Analyzing the Logical Structure of Relativity Theory via Model Theoretic Logic

(Lecture Notes for A'dam Course Spring 1998)

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comments are welcome

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Contents

1 Introduction

1.1 Broad introduction

This is a lecture notes for a course given March 1 - April 20, 1998 in CCSOM of the University of Amsterdam. The course description included below also serves as part of the present introduction.

Course description

This course is mainly designed for logicians and people having a strong logical background who are interested in using logic for gaining a deeper understanding of (at least part of) reality. This might include e.g. people studying the methodology or philosophy of sciences.

Historical perspective. Tarski formalized geometry as a theory of first order logic. The point here is to use only first order logic; no external "devices" or tacit assumptions are allowed to enter the picture. Motivated by Tarski, P. Suppes raised the problem of formalizing the theory of special relativity as a theory purely in first order logic. We will do this, and some more. Our version of relativity theory will consist of a small number of axioms which are not only in first order logic, but are moreover easily comprehensible.

Logical core. In a few papers, Johan van Benthem elaborates the idea of separating out the "logical cores" of certain logics. The idea here is separating out the really essential part (from the logical point of view) from the whole "burden of mathematical machinery attached to the subject in the course of time". In the same "Benthemian" spirit we will hereby clearing the way for asking logical questions, concerning e.g. the logical structure of the theory, the number of non-elementarily-equivalent models, classification of models, etc. Among other things, we will use logic to find out which axioms are responsible for certain surprising predictions of relativity theory like e.g. "no observer can move faster than the speed of light", "the twin paradox" or issues concerning the (im)possibility of time travel.

Temporal logic. Temporal and modal logics of relativity theory have been around in the literature. We will explore the uses of our first order theory of relativity as a foundation for such temporal logics.

Searching for insight. All discussions will be in terms of simple concepts. When formalizing our (language and) axioms we will confine ourselves to a very plain language, using such understandable expressions as "bodies" or "observers". Whenever we need more complex expressions like “energy”, “entropy” or “curvature of space-time”, then we will first define these, as a logician would do, in terms of our plain language. This way we hope to gain real insight into *why* certain exotic predictions of relativity theory are “predicted”. It also allows us to make the axioms with which we started subject to debate: both because of the plain language in which they are expressed and because of the purely logical nature of our reasoning.

Future perspectives. Hawking, Weinberg and others suggested the possibility of a final Theory of Everything. In the literature it is often argued that Gödel’s incompleteness theorem renders such theory impossible. This is a challenge for the logician. After the course, we will be in a better position for progress in this area. Besides first order logic, modal logic can be used to improve the theory. We plan to discuss also temporal logic for the relevant aspects of relativity.

Prerequisites. Before anything else, a good working knowledge of first order logic (FOL) is the most important prerequisite: model theory of FOL, many-sorted FOL. Some knowledge of linear algebra and fields would be desirable. Familiarity with first order modal logic and with the ideas of non-standard analysis would be useful, but not indispensable.

1.2 More specific introduction

About our motivation and aims:

Some works of Tarski and his followers [39] are devoted to the formalization of geometry as a theory of first order logic. The point is to use *only* first order logic, and no "external" devices or tacit assumptions are allowed. In analogy with that paper of Tarski, here we try (among others) to formalize relativity theory purely in first order logic. Using nothing external, no tacit assumptions etc. Actually, the problem of elaborating such a formalization was raised by P. Suppes [41] (based on discussions with Tarski). Cf. also Ax [5], Mundy [32][1]. After formalizing the theory we also develop it to some extent and then use the formalized version to analyze the logical structure of the theory. First we concentrate on special relativity, then we move to accelerating observers, and then explore the possibilities of moving in the direction of general relativity.

An additional motivation is the following. First we quote from the book Matolcsi [30, p.11].

> "Mathematics reached a crisis at the end of the last century when a number of paradoxes came to light. Mathematicians surmounted the difficulties by revealing the origin of the troubles: the obscure notations, the inexact definitions; then the modern mathematical exactness was created and all the earlier notions and results were reappraised. After this great work nowadays mathematics is firmly based upon its exactness.
>
> Theoretical physics — in quantum field theory — reached its own crisis in the last decades. The reason of the troubles is the same. Earlier physics has treated common, visible and palpable phenomena, everything has been obvious."
>
> ...
>
> "It is quite evident, that we have to follow a way similar to that followed by mathematicians to create a firm theory based on mathematical exactness; having mathematical exactness as a guiding principle, we must reappraise physics, its most common, most visible and most palpable notions as well. Doing so we can hope we shall be able to overcome the difficulties."

Mathematics solved the above problem by using *logic*. Here we will experiment with doing the same in relativity theory, that is, build up (at least parts of) relativity

[1] For some of the reasons why we want to stick with first order logic see e.g. Ax [5], Mundy [32] and da Costa et al [8].

theory in first order logic. The above quoted work Suppes [41] asks for a formalization of special relativity in first order logic (e.g. as a kind of continuation of Tarski et al.'s formalization of geometry). (Axiomatizations of parts of relativity had been available before, but they were not developed in purely first order logic.) Among others, we will do this and some more in the following sense. E.g.,

(i) We will also study the possibility of moving beyond special relativity in the direction of general relativity, and will look into having accelerating observers and observers who do not "observe" all events that might be "observed" by other observers (cf. $\mathbf{Ax6}_{00}$ in subsection 3.2). Also,

(ii) We will look into possibilities of making our theory more general by making the axioms more flexible (i.e., weaker), and studying the number and structure of all complete theories extending our flexible theory. Further, to eliminate tacit assumptions and other kinds of "implicit understandings", our theory will have objects in it which are more intuitive than just Minkowskian geometry. (These are things like bodies, inertial bodies, photons etc.) Some purists might think that these are superfluous, but, as it will turn out, they are absolutely needed for our purposes.

So let's get started. We want to develop a kinematics.

- What is kinematics?
- A theory of *motion*.
- What moves?
- Idealization: We assume that there are things called *bodies* (like "heavenly bodies") and they move.
- How do bodies move?
- Idealization: They change their locations.
- What does change of location mean?
- At different *time instances* the same body has different *locations*.

OK, then there are time instances and locations involved (whatever they are). Let us fix that. Our paradigm says that time instances and locations are only relative to something which we will call *observers*. So we assume that there are observers (special bodies). Given an observer m, time instance t and location s, m may "observe" certain body b at $\langle t, s\rangle$ while m may observe other bodies b_1 not at $\langle t, s\rangle$. This simply means that from the point of view of m, b is present at location s at time t. We treat this concept of observing as primitive and denote it as $b \in w_m(t, s)$. That is, $w_m(t, s)$ is the set of bodies present at $\langle t, s\rangle$ from the point of view of m. We should emphasize that this kind of observing has nothing to do with the intuitive notion of observing in the form of measurement.

- What are time instances t and locations s?

- Our first answer is that they are "labels" used by observers. But sooner or later we will have to be more specific. So let us see what t is.

We agree that, for an observer m, time instances are "quantities" like 100, 500, 1/2. To be on the safe side, we *do not decide* what quantities are, we only postulate that they satisfy some simple axioms which in themselves are intuitively convincing. Namely, we assume that quantities form an ordered field $\mathfrak{F} = \langle F, +, \cdot, \leq \rangle$, that is, $\mathfrak{F}$ satisfies the usual axioms (to be recalled in subsection 2.1 from, e.g., [7]). The time scale of observer m is simply $\mathfrak{F}$ itself, the neutral element 0 of $\mathfrak{F}$ means "*now*", $t > 0$ represents "*future*" and $t < 0$ represents "*past*". For simplicity, we agree that locations s are represented by triplets of quantities $s = \langle s_1, s_2, s_3 \rangle \in {}^3F$.

This might suggest that we assume the usual Euclidean geometry structure (or the vector-space structure) of ${}^3\mathfrak{F}$ on our set ${}^3F$ of (spatial) locations. But we are *not* doing that. (At a certain point we might [or might not] make such a decision, but *then* it will be a reasoned decision.) The only thing we agreed on is to represent locations by triplets of quantities, or by triplets of "coordinates" from the field $\mathfrak{F}$. It is pairs $p = \langle t, s \rangle$ of time instances and locations for which we say that a body b occurs there (at $\langle t, s \rangle$) for observer m. We call such pairs points of <u>*space-time*</u>. Therefore points of space-time are elements $p = \langle p_0, \ldots, p_3 \rangle \in {}^4F$. We call p_0 the <u>*time coordinate*</u> and $\langle p_1, \ldots, p_3 \rangle$ the <u>*space coordinates*</u> of p. At this point, we notice that space-time happens to form a vector space ${}^4\mathfrak{F}$ over the field $\mathfrak{F}$. (We are not assuming this, instead, it is a consequence of the convention we made so far.) However, we are *not* assuming that this vector space structure of our space-time ${}^4\mathfrak{F}$ would have any physical meaning or significance. On the other hand, we feel free to *use* this vector space structure for notational purposes as part of our syntax (that is, part of our language). Actually, ${}^4\mathfrak{F}$ forms a Euclidean vector space but we are not heavily relying on that now (cf. Matolcsi [30], Lánczos [25]).

Although space-time is four-dimensional, many of the ideas (and proofs) can be illustrated already in two or three dimensions. We will try to keep our presentation as simple as possible. Therefore we will sometimes *pretend* that space-time is 2-dimensional *but* we will go up to 3 or 4 dimensions as soon as the higher dimensional case would behave differently.

As we said, to each point $p \in {}^4F$ of space-time, an observer m associates a set $w_m(p)$ of bodies which, for m, are present at point p. Therefore, to each observer m, we associate a function $w_m : {}^4F \longrightarrow \mathcal{P}(B)$ mapping space-time ${}^4F$ into the powerset $\mathcal{P}(B)$ of the set B of bodies. We call the elements of $\mathcal{P}(B)$ "<u>*events*</u>". Matolcsi [30] calls them occurrences. For us an event is nothing but information telling us which bodies are present and which are absent. (This is why [30] calls them *occurrences*.) Therefore we can identify an event by a subset of B.

