{mk'} = \mathsf{f}_{k'm}.$$

But by (140) and $\mathbf{Ax9}_0$, we have

$$(141) \qquad \mathsf{f}_{k'k} \in Newt^{\star} .$$

Now by (140), we have

$$\mathsf{f}_{mk} = \mathsf{f}_{m'k'} \circ \mathsf{f}_{k'k} = \mathsf{f}_{k'm} \circ \mathsf{f}_{k'k} = \mathsf{f}_{k'k} \circ \mathsf{f}_{km} \circ \mathsf{f}_{k'k} .$$

This and (141) completes the proof of (ii) of Thm.3.138.

Proof of Thm.3.138(i): By (ii) it is enough to prove that if $n \geq 3$ then

$$Basax(\mathbf{n}) \cup \{\mathbf{Ax8}, \mathbf{Ax9}_0, \mathbf{Ax}(\sqrt{\ })\} \models (\mathbf{Ax\triangle 2} \Rightarrow \mathbf{Ax\triangle 1})\,.$$

To see this, assume $n \geq 3$ and $Basax(\mathbf{n}) \cup \{\mathbf{Ax8}, \mathbf{Ax9}_0, \mathbf{Ax}(\sqrt{\ }), \mathbf{Ax\triangle 2}\}$. To prove $\mathbf{Ax\triangle 1}$, let $m, k \in Obs$. Then by Prop.3.131, we have that

$$\mathsf{f}_{mk} = N \circ \mathsf{f}_{km} \circ N\,, \quad \text{for some } N \in Newt_t\,. \tag{142}$$

Now by $N \in Newt_t$ and by Lemma 3.163, there is $k' \in Obs$ with

$$\mathsf{f}_{k'k} = N \text{ and} \tag{143}$$

$$tr_{k'}(k) = \bar{t}\,. \tag{144}$$

Now

$$\begin{aligned} \mathsf{f}_{mk'} &= \mathsf{f}_{mk} \circ \mathsf{f}_{kk'} \\ &= \mathsf{f}_{mk} \circ N^{-1} \quad \text{(by (143))} \\ &= N \circ \mathsf{f}_{km} \quad \text{(by (142))} \\ &= \mathsf{f}_{k'k} \circ \mathsf{f}_{km} \quad \text{(by (143))} \\ &= \mathsf{f}_{mk'}\,. \end{aligned}$$

The above computation and (144) completes the proof. ∎

4 Toward general relativity: accelerating observers

4.1 Accelerating observers

In this chapter we wish to expand our language and theory to handle non-inertial observers as well. We start general relativity theory by waiving our old assumptions that:

- all observers are inertial bodies (**Ax2**), and
- all geodesics are Euclidean lines (**Ax1**).

The physical intuition is that we do not exclude the existence of observers whose velocity changes in time; and such observers can see inertial bodies moving on geodesics different from Euclidean lines. By waiving (**Ax2**) we allow the existence of *"accelerating observers"*, besides inertial ones. Technically speaking, in a model, *Obs* may contain elements outside of *Ib*. We denote the set of *inertial observers* (that is, observers which are inertial bodies) by *IOb*, that is,

$$IOb \stackrel{\text{def}}{=} Obs \cap Ib\,.$$

Besides waiving old axioms, we need to postulate new ones. More concretely,

- we keep a part of *Newbasax* such that we replace *Obs* by *IOb* in them;
- we postulate a set of axioms referring to *all* observers, including the "new" ones ($Obs \setminus IOb$) as well;
- we postulate some more axioms for treating "real" relativistic effects.

To execute our plan above, we need to modify our first order language and frame models used so far, as follows. In our old frame models, all observers "shared" the same set G of geodesics. In our new frame models, every observer m will have a set G_m of its own geodesics. This means that we have to change our first order language as well. Throughout in this section, G is a sort containing the geodesics and there is a new binary relation symbol Go defined between the "sort" of observers and the sort G. Intuitively, if m is an observer (in some model $\mathfrak{M}$), then the set $G_m = \{\ell : \langle m, \ell\rangle \in Go^{\mathfrak{M}}\}$ is the collection of the geodesics of m. The reason for this decision is the following. The set of geodesics of m is intended to represent the trace of inertial bodies. Since m is not necessarily inertial, it really depends on m, how (s)he observes the movements of inertial bodies. Thus if m_1 is another observer, then the set of traces of inertial bodies from the point of view of m is not necessarily the same as the set of traces of inertial bodies from m_1's point of view. In order to keep the notation simpler, we will use the notation G_m in the formulas below.

Another modification will be that we will introduce *metrics* d_m for each observer $m \in Obs$. This is motivated by Theorem 4.15 below. Before the formulation and proof of this theorem here we include an intuitive explanation of this theorem.

Before discussing Theorem 4.15 in a precise language, we discuss it on a very informal, intuitive level. Theorem 4.15 can be interpreted as predicting certain things that happen in a gravitational field. Roughly, the theorem implies that

(*) In a gravitational field, like that of the Earth, clocks (and hence processes in general) higher up in the field run slower than clocks deeper down.

E.g. in a very high tower, clocks in the attic (on top of the tower) run *faster* than clocks in the basement. Similarly, clocks closer to a black hole run slower than distant clocks[73]. Statement (*) above is called the *tower paradox*. Below we turn to a more precise, more logical and at the same time stronger (but still intuitive) formulation of the tower paradox, i.e. to Thm.4.15. Theorem 4.15 will be stronger than (*) in the sense that it says that no matter how the accelerating observer chooses his coordinatization (of space-time), (*) will remain true.

On the proof. A statement much easier than Thm.4.15 would say something like the following. In the "most natural" or " simplest" models of our theory *Acc* of accelerating observers, statement (*) is true. This can be checked the following way. (1) Look at the examples we give (later) for the world-view transformation f_{mk} for the case when m is inertial and k is accelerating. (2) Try to interpret (*) formally in this context. (3) Try to check that the so interpreted version of (*) holds for the particular example of f_{mk}. Next we turn to discussing Thm.4.15.

Let us choose an observer m and let e_1 and e_2 be events having the same location (that is, having the same space-like coordinates) from the point of view of m. In Special Relativity the time passed between e_1 and e_2 (observed by m) is simply

[73] Actually, for a *distant* observer, clocks on the surface of the event horizon of the black hole stop moving (they "freeze"). We did not yet check how much of this (about black holes) follows directly from Theorem 4.15, or whether extra considerations are needed for deriving this from Thm.4.15. However, we are under the impression that derivations of this effect start out from (basically, something like) Thm.4.15, cf. e.g. Kenyon's book on general relativity (Oxford U.P., 1990).

the difference between the time like coordinates of the events in question. By definition, a time like coordinate line connects simultaneous events. So, in particular, the time passed between two time like coordinate lines does not depend on the location of the "measurement" of m. The same (more precisely, the dual) applies to the space-like coordinate lines too. In the case of accelerating observers the situation is more complicated. Roughly speaking, in Theorem 4.15 below we will prove[74] that in every model whose field reduct is isomorphic to the field of reals, if the distance of any two coordinate lines of an observer m is constant, then the speed of m is constant as well, thus m is not accelerating. This can be illustrated in the following way. Suppose m is an accelerating observer and e_0, e'_0, e_1 and e'_1 are events such that

e_0 is simultaneous with e'_0 and
e_1 is simultaneous with e'_1 and
e_0 has same location as e_1 and
e'_0 has same location as e'_1

(from the point of view of m, of course). Then the distance between e_0 and e'_0 differs from the distance between e_1 and e'_1. This suggest the following experiment. Let us move, say 1 km in a certain direction and then let us wait, say 1 year. Then our space-time coordinates will be different from our partner's space-time coordinates who first waited a year then went 1 km in the same direction as we did. Summing up, for an accelerating observer, simultaneity is not the same as "waiting the same time". Thus, if we want to speak about both of these notions then we have to expand our language distinguishing simultaneities and lengths of time intervals. This is the reason for introducing the metrics d_m. Let us notice that the result of the above mentioned physical experiment coincides with the conclusion of Theorem 4.15, however our decision about expanding the language is motivated by Theorem 4.15.

Turning to the new language, let Do be a new $1 + 2n$ – placed function symbol $Do : Obs \times {}^nF \times {}^nF \to F$. Intuitively, if m is an observer (in some model $\mathfrak{M}$) then the value of the function $d_m(x, y) := Do(m, x, y)$ is the distance between the space-time points x and y observed by m. Note that x and y are points in the space-time (not simply points in the space). As usual, in the formulas below, we will use the notation d_m.

[74]At the moment we have a proof only in the two dimensional case, but this result has implications in higher dimensions as well.

Thus $\mathfrak{M}$ is a model for our expanded language iff it is of the form

$$\mathfrak{M} = \langle (B, Obs, Ph, Ib), \mathfrak{F}, \mathrm{E}, W, G, Go, Do \rangle.$$

Using the above introduced notation we define our new frame models as follows. A model $\mathfrak{M}$ is a *frame model for accelerating observers* iff conditions (1) and (2) below hold.

(1)

$$\mathfrak{M} = \langle (B, Obs, Ph, Ib), \mathfrak{F}, \mathrm{E}, W, G_m, d_m \rangle_{m \in Obs}$$

where

- $B, Obs, Ph, Ib, \mathfrak{F}, \mathrm{E}, W$ are the same as in Definition 2.1;
- for arbitrary $m \in Obs$,
  $G_m \subseteq \mathcal{P}({}^nF)$, and
  $d_m : {}^nF \times {}^nF \longrightarrow F^+$, where F^+ denotes the non-negative part of F.

(2) Frame theory:

- We keep our frame theory used so far. More precisely, we assume $\mathbf{Ax}_{\mathrm{OF}}$ and the straightforward generalization $\mathbf{Ax}_{\mathrm{G}}(m)$ of $\mathbf{Ax}_{\mathrm{G}}$ from G to G_m, for every $m \in Obs$.
- To these we add the new assumptions $\mathbf{Ax}_{\mathrm{met}}(m)$ saying that d_m is a Lorentz–metric[75] for every $m \in Obs$. We note that a Lorentz–metric is not a metric in the usual sense.

We will denote the class of our *frame models for accelerating observers* by $\mathsf{FM}_{\mathrm{acc}}$. That is,

$$\mathsf{FM}_{\mathrm{acc}} \stackrel{\mathrm{def}}{=} \{\mathfrak{M} = \langle (B, Obs, Ph, Ib), \mathfrak{F}, \mathrm{E}, W, G_m, d_m \rangle_{m \in Obs} : \\ \mathfrak{M} \models \mathbf{Ax}_{\mathrm{OF}} \cup \{\mathbf{Ax}_{\mathrm{G}}(m) : m \in Obs\} \cup \{\mathbf{Ax}_{\mathrm{met}}(m) : m \in Obs\}\}.$$

[75]For completeness, we include here the set $\mathbf{Ax}_{\mathrm{met}}$ of axioms of a Lorentz–metric: We call $d : {}^nF \times {}^nF \longrightarrow F^+$ a *Lorentz–metric* on the ordered field $\mathfrak{F}$ iff d satisfies the conditions below, for each $p, q, s \in {}^nF$.

$d(p,q) \geq 0$;
$d(p,q) = d(q,p)$;
$d(p,q) + d(q,s) \leq d(p,s)$ whenever $p_t - q_t$ and $q_t - s_t$ have the same sign (reversed triangle inequality).

According to our Convention 2.2, $\mathsf{FM}_{\mathrm{acc}}(2, \mathfrak{R})$ denotes that subclass of $\mathsf{FM}_{\mathrm{acc}}$ where $n = 2$ and $\mathfrak{F} = \mathfrak{R}$. In this section, unless stated otherwise, we tacitly assume that $\mathfrak{M} \in \mathsf{FM}_{\mathrm{acc}}$ whenever $\mathfrak{M}$ is of the form

$$\langle (B, Obs, Ph, Ib), \mathfrak{F}, \ \mathrm{E}\ , W, G_m, d_m \rangle_{m \in Obs} .$$

Now we start to postulate our set Acc of axioms for accelerating observers. Though we will be able to state meaningful theorems for particular dimensions n only, we are trying to formulate the axioms for arbitrary $n \in \omega$.

As we already indicated, our first group Acc_1 of axioms consists of an appropriate part of *Newbasax* "restricted" to inertial observers (IOb). Concretely,

$$Acc_1 \stackrel{\text{def}}{=} \{\mathbf{Ax1}_g, \mathbf{Ax2}_g, \mathbf{Ax5}_g, \mathbf{AxE}_{0g}\} ,$$

where,

$\mathbf{Ax1}_g$ $m \in IOb \Rightarrow (G_m = \mathsf{Eucl}(\mathbf{n}, \mathbf{F})$ and d_m is the Minkowski metric[76]).

$\mathbf{Ax2}_g$ $Ph \subseteq Ib$.
(Notice that $IOb \subseteq Ib$ automatically holds by the definition of IOb.)

$\mathbf{Ax5}_g$ $m \in IOb \Rightarrow (\forall \ell \in G_m)(ang^2(\ell) < 1 \Rightarrow (\exists k \in Obs)\ell = tr_m(k)$ and

$$ang^2(\ell) = 1 \Rightarrow (\exists ph \in Ph)\ell = tr_m(ph)).$$

$\mathbf{AxE}_{0g}$ $[(m \in IOb \wedge tr_m(ph) \neq \emptyset) \Rightarrow (tr_m(ph) \in G_m \ \wedge \ v_m(ph) = 1)] \ \wedge \ (\exists k \in Obs) tr_k(ph) \neq \emptyset)$.

Our second group Acc_2 of axioms for accelerating observers is defined as follows.

$$Acc_2 \stackrel{\text{def}}{=} \{\mathbf{Ax3}_g, \mathbf{Ax4}_g, \mathbf{Ax6_{00}}, \mathbf{Ax6_{01}}\} ,$$

where, for each $k, m \in Obs$,

$\mathbf{Ax3}_g$ $(\forall h \in Ib)(tr_m(h) \in G_m \cup \{\emptyset\} \ \wedge \ (\exists k \in Obs) tr_k(h) \neq \emptyset)$.

$\mathbf{Ax4}_g$ $tr_m(m) = F \times {}^{n-1}\{0\}$.

$\mathbf{Ax6_{00}}$ $w_m[tr_m(k)] \subseteq Rng(w_k)$.

[76] $d_m(p, q) = \mu(p, q)$ for every $p, q \in {}^nF$ (where μ will be defined later in this section, see Definition 4.5).

Ax6$_{01}$ $Dom(\mathsf{f}_{mk}) \in Open(\mathbf{n}, \mathfrak{F})$,
where $Open(\mathbf{n}, \mathfrak{F})$ denotes the set of all open[77] subsets of ${}^nF$.

PROPOSITION 4.1 *Suppose* $\mathfrak{M} \in \mathsf{FM}_{\mathrm{acc}}, \mathfrak{M} \models Acc_2$ *and* $m \in Obs^{\mathfrak{M}}, k \in IOb^{\mathfrak{M}}$. *Then the world view transformation* f_{mk} *is a function. Note, that* f_{km} *is not necessarily a function.*

Proof. Let $p \in {}^nF^{\mathfrak{M}}$ be arbitrary, and assume $q_0, q_1 \in {}^nF^{\mathfrak{M}}$ are such that $q_0, q_1 \in f_{mk}(p)$ hold. If $q_0 \neq q_1$ would hold, then by **Ax5**$_g$ there would be an inertial observer or a photon, whose trace incident with q_0 but not incident with q_1 in the world view of k. In that case $w_m(p) = w_k(q_0) \neq w_k(q_1) = w_m(p)$ would hold, which is a contradiction. Therefore $q_0 = q_1$. ∎

In our third group of axioms, we will assume that all the world view transformations between a non inertial and an inertial observer are differentiable functions. In order to save space, we omit to write down everywhere the first order formula requiring differentiability. Instead of this, we introduce the following convention. Throughout, in any axiom, whenever we refer to a derivative of a function, we implicitly assume that the axiom postulates the derivability of the function in question as well.

Our third group Acc_3 of axioms for accelerating observers is defined as follows.

$$Acc_3 \stackrel{\text{def}}{=} \{\mathbf{Ax_g i} : 1 \leq \mathbf{i} \leq 5\},$$

where $\mathbf{Ax_g1}$–$\mathbf{Ax_g5}$ are defined below. Since the axioms $\mathbf{Ax_g1}$–$\mathbf{Ax_g5}$ are based on new ideas, below we will explain the intended meanings of some of these axioms.