Besides the aims stated at the beginning of this introduction: building up relativity theory purely in first order logic, we intend to explore the *inductive–logical structure* of relativity theory, too. For example, textbooks on relativity theory often start with claiming that, in relativity theory, space-time (mathematically speaking) is nothing but a "*manifold augmented with a Euclidean metric*". Seeing this statement the first time, it is natural to get puzzled at it, and we ask ourselves: *Why* is space-time *just* a manifold augmented with a Euclidean metric? You could answer: Well, because Nature is such. OK, but we may ask: Is this the most natural mathematical model of space-time? Though, e.g., [30] gives excellent intuitive motivation for accepting space-time to be a manifold augmented with a Euclidean metric, we still ask: How did physicists/mathematicians arrive at this mathematical model? Could one build other, different mathematical models for space-time which would reflect the (known) physics of space-time as correctly as a manifold augmented with a Euclidean metric? Another question: Could one find some *logically natural* axiomatization for a mathematical model of space-time (be it a manifold augmented with a Euclidean metric or not)? Which are the *necessary* logical assumptions for developing relativity theory?

More on our aims and some outline of this work:

As already indicated, we will formalize our theory purely in (many-sorted) first order logic. In particular, all our axioms will be formulas of first order logic.

We do not want to make our axioms generate a complete theory.[2] Our purpose is the opposite: we want to make our axioms as weak (and therefore intuitively acceptable and convincing) as possible while still (strong enough for) proving interesting theorems of relativity theory.[3]

When introducing a new axiom, say **Ax**, we will investigate why **Ax** is plausible, why we (or the student) should believe in **Ax**, why we need it, and what would happen if we would drop it. This way we will obtain a relatively small set, called *Basax*, of axioms (for basic axioms). We will study extra axioms, too, which we will

[2]A theory T is called *complete* iff for every sentence φ in the language of T, exactly one of φ and $(\neg\varphi)$ follows from T.

[3]The situation is somewhat analogous with the difference between classical number theory studying the standard model $\underline{Z} = \langle Z, +, \cdot, -, 0, 1\rangle$ consisting of the set Z of integers, contrasted with, say, a part of abstract algebra, e.g. ring theory (or the theory of fields) where we study a broad class K of all rings of which $\underline{Z}$ is only a very special element. Sometime when we prove theorems about K, we say (or feel) that we understand more (or better) why that theorem is true for $\underline{Z}$. In this analogy, classical, standard special relativity is analogous with the complete theory of $\underline{Z}$ while the version we are describing here is analogous with the algebraic theory of K. (We note, however, that this analogy is imperfect, as often happens with analogies.)

call experimental axioms. We will investigate how many different complete theories $T \supseteq Basax$ exist, which are possible consistent extensions of $Basax$. (In a sense, our study of $Basax$ can be considered as a unified study of all these "possible relativity theories" T.) Such a complete T extending $Basax$ will be called a "possible (special) relativity theory". Equivalently, we will also study how many models $\{\mathfrak{M}_i : i \in I\}$ of $Basax$ exist such that they are mutually not elementarily equivalent, that is, such that $i \neq j \Rightarrow \mathfrak{M}_i \not\equiv_{ee} \mathfrak{M}_j$. We will also attempt a structural description of the essentially different kinds of models of $Basax$.

We will study which interesting theorems of relativity theory can be proved from $Basax$ (e.g., surprisingly, little of our axioms is needed for deriving the so called twin paradox). We will look also into which further theorems need what extra axioms etc. (Of course, we will try to obtain new theorems, too.) After carrying out the above kind of investigation, we will look into the question of how to extend the outlined kind of investigation to accelerating observers (here gravity appears), to very fast accelerating observers (here some kind of "black-hole-ish" features appear), and towards general relativity (and towards ideas up in Gödel's papers on cosmology and relativity, see e.g. Gödel [17]).

As further references we mention [18], [22], [37], [40], [45], [46].

Acknowledgement: Chapter 3 of this work was written by J. Madarász and, together with certain other parts of this work, will be used as material for her dissertation.

2 Special Relativity

2.1 Frame language of relativity theory; world view function

Some set theoretical notation and convention:

ω denotes the set of all <u>natural numbers</u> $\{0, 1, \ldots, n, \ldots\}$. We use von Neumann's concept of natural numbers, that is,

$0 \stackrel{\text{def}}{=} \emptyset$ ($\emptyset$ denotes the empty set) and

$n + 1 \stackrel{\text{def}}{=} n \cup \{n\} = \{0, \ldots, n\}$ for every $n \in \omega$.

R denotes the set of all <u>*real numbers*</u>.

Z denotes the set of all <u>*integers*</u>.

For any set H, $\mathcal{P}(H)$ denotes the <u>*powerset*</u> of H, that is,

$\mathcal{P}(H) = \{X : X \subseteq H\}$.

If g is a function then $Dom(g)$ and $Rng(g)$ denote its <u>*domain*</u> and <u>*range*</u>, respectively. That is:

$Dom(g) \stackrel{\text{def}}{=} \{a : \langle a, b\rangle \in g\}$ and

$Rng(g) \stackrel{\text{def}}{=} \{b : \langle a, b\rangle \in g\}$.

$g : A \longrightarrow B$ or $A \stackrel{g}{\longrightarrow} B$ denote that $Dom(g) = A$ and $Rng(g) \subseteq B$.

For an arbitrary set H and $n \in \omega$, we often identify the set

${}^nH \stackrel{\text{def}}{=} \{g : Dom(g) = n \text{ and } Rng(g) \subseteq H\}$ with the Cartesian power

$\underbrace{H \times \ldots \times H}_{n\text{-times}} \stackrel{\text{def}}{=} \{\langle h_0, \ldots, h_{n-1}\rangle : (\forall i < n) h_i \in H\}$. Thus, in particular,

${}^2H = H \times H$.

If R and S are binary relations (i.e., sets of pairs), then their composition $R \circ S$ is defined as

$R \circ S \stackrel{\text{def}}{=} \{(a, b) : (\exists c)[(a, c) \in R \ \wedge \ (c, b) \in S]\}$.

A function $f : A \to B$ is considered to be the set of pairs $f = \{(a, f(a)) : a \in A\}$. If f and g are functions with $Rng(f) \subseteq Dom(g)$ then we write their *composition* the following way[4]:

$(f \circ g)(x) \stackrel{\text{def}}{=} g(f(x))$ for every $x \in Dom(f)$. $\lhd$

We will start our formal exposition to relativity theory with fixing a 3-sorted first order language. We will call this language the *frame-language of relativity theory*.[5] We will use this language for axiomatizing our mathematical model of relativity.

[4]This is usually used in the reverse order in the literature.

[5]Later we will expand our frame-language with a *metric* $d : {}^nF \times {}^nF \longrightarrow F$, also called *distance*, see subsection 4.1 in section 4.

Before giving the definition of our frame-language, we recall some of the standard notation and terminology used in first order logic from [7].

Let Fm and M denote, respectively, the set of all formulas and the class of all models of an arbitrary first order language. Then $\models$ ($\subseteq \mathsf{M} \times Fm$) denotes the *validity relation* of this language. We extend $\models$ to $\mathcal{P}(\mathsf{M}) \times \mathcal{P}(Fm)$ the usual way: Let $\mathsf{K} \subseteq \mathsf{M}$ and $\Sigma \subseteq Fm$. Then

$$\mathsf{K} \models \Sigma \ \text{ iff } \ (\forall \mathfrak{M} \in \mathsf{K})(\forall \varphi \in \Sigma)\mathfrak{M} \models \varphi\,.$$

We will write $\mathsf{K} \models \varphi$ in place of $\mathsf{K} \models \{\varphi\}$ and $\mathfrak{M} \models \Sigma$ when $K = \{\mathfrak{M}\}$.

$$Th(\mathsf{K}) \stackrel{\text{def}}{=} \{\varphi \in Fm : \mathsf{K} \models \varphi\}$$

is the *theory* of K, and

$$\mathsf{Mod}(\Sigma) \stackrel{\text{def}}{=} \{\mathfrak{M} \in \mathsf{M} : \mathfrak{M} \models \Sigma\}$$

is the *class of all models* of Σ. Let $\varphi \in Fm$. Then we say that φ is a *semantical consequence* of Σ, in symbols $\Sigma \models \varphi$, iff $\mathsf{Mod}(\Sigma) \models \varphi$.

Definition 2.1 (frame-language of relativity theory)
Let B, Q and G denote three sorts called *bodies*, *quantities* and *geodesics* (or *lines* or *geometry*), respectively. Let a natural number $n > 1$ be fixed.[6] We are defining a first order language with set of sorts[7] $\{\mathrm{B}, \mathrm{Q}, \mathrm{G}\}$ by first defining its models, as follows. $\mathfrak{M}$ is a *model (of dimension n) of this language* iff

$$\mathfrak{M} = \langle B^{\mathfrak{M}}, F^{\mathfrak{M}}, G^{\mathfrak{M}};\ Obs^{\mathfrak{M}}, Ph^{\mathfrak{M}}, Ib^{\mathfrak{M}}, +^{\mathfrak{M}}, \cdot^{\mathfrak{M}}, \leq^{\mathfrak{M}},\ \mathrm{E}^{\mathfrak{M}}\ , W^{\mathfrak{M}}\rangle\,,$$

also denoted as

$$\mathfrak{M} = \langle B, F, G;\ Obs, Ph, Ib, +, \cdot, \leq,\ \mathrm{E}\ , W\rangle$$

for brevity, where

- B is a nonempty set, this is $\mathfrak{M}$'s universe of sort B. B is called the set of *bodies* (of $\mathfrak{M}$).

[6] We will be interested only in the case $n \in \{2, 3, 4\}$, but we give definitions and lemmas for arbitrary n if this does not cost any extra effort. Intuitively, n will be the dimension of our space-time.

[7] Many-sorted logic is known to be reducible to one-sorted logic the following way (cf. Monk [31], Enderton [11]). We will use $B \cup F \cup G$ as the universe of our one-sorted model and call B, Q, G unary predicates.

- $\mathfrak{F} := \langle F, +, \cdot, \leq \rangle$ is a linearly ordered field. That is, the following set $\mathbf{Ax}_{\mathrm{OF}}$ of axioms is satisfied by $\langle F, +, \cdot, \leq \rangle$.

  $\mathbf{F} := \langle F, +, \cdot \rangle$ is a field[8]
  $\langle F, \leq \rangle$ is a linear order, and for every $a, b \in F$,
  $a \leq b \Rightarrow (\forall c \in F)(a + c \leq b + c)$ and
  $(a \leq b$ and $c > 0) \Rightarrow (c \cdot a \leq c \cdot b)$ hold.

  F is $\mathfrak{M}$'s universe of sort Q. Intuitively, F serves both to be our "*time scale*" and "*space scale*". We will denote the ordered field $\langle F, +, \cdot, \leq \rangle$ by $\mathfrak{F}$ and its field reduct $\langle F, +, \cdot \rangle$ by $\mathbf{F}$. Sometimes we write $\mathfrak{F}^{\mathfrak{M}}$ for $\mathfrak{F}$ ($\mathbf{F}^{\mathfrak{M}}$ for $\mathbf{F}$) when we want to indicate explicitly that we look at $\mathfrak{F}$ ($\mathbf{F}$) as the "quantity part" of $\mathfrak{M}$. We note that a linearly ordered field is infinite.

- G is a nonempty set, this is $\mathfrak{M}$'s universe of sort G. G is called (the set of) *geodesics* (or *lines*). Intuitively, geodesics represent motion of inertial bodies.

- $Obs, Ph, Ib \subseteq B$ are unary relations (of sort B). Their names are: set of *observers*, set of *photons*, and set of *inertial bodies* (or *lonely bodies*), respectively.

- $\mathrm{E} \subseteq {}^nF \times G$ is an $(n+1)$-ary relation of sort $\langle \mathrm{Q}, \dots, \mathrm{Q}, \mathrm{G} \rangle$. Intuitively, for $p = \langle p_0, \dots, p_{n-1} \rangle \in {}^nF$ and $\ell \in G$, $\mathrm{E}\ (p_0, \dots, p_{n-1}, \ell)$ expresses that the point $p \in {}^nF$ is on the line ℓ. If p and ℓ are as above, we abbreviate $\mathrm{E}\ (p_0, \dots, p_{n-1}, \ell)$ by $p \mathrel{\mathrm{E}} \ell$. We postulate axiom $\mathbf{Ax}_{\mathrm{G}}$ below, called the *extensionality* of lines.

  $\mathbf{Ax}_{\mathrm{G}}$ $(\forall \ell_1, \ell_2 \in G)((\forall p \in {}^nF)(p \mathrel{\mathrm{E}} \ell_1 \Leftrightarrow p \mathrel{\mathrm{E}} \ell_2) \Rightarrow \ell_1 = \ell_2)$.

  Here we note that the axiom of extensionality allows us to identify $\ell \in \mathrm{G}$ with a subset of ${}^nF$. Indeed, we will identify ℓ with the set $\{p \in {}^nF : p \mathrel{\mathrm{E}} \ell\}$ (which is sometimes called the extension of ℓ). With this identification we may assume that $G \subseteq \mathcal{P}({}^nF)$ and E is the real "element-of" relation, $\in$. We will do this from now on.[9]

[8] For completeness, we recall here the definition of a field. $\langle F, +, \cdot \rangle$ is called a *field* iff
$\langle F, + \rangle$ is a commutative group, we let 0 denote its neutral element;
$\langle F \setminus \{0\}, \cdot \rangle$ is a commutative group, we let 1 denote its neutral element;
$\cdot$ distributes over $+$ "on both sides", that is,
$a \cdot (b + c) = a \cdot b + a \cdot c$ and $(b + c) \cdot a = b \cdot a + c \cdot a$ hold for every $a, b, c \in F$.

[9] This is a standard technique for handling higher order objects of a logic.

- $W \subseteq B \times {}^nF \times B$, that is, W is an $n+2$-ary relation of sort $\langle \mathrm{B}, \underbrace{\mathrm{Q}, \ldots, \mathrm{Q}}_{n\text{-times}}, \mathrm{B} \rangle$.

  W is called the <u>*world view relation*</u> (of $\mathfrak{M}$). The most important part of our model is the world view relation, W. Intuitively, $W(m,t,x,y,z,h)$ means that the observer m "observes" or "sees"[10] the body h at time t at location $\langle x,y,z\rangle$. From the $(n+2)$-ary relation W and arbitrary observer $m \in Obs$ we define the <u>*world view function*</u> $w_m : {}^nF \longrightarrow \mathcal{P}(B)$ as follows:

$$w_m(p) \stackrel{\text{def}}{=} \{h \in B : W(m,p,h)\} \text{ for every } p \in {}^nF .$$

  For $p \in {}^nF$, we call the set $w_m(p)$ of bodies the <u>*event*</u> at p as seen by m.[11] Intuitively, w_m defines the "subjective reality" of m. That is, w_m tells us how observer m "arranges" the events (elements of $\mathcal{P}(B)$) in the coordinate-system ${}^nF$; in other words, w_m tells us how m "coordinatizes" the events $\mathcal{P}(B)$.

Thus the similarity type of our first order language consists of

the unary relation symbols Obs, Ph, Ib (most often, their interpretations in models are denoted by Obs, Ph, Ib as well);
the symbols $+, \cdot, \leq$ of the ordered field $\mathfrak{F}$ (sometimes the neutral elements 0 and 1 will also be treated as basic symbols);
the (n+1)-ary relation symbol E ;
the (n+2)-ary relation symbol W.
The reduct $\langle B, Obs, Ph, Ib\rangle$ of $\mathfrak{M}$ is purely of sort B (body);
$\mathfrak{F} = \langle F, +, \cdot, \leq\rangle = \langle \mathbf{F}, \leq\rangle$ is purely of sort Q (quantities);
G is the universe of sort G (geodesics), and there are no relation or function symbols which would be purely of sort G.
E acts between sorts Q and G, while W involves B and Q.

[10] We want to emphasize that here "observing" or "seeing" has nothing to do with the intuitive notion of observing in the form of measurement, or with the everyday notion of seeing via photons. In the present text, "observer" and "observing" are technical expressions which we use for historical reasons. Our "observing" is really a kind of coordinatizing, i.e. when we say that observer m observes event e at coordinates t,x,y,z, we mean only to say that m associates coordinates t,x,y,z to event e. (As opposed to "real observing", this is a very abstract act only.) By the word "observer" we mean what is sometimes called <u>*frame of reference*</u> or "system of reference" (or coordinate system)

[11] Two or more bodies occupying the same space at the same time might contradict the physical intuition. However, presently we abstract away from the sizes of the bodies and therefore we permit two bodies to be at the same place at the same time.

Variables ranging over the universes B, F, G of $\mathfrak{M}$ are most often chosen as follows. For arbitrary $i \in \omega$,

$h, h_i, k, k_i, m, m_i, ph, ph_i \in B$;
$a, a_i, b, b_i, c, c_i, d, d_i, e, e_i, t, t_i, x, x_i, y, y_i, z, z_i, \varepsilon \in F$;
$\ell, \ell_i \in G$.

Let us recall that
$\mathbf{Ax}_{\mathrm{OF}} \cup \{\mathbf{Ax}_{\mathrm{G}}\} =$
{the axioms postulating that $\mathfrak{F}$ is a linearly ordered field, axiom of extensionality}. Now the *frame-language of relativity theory of dimension n* is defined to be the 3-sorted first order language built up from the above symbols the usual way. A model $\mathfrak{M} = \langle B, F, G; Obs, Ph, Ib, +, \cdot, \leq, \quad \mathrm{E} \quad, W\rangle$ is called a *frame model (of relativity theory, of dimension n)* iff

$$\mathfrak{M} \models \mathbf{Ax}_{\mathrm{OF}} \cup \{\mathbf{Ax}_{\mathrm{G}}\} \cup \{W(m,p,h) \to Obs(m)\}\,.$$

We denote the class of all frame models by $\mathsf{M}_{\mathrm{OFG}}$ or simply by FM. We call $\mathbf{Ax}_{\mathrm{OF}} \cup \{\mathbf{Ax}_{\mathrm{G}}\} \cup \{W(m,p,h) \to Obs(m)\}$ the *frame theory of special relativity theory* (or *frame theory* for short). By $\models^{\mathrm{OFG}}$ we denote semantical consequence within our present frame theory $\mathbf{Ax}_{\mathrm{OF}} \cup \{\mathbf{Ax}_{\mathrm{G}}\} \cup \{W(m,p,h) \to Obs(m)\}$. That is, for two sets Σ and Γ of formulas in our frame language,

$$\Sigma \models^{\mathrm{OFG}} \Gamma \iff (\forall \mathfrak{M} \in \mathsf{FM})(\mathfrak{M} \models \Sigma \Rightarrow \mathfrak{M} \models \Gamma)\,.$$

Also we define

$$\mathsf{Mod}_{\mathrm{OFG}}(\Sigma) \overset{\mathrm{def}}{=} \mathsf{FM} \cap \mathsf{Mod}(\Sigma)\,. \qquad \lhd$$

Intuitively, n is the dimension of our space-time. If $n = 2$, then we have one time-dimension, and one space-dimension, i.e. space is one-dimensional. If $n = 3$, then space is two-dimensional, and $n = 4$ represents our usual 4-dimensional space-time, i.e. space is three-dimensional. In the case of $n = 2$ it is rather easy to illustrate things, so we will often use $n = 2$ in our drawings. When $n = 3$, one still can illustrate ideas on drawings quite well. Many ideas can be better seen in the case $n = 2$ and work completely analogously for arbitrary n. Some statements, however, are true for $n = 2$ and not true for $n = 3, 4$. In these cases we will emphasize that $n = 2, n = 3$, or $n = 4$. (Sometimes [but not frequently], the cases of 3 and 4 behave differently. In such cases, of course, one emphasizes this difference. But most of the time, for understanding the key ideas, we will concentrate on the case of $n = 3$.)[12]

[12]Sometimes it is worth contemplating why the proofs are different for different dimensions.