The most important idea what we wish to formalize is that "general relativity locally behaves in the same way as special relativity does". Particularly, suppose m is an accelerating observer and k is an inertial one such that the locations and the velocities of m and k are the same at a certain moment. Then we want to require that the coordinatizations of m and k are the same (at least in a little neighborhood around their common location). One possibility to express this is taking the derivative of the life-line (trace) of m, this is a straight line (from the point of view

[77]Recall from section 3 that $Q \subseteq {}^nF$ is an *open set* iff $(\forall q \in Q)(\exists \varepsilon \in F^+)\big(S(q,\varepsilon) \subseteq Q\big)$, where the *$\varepsilon$-neighborhood $S(p,\varepsilon)$ of p* was defined as

$$S(p,\varepsilon) = \{q \in {}^nF : (q_0 - p_0)^2 + (q_1 - p_1)^2 + \ldots + (q_{n-1} - p_{n-1})^2 < \varepsilon\}.$$

of a third inertial observer), therefore there is an inertial observer k on that line and requiring that the world-view transformation of k coincides with that of m. Carefully analyzing the above heuristic argument one finds that this means the following: for each point $p \in {}^nF$ there are other inertial observers k and k_1 such that $f'_{mk_1}(p) = f_{kk_1}$. Here $f'_{mk_1}(p)$ is the derivative of the function f_{mk_1} at point p, which is a linear transformation. We found this axiom too restrictive (see Theorem 4.2 below), therefore we will require only that the simultaneities of k and m would be the same (and the same for the space-like parts of their world views as well) in a sufficiently small neighborhood around their common location. This can be formulated in the following way:

$\mathbf{Ax_g1}$ $(\forall m \in Obs)(\forall q \in {}^nF)(\forall \varepsilon \in F^+)$
$(\exists p \in S(p, \varepsilon))(\exists m_1 \in IOb)(\exists k \in IOb)(\exists a_0, \ldots, a_{n-1} \in F)$
$(\mathsf{f}_{km_1}(\bar{0}) = \mathsf{f}_{mm_1}(p) \ \wedge \ \bigwedge\{\partial_i \mathsf{f}_{mm_1}(p) = a_i \cdot \mathsf{f}_{km_1}(1_i) : 0 \le i < n\})$.

This axiom expresses that, as seen by an inertial observer $m_1 \in IOb$, every (possibly accelerating) observer $m \in Obs$ "behaves" in every (small) neighborhood of each point $q \in {}^nF$ of space-time the same way as *some* inertial observer $k \in IOb$ with its $\bar{0}$-point at p. ($\partial_i \mathsf{f}_{mm_1}(p)$ is the i-th partial derivative of f_{mm_1} at point p .)

THEOREM 4.2 *If $n = 2$ then the modified version of $\mathbf{Ax_g1}$ where all the constants a_i are just 1 is inconsistent. (More precisely, this version of $\mathbf{Ax_g1}$ excludes uniformly accelerating observers over the field of reals.) In particular, it cannot be the case that both $\mathbf{Ax_g1}^+$ below and $Dom(f_{mm_1}) = {}^2F$ hold.*

$\mathbf{Ax_g1}^+$ $\quad \partial_t \mathsf{f}_{mm_1}(p) = \mathsf{f}_{km_1}(1_t)$ *and* $\partial_x \mathsf{f}_{mm_1}(p) = \mathsf{f}_{km_1}(1_x)$.

(Now, $\mathbf{Ax_g1}^+$ is the same as requiring $f'_{mm_1}(p) = f_{km_1}$ where f' is the derivative of f.)

Proof. The proof can be found in [38]. ∎

Remark 4.3 *Without the condition $Dom(f_{mm_1}) = {}^2F$ the above theorem is* <u>not</u> *true. In fact, there is a model $\mathfrak{M}_K$* [78] *satisfying the remaining conditions of the*

[78] K stands for Kruskal.

above theorem. In this model f_{mk} is uniquely determined by the life-line $tr_m(k)$ of k (which is not the case if we use $\mathbf{Ax_g1}$ *instead of the axiom in Theorem 4.2). This model is based on the coordinate transformation described e.g. in [35] on page 156. Actually,*

$$\mathfrak{M}_K \models Acc_0 + Acc_1 + \mathbf{Ax_g1} + \mathbf{Ax_g2} + \mathbf{Ax_g4} + \mathbf{Ax_g5},$$

where $\mathbf{Ax_g2}$ *etc will be defined soon. However, this model doesn't satisfy* $\mathbf{Ax_g3}$ *(see below)*

The strong version $\mathbf{Ax_g1}^+$ *of* $\mathbf{Ax_g1}$ *(together with Acc_0 , Acc_1) implies that the speed of light is the same for all observers i.e. even if m is an accelerating observer we have $(\forall ph \in Ph)(v_m(ph) = 1)$. On the other hand, our axiom system Acc (see below) does not imply anything like this for accelerating observers. In fact, in the subsection "Constructing models for accelerating observers" there is a model $\mathfrak{M}$ of the quoted axioms such that $v_m(ph) \neq 1$ for some $m \in Obs^{\mathfrak{M}}$ and $ph \in Ph^{\mathfrak{M}}$. Moreover, the velocity of ph as seen by m is not constant (ph accelerates) in that $\mathfrak{M}$. We conjecture that this behavior of photons might be strongly connected to the present version of* $\mathbf{Ax_g3}$.

Conjecture 4.4 *Theorem 4.2 is provable from Theorem 4.15 below.*

Our following axiom, $\mathbf{Ax_g2}$, is concerned with properties of the distance d_m of (possibly accelerating) observers $m \in Obs$. We will express d_m via the so called *Minkowski distance* μ of some inertial observer $m_1 \in IOb$. So let us define the Minkowski distance first.

Definition 4.5 Assume $\mathfrak{M} \models Newbasax \wedge \mathbf{Ax\triangle 2}$ (see section 3.7) and assume that *square roots exist in* $\mathfrak{F}$. Then the Minkowski distance $\mu : {}^nF \times {}^nF \longrightarrow F^+$ of two points $p, q \in {}^nF$ is defined as follows. Consider the world view of some inertial observer $m_1 \in IOb$.

- If the (Euclidean) straight line $\overline{pq}$ connecting p and q is parallel with the time coordinate axis or $p_t = q_t$ then $\mu(p,q) \stackrel{\text{def}}{=}$ (the Euclidean distance of p and q) $= \sqrt{\sum_{i=0}^{n-1} (p_i - q_i)^2}$.
- If both p and q are on the light cone, that is, if $ang^2(\overline{pq}) = 1$, then $\mu(p,q) \stackrel{\text{def}}{=} 0$.
- If $ang^2(\overline{pq}) < 1$ then, by $\mathbf{Ax5}$, there is an observer $k \in Obs$ such that $\overline{pq} = tr_{m_1}(k)$. Then $\mu(p,q) \stackrel{\text{def}}{=} \mu(f_{m_1k}(p), f_{m_1k}(q))$.

- If $ang^2(\overline{pq}) > 1$ then, by **Ax5** again, there is an observer $k \in Obs$ such that the reflection of $\overline{pq}$ to the light cone is $tr_{m_1}(k)$, and $\mu(p,q) \stackrel{\text{def}}{=} \mu(f_{m_1k}(p), f_{m_1k}(q))$ for this k.

Exercise 4.6 Prove that μ is a Lorentz – metric (that is, μ satisfies $\mathbf{Ax}_{\mathrm{met}}$).

Exercise 4.7 Working over the field of reals, prove that for any two points $p, q \in {}^n\Re$, the distance between p and q according to the Lorentz – metric is

$$\sqrt{(p_0 - q_0)^2 - \sum_{i=1}^{n-1} (p_i - q_i)^2}$$

when $ang^2(\overline{pq}) < 1$. What changes when $ang^2(\overline{pq}) \geq 1$?

The <u>*Minkowski length*</u> $\mu(\overline{pCq})$ of some curve C connecting p and q is defined to be the length of C from p to q according to the Minkowski distance. Suppose we are working over the field of reals. Then one can express the length of C in terms of an integral.

Now we intend to postulate the following.

$\mathbf{Ax_g2'}$ $(\forall m \in Obs)(\forall m_1 \in IOb)(\forall p, q \in {}^nF)$
$((p_0 = q_0 \text{ or } (\forall 0 < i < n) p_i = q_i) \wedge ([p,q] \subseteq Rng(f_{m_1m}))) \Rightarrow d_m(p,q) = \mu(\mathsf{f}_{mm_1}[\overline{pq}]))$.

There are two problems to be solved here.

Item 4.8 **$\mathbf{Ax_g2}$**' should function as the definition of d_m, therefore we need that

$$(\forall m \in Obs)(\forall m_1, m_2 \in IOb)\mu(\mathsf{f}_{mm_1}[\overline{pq}]) = \mu(\mathsf{f}_{mm_2}[\overline{pq}]) \,.$$

Item 4.9 **$\mathbf{Ax_g2}$**' should be translated to our first order frame language (one way or another).

The problem in item 4.9 is that we haven't yet described integration in first order logic. First suppose that we are working over the field of reals and let us try to postulate "a first order approximation of the notion of integral". This can be done by the following two steps. First we postulate that the integral is additive, that is, its value does not changes when we cut the curve in question into two (or equivalently, finitely many) parts and compute the sum of the integrals taken in the new (small) curves. Second, we postulate, that in a sufficiently small curve, the value of the integral and the value of μ almost coincide (this can be done by a standard, first order limit process). After this, one can derive $\mathbf{Ax_g2'}$ over the field of reals as follows. Choose an arbitrary (small) positive number, say ε. Using the first condition ($\mathbf{Ax_g2_0}$ below) the curve can be cut into sufficiently small parts, in which $\mathbf{Ax_g2_1}$ below guarantees that the value of the integral differs from the value of μ with at most ε (relative to the length of the small parts). Thus, for any positive ε the difference between the value of the integral and the value of μ is at most εc (c is an appropriate constant depending on the curve in question). So the difference is smaller than any positive number, thus it is equal to 0. Now we formalize $\mathbf{Ax_g2_0}$ and $\mathbf{Ax_g2_1}$.

Below we use the following notation: if $p, q \in {}^nF$ then $[p, q]$ is defined to be $[p, q] = \{x \in \overline{pq} :$ x is between p and $q\}$

$\mathbf{Ax_g2_0}$ $(\forall m \in Obs)(\forall p, q \in {}^2F)(p_0 = q_0$ or $(\forall 0 < i < n)(p_i = q_i)) \Rightarrow (\forall r \in [p, q])(d_m(p, r) + d_m(r, q) = d_m(p, q))$,

$\mathbf{Ax_g2_1}$ $(\forall m \in Obs)(\forall m_1 \in IOb)(\forall p, q \in {}^nF)(\forall \varepsilon \in F^+)(\exists \delta \in F^+)$
$(\forall a, c \in [p, q])(|a - c| \leq \delta \ \wedge \ [a, c] \subseteq Rng(f_{m_1m}) \Rightarrow$
$(\forall c \in S(a, \delta)(|(d_m(a, c) - \mu(f_{mm_1}(a), f_{mm_1}(c)))/(a - c)| \leq \varepsilon)$.

Let us notice that these formulas are first order formulas in our frame language. However, considering these formulas over some non-archimedean field, one can find that the behaviour of $\mathbf{Ax_g2_0}$ and $\mathbf{Ax_g2_1}$ can deviate from their behaviour in the "standard" case of $\mathbf{F} = \mathrm{R}$. We will illustrate this by constructing a non-archimedean field and a frame model over it in which the function d can be defined in two different ways such that both definitions satisfy $\mathbf{Ax_g2_0}$ and $\mathbf{Ax_g2_1}$ (see Example 4.11 below).

In order to prove $\mathbf{Ax_g2'}$ over the field of reals, we formulated $\mathbf{Ax_g2_1}$ such that δ (the size of the neighbourhood of the point in question) does not depend on "a" (the point in question). Thus, $\mathbf{Ax_g2_1}$ requires a kind of uniform approximability of the integral. Therefore we found $\mathbf{Ax_g2_1}$ too restrictive.

The reader who doesn't want to "worry" too much at this point on finding another formalization of $\mathbf{Ax_g2}$ (improved) may skip the following items $\mathbf{Ax_g2(2)}$ and $\mathbf{Ax_g2}$ at the first reading of this section. Such a reader is advised to consider axiom $\mathbf{Ax_g2}$ to be $\{\mathbf{Ax_g2_0}, \mathbf{Ax_g2_1}\}$.

Another possibility for expressing $\mathbf{Ax_g2'}$ (over the field of reals) is to "switch" the equation containing an integral to its derivative form: the latter can be expressed in first order logic. This "switch" will not give us a form completely equivalent to $\mathbf{Ax_g2}$', but $\mathbf{Ax_g2}$' will follow from it, see our next theorem to come.

First suppose $n = 2$. Let $p, q \in {}^2F$ and $m \in Obs$ be arbitrary but fixed. Consider the *unary* function $d_m(p, p+a\cdot q) : F \longrightarrow F$. (In λ-calculus notation, this is $\lambda a.d_m(p, p+a\cdot q)$.) We denote the derivative of $d_m(p, p+a\cdot q)$ by $\partial_a d_m(p, p+a\cdot q)$. Thus

$$\partial_a d_m(p, p+a\cdot q) = b \Longleftrightarrow$$
$$(\forall \varepsilon > 0)(\exists \delta > 0)(\forall a_1 \in S(a,\delta))\Big(\Big|\frac{d_m(p, p+a_1\cdot q) - d_m(p, p+a\cdot q)}{a_1 - a} - b\Big| < \varepsilon\Big).$$

Now we let

$\mathbf{Ax_g2(2)}$ $(\forall m \in Obs)(\forall p \neq q \in {}^nF)(\exists m_1 \in IOb)(\exists \varepsilon \in F^+)(\forall a \in F, -\varepsilon \le a \le \varepsilon)$

$$\Big[([p,q] \subseteq Rng(f_{m_1 m})) \wedge$$
$$\Big((ang^2(\overline{pq}) \le 1 \Longrightarrow$$
$$[(\partial_a d_m(p, p+a\cdot(q-p)))^2 =$$
$$(\partial_a \mathsf{f}_{mm_1}(p+a\cdot(q-p))_t)^2 - ((\partial_a \mathsf{f}_{mm_1}(p+a\cdot(q-p)))_x)^2])$$
$$\wedge$$
$$(ang^2(\overline{pq}) > 1 \Longrightarrow$$
$$[(\partial_a d_m(p, p+a\cdot(q-p)))^2 =$$
$$((\partial_a \mathsf{f}_{mm_1}(p+a\cdot(q-p)))_x)^2 - (\partial_a \mathsf{f}_{mm_1}(p+a\cdot(q-p))_t)^2])\Big)\Big].$$

The generalization of $\mathbf{Ax_g2(2)}$ for arbitrary n might be:

$\mathbf{Ax_g2}$ $(\forall m \in Obs)(\forall p \neq q \in {}^nF)(\exists m_1 \in IOb)(\exists \varepsilon \in F^+)(\forall a \in F, -\varepsilon \le a \le \varepsilon)$

$$\Big[([p,q] \subseteq Rng(f_{m_1 m})) \wedge$$
$$\Big((ang^2(\overline{pq}) \le 1 \Longrightarrow$$
$$[(\partial_a d_m(p, p+a\cdot(q-p)))^2 =$$
$$(\partial_a \mathsf{f}_{mm_1}(p+a\cdot(q-p))_t)^2 - \Sigma_{i=1}^{n-1}((\partial_a \mathsf{f}_{mm_1}(p+a\cdot(q-p)))_i)^2])$$
$$\wedge$$
$$(ang^2(\overline{pq}) > 1 \Longrightarrow$$
$$[(\partial_a d_m(p, p+a\cdot(q-p)))^2 =$$
$$\Sigma_{i=1}^{n-1}((\partial_a \mathsf{f}_{mm_1}(p+a\cdot(q-p)))_i)^2 - (\partial_a \mathsf{f}_{mm_1}(p+a\cdot(q-p))_t)^2])\Big)\Big].$$

THEOREM 4.10 $\mathsf{FM}_{\mathrm{acc}}(2, \mathfrak{R}) \models \mathbf{Ax_g2(2)} \Rightarrow \mathbf{Ax_g2'(2)}$.

Proof. The proof can be found in [38]. ∎

The further axioms are the following.

$\mathbf{Ax_g3}$ $(\forall 0 \leq i < n)(\forall p \in \bar{i})(d_m(\bar{0}, p) = p_i)$, where $\bar{i}$ denotes the i-th coordinate axis.[79]

$\mathbf{Ax_g4}$ $(\forall m \in Obs)(\forall a \in {}^{n-1}F)(\exists k \in Obs)$

$$\begin{aligned}&(tr_m(k) = F \times \{a\} \;\wedge\; \mathsf{f}_{mk} \text{ is a bijection } \wedge\\ &(\forall p, q \in {}^nF)(p_t = q_t \to \mathsf{f}_{mk}(p)_t = \mathsf{f}_{mk}(q)_t) \;\wedge\\ &(\textstyle\bigwedge_{0<i<n} p_i = q_i \to \bigwedge_{0<i<n} \mathsf{f}_{mk}(p)_i = \mathsf{f}_{mk}(q)_i)).\end{aligned}$$

$\mathbf{Ax_g5}$ $(\forall m \in Obs)(\forall p \in {}^nF)(\forall \varepsilon \in F^+)(\exists q \in S(p, \varepsilon))(\exists k \in IOb)(k \in w_m(q))$.

Now

$$Acc \stackrel{\text{def}}{=} Acc_1 \cup Acc_2 \cup Acc_3\,,$$

$$Acc' = Acc - \{\mathbf{Ax_g2}\} \cup \{\mathbf{Ax_g2_0}, \mathbf{Ax_g2_1}\}.$$

Of course, Acc is consistent, because no axiom requires the existence of a non inertial observer, so the model over $\mathfrak{R}$ described in Theorem 3.139 can be extended to a model for Acc. Our next main goal is to build models containing non inertial (accelerating) observers. We will do this in two steps. First of all, we will restrict ourself to dimension $n = 2$ and we will always work over the field of reals. First we will study some functions of form ${}^2\mathfrak{R} \to {}^2\mathfrak{R}$ which describe "potential" world view transformations between an inertial and an accelerating observer. These investigations can be seen simply as "playing" with differential geometry (similarly to section 2.4, where we played with linear algebra). After this, we show the relative consistency of $Acc \cup \{(\exists m)(m \in Obs - IOb)\}$ over *Newbasax*. That is, we will choose a model of *Newbasax* and will "put into" that model an accelerating observer. When we construct the world view transformations of the new observer, we will use our results obtained by playing with differential geometry. After this, we will study fancy accelератings, for example such accelerations that the speed of an accelerating observer m converges to the speed of light in such a way, that an inertial observer m_1 never

[79]We are not sure that we want to keep this axiom because it fails in one of the important models of Acc, namely in the model discussed in 4.3. On the other hand, some weaker version of $\mathbf{Ax_g3}$ can be retained. For example we could require that the numbers of the time-like coordinate lines follow in the same order as $\mathbf{Ax_g3}$ would require, i.e., for example

$$(\forall p, q \in \bar{t})(p_t \leq q_t \Leftrightarrow d_m(\bar{0}, p) \leq d_m(\bar{0}, q)).$$

sees the event when m's clock reaches say, the time instant 1. Thus, the events seen by m and the events seen by m_1 must be different (that is, $\mathbf{Ax_6}$ fails).

Discussing the axiom system *Acc*.

First we give an example showing that over some non-archimedean field, the axioms $\mathbf{Ax_g2_0}, \mathbf{Ax_g2_1}$ do not define the metric. Namely, we will construct a non-archimedean field and a frame model over it in which exists an observer, whose function d can be defined in two different ways such that both definitions satisfy $\mathbf{Ax_g2_0}$ and $\mathbf{Ax_g2_1}$.

Example 4.11 Let $\mathfrak{M}$ be a frame model over the field of reals such that

$$\mathfrak{M} \models Newbasax \ \wedge \ \mathbf{Ax_g2_0} \ \wedge \ \mathbf{Ax_g2_1}.$$

(It is not hard to see at this point that such a model exists, because no axiom forces the existence of an accelerating observer.) Let $\mathfrak{M}_1$ be an ultrapower of $\mathfrak{M}$ with respect to a non-principal ultrafilter. First we concentrate on the field reduct of $\mathfrak{M}_1$. We will identify the real numbers with their images according to the natural (or diagonal) embedding into the corresponding ultrapower of $\mathfrak{R}$.

We define an equivalence relation ϱ on $\mathfrak{F}^{\mathfrak{M}_1}$ as follows: for any two elements $a, b \in F^{\mathfrak{M}_1}$ $\ a\varrho b$ iff there exists a real number r such that $a - r \leq b \leq a + r$. This is an equivalence relation and we will denote the equivalence class of a by $[a]$. In fact, ϱ is a congruence relation of the abelian group $\langle F^{\mathfrak{M}_1}; +\rangle$, that is, for any elements $a, a', b, b' \in F^{\mathfrak{M}_1}$ we have $(a\varrho a' \ \wedge \ b\varrho b') \ \Rightarrow (a+b)\varrho(a'+b')$. Note that $[0]$ is a group itself (that is, the set $[0]$ is closed under addition $+^{\mathfrak{M}_1}$). To keep notation simpler, we will denote this group by $[0]$ as well. The abelian group $\langle F^{\mathfrak{M}_1}; +\rangle$ is decomposable into the direct product of $[0]$ and $\langle F^{\mathfrak{M}_1}; +\rangle/\varrho$ (this is possibly familiar to the reader, but for completeness we include here a proof).

Claim 4.12 *The abelian group $\langle F^{\mathfrak{M}_1}; +\rangle$ is isomorphic to the direct product $[0] \times \langle F^{\mathfrak{M}_1}; +\rangle/\varrho$. We will denote by φ the isomorphism*

$$\langle F^{\mathfrak{M}_1}; +\rangle \cong [0] \times \langle F^{\mathfrak{M}_1}; +\rangle/\varrho$$

constructed below.

Proof. *We say, that a subgroup H of $\langle F^{\mathfrak{M}_1}; +\rangle$ is a (ϱ, Q) subgroup, iff the following two conditions hold for H:*

- *for all rational numbers q and for all $h \in H$ $qh \in H$,*
- *for all $h_1, h_2 \in H$ we have $h_1 \varrho h_2 \Rightarrow h_1 = h_2$.*

Clearly, there exists a (ϱ, Q) subgroup, for example the subgroup $\{0\}$ containing only one element is such. The set of all (ϱ, Q) subgroups of $\langle F^{\mathfrak{M}_1}; +\rangle$ is closed under directed unions, so applying Zorn's lemma we conclude that there exists a maximal (ϱ, Q) subgroup G.

Now we show that for all $a \in F^{\mathfrak{M}_1}$ there exists $g \in G$ such that $a\varrho g$. Suppose for contradiction that there exists an element $a \in F^{\mathfrak{M}_1}$ whose ϱ equivalence class $[a]$ is disjoint from G. Let $G_0 = \{g + qa : q$ is rational and $g \in G\}$. It is easy to see that G_0 is a subgroup of $\langle F^{\mathfrak{M}_1}; +\rangle$, strictly bigger than G and satisfies the first condition of being a (ϱ, Q) subgroup. We will show that G_0 is a (ϱ, Q) subgroup. To do this, suppose q_1 and q_2 are rational numbers, $g_1, g_2 \in G$ such that $(g_1 + q_1 a)\varrho(g_2 + q_2 a)$. We have to show $(g_1 + q_1 a) = (g_2 + q_2 a)$.

Suppose first $q_1 \neq q_2$. Observe that ϱ is compatible with multiplying by a rational number, that is, if q is a rational number and $u\varrho v$ then $qu \ \varrho \ qv$. Therefore from $(g_1 + q_1 a)\varrho(g_2 + q_2 a)$ we conclude $(g_1 - g_2)/(q_2 - q_1)\varrho a$. The left hand side belongs to G (because G is a (ϱ, Q) subgroup) contradicting to the choice of a. Therefore we have $q_1 = q_2$.

If $q_1 = q_2$ then $g_1 \varrho g_2$ since ϱ a congruence relation. But G is a (ϱ, Q) subgroup so $g_1 = g_2$ and therefore $g_1 + q_1 a = g_2 + q_1 a = g_2 + q_2 a$.

Therefore G_0 is a (ϱ, Q) subgroup, so G cannot be a maximal (ϱ, Q) subgroup, which is a contradiction. Therefore G contains an element ϱ-equivalent to a.

Summing up, G is such a subgroup that for all $a \in F^{\mathfrak{M}_1}$ we have $|G \cap [a]| = 1$. We will denote the element in $G \cap [a]$ by $[a]_G$. Finally we define the function $\varphi : F^{\mathfrak{M}_1} \to [0] \times F/\varrho$ as follows:

$$(\forall a \in F^{\mathfrak{M}_1}) \ \varphi(a) = \langle a - [a]_G, [a]\rangle.$$

Note that for any two elements $a, b \in F^{\mathfrak{M}_1}$ we have $[a + b]_G = [a]_G + [b]_G$ because $a \ \varrho \ [a]_G$ and $b \ \varrho \ [b]_G$ so both $[a + b]_G$ and $[a]_G + [b]_G$ are the unique element of $G \cap [a + b]$.

Using this fact, it is easy to see that φ is an isomorphism as required. ∎

Using the notation of the above Claim we define a function $\psi : F^{\mathfrak{M}_1} \to F^{\mathfrak{M}_1}$ as follows:

$$(\forall a \in F^{\mathfrak{M}_1})\ \psi(a) = \varphi^{-1}(\langle a - [a]_G, [2a] \rangle).$$

Now we show that ψ preserves addition of $F^{\mathfrak{M}_1}$. Indeed, for all $a, b \in F^{\mathfrak{M}_1}$ we have $\psi(a+b) = \varphi^{-1}(\langle (a+b) - [a+b]_G, [2(a+b)] \rangle) = \varphi^{-1}(\langle a - [a]_G, [2a] \rangle + \langle b - [b]_G, [2b] \rangle) = \varphi^{-1}(\langle a - [a]_G, [2a] \rangle) + \varphi^{-1}(\langle b - [b]_G, [2b] \rangle) = \psi(a) + \psi(b)$.

Moreover ψ leaves the set $[0]$ fixed pointwise, that is, $(\forall a \in [0])(\psi(a) = a)$.

Now we are ready to turn to finish our frame model construction. Let m be an inertial observer of $\mathfrak{M}_1$ and let m' be a new symbol. We define a new frame model $\mathfrak{M}_1'$ as follows. The universe of $\mathfrak{M}_1'$ is the same as that of $\mathfrak{M}_1$ expanded with a new observer m', and $B^{\mathfrak{M}_1'} = B^{\mathfrak{M}_1} \cup \{m'\}$, $Obs^{\mathfrak{M}_1'} = Obs^{\mathfrak{M}_1} \cup \{m'\}, Ph^{\mathfrak{M}_1'} = Ph^{\mathfrak{M}_1}, Ib^{\mathfrak{M}_1'} = Ib^{\mathfrak{M}_1}, \mathfrak{F}^{\mathfrak{M}_1'} = \mathfrak{F}^{\mathfrak{M}_1}$, $\mathrm{E}^{\mathfrak{M}_1'} = \mathrm{E}^{\mathfrak{M}_1}$ and for all $k \in Obs^{\mathfrak{M}_1}$ we define $G_k^{\mathfrak{M}_1'} = G_k^{\mathfrak{M}_1}, d_k^{\mathfrak{M}_1'} = d_k^{\mathfrak{M}_1}, G_{m'}^{\mathfrak{M}_1'} = G_m^{\mathfrak{M}_1}, d_{m'}^{\mathfrak{M}_1'} = d_m^{\mathfrak{M}_1} \circ \psi$. Finally $W^{\mathfrak{M}_1'}$ is defined as follows:

$$(\forall k \in Obs^{\mathfrak{M}_1})(\forall x \in {}^n\mathfrak{F})\quad w_k^{\mathfrak{M}_1'}(x) = \begin{cases} w_k^{\mathfrak{M}_1}(x) & \text{if } m \notin w_k^{\mathfrak{M}_1}(x), \\ w_k^{\mathfrak{M}_1}(x) \cup \{m'\} & \text{otherwise.} \end{cases}$$

and finally $(\forall x \in {}^n\mathfrak{F})(w_{m'}^{\mathfrak{M}_1'}(x) = w_m^{\mathfrak{M}_1'}(x))$.

By this we have defined the frame model $\mathfrak{M}_1'$ which is, intuitively, nothing more than the result of "putting m' into $\mathfrak{M}_1$". Moreover, m and m' are basically the same, except of their metrics. We now verify that $\mathfrak{M}_1' \models \mathbf{Ax_g2_0} \wedge \mathbf{Ax_g2_1}$. To do this, let us choose an observer $m \in Obs^{\mathfrak{M}_1'}$. By the construction, if $m \in Obs^{\mathfrak{M}_1}$ then the axioms $\mathbf{Ax_g2_0} \wedge \mathbf{Ax_g2_1}$ indeed hold, because here $\mathfrak{M}_1'$ and $\mathfrak{M}_1$ coincide, and $\mathfrak{M}_1$ is an ultrapower of $\mathfrak{M}$ which was chosen such that $\mathfrak{M} \models \mathbf{Ax_g2_0} \wedge \mathbf{Ax_g2_1}$. So we may suppose $m = m'$.

$\mathbf{Ax_g2_0}$ *holds because ψ preserves addition. More concretely, for any points p_1, p_2, p_3 satisfying the conditions of $\mathbf{Ax_g2_0}$ we have $d_{m'}(p_1, p_3) = \psi(d_m(p_1, p_3)) = \psi(d_m(p_1, p_2) + d_m(p_2, p_3)) = \psi(d_m(p_1, p_2)) + \psi(d_m(p_2, p_3)) = d_{m'}(p_1, p_2) + d_{m'}(p_2, p_3)$.*

To show $\mathbf{Ax_g2_1}$ *let us choose* $m_1 \in IOb^{\mathfrak{M}'_1}, p, q \in {}^nF$ *and* $\varepsilon \in F^+$. *In* $\mathfrak{M}_1$ *axiom* $\mathbf{Ax_g2_1}$ *gives a* δ *for that choice. Let* $\delta' = min\{\delta, 1\}$. *We claim that for all* $a, c \in {}^nF$, *if* $|a - c| \leq \delta'$ *then* $d_m(a, c) = d_{m'}(a, c)$. *This is true, because*

$$\mathfrak{M} \models (\forall m \in IOb)(\forall p, q \in {}^nF)(d_m(p, q) \leq |p - q|),$$

and therefore $\mathfrak{M}_1 \models (\forall m \in IOb)(\forall p, q \in {}^nF)(d_m(p, q) \leq |p - q|)$. *Hence,* $d_m(a, c) \leq \delta' \leq 1$, *so* $d_{m'}(a, c) = \psi(d_m(a, c))) = d_m(a, c)$ *since* φ *leaves* $[0]$ *fixed pointwise.*

Finally observe that in $\mathbf{Ax_g2_1}$ *the metric* $d_{m'}$ *is used in such a way that (using the notation of* $\mathbf{Ax_g2_1}$*)* $|a - c| \leq \delta'$. *So* $d_{m'}(a, c) = d_m(a, c)$ *and* $\mathfrak{M}_1 \models \mathbf{Ax_g2_1}$ *therefore* $\mathfrak{M}'_1 \models \mathbf{Ax_g2_1}$ *as desired.*

This completes the example. ∎

Now we turn to the proof of Theorem 4.15 announced at the beginning of this section. From now on we restrict ourself considering only two dimensional models having field reduct isomorphic to the field of reals. We start with introducing some notation.
By definition, for any observer m the time like coordinate-lines $\ell_t := \{(t, x) : x \in \Re\}$ of m connects simultaneous events[80] (as seen by m). Let k be another observer. Applying the world view transformation f_{mk} to the sets ℓ_t one gets curves in the coordinate system of k connecting the events simultaneous wrt m. The same applies to space like coordinates.

Definition 4.13 Let m and k be observers. The f_{mk}-lines, or f_{mk}-coordinates are the sets $f_{mk}[\ell_t]$, $f_{mk}[\ell_x]$, where $\ell_t := \{(t, x) : x \in \Re\}$ and $\ell_x := \{(t, x) : t \in \Re\}$.

Using the notation of the previous definition, the coordinate line ℓ_x (for some x) of observer m can be seen as a trace of an observer m' who is at rest as seen by m. The existence of m' is guarantied by axiom $\mathbf{Ax_g4}$. We define the velocity (or speed) of the f_{mk}-line $f_{mk}[\ell_x]$ to be the velocity of m' as seen by k. It is easy to show that this definition does not depend on the particular choice of m'.

Recall that if (t_1, x_1) and (t_2, x_2) are two points in space-time, then the set of points

[80]two events e_0, e_1 are simultaneous wrt m iff e_0, e_1 have the same time-coordinate in m's coordinate system.

between them is denoted by $[(t_1, x_1), (t_2, x_2)]$, that is,

$$[(t_1, x_1), (t_2, x_2)] = \{s(t_1, x_1) + (1 - s)(t_2, x_2) : \ 0 \leq s \leq 1\}.$$

Recall moreover, that if $f : {}^2F \to {}^2F$ is a function, then the coordinate functions of f are denoted by f_t, f_x respectively, that is, for every $(t, x) \in Dom(f)$ we have

$$f(t, x) = (f_t(t, x), f_x(t, x)).$$

Now we are ready to formulate the first version of our theorem.