There is another reason why it may be useful to allow the dimension of space-time to vary. Later we may devise models in which not all observers coordinatize events with the same dimensional vector-space. E.g. we could allow that most of the observers coordinatize events in 4-dimension, while some special (e.g. faster-than-light) observers coordinatize events with 3-dimensional space-time only.

Figures 1 - 3 illustrate the structure of an arbitrary model $\mathfrak{M}$ (of dimension 2) in the sense of Definition 2.1. Consider the coordinate system (i.e. the vector space ${}^nF$) on the right hand side of Figure 1 (or on the left-hand sides of Figures 2, 3). Intuitively, the first (vertical) axis is the *time scale* while the second (horizontal) axis represents *space*. The straight lines ℓ and ℓ_1 represent geodesics. W, which is the heart of our model, is illustrated on Figures 2, 3. W is represented by the set of world-view functions $\{w_m : m \in Obs\}$. On Figure 2, $w_m(p) = \{h, ph\}$ means that m "sees" at time p_0 at location p_1 two bodies: h and ph. I.e, $W(m,p,h)$, $W(m,p,ph)$ are true, while e.g. $W(m,p,m)$ is not true. Spacetime for us now is a vector-space on which we have a geometry G. Later there will be a metric on spacetime, too. At the time being we do not have a structure on the set $\mathcal{P}(B)$ of events. Later we will put some structure there, too. (Sometimes, the structure in Figure 3 is represented by a so called manifold.[13])

We will use the following notation. For $Obs(h), Ph(h), Ib(h)$ we often write $h \in Obs$, $h \in Ph$, $h \in Ib$, respectively. Moreover, we will reserve the variables m, m_i, k, k_i to denote observers; we reserve ph, ph_i for photons; finally we use the symbols p, q, r, s to denote elements of ${}^nF$. Thus we have

$m, m_i, k, k_i \in Obs$;
$ph, ph_i \in Ph$;
$p, q, r, s \in {}^nF$.

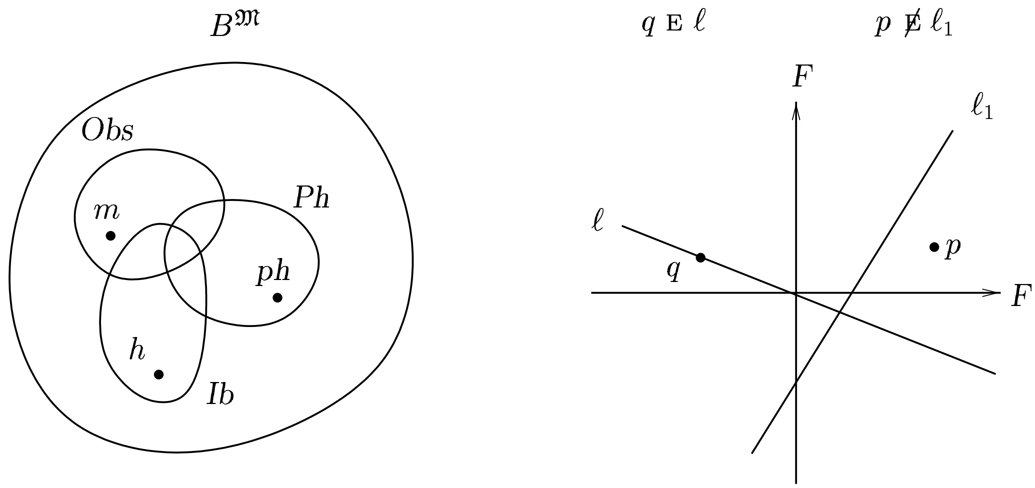
Using the terminology of vector spaces, elements of ${}^nF$ will often be referred to as *vectors*. For $\ell \in G$ we will alternatively write $line(\ell)$. An alternative notation for the long tuple

$$\langle B, F, G;\ Obs, Ph, Ib, +, \cdot, \leq,\ \mathrm{E}\ , W\rangle$$

will be the somewhat more concise

$$\langle (B, Obs, Ph, Ib), \mathfrak{F}, G;\ \mathrm{E}\ , W\rangle\,.$$

[13]The manifold structure is not particularly relevant at the present point, but it will be relevant in later developments.



bodies of $\mathfrak{M}$ quantities and lines of $\mathfrak{M}$

Figure 1: Illustration of a model $\mathfrak{M}$.

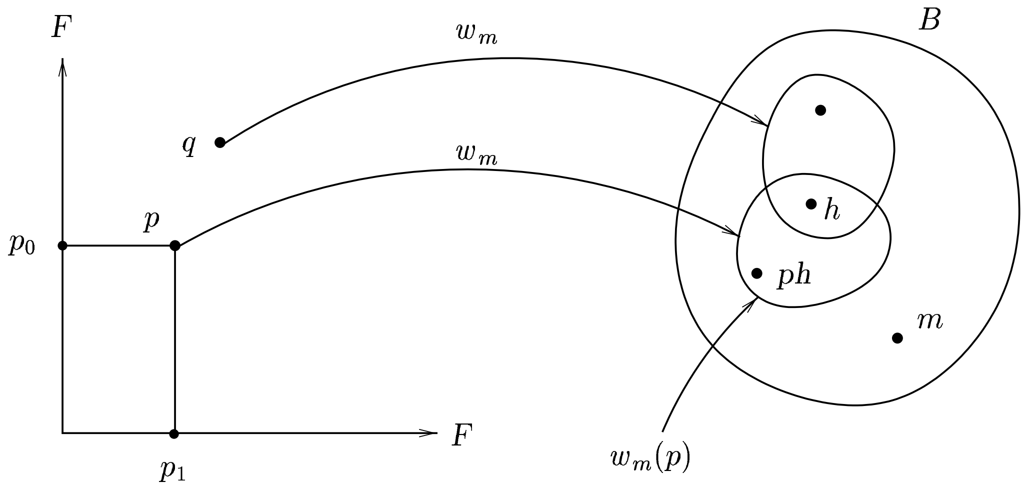
We will often consider the special class of models when $\mathfrak{F}$ is chosen to be the ordered field $\mathfrak{R} = \langle \mathrm{R}, +, \cdot, \leq \rangle$ of real numbers (where $+, \cdot, \leq$ are, respectively, the usual addition, multiplication and ordering of real numbers). In some cases we will consider only a reduct of the ordered field $\mathfrak{F}$, e.g. a linear ordering $\langle F, \leq \rangle$ only, in place of the whole of $\mathfrak{F}$ (e.g. because this will be enough for proving some of our results).

We close this subsection with giving a possible formulation of the so called "*twin paradox*", as an example for a formula in our frame language.[14] Intuitively, the twin paradox says that if one of two twin brothers leaves the other accelerating and returns to him later, then the brother who stayed behind will be *older* at the time of their reunion. That is, more time has passed for the "non-moving" brother than for the traveling one.

(TwP) $(\forall m \in Obs \cap Ib)(\forall k \in Obs \setminus Ib)(\forall p, q, p', q' \in {}^4F)$
$(m, k \in w_m(p) \cap w_m(q) \ \land \ w_m(p) = w_m(p') \ \land \ w_m(q) = w_m(q')) \Rightarrow$
$|p_0 - q_0| < |p'_0 - q'_0|$.

See Figure 4.

[14]We use natural abbreviations here, as well as later. E.g. we write "$m, k \in w_m(p) \cap w_m(q)$" in place of the longer "$W(m, p, m) \ \land \ W(m, q, m) \ \land \ W(m, p, k) \ \land \ W(m, q, k)$".



The world-view relation W and world-view functions w_m.
m “sees” at time p_0 at location p_1 two bodies: h and ph.

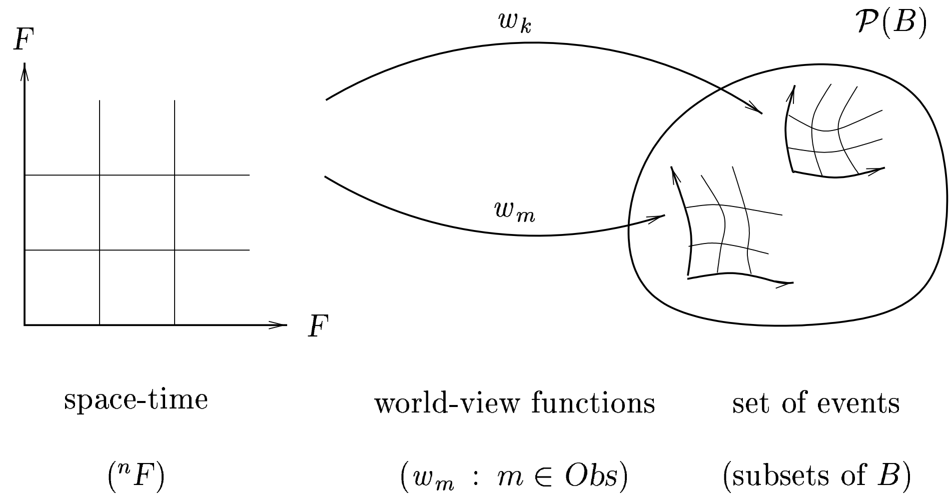
Figure 2: Illustration of the world-view function w_m.