THEOREM 4.14 *Let $\mathfrak{M} \in \mathsf{FM}_{\mathrm{acc}}$ be a frame model satisfying the symmetry axiom* **Ax△2** *(see in section 3.7). Suppose $\mathfrak{M} \models Acc$ and the field reduct of $\mathfrak{M}$ is isomorphic to the field of reals. Let k be an inertial observer in $\mathfrak{M}$ and let m be another (arbitrary, i.e., possibly non-inertial) observer in $\mathfrak{M}$. Suppose $f : {}^2\Re \to {}^2\Re$ is the world view transformation from m to k, thus f is a partial function. Assume moreover:*

(a) *the domain of f is open and convex,*
(b) *f is twice continuously differentiable,*
(c) *the velocity of every space-like f_{mk}-line is slower than the speed of light,*
(d) *the μ-distance between parallel f-coordinate lines is constant, that is*
$(\forall t_0, t_1, x_0, x_1 \in \Re)((t_0, x_0), (t_0, x_1), (t_1, x_0), (t_1, x_1) \in Dom(f) \Rightarrow$
$\mu(f[(t_0, x_0), (t_0, x_1)]) = \mu(f[(t_1, x_0), (t_1, x_1)]) \ \wedge$
$\mu(f[(t_0, x_0), (t_1, x_0)]) = \mu(f[(t_0, x_1), (t_1, x_1)]))$.

Then m does not accelerate.

Proof. It follows from Theorem 4.15, see below. ∎

Seeing the above theorem, one can argue, that o.k., then probably, the conditions of Theorem 4.14 are too strong, so let us try to find models, in which an accelerating observer whose trace is not differentiable twice from any inertial observers view, and probably this strange accelerating observer can coordinatize the spacetime such that the distances between his/her coordinate lines are constant. However, we **don't want to exclude** the model of simplest uniformly accelerating observers over a real field, for example. This, and many other possibly interesting frame model do satisfy the conditions of Theorem 4.14.

Theorem 4.14 is easily seen to be equivalent with Theorem 4.15 below. The connections between the conditions of these theorems are as follows: $(a), (b), (c), (d)$ correspond to $(0), (1), (2), (4)$ respectively, while (3) corresponds to the assumption of $\mathfrak{M} \models \mathbf{Ax_g1}$.

THEOREM 4.15 *Suppose $f : {}^2\Re \to {}^2\Re$ is a partial function such that*

(0) *The domain of f is open and convex,*
(1) *f is twice continuously differentiable,*
(2) $(\forall (t,x) \in Dom(f))((\partial_t f_x(t,x))^2 < (\partial_t f_t(t,x))^2 \wedge$
$(\partial_x f_t(t,x))^2 < (\partial_x f_x(t,x))^2)$,
(3) *f is locally Lorentz, that is,* $(\forall (t,x) \in Dom(f))\left(\dfrac{\partial_t f_x(t,x)}{\partial_t f_t(t,x)} = \dfrac{\partial_x f_t(t,x)}{\partial_x f_x(t,x)} \right)$,
(4) *The μ-distance between parallel f-coordinate lines is constant, that is*
$(\forall t_0, t_1, x_0, x_1 \in \Re)((t_0,x_0),(t_0,x_1),(t_1,x_0),(t_1,x_1) \in Dom(f) \Rightarrow$
$\mu(f[(t_0,x_0),(t_0,x_1)]) = \mu(f[(t_1,x_0),(t_1,x_1)]) \wedge$
$\mu(f[(t_0,x_0),(t_1,x_0)]) = \mu(f[(t_0,x_1),(t_1,x_1)]))$.

Then f is a partial Lorentz transformation.

Proof. First we introduce functions measuring the distances between parallel coordinate lines induced by f. Let $t_0 \le t_1 \in \Re$ be such that the set

$$D_{t_0,t_1} := \{x \in \Re :\ (t_0,x),(t_1,x) \in Dom(f)\}$$

is non empty. Then the function $g_{t_0,t_1} : D_{t_0,t_1} \to \Re$ is defined to be

$$(\forall x \in D_{t_0,t_1})(g_{t_0,t_1}(x) = \textstyle\int_{t_0}^{t_1} \sqrt{(\partial_t f_t(s,x))^2 - (\partial_t f_x(s,x))^2 ds}).$$

Note that this is a correct definition because of our conditions. Intuitively, the function g_{t_0,t_1} measures the time passed between the time-like coordinate lines labeled by t_0 and t_1 at location x.

Similarly, for the space-like coordinate lines, we define the functions h_{x_0,x_1} as follows. Suppose $x_0 \le x_1 \in \Re$ are such, that the set $D_{x_0,x_1} := \{t \in \Re :\ (t,x_0),(t,x_1) \in Dom(f)\}$ is non empty. Then

$$(\forall t \in D_{x_0,x_1})(h_{x_0,x_1}(t) = \int_{x_0}^{x_1} \sqrt{(\partial_x f_x(t,s))^2 - (\partial_x f_t(t,s))^2} ds).$$

Now condition (4) states, that the above defined functions g_{t_0,t_1} and h_{x_0,x_1} are constants. All of these functions are differentiable because of condition (1). Thus, their derivatives are identically zero:

$$(5)\quad (\forall t_0, t_1)(\forall x \in D_{t_0,t_1})\ \ 0 = g'_{t_0,t_1}(x) = \partial_x \int_{t_0}^{t_1} \sqrt{(\partial_t f_t(s,x))^2 - (\partial_t f_x(s,x))^2} ds,$$
$$(6)\quad (\forall x_0, x_1)(\forall t \in D_{x_0,x_1})\ \ 0 = h'_{x_0,x_1}(t) = \partial_t \int_{x_0}^{x_1} \sqrt{(\partial_x f_x(t,s))^2 - (\partial_x f_t(t,s))^2} ds.$$

By condition (1) the integration and the derivation in (5) (and in (6) as well) commute, so executing the derivation in (5), one has

$$(8)\quad (\forall t_0, t_1)(\forall x \in D_{t_0,t_1})\ 0 = \int_{t_0}^{t_1} \frac{2\partial_t f_t(s,x)\partial_x\partial_t f_t(s,x) - 2\partial_t f_x(s,x)\partial_x\partial_t f_x(s,x)}{2\sqrt{(\partial_t f_t(s,x))^2 - (\partial_t f_x(s,x))^2}}\ ds.$$

It follows, that for all $(s,x) \in Dom(f)$

$$(9)\quad 0 = \frac{2\partial_t f_t(s,x)\partial_x\partial_t f_t(s,x) - 2\partial_t f_x(s,x)\partial_x\partial_t f_x(s,x)}{2\sqrt{(\partial_t f_t(s,x))^2 - (\partial_t f_x(s,x))^2}}$$

because of the following. Let us denote the right hand side of (9) by $j(s,x)$. Suppose for contradiction that there exists a point $(s,x) \in Dom(f)$ such that $j(s,x) \neq 0$. By condition (1) the function j is continuous, therefore there exists a neighbourhood $N(s,x)$ of (s,x) such that $N(s,x) \subseteq Dom(f)$ and j does not vanish on $N(s,x)$. Now choose $t_0 < t_1 \in \Re$ such that $(t_0,x),(t_1,x) \in N(s,x)$ hold. Clearly, $0 \neq \int_{t_0}^{t_1} j(s,x)ds$ contradicting to (8). Therefore (9) is true; which can be reduced to the following partial differential equation for f:

$$(10)\quad 0 = \partial_t f_t \partial_x \partial_t f_t - \partial_t f_x \partial_x \partial_t f_x.$$

Similarly to the previous paragraph, from (6) one obtains the following partial differential equation for f:

$$(11)\quad 0 = \partial_x f_x \partial_t \partial_x f_x - \partial_x f_t \partial_t \partial_x f_t.$$

Note that by condition (2) we have $(\forall (t,x) \in Dom(f))(\partial_t f_t(t,x) \neq 0 \wedge \partial_x f_x(t,x) \neq 0)$. By condition (1) $\partial_t \partial_x f_t = \partial_x \partial_t f_t$ and $\partial_t \partial_x f_x = \partial_x \partial_t f_x$ also hold. From (10) expressing $\partial_t \partial_x f_t$ and substituting the result into (11) we have

$$(12)\quad \partial_x f_x \partial_t \partial_x f_x = \partial_x f_t \frac{\partial_t f_x}{\partial_t f_t} \partial_x \partial_t f_x.$$

Similarly, from (11) expressing $\partial_x \partial_t f_x$ and substituting it into (10) we have

$$(13)\quad \partial_t f_t \partial_x \partial_t f_t = \partial_t f_x \frac{\partial_x f_t}{\partial_x f_x} \partial_t \partial_x f_t.$$

Now we show that $(\forall (t,x) \in Dom(f))(\partial_t \partial_x f_x(t,x) = 0 \ \wedge \ \partial_t \partial_x f_t(t,x) = 0)$. Suppose for contradiction, that there exists a point $(t,x) \in Dom(f)$ such that $\partial_t \partial_x f_x(t,x) \neq 0$. Now it follows from (12) that

$$(14)\quad \partial_x f_t \partial_t f_x = \partial_x f_x \partial_t f_t$$

holds in a sufficiently small neighbourhood of (t,x). Using condition (3) one concludes from (14) that $(\partial_t f_x(t,x))^2 = (\partial_t f_t(t,x))^2$ contradicting condition (2).

Similarly, if $\partial_t \partial_x f_t(t,x) \neq 0$ would hold for some $(t,x) \in Dom(f)$ then using (13) one can deduce (14) yielding the above contradiction.

So, we proved $(\forall (t,x) \in Dom(f))(\partial_t \partial_x f_x(t,x) = 0 \ \wedge \ \partial_t \partial_x f_t(t,x) = 0)$. This implies, that there exist differentiable, real valued functions $C_1, C_2, D_1, D_2 : \Re \rightarrow \Re$ such that

$$(15)\quad (\forall (t,x) \in Dom(f))(f_t(t,x) = C_1(t) + C_2(x) \ \wedge \ f_x(t,x) = D_1(t) + D_2(x))$$

and therefore $(\forall (t,x) \in Dom(f))$

$$\partial_t f_t(t,x) = C_1'(t),\ \partial_t f_x(t,x) = D_1'(t),\ \partial_x f_t(t,x) = C_2'(x),\ \partial_x f_x(t,x) = D_2'(x).$$

Using condition (3) it follows that

$$(16)\quad \forall (t,x) \in Dom(f) \ \ \frac{D_1'(t)}{C_1'(t)} = \frac{C_2'(x)}{D_2'(x)}$$

which is possible only when there exists a constant $\lambda \in \Re$ such that both sides of (16) are equal with λ, that is,

$$(17)\quad \forall (t,x) \in Dom(f) \ \ \lambda = \frac{D_1'(t)}{C_1'(t)} = \frac{C_2'(x)}{D_2'(x)}.$$

But (17) means that the tangent lines of the f-coordinate lines are the same in every point, that is, the f-coordinate lines are straight lines themselves, so (using again condition (3) one can conclude that) f is a partial Lorentz transformation, as desired. ∎

Now we turn to constructing models in which there exists a non-inertial observer. To do this, first we develop a little differential geometry.

Preliminaries from Differential Geometry

From now on we are working over the field of reals and in dimension $n = 2$ and when speaking about relativity models, we always assume the symmetry axiom **Ax△2**.

Definition 4.16 By a curve we mean a twice continuously differentiable function $f : \Re \to {}^2\Re$.

Remark 4.17 *Recall that if f is a curve, then the coordinate functions of f are denoted by f_t, f_x respectively. Moreover, if the function f'_t satisfies the condition $(\forall t \in Dom(f))(f'_t(t) \neq 0)$ then f_t is strictly monotone therefore $(f_t(t), f_x(t))$ is the unique point in $Ran(f)$ having first coordinate $f_t(t)$, so $Ran(f)$ can be considered as a graph of the one variable function $f_t(t) \mapsto f_x(t)$, we will denote this function by f_{*t}. In this case the first and second derivatives of f_{*t} can be computed as follows:*

$$f'_{*t}(t) = f'_x(t)/f'_t(t) \text{ and}$$
$$f''_{*t} = (f''_x(t)f'_t(t) - f''_t(t)f'_x(t))/(f'_t)^3.$$

Our intuition about the uniform acceleration is that the uniformly accelerating observer observes his/her change of speed in such a way that this change does not depend on the time instant when the observation is made. However, when velocity is changing, it is not clear, what are the unit vectors of the accelerating observer at a certain moment. By $\mathbf{Ax_g1}$ locally we can approximate the trace of an accelerating observer by an inertial one, and we can ask this new inertial observer about the acceleration of the accelerating observer.

To be more concrete let us fix a model $\mathfrak{M} \in \mathsf{FM}_{\text{acc}}$ such that $\mathfrak{M} \models Acc \wedge$ **Ax△2**. Suppose m is an uniformly accelerating observer, k is an inertial observer, and for any $t \in \Re$ k_t is such an inertial observer, whose location and velocity coincide with that of m at time instant t as seen by k. Suppose moreover that the function $f : \Re \to \Re$ is such that $f = tr_k(m)$ (as set of pairs). Then

$$tr_{k_t}(m) = \{\langle \frac{1}{\sqrt{1 - v_k(k_t)}} (t - v_k(k_t) f(t),\ f(t) - v_k(k_t)t) \rangle :\ t \in \Re\}.$$

So the trace of m as seen by k_t is the range of a curve. Note that $v_k(k_t)$ is equal to the velocity of m at time instance t, as seen by k. The acceleration of m observed by k_t is simply the second derivative of the above curve. Thus, the acceleration is uniform (in relativistic sense) iff this second derivative at (time instant) t does not depend on t. After a straightforward computation based on Remark 4.17 this motivates the following definition.

Definition 4.18 A twice derivable function $f : \Re \to \Re$ is called UAT iff it satisfy the differential equation

$$f'' = \alpha(1 - (f')^2)^{3/2}$$

for some $0 \neq \alpha \in \Re$. UAT stands for uniform accelerating trace.

Thus, the range of an UAT function can be the trace of an uniformly accelerating observer from an inertial observer's view.

THEOREM 4.19 *Let $f : \Re \to \Re$ be a function. Then f is UAT iff there are constants $\alpha, \beta, \gamma \in \Re$ such that* $(\forall t \in \Re)$

$$f(t) = \frac{\sqrt{1 + \alpha^2 (t + \beta)^2} + \gamma}{\alpha}$$

Pfoof. By solving the differential equation given in Definition 4.18. ∎

Remark 4.20 The constants α, β, γ in Theorem 4.19 coincide with the "initial" acceleration, velocity and location of the accelerating observer m as seen by k. For instance, when $m \in w_k(\bar{0})$ and the velocity of m at time instant 0 is equal to 0 (as seen by k) then the trace of m as seen by k is the range of the function

$$t \mapsto \frac{\sqrt{1 + \alpha^2 t^2} - 1}{\alpha}.$$

Now we have computed the trace of an accelerating observer. In order to build a model, the next step is to determine the coordinate system of the accelerating observer m. Let k be an inertial observer. By axiom $\mathbf{Ax4_g}$, the events in $tr_k(m)$ are the same as the events in the time axis of m. The other coordinate lines of m are also visualizable in the coordinate system of k: these are the f_{mk} lines, and this is what we will use.

If m would be an inertial observer, then the time like f_{mk}-lines were parallel to $tr_k(m)$. Therefore, in the case of an accelerating observer, we define the time like f_{mk}-lines to be parallel to $tr_k(m)$. More precisely, we would like to find a world view transformation f_{mk} which satisfies the above property. Let us notice, that this is an ad hoc decision, that is, no axiom forces us to go this way. We will return to this question after constructing a model of Acc.

Thus, we already know what events have same location from the viewpoint of m. To determine simultaneities, we will use $\mathbf{Ax_g1}$.

Definition 4.21 Let $g : \Re \to \Re$ be a twice differentiable function such that $(\forall t \in Dom(g))(g'(t) < 1)$. Then by an inverse simultaneity of g we mean a function h which satisfies the differential equation

$$(\forall x)(h'(x) = g'(h(x))).$$

By a simultaneity we mean an invertible function whose inverse is an inverse simultaneity.

An inverse simultaneity is intended to be the function h associating a spacetime point $h(p)$ to each space point p. Intuitively, this function tells us "when" showed the clock at p a certain (fixed) value. A simultaneity parametrizes the range of an inverse simultaneity, but wrt the time axis. Both of the notions of inverse simultaneity and simultaneity will be useful.

THEOREM 4.22 *The simultaneities of the UAT function*

$$f(t) = \frac{\sqrt{1 + \alpha^2(t+\beta)^2} + \gamma}{\alpha}$$

are the functions:

$$\frac{\sqrt{1 + \alpha^2(t+\beta)^2} + 0.5ln(\frac{e^{arsh(\alpha(t+\beta))} - 1}{e^{arsh(\alpha(t+\beta))} + 1})}{\alpha}.$$

Proof. By solving the differential equation in Definition 4.21. ∎

COROLLARY 4.23 For every UAT function f and for each point $p \in {}^2\Re$ there is a simultaneity (or inverse simultaneity) of f incident with p.

By this we determined all the f_{mk} lines of the uniformly accelerating observer m wrt inertial observer k.

Before building a model we describe here another acceleration motivated by the following. In special relativity, if an observer m moves faster and faster relative to k, then m's clock ticks slower and slower (relative to k's clock). So is it possible to accelerate so fast that k doesn't see the event when the internal clock of m shows, say, the time instant 1 ? The answer is yes. Intuitively, such a movement can be constructed as follows. For each $n \in \omega$ there is a velocity v_n such that the clock of an inertial observer m_n moving velocity v_n relative to k shows $1/2^n$ when the clock of k shows 1. Now if the accelerating observer m, moves in the first second with velocity v_1 (relative to k) and then with velocity v_2,... and so on, then k never observes the event when m's clock reaches 1. The problem is that the above constructed trace of m is not differentiable.

Definition 4.24 We say that a function $f : \Re \to \Re$ is CCT if the Minkowski lenght of (the range of) f between 0 and the infinity is finite. (CCT stands for convergent clock trace.)

THEOREM 4.25 *For any positive $\alpha \in \Re$ the function $f_\alpha : \Re \to \Re$,*

$$(\forall t)(f_\alpha(t) = arsh(\alpha e^x))$$

is CCT. For any point $p \in {}^2\Re$ there is an f_α simultaneity incident with p.

Proof. The proof can be found in [38]. ∎

LEMMA 4.26 *If f is a CCT function and there is a function g such that*

$$(\exists a \in \Re)(\forall x \geq a)(f(x) = g(x))$$

then g is a CCT function as well.

Proof. Obvious. ∎

Constructing Models for Accelerating Observers

Now we are ready to build a model in which there exists an accelerating observer. Instead of giving one model, we are giving a construction which expands certain special relativity models to a model of *Acc*. Thus, we will show the relative consistency of $Acc \cup \{(\exists m)(m \in Obs - IOb)\}$.

Let $\mathfrak{M}$ be a two dimensional special relativity model such that $\mathfrak{F}^{\mathfrak{M}} \cong \Re$ and $\mathfrak{M} \models Newbasax \wedge \mathbf{Ax\triangle 2}$. Let $n \in Obs^{\mathfrak{M}}$ be an (inertial) observer. Intuitively, we will put into the world view of n the new accelerating observers according to an UAT function g. So let us choose an UAT function g such that $g(0) = 0$. We will denote the new model $\mathfrak{M}^g$ (in fact, this model depends on the choice of n,$\mathfrak{M}$ and g). Fix a set of new symbols $\{s_r : r \in \Re\}$ disjoint from the universes of $\mathfrak{M}$. This set consists of the new, accelerating observers.

Let $\mathfrak{F}^g = \Re$.
Let $B^g = B^{\mathfrak{M}} \cup \{s_r : r \in \Re\}$.
Let $Obs^g = Obs^{\mathfrak{M}} \cup \{s_r : r \in \Re\}$.
Let $Ib^g = Ib^{\mathfrak{M}}$.
Let $Ph^g = Ph^{\mathfrak{M}}$.
Let $\mathrm{E}^g = \mathrm{E}^{\mathfrak{M}} = \in \cap {}^2\Re \times \mathcal{P}({}^2\Re)$.

To define the world view relation W^g first we introduce the functions $f_{s_r n} : {}^2\Re \to {}^2\Re$ as follows (see the figure below). Choose an arbitrary point $p = (t, x) \in {}^2\Re$. Let us denote by h the g-(inverse) simultaneity incident with $\bar{0}$. Find the point q such that the Minkowski length of h between $\bar{0}$ and q is equal to r[81]. Find another point q_1 such that the Minkowski length of h between q and q_1 is equal to x. Let g_0 and g_1 be curves parallel to g and incident with q,q_1 respectively. Let q_2 be the point incident with g_0 such that the Minkowski length of g_0 between q and q_2 is equal to t. Let g_3 be the g (inverse-)simultaneity incident with q_2. Finally, let p' be the intersection of g_3 and g_1. We define $f_{s_r n}(p) = p'$. It is easily seen that the above defined points exist, are unique, and $f_{s_r n}$ is a bijection.

Let $w^g_n = w^{\mathfrak{M}}_n \cup \{\langle f_{s_r n}(t, 0), s_r\rangle : \ t, r \in \Re\}$.
For all $m \in Obs^{\mathfrak{M}} - \{n\}$ let $w^g_m = w^{\mathfrak{M}}_m \cup \{\langle f^{\mathfrak{M}}_{nm}(f_{s_r n}(t, 0)), s_r\rangle : t, r \in \Re\}$..

[81] the sign of r determines the direction of the measurement

Figure 46:

For all $r \in \Re$ let $w^g_{s_r}(t,x) = w^g_n(f_{s_r n}(t,x))$.

By this we defined the world view relation W^g of the new model.

For all $m \in Obs^{\mathfrak{M}}$ let $G^g_m = G^{\mathfrak{M}}_m$.
For all $r \in \Re$ let $G^g_{s_r} = \{f^{-1}_{s_r n}[\ell] : \ \ell \in G^{\mathfrak{M}}_n \ \}$.

For all $m \in Obs^{\mathfrak{M}}$ let $d^g_m = \mu^{\mathfrak{M}}$ (where $\mu^{\mathfrak{M}}$ is the Minkowski metric in $\mathfrak{M}$).
For all $r \in \Re$ let $d^g_{s_r}(p,q)$ be the Minkowski length of $f_{s_r n}[p,q]$.

Finally, let $\mathfrak{M}^g = \langle B^g, Obs^g, Ph^g, Ib^g, \mathfrak{F}^g, \ \mathrm{E}\ ^g, W^g, G^g_m, d^g_m \rangle_{m \in Obs^g}$.

THEOREM 4.27 $\mathfrak{M}^g \models$ *Acc.*

Recall that $\mathfrak{M} \models$ *Newbasax* $\wedge$ **Ax△2**.
Proof. By checking the axioms. ∎

Let us notice that this construction is ad hoc in the sense, that no axiom forces us to go this way. Indeed, instead of the trace of the "new" observer, if we would know another function which can be seen as a simultaneity of the "new" accelerating observer, then using an analogue idea to Definition 4.21 we would determine the space-like f_{mk} lines and would repeat the above construction. This construction will be called the dual construction.

Exercise 4.28 Prove that the formula TwP (see section 2.1) describing the twin paradox is valid in $\mathfrak{M}^g$ (for arbitrary $\mathfrak{M}, g$ and $n \in Obs^{\mathfrak{M}}$ which does not excluded by the construction).

Let us notice that using the dual construction, the "reason" for the twin paradox is much more complicated.

Remark 4.29 *It is interesting to investigate how the speed of light behaves in models of Acc. In particular, if $m \in Obs$ is not inertial then there is no axiom requiring that m would observe the speed of light to be similar to what an inertial observer would see (the latter is "constantly" 1 by $\mathbf{AxE}_{0g}$). Let us consider the particular models $\mathfrak{M}^g$. Let us assume that $k, m \in Obs$ and k is inertial. Throughout, we assume that m accelerates with a uniform acceleration. First, assume that the speed of m at time 0 is 0 when observed by k. For simplicity assume $\bar{0} \in tr_k(m)$ too. Let $ph \in Ph$ be such that $\bar{0} \in tr_k(ph)$. Then the speed of ph as observed by m is not constant, actually the photon ph is accelerating (increasing its velocity) as seen by m. This behaviour does not depend on the direction in which ph is moving (i.e. ph may be "falling in the direction in which gravity[82] is pulling it" or may be moving in the other direction). We do not prove this here but we refer to [38].*

Next, let us assume that the speed of m at time 0 is large i.e. it is close to 1 (e.g. it may be 0.8 or 0.9). Now we may investigate 6 kinds of photons $ph \in Ph$. First assume that /ph moves in the direction against that of gravity. Assume that ph starts out (sometime) on the lifeline of m. Now, if ph starts out sometime > 0 then we conjecture that it may accelerate. However if ph starts out at time 0 then we think that it will decelerate (as opposed to accelerating). Moreover if ph starts out sometime in the sufficiently distant past then we think that it will "strongly decelerate", in some sense. If ph moves in the other direction (i.e. it is "falling") then we do not know whether it accelerates, decelerates or what exactly it does.

[82]We apologize for mentioning gravity in an undefined way. In the present approach gravity is a carefully defined notion, but for lack of time we do not recall the definition here. Intuitively, what we call gravity is the usual side-effect of acceleration "experienced" by the accelerating observer m, and to every point p of the spacetime ${}^nF$ of m the gravity at p experienced by m is a vector pointing in the direction opposite to that of the acceleration of m.

All these seem to point in the direction that the speed of light in models like $\mathfrak{M}^g$ behaves in somewhat complicated way. As a contrast we note that in the kind of "goldfish" models (or Kruskal-models) referred to in Remark 4.3 the absolute value of the speed of light seems to be constant (and is 1*) for even the accelerating observers.*

We close this remark by mentioning that what we said here is somewhat tentative and that we did not check all the details.

Now we turn to constructing a model, containing an observer, whose trace as seen by another inertial observer is the range of some CCT function. Roughly, we repeat the previous construction, but we have to do something more, because a CCT function does not determine the whole world view of the new observer m, i.e. no conditions about events seen by m in a time instant which is bigger than the Minkowski length of (the range of) the CCT function in question. Therefore we start with two models.

Let us choose a CCT function g described in Lemma 4.26 such that g is twice continuously differentiable, $(\forall x \leq 0)(g(x) = 0)$ and $g'(0) = 0$ (clearly, there exists such a function). Let $\mathfrak{M}_1$ and $\mathfrak{M}_2$ be two models over the field of reals, such that $\{\mathfrak{M}_1, \mathfrak{M}_2\} \models \mathit{Newbasax} \wedge \mathbf{Ax\triangle 2}$ and $B^{\mathfrak{M}_1} \cap B^{\mathfrak{M}_2} = \emptyset$. Choose an observer $n \in Obs^{\mathfrak{M}_1}$ and $n_0 \in Obs^{\mathfrak{M}_2}$. Fix a set of new symbols $\{s_r : r \in \Re\}$ disjoint from the universes of $\mathfrak{M}_1, \mathfrak{M}_2$. This set consists of the new, accelerating observers. We will denote the new model constructed below by $\mathfrak{M}^g$. In fact, $\mathfrak{M}^g$ depends on the choice of $g, \mathfrak{M}_1, \mathfrak{M}_2, n$ and n_0.

Let $\mathfrak{F}^g = \Re$.
Let $B^g = B^{\mathfrak{M}_1} \cup B^{\mathfrak{M}_2} \cup \{s_r : r \in \Re\}$.
Let $Obs^g = Obs^{\mathfrak{M}_1} \cup Obs^{\mathfrak{M}_2} \cup \{s_r : r \in \Re\}$.
Let $Ib^g = Ib^{\mathfrak{M}_1} \cup Ib^{\mathfrak{M}_2}$.
Let $Ph^g = Ph^{\mathfrak{M}_1} \cup Ph^{\mathfrak{M}_2}$.
Let $\mathrm{E}^g = \mathrm{E}^{\mathfrak{M}} = \in \cap {}^2\Re \times \mathcal{P}({}^2\Re)$.

Let τ be the Minkowski length of g between 0 and infinity. Let $T = \{(t,x) \in {}^2\Re : \ t < \tau\}$. Analogously to the previous construction, to define the world view relation W^g first we introduce the functions $f_{s_r n} : {}^2\Re \to {}^2\Re$ as follows (see the figure below). Choose an arbitrary point $p = (t,x) \in T$. Let us denote by h the inverse simultaneity of g incident with $\bar{0}$. By the choice of g this is simply the x axis. Find the point q such that the Minkowski length of h between $\bar{0}$ and q is equal to r[83].

[83]Again, the sign of r determines the direction of the measurement.

Find another point q_1 such that the Minkowski length of h between q and q_1 is equal to x. Let g_0 and g_1 be curves parallel to g and incident with q,q_1 respectively. Let q_2 be the point incident with g_0 such that the Minkowski length of g_0 between q and q_2 is equal to t. Let g_3 be the g (inverse-) simultaneity incident with q_2. Finally, let p' be the intersection of g_3 and g_1. We define $f_{s_r n}(p) = p'$. It is easily seen that the above defined points exist, are unique, and $f_{s_r n}$ is injective.

Figure 47:

Let $w_n^g = w_n^{\mathfrak{M}_1} \cup \{\langle f_{s_r n}(t,0), s_r\rangle : \ (t,r) \in T\}$.
For all $m \in Obs^{\mathfrak{M}_1} - \{n\}$ let $w_m^g = w_m^{\mathfrak{M}_1} \cup \{\langle f_{nm}^{\mathfrak{M}_1}(f_{s_r n}(t,0)), s_r\rangle : (t,r) \in T\}$.

$$(\forall m \in Obs^{\mathfrak{M}_2})(\forall (t,r) \in {}^2\Re) w_m^g(t,r) = \begin{cases} w_m^{\mathfrak{M}_2}(t,r) & \text{if } t < \tau \\ w_m^{\mathfrak{M}_2}(t,r) \cup \{s_r\} & \text{if } t > \tau. \end{cases}$$

In order to define the world view of s_r, for all $r \in \Re$ let us fix an observer $n_r \in Obs^{\mathfrak{M}_2}$ such that $tr_{n_0}^{\mathfrak{M}_2}(n_r) = \Re \times \{r\}$.

$$(\forall r \in \Re)(\forall (t,x) \in {}^2\Re) w^g_{s_r}(t,x) = \begin{cases} w^g_n(f_{s_r}(t,x)) & \text{if } (t,x) \in T \\ w^{\mathfrak{M}_2}_{n_r}(t,x) \cup \{s_{r+x}\} & \text{if } t > \tau \\ \emptyset & \text{if } t = \tau, x \neq r \\ \{s_r\} & \text{if } t = \tau, x = r. \end{cases}$$

By this we defined the world view relation W^g of the new model.

For all $m \in Obs^{\mathfrak{M}_1}$ let $G^g_m = G^{\mathfrak{M}_1}_m$.
For all $m \in Obs^{\mathfrak{M}_2}$ let $G^g_m = G^{\mathfrak{M}_2}_m$.
For all $r \in \Re$ let $G^g_{s_r} = \{f^{-1}_{s_r n}[\ell] :\ \ell \in G^{\mathfrak{M}}_n\} \cup \{f^{-1}_{s_r n_r}[\ell] \cap ({}^2\Re \setminus T) :\ \ell \in G^{\mathfrak{M}_2}_{n_r}\}$.

For all $m \in Obs^{\mathfrak{M}_1}$ let $d^g_m = \mu^{\mathfrak{M}_1}$
For all $m \in Obs^{\mathfrak{M}_2}$ let $d^g_m = \mu^{\mathfrak{M}_2}$

$$(\forall r \in \Re)(\forall p, q \in {}^2\Re) d^g_{s_r}(p,q) = \begin{cases} \text{the Minkowski length of } f_{s_r n}[p,q] & \text{if } p, q \in T \\ \text{the Minkowski length of } f_{s_r n_r}[p,q] & \text{if } p, q \notin T \\ 0 & \text{Otherwise.} \end{cases}$$

Finally, let $\mathfrak{M}^g = \langle B^g, Obs^g, Ph^g, Ib^g, \mathfrak{F}^g, \mathrm{E}^g, W^g, G^g_m, d^g_m \rangle_{m \in Obs^g}$.

THEOREM 4.30 $\mathfrak{M}^g \models Acc - \{\mathbf{Ax_g2}, \mathbf{Ax_g3}\}$.

Proof. By checking the axioms. ∎

The axioms $\mathbf{Ax_g2}, \mathbf{Ax_g3}$ fail to hold because there are points separated with the border of T (which we will call an event horizon), and somehow the observers $s_r, r \in \Re$ "changed their universe". The same situation appears around black holes. Finally we start to study such situations.