2.2 Basic axioms *Basax*

Our next task is to postulate axioms in our frame language, to express our intuition about physical reality. Our first set of axioms to be proposed shortly will be called *Basax*, and it serves *special* relativity theory. In section 4.1 we will give another (actually a more advanced) set of axioms, *Acc*, in which we will allow accelerating observers, and accordingly, in *Acc* we will modify some of the postulates of *Basax* (e.g. we will modify item 7 below).[15] In subsection 3.3 we will define variants of the axioms of *Basax*, and variants of *Basax* itself (e.g. *Newbasax*).[16] These new versions will be more “balanced” in a sense, and will make it easier to move toward having accelerating observers, i.e. toward *Acc*. However, the present *Basax* has the advantage that its axioms are easy to formulate and understand, so it might be considered as a good starting point.

[15] *Acc* (and its theory) can be considered as a first step in the direction of treating general relativity (there we will have gravity, event horizons etc).

[16] It belongs to the spirit of the axiomatic method that we start out with a simple set of axioms (like *Basax*), investigate its properties, prove some theorems from it, and then we use our so obtained experience for modifying this axiom system. After that, we restart the “cycle”, i.e. we start investigating the new axiom system etc.



The heart of our model is W, which is represented by functions $w_m : {}^nF \to \mathcal{P}(B)$ for each $m \in Obs$.

Figure 3: Illustration of a model $\mathfrak{M}$.

Informally, about a model $\mathfrak{M} = \langle (B, Obs, Ph, Ib), \mathfrak{F}, G; \ \mathrm{E} \ , W \rangle$, *Basax* will postulate the following.[17]

1. $\mathfrak{F}$ is a linearly ordered field; we can thus define straight lines (which, intuitively, are "*life-lines*" or "*traces*" of the motions of inertial bodies), and we can define angles of straight lines (which represent "*speed*" or "*velocity*" of inertial bodies). G is the set of straight lines of $\mathfrak{F}$.

2. Observers and photons are inertial bodies.

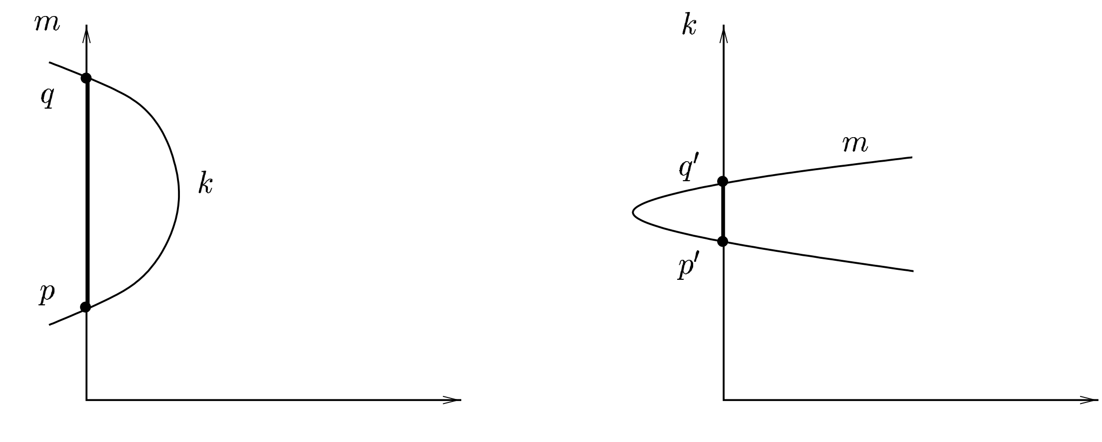
3. The "trace" of an inertial body h as seen by any observer m is a straight line.

4. Any observer m sees itself as "standing still" (at rest) in the origin.

5. Any observer sees some observer on each "slow" line.

6. Each line which could be the life-line of a photon (according to item 8 below) is indeed the life-line of a photon.

[17]Below, as later on, we will use the word "see" as a kind of "animation" of the primitive notion of the world-view function, see remark in the previous section.



m: "non-moving" (inertial) brother $\qquad$ k: traveling (accelerating) brother

Figure 4: The "twin paradox" written in our frame language.

7. Any two observers see the same events.

8. All observers "see" all photons moving with the same speed.

For the formal definition of *Basax*, we need some preparation. We start with recalling some basic notions of linear algebra e.g. from Halmos [21] or Gyapjas [20] or Rózsa [36] (or any other textbook on linear algebra).

If $p \in {}^nF$ for some set F and $n \in \omega$ then, for any $i < n$, p_i denotes the i-th component (projection) of p. Thus $p = \langle p_0, \dots, p_i, \dots, p_{n-1} \rangle = \langle p_i \rangle_{i<n}$.

Recall from any textbook on vector spaces (e.g. [20, p.49, Example 4]) that, to any field $\mathbf{F} = \langle F, +, \cdot \rangle$ and natural number $n \in \omega$, an n-dimensional *vector space* ${}^n\mathbf{F}$ can be associated the following natural way. Defining $+^V : {}^nF \longrightarrow {}^nF$ by

$$(\forall p, q \in {}^nF) p +^V q \stackrel{\text{def}}{=} \langle p_i + q_i \rangle_{i<n} \,,$$

$\langle {}^nF, +^V \rangle$ turns out to be a commutative group (with neutral element $\bar{0} = \langle 0 \rangle_{i<n}$ and inverse $-^V p = \langle -p_i \rangle_{i<n}$ for any $p \in {}^nF$). With defining the "scalar product" $\cdot^V : F \times {}^nF \longrightarrow {}^nF$ by

$$r \cdot^V p \stackrel{\text{def}}{=} \langle r \cdot p_i \rangle_{i<n} \text{ for each } r \in F \text{ and } p \in {}^nF \,,$$

$\langle {}^nF, +^V \rangle$ becomes a vector space over the field $\mathbf{F}$. We denote this vector space by ${}^n\mathbf{F}$. We note that any n–dimensional vector space over $\mathbf{F}$ is isomorphic to ${}^n\mathbf{F}$ (see e.g. [20, p.67]).

As usual, we will often write $p -^V q$ in place of $p +^V (-^V q)$ for simplicity. Further, we will often omit the index V from $\cdot^V$, $+^V$ and $-^V$, and hope that context will always save us from misunderstandings.

CONVENTION 2.2 (i) When we work in ${}^n\mathbf{F}$ $(2 \le n \le 4)$, to match the physical intuition, we call the 0-th coordinate p_0 of a point $p = \langle p_0, \ldots, p_{n-1} \rangle \in {}^nF$ the *time coordinate* of p. Accordingly, when drawing coordinate systems, we call the 0-th axis of it the *time axis* or *t-axis*. The rest of the coordinates are the *space coordinates*. On our figures we denote the 0-th, first, second and third axes, respectively, by t, x, y and z. In formulas we use the "bar version" of these, e.g. $\bar{x}$ for x because, sometimes, x stands for an element of F, so there is a chance for confusion. Also we put $p_t \stackrel{\text{def}}{=} p_0$, $p_x \stackrel{\text{def}}{=} p_1$, $p_y \stackrel{\text{def}}{=} p_2$, $p_z \stackrel{\text{def}}{=} p_3$ for each $p \in {}^nF$.

(ii) Throughout this work, the dimension $n (\in \omega)$ is a parameter of almost all of our concepts. Therefore a possibility for a rigorous presentation would be to indicate n in the name of each concept we introduce, e.g., by putting something like "(n)" after it. But then the text would become too complicated. Therefore we chose omitting the "(n)"-s except when this would lead to misunderstanding or when we want to emphasize the presence of n.

But sometimes we will define or state things for one particular n only (e.g., for just $n = 2$). In these cases we will indicate this fact by putting the particular number, in parenthesis, after the name of the concept involved. For example, we will formulate an axiom **Ax1**, where n will be a parameter of **Ax1**. Then the instance of **Ax1** for the case $n = 2$ will be denoted by **Ax1(2)**.

We will treat some other parameters likewise. E.g., we will sometimes state things for a collection of models from FM such that all $\mathfrak{M} \in \mathsf{FM}$ share the same ordered field $\mathfrak{F}$ as their "quantity part". Then we will denote this collection by $\mathsf{FM}(\mathfrak{F})$.

In cases when we will need more than one parameter we will list them in parentheses, separated by commas. For example,

$$\mathsf{FM}(\mathbf{3}, \mathfrak{R}) = \{\mathfrak{M} \in \mathsf{FM}(\mathbf{3}) : \mathfrak{F}^{\mathfrak{M}} = \mathfrak{R}\}.$$

That is, $\mathfrak{M} \in \mathsf{FM}(\mathbf{3}, \mathfrak{R})$ iff $\mathfrak{M}$ is of dimension 3 and the quantity part of $\mathfrak{M}$ is the ordered field $\mathfrak{R}$ of real numbers. $\lhd$

Besides our frame language introduced in subsection 2.1, we will sometimes use the language of ${}^n\mathbf{F}$ for expressing ideas concisely. (E.g., for $r, s \in {}^nF$ we may mention the vectors $r+s$ or $3\cdot r$.) But we will always make sure that the ${}^n\mathbf{F}$ formulas we use can be translated into the frame language of relativity theory (given in subsection 2.1). As a first example for this (and for the other natural abbreviations we will use), we introduce our first axiom **Ax1** both as a ${}^n\mathbf{F}$ formula and, equivalently, as a (longer) formula written purely in the frame language.

The set of straight lines of ${}^n\mathbf{F}$ in the usual Euclidean sense is denoted by $\mathsf{Eucl}(\mathbf{n},\mathbf{F})$, that is,

$$\ell \in \mathsf{Eucl}(\mathbf{n},\mathbf{F}) \stackrel{\text{def}}{\Longleftrightarrow} (\exists r, s \in {}^nF)(s \neq 0 \ \wedge\ \ell = \{r + a\cdot s : a \in F\}).$$

Ax1 in a concise language:
$G = \mathsf{Eucl}(\mathbf{n},\mathbf{F})$.