Definition 4.31 Let $\mathfrak{M} \in \mathsf{FM}_{\mathrm{acc}}$ be a model, $m \in Obs^{\mathfrak{M}}$ and $r \in \Re$. We say that in $\mathfrak{M}$ observer m has an event horizont at r iff

$$(\exists n \in IOb^{\mathfrak{M}})(\forall \varepsilon, k \in (F^{\mathfrak{M}})^+)(\exists p \in {}^2F^{\mathfrak{M}})(|p_x| \geq k \wedge r - \varepsilon \leq p_t \leq r + \varepsilon \wedge n \in w^{\mathfrak{M}}_m(p)).$$

THEOREM 4.32 *Let $\mathfrak{M} \in \mathsf{FM}_{\rm acc}$ be a two dimensional model such that $\mathfrak{F}^{\mathfrak{M}} \cong \Re$ and $\mathfrak{M} \models Acc - \{\mathbf{Ax_g2}, \mathbf{Ax_g3}\}$. Let $m \in Obs^{\mathfrak{M}}$ such that there are two points $p_0, p_1 \in {}^2\Re$ and an inertial observer $k \in IOb^{\mathfrak{M}}$ satisfying $k \in w_m^{\mathfrak{M}}(p_0) = w_m^{\mathfrak{M}}(p_1)$. Suppose moreover that the set $tr_m^{\mathfrak{M}}(k)$ is closed (in the topology induced by the ordering), f_{mk} is locally injective and whenever $q_0 \neq q_1 \in {}^2\Re$ we have*

$$k \in w_m^{\mathfrak{M}}(q_0) \cap w_m^{\mathfrak{M}}(q_1) \Rightarrow q_{0t} \neq q_{1t}.$$

Then there exists an $r \in \Re$ between p_{0t} and p_{1t} such that m has an event horizon at r.

Proof. The proof can be found in [38]. ∎

5 Appendix

5.1 Cartoon models (or virtual realities)

In this section we introduce cartoon models, CM's (or virtual realities, VR's). Our purpose is to give a different kind of models, models with a different flavor.[84] We then will show how to "translate" CM's to frame models, FM's, and vica versa. One can use CM's to "visualize" frame models, and with the two translation functions in place, one can prove theorems about FM's via using CM's, or vica versa.

First we describe the new kind of models intuitively, and then we will give a formal definition.

One can view a CM as a two-player game (or as a simulation of a physical world on a computer). One of the players we will call "the player", the other player is the computer. The player can put characters or photons on the screen, all characters have their own "internal" or "inner" clocks; photons do not have clocks. The player can set velocities of the characters and he can initialize the clocks of the characters; he can only set the direction of the movement of the photons, he cannot set their actual speeds. The player also picks one of the characters as the main one, and then he can start the screen, he can start the movements forward in time, or backward in time. At any time he can freeze the screen and put in new characters or photons, or click on another character making it the main character from then on.

During the play, the computer chooses speeds of the photons (in the directions specified by the player), the computer chooses how the inner clocks of the characters on the screen tick (when the screen is run forward or backward). When the player clicks on a new character, the computer is allowed to rearrange the screen, it can change locations, clock readings, and velocities of the bodies on the screen. When the player puts a character or photon on the screen, the computer may refuse it. The computer has to obey some rules when doing all the above. First we summarize the moves of the play, and then we outline the rules of the game.

The player's possible moves:

– put characters on the screen
– set velocities of characters (computer may refuse some)
– initialize internal clocks of characters
– put photons (with directions) on the screen
– define main character (click on)

[84]The presently described idea of cartoon models (or virtual realities) grew out from the illuminating cartoon-like drawings (and explanations) in Epstein [12], cf. e.g. pp.12-16,58 therein.

– start forward (run the screen forward in time)
– start backward (run the screen backward in time)
– freeze the screen.

The computer's free choices:

– speeds of photons
– ticking of internal clocks
– rearrangement of screen during a "click", i.e. during change of main character. (Locations of bodies, clock readings, velocities may change.)
– computer may refuse certain velocities (proposed for characters).

The rules of the game (the rules the computer has to obey):

(R1) Main character is at rest, it is standing in the origo.
(R2) The clock of the main character ticks with the ratio of real time.
(R3) Distances are real distances (i.e. distances measured by player on the screen by "real centimeter"). We assume that the screen is infinite in both directions.
(R4) Velocities are real velocities on screen, i.e $\Delta(t)$ (measured by main character) / distance measured on screen by real centimeter.
(R5) When the player clicks on NEW main character k, a new window opens, k will stand in the origo, and everything (e.g. its clock) on k's place will be unchanged except for velocities, k's velocity will be $\bar{0}$. Characters and photons on the screen will be the same (no new ones will appear, no one of them will disappear). All else may change.
(R6) Commutativity: If main character is the same and its clock shows same time at two time instances in the game, then we will have the same pictures on the screen relative to the common characters and photons.
(R7) When playing forward and backward, the clocks of all characters run through all real numbers.[85]

Now we begin to give a formal definition for cartoon models. Let $n \in \omega$ be a number, we will have an n-dimensional screen. I.e. we will identify the screen with ${}^n\mathrm{R}$. Then both "locations (or positions) on the screen" and "velocities" will be elements of ${}^n\mathrm{R}$, and time instances will be elements of R.

A *cartoon model* is a triple $\langle Obs, Ph, \delta\rangle$, satisfying the following. Obs and Ph are two disjoint sets (the *names* the computer will give to characters, and photons respectively). δ is the "*state-transition function*". It is a partial function mapping $I \times S$ to the set of all functions mapping ${}^+\mathrm{R}$ to S. Here I is the set of possible

[85] Does (R7) follow from (R1)-(R6)?

inputs (to be defined below), S is the set of all possible *states* (of the screen), ${}^{+}\mathrm{R}$ is the set of non-negative reals, and then a function mapping ${}^{+}\mathrm{R}$ to S is a possible run of the screen in the future. In more detail, S is the set of all triples (γ, π, m) which satisfy the following:[86]

$$\gamma : Obs \overset{\circ}{\longrightarrow} {}^{n}\mathrm{R} \times {}^{n}\mathrm{R} \times R, \pi : Ph \overset{\circ}{\longrightarrow} {}^{n}\mathrm{R} \times {}^{n}\mathrm{R}, \text{ and } m \in Dom(\gamma), \gamma(m) = (\bar{0}, \bar{0}, t).$$

Intuitively, $s = (\gamma, \pi, m) \in S$ describes "state of the screen", $Dom(\gamma)$ is the set of characters on the screen, $Dom(\pi)$ is the set of photons on the screen, γ tells about each character on the screen its location, velocity, and time of its clock, π tells about each photon on the screen its location and velocity, m is the "main character". The main character is at rest (its velocity is $\bar{0}$) and is standing in the origo. This last condition is the formalization of rule $(R1)$.

I is the set of possible inputs, i.e.

$$I \overset{\text{def}}{=} ({}^{n}\mathrm{R} \times {}^{n}\mathrm{R} \times \mathrm{R}) \cup ({}^{n}\mathrm{R} \times {}^{n}\mathrm{R}) \cup Obs \cup \{sf, sb, fr\}.$$

Intuitively,

– $(p, v, t) \in ({}^{n}\mathrm{R} \times {}^{n}\mathrm{R} \times \mathrm{R})$ means "*put a character* onto the screen to location p with velocity v and clock showing t". The computer then will put such a character on the screen and gives name to it.

– $(p, v) \in ({}^{n}\mathrm{R} \times {}^{n}\mathrm{R})$ means "*put a photon* onto the screen to place p with direction that of v". The computer will then put a photon on the screen, gives name to it, and chooses velocity for this photon, with direction that of v.

– $k \in Obs$ means "*click on observer* k and make it to be the main character from now on". The computer then will rearrange the screen such that k becomes the main character. The set of characters and photons on the screen remains the same, only their positions, velocities, and inner clock readings may change. The characters and photons that were at k's place will all go to $\bar{0}$ with the same inner clock readings what they had.

– sf means "*start forward*". The computer then begins to move the objects on the screen with the specified velocities (velocity is meant from the point of view of "player", or "real world"). The inner clock of the main character ticks as real time, and the computer determines how the inner clocks of the other characters tick.

– sb means "*start backward*". As the above, just the computer plays backward in time.

[86] $f : A \overset{\circ}{\longrightarrow} B$ means that f is a partial function from A into B, i.e. $Dom(f) \subseteq A$ and $f : Dom(f) \longrightarrow B$.

– fr means "*freeze the screen*".

We now begin to formalize the requirements on the state-transition function δ (described informally above as the responses of the computer to the moves of the player).

As we said earlier,

$$^{+}\mathrm{R} \stackrel{\mathrm{def}}{=} \{r \in \mathrm{R} : r \geq 0\},$$

and $^{+\mathrm{R}}S$ denotes the set of all (total) functions from $^{+}\mathrm{R}$ to S. We require that

$$\delta : I \times S \xrightarrow{\circ} {}^{+\mathrm{R}}S$$

is a partial function which satisfies the following.

– If $\delta(i, s)$ is undefined, then $i \notin \{sf, sb, fr\}$. If $i \in Obs$, then [$\delta(i, s)$ is defined iff $i \in Dom(s_0)$]. Note that if $s \in S$, then s_0, its first component, is a partial mapping from Obs to $^{n}\mathrm{R} \times {}^{n}\mathrm{R} \times \mathrm{R}$, and $Dom(s_0)$ is then the set of characters on the screen in state s. This condition then says that the computer can refuse putting a character or a photon on the screen, and the player can click on a character k iff k is already on the screen.[87]

Let $(\gamma, \pi, m) \in S$. We now define $\delta(i, (\gamma, \pi, m))$ for all $i \in I$. In the first three cases, when defined, we always get a constant state as a result (there $^{+}\mathrm{R} \times \{s\}$ denotes the constant state s).

– $\delta((p, v, t), (\gamma, \pi, m)) = {}^{+}\mathrm{R} \times \{(\gamma', \pi, m)\}$, where $\gamma' \supset \gamma$, $Dom(\gamma') = Dom(\gamma) \cup \{k\}$, $k \notin Dom(\gamma)$, and $\gamma'(k) = (p, v, t)$, if $\delta((p, v, t), (\gamma, \pi, m))$ is defined. (The computer gave the name k to the character.)

– $\delta((p, v), (\gamma, \pi, m)) = {}^{+}\mathrm{R} \times \{(\gamma, \pi', m)\}$, where $\pi' \supset \pi$, $Dom(\pi') = Dom(\pi) \cup \{ph\}$, $ph \notin Dom(\pi)$, and $\pi'(ph) = (p, \lambda \cdot v)$ for some $\lambda \in {}^{+}\mathrm{R}, \lambda \neq 0$, if $\delta((p, v), (\gamma, \pi, m))$ is defined. (The computer gave the name ph to the photon, and it gave it velocity $\lambda \cdot v$. $\lambda > 0$ ensures that this velocity has the same direction as v.)

– Assume that $k \in Dom(\gamma)$. Then $\delta(k, (\gamma, \pi, m)) = {}^{+}\mathrm{R} \times \{(\gamma', \pi', k)\}$, where $(\gamma', \pi', k) \in S$ such that $Dom(\gamma') = Dom(\gamma)$, $Dom(\pi') = Dom(\pi)$. Further, for all $k' \in Dom(\gamma)$, if $\gamma(k')_0 = \gamma(k)_0$, then $\gamma'(k')_0 = \bar{0}$ and $\gamma'(k')_2 = \gamma(k')_2$; and if

[87] It will be useful to make some requirements about the domain of δ. E.g. we can require that whether $\delta((p, v, t), s)$ is defined or not depends only on v, the input character speed; and that the "accepted" speeds form an open set. Or we could require that to an input character speed v the computer is permitted to say "no" only if there is no photon on the screen with bigger velocity.

$\gamma(k')_0 \neq \gamma(k)$, then $\gamma'(k') \neq \bar{0}$. A similar condition holds for all $ph \in Dom(\pi)$: $\pi'(ph) = \bar{0}$ iff $\pi(ph)_0 = \gamma(k)_0$. By this we ensured that the computer obeys rule $(R5)$. (Note that $\gamma(k)_2$ is the clock reading of k on the screen.)

– Let $f \stackrel{\text{def}}{=} \delta(sf, (\gamma, \pi, m))$. Then $f : {}^+\mathrm{R} \to S$ is such that for all $t \in {}^+\mathrm{R}$ we have the following.

$f(t) = (\gamma^t, \pi^t, m)$ where $Dom(\gamma^t) = Dom(\gamma)$, $Dom(\pi^t) = Dom(\pi)$,

for all $k \in Dom(\gamma)$ there is $ic^+(k,t) \in \mathrm{R}$ such that
$\gamma^t(k) = (\gamma(k)_0 + t \cdot \gamma(k)_1, \gamma(k)_1, ic^+(k,t))$,

$\gamma^t(m) = (\bar{0}, \bar{0}, \gamma(m)_2 + t)$, i.e. $ic^+(m,t) = \gamma(m)_2 + t$,

for all $ph \in Dom(\pi)$ we have that $\pi^t(ph) = (\pi(ph)_0 + t \cdot \pi(ph)_1, \pi(ph)_1)$.

Here we took care of rules (R2), (R3), (R4).

– Let $f \stackrel{\text{def}}{=} \delta(sb, (\gamma, \pi, m))$. Then $f : {}^+\mathrm{R} \to S$ is such that for all $t \in {}^+\mathrm{R}$ we have the following.

$f(t) = (\gamma^t, \pi^t, m)$ where $Dom(\gamma^t) = Dom(\gamma)$, $Dom(\pi^t) = Dom(\pi)$,

for all $k \in Dom(\gamma)$ there is $ic^-(k,t) \in \mathrm{R}$ such that
$\gamma^t(k) = (\gamma(k)_0 - t \cdot \gamma(k)_1, \gamma(k)_1, ic^-(k,t))$,

$\gamma^t(m) = (\bar{0}, \bar{0}, \gamma(m)_2 - t)$, i.e. $ic^-(m,t) = \gamma(m)_2 - t$,

for all $ph \in Dom(\pi)$ we have that $\pi^t(ph) = (\pi(ph)_0 - t \cdot \pi(ph)_1, \pi(ph)_1)$.

We took care of rules (R2), (R3), (R4).

– $\delta(fr, (\gamma, \pi, m)) = {}^+\mathrm{R} \times \{(\gamma, \pi, m)\}$.

By this, the notion of a cartoon model has been defined. CM^- will denote the class of all such cartoon models. We denote this class with CM^- because we did not ensure that conditions (R6) and (R7) be satisfied. A cartoon model which satisfies (R6) will be called a commuting cartoon model. Sometimes we will call cartoon models as cartoon machines (because its definition is very much like that of an automaton). Next we define the set of games of a cartoon model.

Let $\underline{C} = \langle Obs, Ph, \delta \rangle \in CM^-$. A <u>game</u> of $\underline{C}$ is a function $g : \omega \to (\mathrm{R} \times I \times S)$ such that by denoting $g(n) = (t_n, i_n, s_n)$ for all $n \in \omega$, we have

$t_0 = 0$, $t_n < t_{n+1}$,

$s_0 = (\gamma, \emptyset, m)$ for some m and γ such that $Dom(\gamma) = \{m\}$ (i.e. in the first step there is only a main character on the screen),

$s_{n+1} = \delta(i_n, s_n)(t_{n+1} - t_n)$. (Here we also require that $\delta(i_n, s_n)$ be defined.)

Intuitively: we start at time 0 with main character m showing some inner time, then at (real) time t_1 we give the input i_1, we let the machine run till (real) time t_2, then we give the input i_2, etc. In this sequence, s_n is the state of the machine at time t_n in this game. $Games(\underline{C})$ denotes the set of all games of the cartoon model $\underline{C}$.

Let $\underline{C}$ be a cartoon model. We say that $\underline{C}$ satisfies commutativity, or $\underline{C}$ is *commuting*, or $\underline{C}$ is deterministic, if in each game, whenever at two time-instances the main characters are the same their internal clocks showing the same time, the whole screen is the same relative to the joint characters and photons (i.e. main character and its inner clock determine the screen). Formally,

$\underline{C}$ commutes iff for all $g \in Games(\underline{C})$, for all $m, n \in \omega$, if $g(n) = (t_n, i_n, (\gamma_n, \pi_n, k_n))$, $g(m) = (t_m, i_m, (\gamma_m, \pi_m, k_m))$, and $k_n = k_m$, $\gamma_n(k_n) = \gamma_m(k_m)$, then $\gamma_n(k) = \gamma_m(k)$ and $\pi_n(ph) = \pi_m(ph)$ for all $k \in (Dom(\gamma_n) \cap Dom(\gamma_m))$ and $ph \in (Dom(\pi_n) \cap Dom(\pi_m))$.

Before going on, we give some examples of CM^-'s.

EXAMPLE 1 (Newtonian Cartoon Model) Let Obs, Ph be two arbitrary disjoint sets, and let $<$ be a well-ordering of $Obs \cup Ph$. We define $\underline{C}_1 = \langle Obs, Ph, \delta_1 \rangle$ as follows.