Ax1 in the frame language of relativity theory:

Ax1' $(\forall r_0,\dots,r_{n-1},s_0,\dots,s_{n-1} \in F)$
$(\{s_0,\dots,s_{n-1}\} \neq \{0\} \Rightarrow (\exists \ell \in G)(\forall p_0,\dots,p_{n-1} \in F)(\ \mathrm{E}\ (p_0,\dots,p_{n-1},\ell) \Leftrightarrow$
$(\exists a \in F) \wedge_{i<n} p_i = r_i + a\cdot s_i))$

and

$(\forall \ell \in G)(\exists r_0,\dots,r_{n-1},s_0,\dots,s_{n-1} \in F)$
$(\{s_0,\dots,s_{n-1}\} \neq \{0\}\ \wedge\ (\forall p_0,\dots,p_{n-1} \in F)(\ \mathrm{E}\ (p_0,\dots,p_{n-1},\ell) \Leftrightarrow$
$(\exists a \in F) \wedge_{i<n} p_i = r_i + a\cdot s_i))$.

Here we emphasize that **Ax1** is designed to serve the purposes of special relativity only. In later parts when dealing with more general theories of relativity, **Ax1** will be changed.

If $\ell \in \mathsf{Eucl}(\mathbf{n},\mathbf{F})$, then we can consider the angle between ℓ and the time axis. By $ang^2(\ell)$ we denote the square of the tangent of the angle between ℓ and the time axis.[18] Thus, for $\ell = \{r + a\cdot s : a \in F\} \in \mathsf{Eucl}(\mathbf{n},\mathbf{F})$, if $s_0 \neq 0$ then

$$ang^2(\ell) \stackrel{\text{def}}{=} \frac{s_1^2 + s_2^2 + \ldots + s_{n-1}^2}{s_0^2};$$

if $s_0 = 0$ then we consider $ang^2(\ell)$ to be infinite. (It will cause no problem that "infinity" is not an element of F.) Thus $0 \leq ang^2(\ell) \leq \infty$. $ang^2(\ell) = 0$ means that ℓ is vertical, $ang^2(\ell) = 1$ means that the angle between ℓ and the time axis is 45^o, and $ang^2(\ell) = \infty$ means that ℓ is horizontal. The definition of $ang^2(\ell)$ is illustrated on Figure 5.

[18]We consider the square of the tangent (instead of the tangent itself) of this angle because, in general, we do not assume that square-roots exist in $\mathbf{F}$.

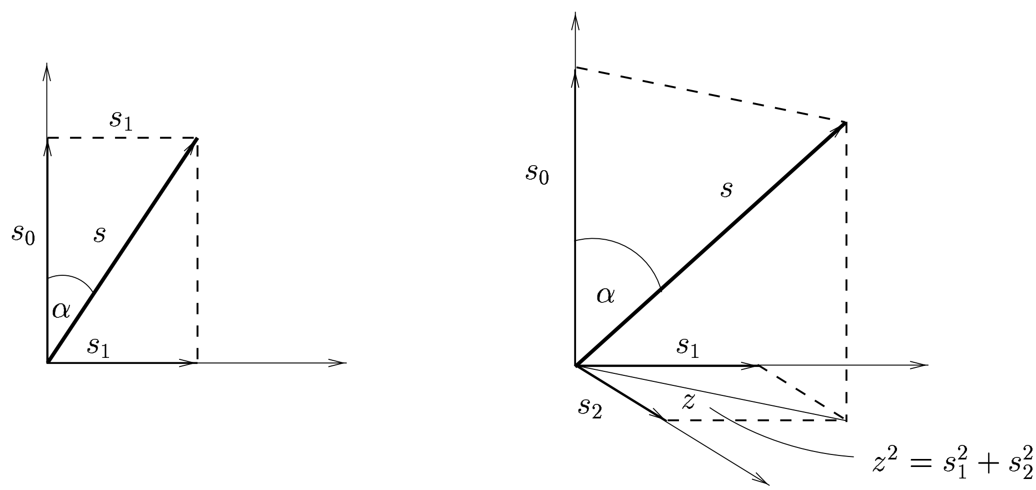


Figure 5: Illustration of angle of a line.

Definition 2.3 (life line (or trace), velocity)
Let $\mathfrak{M}$ be a model (of dimension n) as in Definition 2.1. Let $m \in Obs$ and $h \in B$ be arbitrary but fixed. Recall from Definition 2.1 that the world view function $w_m : {}^nF \longrightarrow \mathcal{P}(B)$ of m was defined as follows:

$$w_m(p) = \{h \in B : \langle m, p, h\rangle \in W\} \text{ for every } p \in {}^nF .$$

(i) By the *life line (*or *trace) of h as seen by m* (or *life line (*or *trace) of h by the world view of m*) we mean the set

$tr_m(h) \stackrel{\text{def}}{=} \{p \in {}^nF : h \in w_m(p)\} = \{p \in {}^nF : W(m, p, h)\}.$

(ii) If $tr_m(h) \in \mathsf{Eucl}(\mathbf{n}, \mathbf{F})$, then by the *velocity* or *speed of h as seen by m* we mean

$v_m(h) \stackrel{\text{def}}{=} ang^2(tr_m(h)).$

The formula $v_m(h) = a$ will abbreviate that

$tr_m(h) \in \mathsf{Eucl}(\mathbf{n}, \mathbf{F})$ and $ang^2(tr_m(h)) = a.$ $\quad\triangleleft$

On Figure 6, the line $tr_m(h)$ illustrates the life line of a body h (in case $n = 2$). The acronym "*tr*" stands for "*trace*". If $tr_m(h) = \{\langle t, x, y, z\rangle : t \in F\}$, then m always, at each time instance $t \in F$, sees h at location $\langle x, y, z\rangle$, i.e. m sees the body

h at rest at location $\langle x, y, z\rangle$. Thus, $tr_m(h)$ is a horizontal line (a line parallel with the time axis), i.e. $v_m(h) = 0$, means that "h is at rest, as seen by m". Similarly, the bigger $v_m(h)$ is, the more "speed" h is moving with, as seen by m.

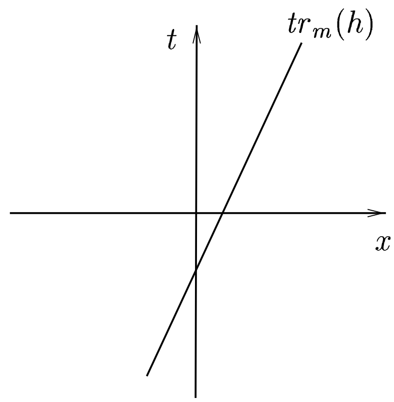


Figure 6: Illustration of life line.

We are ready to postulate axioms **Ax2**–**Ax6**.

Ax2 $Obs \cup Ph \subseteq Ib$.

That is, observers are inertial bodies; and so are photons.

Ax3 $(\forall h \in Ib)(\forall m \in Obs)(tr_m(h) \in G)$.

That is, the life line of any inertial body h as seen by any observer m must be a geodesic.

Ax4 $(\forall m \in Obs)(tr_m(m) = F \times \{0\} \times \ldots \times \{0\}(= F \times {}^{n-1}\{0\}))$.

Ax4 states that the life line $tr_m(m)$ of an observer as seen by itself is the 0-th axis (the time axis). Thus **Ax4** says that each observer sees itself to be a body at rest (not moving) at location $\langle 0, \ldots, 0\rangle$. In particular, $v_m(m) = 0$. This is one of the basic axioms of relativity theory. This was a "relativistic axiom" already before Einstein.[19] It expresses that each observer can "think" that he is at rest and all other bodies are moving. The first step toward general relativity

[19]Sometimes it is referred to as Galileo's relativity principle, cf. e.g. Geroch [16], pp.32-39.

theory will be that we will extend **Ax4** to accelerating observers, too[20]: then even accelerating observers can "think" that they are at rest (and then, in a poetic language, gravity will come into the picture to explain certain strange behavior of other bodies). In an intuitive level, the principle on which special relativity is based is quoted as "all inertial observers [or reference frames] are equivalent" (at this point our future **AxE** will play a role, too); and the principle of general relativity is quoted as "all observers [including the accelerated ones] are equivalent". (Here "equivalent" means only that each of these observers may imagine that he is not moving and it is the rest of the universe which moves, accelerates etc.)

Ax5 $(\forall m \in Obs)(\forall \ell \in G)(ang^2(\ell) < 1 \Rightarrow (\exists k \in Obs)\ell = tr_m(k)$ and

$$ang^2(\ell) = 1 \Rightarrow (\exists ph \in Ph)\ell = tr_m(ph)).$$

Ax5 makes sense only in the presence of **Ax1** (because $ang^2(\ell)$ is not defined otherwise). Then it states on one hand that each straight line with angle less than 45^o is the life line of some observer. On the other hand, straight lines with angle 45^o are life lines of photons.[21]

Ax6 $(\forall k, m \in Obs)(Rng(w_m) = Rng(w_k))$.

Ax6 states that all observers see the same set of events. I.e. whenever an observer m sees a set H of bodies at some time point t and space point s, any other observer k must see the same set H of bodies at *some* time point t_1 and space point s_1 (where $t \neq t_1$ or $s \neq s_1$ are permitted). **Ax6** seems to be quite strong. In particular, it will not be true in our theory of accelerating observers.[22] Later we will weaken **Ax6** to **Ax6$_{00}$** such that the new version will be true for our accelerating observers, too. The new version **Ax6$_{00}$** will say that if m sees an event H on the trace of the observer k, then k itself sees this event H.

---

[20] According to e.g. Friedman [13], p.5, general relativity begins with the study of accelerated observers (or accelerated reference frames) (at least when they are treated "equivalently" with inertial reference frames). In this sense, our section 4.1 deals with the (first steps of the) generalization of our (logic based) method from special relativity to general relativity.

[21] We will see a (first order) modal logic refinement (or variant) of our axioms (and formalism) in which **Ax5** sounds "less radical" (that is, sounds more convincing intuitively).