$\delta_1((p, v, t), (\gamma, \pi, m))$ is defined iff $\mathcal{H} \stackrel{\text{def}}{=} Obs \setminus Dom(\gamma) \neq \emptyset$, and if $\mathcal{H} \neq \emptyset$, then the computer gives the name $min(\mathcal{H})$.

$\delta_1((p, v), (\gamma, \pi, m))$ is defined iff $\mathcal{G} \stackrel{\text{def}}{=} Ph \setminus Dom(\pi) \neq \emptyset$, and if $\mathcal{G} \neq \emptyset$, then the computer gives the name $min(\mathcal{G})$ and it gives velocity v.

$\delta_1(k, (\gamma, \pi, m)) = {}^{+}\mathrm{R} \times \{(\gamma', \pi', m)\}$, where for all $k' \in Dom(\gamma)$ and $ph \in Dom(\pi)$, if $\gamma(k) = (p, v, t)$, $\gamma(k') = (p', v', t')$ and $\pi(ph) = (p'', v'')$, then $\gamma'(k') = (p' - p, v' - v, t')$, $\pi'(ph) = (p'' - p, v'' - v)$. (I.e. the rearrangement is that the whole picture on the screen gets shifted by p (so that k gets into the origo) and also v gets subtracted from all velocities (so that k becomes non-moving).) Clocks do not change during the rearrangement.

In $\delta_1(sf, (\gamma, \pi, m))$ and $\delta_1(sb, (\gamma, \pi, m))$ we have $ic^+(k, t) \stackrel{\text{def}}{=} \gamma(k)_2 + t$, $ic^-(k, t) \stackrel{\text{def}}{=} \gamma(k)_2 - t$. (I.e. all clocks move according to "real" time.)

EXAMPLE 2 (Einstenian Cartoon Model) The screen is one-dimensional. $Obs_2 \stackrel{\text{def}}{=} \{\ell \in \mathsf{Eucl}(\mathbf{2}, \mathfrak{R}) : ang^2(\ell) \neq 1\}$, $Ph_2 \stackrel{\text{def}}{=} \{\ell \in \mathsf{Eucl}(\mathbf{2}, \mathfrak{R}) : ang^2(\ell) = 1\}$. $\underline{C}_2 \stackrel{\text{def}}{=} \langle Obs_2, Ph_2, \delta_2 \rangle$ is defined as follows. Take the *Basax* model $\mathfrak{M} \stackrel{\text{def}}{=} \mathfrak{M}_1^P$ from section 2.4, and describe directly the cartoon model $\mathbf{cm}(\mathfrak{M})$ to be defined shortly.

EXAMPLE 3 This will be a kind of mixture of the previous two models. Let $<$ be a well-ordering on $Obs_2 \cup Ph_2$, and let $m_0 \in Obs_2$ be fixed. We define $\underline{C}_3 \stackrel{\text{def}}{=} \langle Obs_2, Ph_2, \delta_3 \rangle$ as follows. Let $s \stackrel{\text{def}}{=} (\gamma, \pi, m)$.

$\delta_3(i, s) \stackrel{\text{def}}{=} \delta_2(i, s)$ if $m_0 \in Dom(\gamma)$.

$\delta_3((p, v, t), s)$ is defined iff $\mathcal{H} \stackrel{\text{def}}{=} Obs_2 \setminus (Dom(\gamma) \cup \{m_0\}) \neq \emptyset$, and if $\mathcal{H} \neq \emptyset$, then the computer gives the name $min(\mathcal{H})$.

$\delta_3(i, s) = \delta_1(i, s)$ in all other cases, i.e. if $m_0 \notin Dom(\gamma)$ and $i \notin {}^n\mathrm{R} \times {}^n\mathrm{R} \times \mathrm{R}$.

Intuitively, $\underline{C}_3$ behaves like $\underline{C}_2$ if m_0 is on the screen, and if m_0 is not on the screen, then $\underline{C}_3$ behaves like $\underline{C}_1$, except that it never puts m_0 on the screen.

EXAMPLE 4 Let $\underline{C} = \langle Obs, Ph, \delta \rangle$ be any CM^- We define $\underline{C}_4 = \langle Obs, Ph, \delta_4 \rangle$ as behaving like $\underline{C}$ except when clicking: let $p \in {}^n\mathrm{R}$, $p \neq \bar{0}$.

$\delta_4(k, (\gamma, \pi, m)) = {}^+\mathrm{R} \times \{(\gamma', \pi, k)\}$, where for all $k' \in Dom(\gamma)$,

$\delta'(k') = (\bar{0}, \bar{0}, \gamma(k')_2)$ if $\gamma(k')_0 = \gamma(k)_0$, and

$\gamma'(k') = (p, \bar{0}, 0)$ if $\gamma(k') \neq \gamma(k)_0$.

Examples 1-3 are commuting, Example 4 is not commuting. Our next theorem says that, assuming commutativity, "time is abslute" implies our usual old "Newtonian" physical model, i.e. having Example 1.

Definition 5.1 We call $\underline{C} = \langle Obs, Ph, \delta \rangle \in CM^-$ *<u>Newtonian</u>* if in $\underline{C}$ all clocks tick alike, i.e. according to real time. (In more detail, $\underline{C}$ is Newtonian, if for all state $s = (\gamma, \pi, m)$, in the definitions of $\delta(sf, s)$ and $\delta(sb, s)$ we have $ic^+(k, t) = \gamma(k)_2 + t$, $ic^-(k, t) = \gamma(k)_2 - t$.) $\quad \lhd$

THEOREM 5.2 *Let $\underline{C}$ be Newtonian. Then $\underline{C}$ is commuting iff [when clicking on a new character, distances and clock-readings do not change, and velocities "add together", i.e. the new velocities will be the old ones minus the velocity of the clicked-on character].*

Now we turn to building connections between our old frame models and between our new cartoon models.

CONSTRUCTING CARTOON MODELS TO FRAME MODELS

Recall that $\mathbf{Ax6_{00}}$ makes possible to define inner clocks in frame models. Namely, if $m, k \in Obs$, then $ic_m(k) : F \to F$ such that $ic_m(k)(t) = t_0$ if $(\exists p) w_k(t_0, \bar{0}) = w_m(t,p)$.

Let $\mathfrak{M} \in \mathsf{Mod}(\{\mathbf{Ax1}, \mathbf{Ax2}, \mathbf{Ax3}, \mathbf{Ax4}, \mathbf{Ax6_{00}}\})$ and assume that $\mathfrak{F}^{\mathfrak{M}} = \mathfrak{R}$. Let $<$ be any well-ordering of $Obs \cup Ph$. We define $\mathbf{cm}(\mathfrak{M})$ as follows. Assume that $\mathfrak{M}$ is $\underline{n}+1$-dimensional. Then the screen of the associated cartoon model $\mathbf{cm}(\mathfrak{M})$ will be $\underline{n}$-dimensional.

For any Euclidean line $\ell \in \mathsf{Eucl}(\mathbf{n}, \mathbf{F})$ we define the *velocity of the line* ℓ as follows:

$$\bar{v}(\ell) \stackrel{\text{def}}{=} v \quad \text{iff} \quad (\exists t, p)\{(t,p), (t+1, p+v)\} \subseteq \ell.$$

Now,

$$\mathbf{cm}(\mathfrak{M}) = \langle Obs^{\mathfrak{M}}, Ph^{\mathfrak{M}}, \delta \rangle$$

where δ is defined as follows. Let $(\gamma, \pi, m) \in S$, $p, v \in {}^{\underline{n}}\mathrm{R}$, $t \in \mathrm{R}$.

To define $\delta((p,v,t),(\gamma,\pi,m))$ we only have to specify when this is defined, and what name the computer will give to the character. Let $t_0 = \gamma(m)_2$, let $\ell = \{(t_0,p) + t \cdot (1,v) : t \in \mathrm{R}\}$, and let

$$\mathcal{H} = \{k \in Obs \setminus Rng(\gamma) : tr_m k = \ell, ic_m(k)(t_0) = t\}.$$

(I.e. $\mathcal{H}$ is the set of observers that are not yet on the screen and whose lifeline is ℓ with inner clock showing t when m's time is t_0.) Now, $\delta((p,v,t),(\gamma,\pi,m))$ is defined iff $\mathcal{H} \neq \emptyset$, and if $\mathcal{H} \neq \emptyset$, then the computer gives the name $min(\mathcal{H})$, i.e.

$$\delta((p,v,t),(\gamma,\pi,m)) = {}^{+}\mathrm{R} \times \{(\gamma',\pi,m)\} \quad \text{where} \quad \gamma'(min(\mathcal{H})) = (p,v,t).$$

The definition of $\delta((p,v),(\gamma,\pi,m))$ is completely analogous: we only have to specify when this is defined, and when it is defined, what name and velocity the computer will give the new photon. Let $\mathcal{G} = \{ph \in Ph \setminus Rng(\pi) : \{(t_0,p),(t_0+1, \lambda \cdot v)\} \subseteq tr_m(ph)$ for some $\lambda > 0\}$. (I.e., $\mathcal{G}$ is the set of photons not yet on the screen whose speed is $\lambda \cdot v$ for some $\lambda > 0$, and that m sees at place p at time t_0.) Now,

Figure 48: .

$\delta((p,v),(\gamma,\pi,m))$ is defined iff $\mathcal{G} \neq \emptyset$, and if $\mathcal{G} \neq \emptyset$, then let $ph = min\mathcal{G}$. The computer will give the name ph to the photon, and it will give it the same speed that ph has (in m's worldview, i.e. the speed of $tr_m(ph)$). See Figure 48.

Let $k \in Dom(\gamma)$ and $t_0 = \gamma(k)_2$. Then $\delta(k,(\gamma,\pi,m)) = {}^+\mathrm{R} \times \{(\gamma',\pi',m)\}$, where

$$\gamma'(k') = (tr_k(k')(t_0), \bar{v}(tr_k(k')), ic_k(k')(t_0))$$

for each $k' \in Dom(\gamma)$ and

$$\pi'(ph) = (tr_k(ph)(t_0), \bar{v}(tr_k(ph)))$$

for all $ph \in Dom(\pi)$. See Figure 49.

To define $\delta(sf,(\gamma,\pi,m))$ we only have to define the real numbers $ic^+(k,t)$ for each $k \in Dom(\gamma)$ and $t \in \mathrm{R}$. Let $t_0 = \gamma(m)_2$. Then

$$ic^+(k,t) \stackrel{\text{def}}{=} ic_m(k)(t_0+t).$$

The definition of $\delta(sb,(\gamma,\pi,m))$ is similar:

$$ic^-(k,t) \stackrel{\text{def}}{=} ic_m(k)(t_0-t).$$

See Figure 50.

By the above, $\mathbf{cm}(\mathfrak{M})$ has been defined.

PROPOSITION 5.3 *Let $\mathfrak{M}, \mathfrak{M}'$ be frame models with $\mathfrak{F}^{\mathfrak{M}} = \mathfrak{F}^{\mathfrak{M}'} = \mathfrak{R}$.*

(i) $\mathbf{cm}(\mathfrak{M})$ *is a commuting cartoon model and* $\mathbf{cm}(\mathfrak{M})$ *satisfies (R7) if*[88] $\mathfrak{M} \models \mathbf{Ax6}$.

(ii) $\mathbf{cm}(\mathfrak{M}) = \mathbf{cm}(\mathfrak{M}') \Longrightarrow \mathfrak{M} = \mathfrak{M}'$, *if* $B = Ib = Obs \cup Ph$ *and* $W(m,t,p,k) \to Obs(m)$ *hold both in* $\mathfrak{M}$ *and in* $\mathfrak{M}'$. *I.e.* $\mathbf{cm} : \mathsf{FM}' \longrightarrow CM^-$ *is an injective function, where* FM' *is the above specified subclass of* FM.

Proof: To prove commutativity of $\mathbf{cm}(\mathfrak{M})$, one proves that in each game g, for each $n \in \omega$, if $g(n) = (t_n, i_n, s_n)$, then $s_n = (\gamma, \pi, m)$ where $\gamma(k) = (tr_m(k)(t_0), \bar{v}(tr_m k), ic_m(k)t_0)$ and $t_0 = \gamma(m)_2$, and the analogous statement for π. The rest of the proof is straightforward, we omit it. ∎

CONSTRUCTING FRAME MODELS FROM CARTOON MODELS

Let $\underline{C} = \langle Obs, Ph, \delta \rangle$ be an $\underline{n}$-dimensional cartoon model, and let $g \in Games(\underline{C})$. We define an $\underline{n}+1$-dimensional frame model $\mathfrak{M} \stackrel{\text{def}}{=} \mathbf{gm}(\underline{C}, g)$ as follows.

For every $n \in \omega$ let $g(n) = (t_n, i_n, s_n)$, and $s_n = (\gamma_n, \pi_n, m_n)$. Now

$Obs^{\mathfrak{M}} \stackrel{\text{def}}{=} \bigcup\{Dom(\gamma_n) : n \in \omega\}$,

$Ph^{\mathfrak{M}} \stackrel{\text{def}}{=} \bigcup\{Dom(\pi_n) : n \in \omega\}$,

$B^{\mathfrak{M}} \stackrel{\text{def}}{=} Ib^{\mathfrak{M}} \stackrel{\text{def}}{=} Obs^{\mathfrak{M}} \cup Ph^{\mathfrak{M}}$,

$\mathfrak{F}^{\mathfrak{M}} \stackrel{\text{def}}{=} \mathfrak{R}$, $G^{\mathfrak{M}} \stackrel{\text{def}}{=} \mathsf{Eucl}(\underline{n}+1, \mathfrak{R})$.

I.e. $Obs^{\mathfrak{M}}$ is the set of characters that appear on the screen sometime during the game, and similarly for $Ph^{\mathfrak{M}}, B^{\mathfrak{M}}$. To define $W^{\mathfrak{M}}$, first we define $tr_m k$ for all $m \in Obs^{\mathfrak{M}}$ and $k \in B^{\mathfrak{M}}$.

Let $m, k \in Obs^{\mathfrak{M}}$. Let $n \in \omega$ be the smallest number such that $m, k \in Dom(\gamma_n)$. There is such an n (by the definition of $Obs^{\mathfrak{M}}$, and since in each game the set of characters on the screen can only grow). For $t \in \mathrm{R}$ we define γ^t as follows. If $t \geq 0$, then

$$(\gamma^t, \pi^t, m) \stackrel{\text{def}}{=} \delta(sf, \delta(m, s_n)(0))(t),$$

and if $t < 0$, then

$$(\gamma^t, \pi^t, m) \stackrel{\text{def}}{=} \delta(sb, \delta(m, s_n)(0))(-t).$$

[88] Actually, the other direction also holds if e.g. $\mathfrak{M} \models \mathbf{Ax5}$, or if $(\forall m)(\forall H \in Rng(w_m)) H \cap Obs \neq \emptyset$ (i.e. if on each point of space-time there is at least one observer).

Notice that $\gamma^t(m)_2 = t_0 + t$, where $t_0 = \gamma_n(m)_2$ (i.e. t_0 is what m's clock shows in the state s_n). Also note that $\gamma^t(k)_0$ is the "position" of k according to γ^t. Now,

$$tr_m k \stackrel{\text{def}}{=} \{\langle t_0 + t, \gamma^t(k)_0\rangle : t \in \mathrm{R}\}.$$

Assume now that $ph \in Ph^{\mathfrak{M}}$. The definition of $tr_m(ph)$ is completely analogous: Let $n \in \omega$ be the smallest number such that $m \in Dom(\gamma_n)$, $ph \in Dom(\pi_n)$. For any $t \in \mathrm{R}$ we define γ^t, π^t exactly as in the previous case, and we define

$$tr_m(ph) \stackrel{\text{def}}{=} \{\langle t_0 + t, \pi^t(ph)_0\rangle : t \in \mathrm{R}\}.$$

We are ready to define $W^{\mathfrak{M}}$:

$$W^{\mathfrak{M}} \stackrel{\text{def}}{=} \{(m, t, p, k) : m \in Obs^{\mathfrak{M}}, k \in B^{\mathfrak{M}}, (t, p) \in tr_m k\}.$$

By the above, $\mathbf{gm}(\underline{C}, g) \stackrel{\text{def}}{=} \mathfrak{M} \stackrel{\text{def}}{=} \langle B^{\mathfrak{M}}, \ldots, W^{\mathfrak{M}}\rangle$ has been defined.