[22] One reason for this is that if observer k accelerates (in m's world) so fast that its clock will never reach 12 o'clock as seen by m, then the "event" seen by k at 12 o'clock (or after 12) will not be "seen" by m.

Our last axiom in the present subsection is the most distinguished one in relativity theory:[23]

AxE $(\forall m \in Obs)(\forall ph \in Ph) v_m(ph) = 1$.

> **AxE** ("*Einstein's axiom*") states that the speed of a photon ph, as seen by any observer m, is always 1. In *Basax*, we choose the "speed of light" to be 1. This is a rather ad-hoc decision, the important part of **AxE** is that all observers see all photons as having the same speed. Later we will weaken **AxE** even to "each observer m sees all photons having the same speed" (thus observer m may see photons "faster" than observer k). We will see that already this weak form of **AxE** will be enough for proving most of the important consequences of *Basax*.

Definition 2.4 **(*Basax*)** We define

$$Basax \stackrel{\text{def}}{=} \{\mathbf{Ax1}, \mathbf{Ax2}, \mathbf{Ax3}, \mathbf{Ax4}, \mathbf{Ax5}, \mathbf{Ax6}, \mathbf{AxE}\},$$

where the axioms **Ax1**–**Ax6**, **AxE** were defined above.

Here is a summary of the axioms in *Basax*:

Ax1 $G = \mathsf{Eucl}(\mathbf{n}, \mathbf{F})$.

Ax2 $Obs \cup Ph \subseteq Ib$.

Ax3 $(\forall h \in Ib)(\forall m \in Obs)(tr_m(h) \in G)$.

Ax4 $(\forall m \in Obs)(tr_m(m) = F \times {}^{n-1}\{0\})$.

Ax5 $(\forall m \in Obs)(\forall \ell \in G)(ang^2(\ell) < 1 \Rightarrow (\exists k \in Obs)\ell = tr_m(k)$ and

$$ang^2(\ell) = 1 \Rightarrow (\exists ph \in Ph)\ell = tr_m(ph)).$$

Ax6 $(\forall k, m \in Obs)(Rng(w_m) = Rng(w_k))$.

AxE $(\forall m \in Obs)(\forall ph \in Ph) v_m(ph) = 1$. $\lhd$

It follows from **Ax2**,**Ax3** that the trace of any observer is a geodesic. Since we will often use this conclusion, we are giving it a name:

[23] One could refer to e.g. the Michelson-Morley experiment for motivation, but instead of doing that, we refer to the introduction of Friedman [13].

(geod) $(\forall m, k \in Obs) tr_m(k) \in G$.

Statement **(geod)** together with **Ax1** imply that $tr_m(k)$ is a *Euclidean* straight line. Later **Ax1** will be generalized so that G will be a more general geometry-like structure, e.g. G might consist of the geodesics of some structure. **(geod)** will then be restricted to inertial observers.

Assuming *Basax*, we can (and will often) draw the "world view" of an observer m as shown in Figure 7. On this figure and on similar pictures, most often we simply write the name of a body h instead of writing out the long expression $tr_m(h)$, when indicating the life line of h (as seen by m).

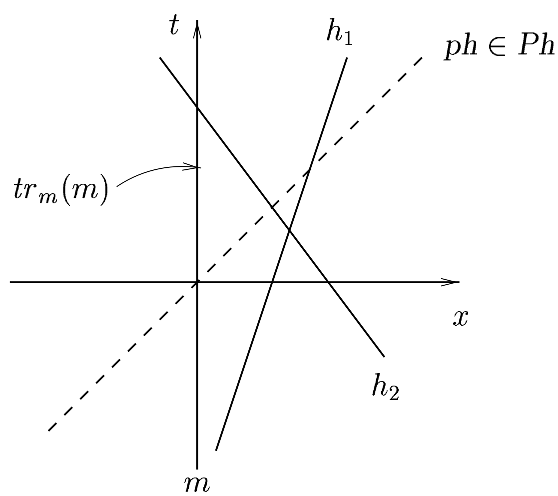


Figure 7: The world view of an observer m.

2.3 Some properties of $Basax$, world view transformation

In this section we prove some simple consequences of our basic axioms, $Basax$.

Definition 2.5 (world view transformation) Given $m, k \in Obs$, we define the *world view transformation* f_{mk} as follows:

$$\mathsf{f}_{mk} \stackrel{\text{def}}{=} w_m \circ w_k^{-1} .$$

$\lhd$

We note that ${w_k}^{-1}$ is a relation, hence the composition $w_m \circ {w_k}^{-1}$ is again a relation, cf. the definition of composition on the first page of subsection 2.1. Thus $\mathsf{f}_{mk} \subseteq {}^nF \times {}^nF$ and $\mathsf{f}_{mk} = \{\langle p, q\rangle \in {}^2({}^nF) : w_m(p) = w_k(q)\}$.

If $g : U \longrightarrow V$ is a function, then for all $X \subseteq U$ we define

$$g[X] \stackrel{\text{def}}{=} \{g(x) : x \in X\}.$$

PROPOSITION 2.6 *Let $\mathfrak{M}$ be a frame model of Basax. Then the following are true for all $m, k, h \in Obs$ and $b \in B$.*

(i) $Obs \cap Ph = \emptyset$.

(ii) $v_m(k) \neq 1$.

(iii) *The world-view function w_m is an injection (i.e. one-one).*

(iv) *The world-view transformation f_{mk} is a bijection (i.e. one-one, defined on ${}^nF$ and onto ${}^nF$).*

(v) $\mathsf{f}_{mm} = \mathrm{Id}$, $\mathsf{f}_{mk} = \mathsf{f}_{km}^{-1}$, *and* $\mathsf{f}_{mk} = \mathsf{f}_{mh} \circ \mathsf{f}_{hk}$.

(vi) $w_m = \mathsf{f}_{mk} \circ w_k$.

(vii) f_{mk} *takes the trace of a body as seen by m to the trace of the body as seen by k, i.e.* $\mathsf{f}_{mk}[tr_m(b)] = tr_k(b)$.

(viii) f_{mk} *takes slow lines to lines, i.e. if* $\ell \in \mathsf{Eucl}(\mathbf{n}, \mathbf{F})$ *and* $\mathrm{ang}^2(\ell) < 1$, *then* $\mathsf{f}_{mk}[\ell] \in \mathsf{Eucl}(\mathbf{n}, \mathbf{F})$.

Later, in subsection 3.2, we will prove that f_{mk} takes all straight lines to straight lines. All the statements in Proposition 2.6 can be expressed with (first-order) formulas in our frame-language. We will prove the items in Prop.2.6 one-by-one, so that we can single out the axioms we need for proving them.

Claim 2.7 $\{\mathbf{Ax1}, \mathbf{(geod)}, \mathbf{Ax4}, \mathbf{AxE}\} \models^{\text{OFG}} Obs \cap Ph = \emptyset$.

Proof: Assume that $m \in Obs \cap Ph$. Then $tr_m(m) \in \mathsf{Eucl}(\mathbf{n}, \mathbf{F})$, by **Ax1**, **(geod)** and $m \in Obs$. Look at $v_m(m)$. By **Ax4** we have that $v_m(m) = 0$, and by **AxE** and $m \in Ph$ we have that $v_m(m) = 1$. Since in all fields 0 and 1 are different elements, we reached a contradiction. ∎

Claim 2.8 $\{\mathbf{Ax4}, \mathbf{Ax5}, \mathbf{Ax6}, \mathbf{AxE}\} \models^{\text{OFG}} v_m(k) \neq 1$.

Proof: Assume that $v_m(k) = 1$ for some $m, k \in Obs$. Then $ang^2(tr_m(k)) = 1$, thus by **Ax5**, $tr_m(k) = tr_m(ph)$ for some $ph \in Ph$. By **Ax4** and **AxE** we have that $tr_k(k) \neq tr_k(ph)$. Say, $q \in tr_k(k) \setminus tr_k(ph)$. Let $H \stackrel{\text{def}}{=} w_k(q)$. Then $H \in Rng(w_k)$ and $k \in H, ph \notin H$. By **Ax6**, $H \in Rng(w_m)$, say $H = w_m(p)$. Then $p \in tr_m(k)$ by $k \in H$, and $p \notin tr_m(ph)$ by $ph \notin H$. This contradicts $tr_m(k) = tr_m(ph)$. ∎

Claim 2.9 $\{\mathbf{Ax1}, \mathbf{Ax5}\} \models^{\text{OFG}} (\forall m \in Obs)(w_m$ *is an injection*$)$.

Proof: Let $m \in Obs$ and assume that $p, q \in {}^nF$, $p \neq q$. Then, by **Ax1** and by the properties of $\mathsf{Eucl}(\mathbf{n}, \mathbf{F})$, $(\exists \ell \in G)(p \in \ell \ \wedge \ q \notin \ell \ \wedge \ ang^2(\ell) < 1)$. By **Ax5**, $(\exists k \in Obs)\ell = tr_m(k)$. For such a k, $k \in w_m(p)$ but $k \notin w_m(q)$. ∎

Claim 2.10

(i) $\{\mathbf{Ax1}, \mathbf{Ax5}, \mathbf{Ax6}\} \models^{\text{OFG}} (\mathsf{f}_{mk}$ *is a bijection* $\mathsf{f}_{mk} : {}^nF \longrightarrow {}^nF)$.

(ii) $\{\mathbf{Ax1}, \mathbf{Ax5}\} \models^{\text{OFG}} (\mathsf{f}_{mk}$ *is a (possibly) partial one-to-one function* $)$.

(iii) $\{\mathbf{Ax1}, \mathbf{Ax5}, \mathbf{Ax6}\} \models^{\text{OFG}} (\mathsf{f}_{mm} = \mathrm{Id}, \mathsf{f}_{mk} = \mathsf{f}_{km}^{-1}, \mathsf{f}_{mk} = \mathsf{f}_{mh} \circ \mathsf{f}_{hk})$.