Let g, g' be games of $\underline{C}$. We say that g' is a <u>*refinement*</u> of g if there is an injective monotonic function $f : \omega \to \omega$ such that $g'(f(n)) = (t'_n, i_n, s_n)$ where $g(n) = (t_n, i_n, s_n)$ for all $n \in \omega$. I.e. for all n, in the $f(n)$-th step of the game g' we give the same input i_n in the same state s_n as in the n-th step of the game g (but maybe at a different time).

PROPOSITION 5.4 *Assume that $\underline{C}$ is a commuting cartoon model. Assume that m, k occur on the screen during the game g of $\underline{C}$. Then $k \in w_m(t, p)$ in $\mathbf{gm}(\underline{C}, g)$ iff there is a refinement g' of g such that at some step in g' the main character on the screen is m with clock showing t, and k is also on the screen, at position p.*

We omit the proof of the above proposition.[89]

We are going to prove that the function $\mathbf{gm}$ is in some sense the inverse of the function $\mathbf{cm}$. For any game g of $\underline{C}$, frame model $\mathfrak{M}$, and set H we define

$B(g) \stackrel{\text{def}}{=} \bigcup\{Dom(g(n)_0) \cup Dom(g(n)_1) : n \in \omega\}$, and

$\mathfrak{M}\lceil H \stackrel{\text{def}}{=} \langle B \cap H, \mathfrak{F}, G; Obs \cap H, Ph \cap H, Ib \cap H, E, W \cap (H \times B \times \ldots \times H)\rangle$.

$B(g)$ is the set of characters and photons (bodies) occurring on the screen during the game g, and $\mathfrak{M}\lceil H$ is the usual restriction of $\mathfrak{M}$ to H.

[89] We could have taken Prop. 5.4 as the definition of $\mathbf{gm}(\underline{C}, g)$.

THEOREM 5.5 *(i) and (ii) below hold.*

(i) $\mathbf{gm}(\underline{C}, g) \models \{\mathbf{Ax1} - \mathbf{Ax4}, \mathbf{Ax6_{00}}, B = Obs \cup Ph\}$ *if* $\underline{C}$ *is commutative, and* $g \in Games(\mathbf{cm}(\mathbf{gm}(\underline{C}, g)))$.

(ii) *Let* $\mathfrak{M} \in \mathsf{Mod}(\{\mathbf{Ax1} - \mathbf{Ax4}, \mathbf{Ax6_{00}}, B = Obs \cup Ph, W(m, \ldots) \to Obs(m)\})$ *and* $g \in Games(\mathbf{cm}(\mathfrak{M}))$. *Then*

$$\mathbf{gm}(\mathbf{cm}(\mathfrak{M}), g) = \mathfrak{M} \lceil B(g).$$

Proof: First we prove (i). $\mathbf{Ax1}, \mathbf{Ax2}$ hold trivially by our construction. To check $\mathbf{Ax3}$, let $m \in Obs^{\mathfrak{M}}$, $k \in B^{\mathfrak{M}}$. Then the trace of k in m's world-view is $tr_m k$ as we defined it in the construction of $\mathbf{gm}(\underline{C}, g)$. We have to show that $tr_m k \in \mathsf{Eucl}(\underline{n} + 1, \mathfrak{R})$. This holds by the properties of the effects of the inputs sf, sb in a cartoon model. $\mathbf{Ax4}$, i.e. $tr_m m = \bar{t}$, holds because the main character is standing in the origo (this is $(R1)$). To check $\mathbf{Ax6_{00}}$ let $m, k \in Obs^{\mathfrak{M}}$ and let $H \stackrel{\text{def}}{=} w_m(t, p)$ for some $(t, p) \in tr_m k$. We have to show that $H = w_k(t', \bar{0})$ for some t'. We will need commutativity of $\underline{C}$ here.

Notice first that $tr_m k$ was defined so that if $\underline{C}$ is commutative then $(t, p) \in tr_m k$ iff the game g can be refined so that for some $n \in \omega$, at step n, m is the main character with its clock showing t and k is on the screen, at position p. (This was stated as Proposition 5.4.) Now $b \in H = w_m(t, p)$ means that $(t, p) \in tr_m(b)$, i.e. there is a step in the game when m is the main character with clock t, and both k, b are at position p. Let k's clock show t' at this occasion. Then if we click on k, then b will get to position $\bar{0}$ (together with k), and k's clock will show t'. Thus $(t', \bar{0}) \in tr_k(b)$, i.e. $b \in w_k(t', \bar{0})$. This shows $H \subseteq w_k(t', \bar{0})$. Assume now $b \in w_k(t', \bar{0})$, i.e. $(t', \bar{0}) \in tr_k(b)$. Let n be a step when m, b, k are all on the screen, and m's clock shows t. Then, by commutativity, k is at position p and its clock shows t'. If we click on k now, then k's clock will show t' and, by commutativity again, b will be at the same position as k, so, by rule $(R5)$, b was at the same position as k before clicking, i.e. b was at position p. Thus $b \in w_m(t, p)$ as was desired. We showed that $w_m(t, p) = w_k(t', \bar{0})$, i.e. $w_m[tr_m(k)] \subseteq Rng(w_k)$. The first part of (i) has been proved. The second part of (i) is not difficult to check. The proof of (ii) is straightforward but lengthy, we omit it. ∎

If we assume only commutativity of $\underline{C}$, then $\mathbf{cm}(\mathbf{gm}(\underline{C}, g))$ can be (rather) different from $\underline{C}$. (This is the case e.g. with Example 3.) We are going to define a property of cartoon models (which we will call "context-free"), we will show that $\mathbf{cm}(\mathfrak{M})$ is such for all frame models $\mathfrak{M}$, and if $\underline{C}$ is context-free (and commuting) then $\mathbf{cm}(\mathbf{gm}(\underline{C}, g))$ is the same as $\underline{C}$ (on the characters involved in g). Moreover, if $\underline{C}$ is context-free, then we will be able to define $\mathbf{gm}(\underline{C})$ without referring to a game g. (In some sense, $\mathbf{gm}(\underline{C})$ is the "union" of all the $\mathbf{gm}(\underline{C}, g)$'s.) We begin with defining the property "context-free".

Definition 5.6 **(I)** Let $H \subseteq Obs \cup Ph$, and let $(\gamma, \pi, m) \in S$, $f \in {}^{+\mathrm{R}}S$. Then

$(\gamma, \pi, m)\lceil H \overset{\text{def}}{=} (\gamma\lceil(H \cup \{m\}), \pi\lceil(H \cup \{m\}), m)$, and

$f\lceil H \overset{\text{def}}{=} \langle f(t)\lceil H : t \in {}^{+}\mathrm{R}\rangle$.

(II) Let $\underline{C} = \langle Obs, Ph, \delta\rangle$ be a cartoon model. We say that $\underline{C}$ is <u>*context-free*</u> iff there are functions $\mathcal{H} : Obs \times {}^{\underline{n}}\mathrm{R} \times {}^{\underline{n}}\mathrm{R} \times \mathrm{R} \longrightarrow \mathcal{P}(Obs)$ and $\mathcal{G} : Obs \times {}^{\underline{n}}\mathrm{R} \times {}^{\underline{n}}\mathrm{R} \longrightarrow \mathcal{P}(Ph)$ and a well-ordering $<$ on $Obs \cup Ph$ such that conditions (i)-(iv) below hold for all $m, k \in Obs$ and $s = (\gamma, \pi, m) \in S$ with $\gamma(m) = (\bar{0}, \bar{0}, 0)$; and for all $p, v \in {}^{\underline{n}}\mathrm{R}, t \in \mathrm{R}$.

(i) $Obs = \bigcup\{\mathcal{H}(m, p, v, t) : (p, v, t) \in I\}$, $Ph = \bigcup\{\mathcal{G}(m, p, v) : (p, v) \in I\}$.

(ii) $\delta((p, v, t), s)$ is defined iff $\mathcal{H}_0 \overset{\text{def}}{=} \mathcal{H}(m, p, v, t) \setminus \gamma^{-1}(p, v, t) \neq \emptyset$, and if $\mathcal{H}_0 \neq \emptyset$, then $\delta((p, v, t), s) = (\gamma \cup \{\langle min\mathcal{H}_0, (p, v, t)\rangle\}, \pi, m)$.

(iii) $\delta((p, v), s)$ is defined iff $\mathcal{G}_0 \overset{\text{def}}{=} \bigcup\{\mathcal{G}(m, p, \lambda \cdot v) : \lambda > 0\} \setminus \bigcup\{\pi^{-1}(p, \lambda \cdot v) : \lambda > 0\} \neq \emptyset$, and if $\mathcal{G}_0 \neq \emptyset$, then $\delta((p, v), s) = (\gamma, \pi \cup \{\langle min\mathcal{G}_0, (p, \lambda \cdot v)\rangle\}, m)$ for some $\lambda > 0$.

(iv) $\delta(i, s)\lceil\{k\} = \delta(i, s\lceil\{k\})$, for all $i \in \{sf, sb\}$, and $\delta(k', s)\lceil\{k', k, s_2\} = \delta(k', s\lceil\{k', k, s_2\})$.

$\triangleleft$

Examples 1,2,4 are context-free, Example 3 is not context-free.

LEMMA 5.7 *Assume that $\underline{C} = \langle Obs, Ph, \delta\rangle$ is context-free, and let $m \in Obs$. Then (i)-(ii) below hold.*

(i) *For every $k \in Obs$ there is a unique (p, v, t) such that $k \in \mathcal{H}(m, p, v, t)$. (I.e. $\langle\mathcal{H}(m, p, v, t) : (p, v, t) \in I\rangle$ is a partition of Obs.) Further, $m \in \mathcal{H}(m, \bar{0}, \bar{0}, 0)$.*

(ii) *For every $ph \in Ph$ there is a unique (p, v) with $|v| = 1$ such that $ph \in \bigcup\{\mathcal{G}(m, p, \lambda \cdot v) : \lambda > 0\}$. I.e. if $ph \in \mathcal{G}(m, p, v) \cap \mathcal{G}(m, p', v')$, then $p = p'$ and $v = \lambda \cdot v'$ for some $\lambda > 0$.*

Proof. For any $k \in Obs$ there is (p, v, t) such that $k \in \mathcal{H}(m, p, v, t)$, by item (i) in Def.5.6. Assume that $m \in \mathcal{H}(m, p, v, t)$ with $(p, v, t) \neq (\bar{0}, \bar{0}, 0)$. Let $Dom(\gamma) = \{k \in \mathcal{H}(m, p, v, t) : k \leq m\}$, and let γ be such that $\gamma(m) = (\bar{0}, \bar{0}, 0)$, $\gamma(k) = (p, v, t)$ for $k \in Dom(\gamma), k \neq m$. Then $m = min(\mathcal{H}(m, p, v, t) \setminus \gamma^{-1}(p, v, t))$ by $(\bar{0}, \bar{0}, 0) \neq (p, v, t)$, thus $\delta((p, v, t), (\gamma, \emptyset, m)) = (\gamma \cup \{\langle m, (p, v, t)\rangle\}, \emptyset, m)$ by item (ii) in Def.5.6. Now $\gamma(m) = (\bar{0}, \bar{0}, 0) \neq (p, v, t)$ gives a contradiction. Thus $m \in \mathcal{H}(m, \bar{0}, \bar{0}, 0)$.

The rest of the proof of Lemma 5.7 is analogous, we omit it. ∎

Context-freeness and commutativity together mean a strong kind of commutativity, where "commutativity holds between different games also". We include here, without proof, a lemma stating this.

LEMMA 5.8 *Let* $\underline{C} \in CM^-$. *Then (i),(ii) below are equivalent.*

(i) $\underline{C}$ *is commutative and context-free.*

(ii) *For all* $g, g' \in Games(\underline{C})$, *for all* $m, n \in \omega$, *if* $g(n) = (t_n, i_n, (\gamma_n, \pi_n, k_n))$, $g'(m) = (t_m, i_m, (\gamma_m, \pi_m, k_m))$, *and* $k_n = k_m$, $\gamma_n(k_n) = \gamma_m(k_m)$, *then* $\gamma_n(k) = \gamma_m(k)$ *and* $\pi_n(ph) = \pi_m(ph)$ *for all* $k \in (Dom(\gamma_n) \cap Dom(\gamma_m))$ *and* $ph \in (Dom(\pi_n) \cap Dom(\pi_m))$.

CM denotes the class of all commuting and context-free cartoon models.

Lemma 5.7 enables us to define a frame model to each context-free cartoon model, without referring to a specific game $g \in Games(\underline{C})$.

Definition 5.9 **(i)** Let $m \in Obs$. First we define $p_m(ph)$ and $v_m(ph)$ for all $ph \in Ph$.

$$p_m(ph) \stackrel{\text{def}}{=} p \quad \textit{iff} \quad ph \in \mathcal{G}(m, p, v) \quad \textit{for some } v.$$

We will define $v_m(ph)$ by induction (along $<$). Assume that $v_m(pht')$ has already been defined for all $ph' < ph, ph' \in Ph$. Let $\pi : Ph \xrightarrow{\circ} ({}^{\underline{n}}\mathrm{R} \times {}^{\underline{n}}\mathrm{R})$ be as follows. $Dom(\pi) = \{ph' \in Ph : ph' < ph\}$ and $\pi(ph') = (p_m(ph'), v_m(ph'))$. Let $s \stackrel{\text{def}}{=} (\{\langle m, (\bar{0}, \bar{0}, 0)\rangle\}, \pi, m)$ (i.e. s is the state in which m is the main character with clock showing 0, and where all the other characters on the screen are photons, exactly those preceding (in $<$) ph, and with the already specified positions and velocities). Assume that $ph \in \mathcal{G}(m, p, v)$, and let $\delta((p, v)), s) = (\gamma, \pi', m)$. Then $ph \in Dom(\pi')$ by item (iii) in Def.5.6. We define

$$v_m(ph) \stackrel{\text{def}}{=} \lambda \cdot v \quad \textit{where} \quad \pi'(ph) = (p, \lambda \cdot v).$$

(ii) We are ready to define $\mathfrak{M} \stackrel{\text{def}}{=} \mathbf{gm}(\underline{\mathrm{C}})$ as follows.

$Obs^{\mathfrak{M}} \stackrel{\text{def}}{=} Obs$, $Ph^{\mathfrak{M}} \stackrel{\text{def}}{=} Ph$, $B^{\mathfrak{M}} \stackrel{\text{def}}{=} Ib^{\mathfrak{M}} \stackrel{\text{def}}{=} Obs \cup Ph$,

$\mathfrak{F}^{\mathfrak{M}} \stackrel{\text{def}}{=} \mathfrak{R}$, $G^{\mathfrak{M}} \stackrel{\text{def}}{=} \mathsf{Eucl}(\underline{\mathrm{n}}+1, \mathfrak{R})$,

$$W^{\mathfrak{M}} \stackrel{\text{def}}{=} \{(m, t', p', k) : m \in Obs, (\exists p, v, t)[k \in \mathcal{H}(m, p, v, t), t' \in \mathrm{R}, p' = p + t' \cdot v]\} \cup \{(m, t', p', ph) : m \in Obs, t' \in \mathrm{R}, p' = p_m(ph) + t' \cdot v_m(ph)\}.$$

$\lhd$

Let $CM^+ \stackrel{\text{def}}{=} \{\underline{\mathrm{C}} \in CM^- : \underline{\mathrm{C}}$ *is commuting and context-free*$\}$ and $FM^+ \stackrel{\text{def}}{=} \mathsf{Mod}(\{\mathbf{Ax1} - \mathbf{Ax4}, \mathbf{Ax6_{00}}, B = Ib = ...\})$. The next theorem says that $\mathbf{gm} : CM^+ \longrightarrow FM^+$ and $\mathbf{cm} : FM^+ \longrightarrow CM^+$ are inverses of each other, i.e. $\mathbf{gm} \circ \mathbf{cm} = \mathrm{Id}$ and $\mathbf{cm} \circ \mathbf{gm} = \mathrm{Id}$. See Figure 51.

THEOREM 5.10 *(i) and (ii) below hold.*

(i) $\mathbf{cm}(\mathfrak{M})$ *is commuting and context-free.*

(ii) *Let $\underline{C}$ be a commuting and context-free cartoon model. Then*

$\mathbf{cm}(\mathbf{gm}(\underline{C})) = \underline{C}$, *and*

$\mathbf{gm}(\underline{C}, g) = \mathbf{gm}(\underline{C}) \lceil B(g)$ *for every* $g \in Games(\underline{C})$.

Proof. The proofs are straightforward but tedious, we omit them. ∎

Theorem 5.10 also gives a characterization of the cartoon models obtainable from frame ones. In this sense, it tells us about some implicit (or hidden) assumptions which we make about the physical world when we model it with frame models. (Cartoon models are intended for providing a kind of conceptual analysis of our relativity theory.)

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Figure 49: .

Figure 50: .

Figure 51: .