Proof: That f_{mk} is one-to-one follows from Claim 2.9. That f_{mk} is defined everywhere and is onto ${}^nF$ follows from **Ax6**. $\mathsf{f}_{mm} = \mathrm{Id}$, $\mathsf{f}_{mk} = \mathsf{f}_{km}^{-1}$ and $\mathsf{f}_{mk} \supseteq \mathsf{f}_{mh} \circ \mathsf{f}_{hk}$ follow from the definition of the world-view transformation functions. Assume $\mathbf{Ax1}, \mathbf{Ax5}, \mathbf{Ax6}$, let $p \in {}^nF$, and $\mathsf{f}_{mk}(p) = q$, i.e. $w_m(p) = w_k(q)$. By **Ax6** there is $p' \in {}^nF$ such that $w_m(p) = w_h(p')$. Now $\mathsf{f}_{mh}(p) = p'$ by $w_m(p) = w_h(p')$ and $\mathsf{f}_{hk}(p') = q$ by $w_h(p') = w_m(p) = w_k(q)$. Thus $\mathsf{f}_{mk}(p) = \mathsf{f}_{hk}(\mathsf{f}_{mh}(p))$. ∎

Remark 2.11 By Claim 2.10 (i),(ii) we have that if the set $Tr^{\mathfrak{M}} \stackrel{\text{def}}{=} \{\mathsf{f}_{mk} : m, k \in Obs^{\mathfrak{M}}\}$ of transformations is closed under composition $\circ$, then $\langle Tr, \circ, {}^{-1}, \mathrm{Id}\rangle$ forms a group. In section 3.6 we will see models of *Basax* where Tr is not closed under composition. However, in section 3.7 we will introduce the "symmetry axiom" $\mathbf{Ax\triangle1}$ and we will see that if $\mathfrak{M} \models Basax \cup \{\mathbf{Ax\triangle1}\}$, then $\langle Tr^{\mathfrak{M}}, \circ, {}^{-1}, \mathrm{Id}\rangle$ is a group. $\triangleleft$

The proof of Prop.2.6 (viii) consists of noting that every slow line is the trace of some observer k_1 as seen by m, and that $tr_k(k_1)$ is a line. The proofs of Proposition 2.6 (vi)-(vii) are similar to those of Proposition 2.6 (i)-(v). We leave them to the reader.

Assuming *Basax*, we have that

$$\mathsf{f}_{mk} : {}^nF \longrightarrow {}^nF \text{ is a bijection.}$$

Figure 8 illustrates a world view transformation f_{mk}, for the 2-dimensional case. Since $\mathsf{f}_{mk} : {}^2F \longrightarrow {}^2F$, we indicated two copies of ${}^2F$, the usual coordinate system way. The world view of m is illustrated in the top coordinate system and the world view of k is in the bottom coordinate system. We drew the picture under the assumption *Basax*. Therefore, the f_{mk}–image of the only photon f drawn in the world view of m will be a photon in the world view of k as well (cf. **AxE** and **Ax5**). On Figure 9 the same f_{mk} is illustrated in such a way that the world views of both k and m are drawn in the same copy of ${}^2F$.

According to our Convention 2.2 (ii), *Basax*$(\mathbf{2})$ denotes *Basax* in the 2-dimensional case. Next, in subsection 2.4, we will see that *Basax*$(\mathbf{2})$ is *consistent*, that is, there exist frame models satisfying *Basax*$(\mathbf{2})$. In subsection 3.5 (cf. Definition 3.98, Theorem 3.99) we will see that *Basax*$(\mathbf{3})$ is consistent, and that generally, *Basax*$(\mathbf{n})$ is consistent for all $n \geq 3$.

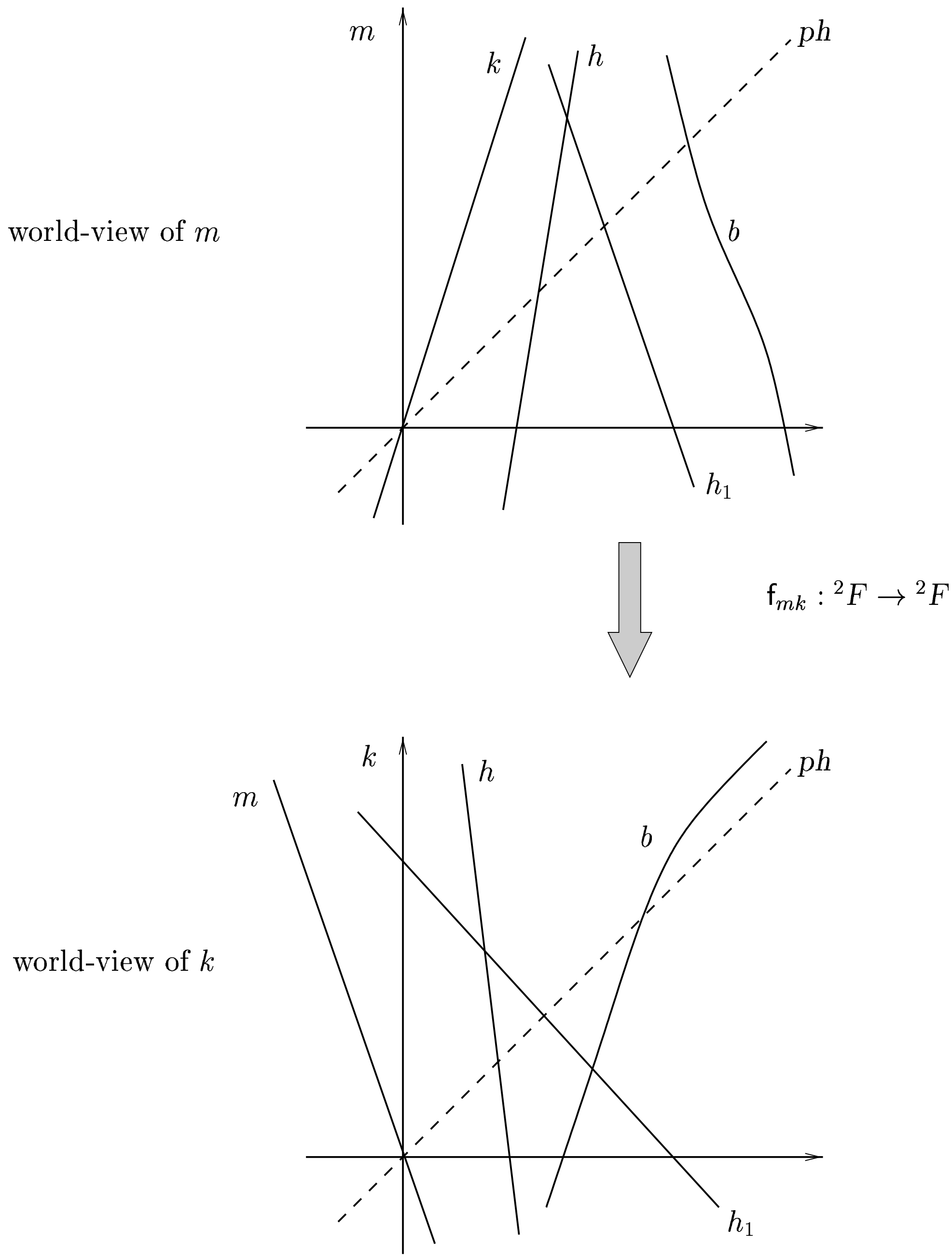


Figure 8: World view transformation 1.

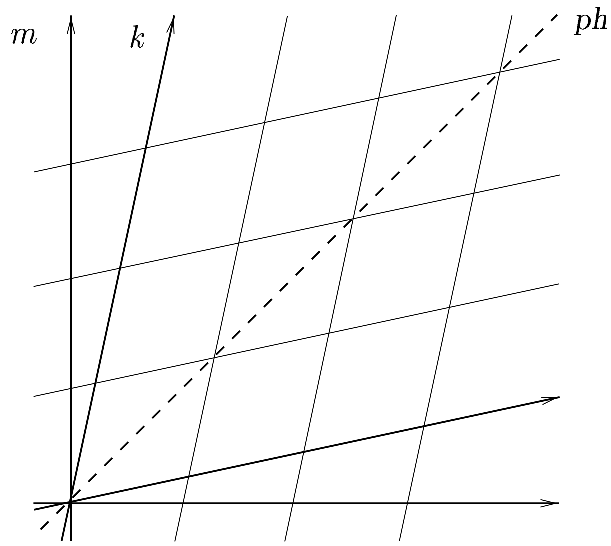


Figure 9: World view transformation 2.

We will see, in section 2.4, that there are models of $Basax(\mathbf{2})$ in which there are observers moving faster than light, while one can prove from $Basax(\mathbf{3})$ that there are no observers moving faster than light (i.e. $Basax(\mathbf{3}) \models^{\text{OFG}} (\forall m, k \in Obs) v_m(k) < 1$, see Theorem 3.28, while $Basax(\mathbf{2}) \models^{\text{OFG}} (\forall m, k \in Obs) v_m(k) < 1$ does not hold.) We do not know whether $Basax(\mathbf{4})$ allows faster than light (FTL) observers or not. We know that $Basax(\mathbf{4})$ together with the axiom stating that square roots exist in $\mathfrak{F}$ does imply that there are no FTL observers (see Theorem 3.28).

We now list some logical properties of *Basax*. We already stated that *Basax* is consistent. We will classify all the models of *Basax*, and we will see that there are continuum many non-elementarily equivalent models of *Basax*. I.e. *Basax* is non-categorical (it has non-isomorphic models of the same cardinality). We will prove that the theory of *Basax*, i.e. the set of first-order consequences of *Basax*, is undecidable. This also proves that *Basax* is non-categorical, because *Basax* is finite. We will define some natural axioms, say **Axnb** and **Axb** and we show that $Basax \cup \{\mathbf{Axnb}\}$ is categorical, while $Basax \cup \{\mathbf{Axb}\}$ is hereditarily undecidable[24], thus no finite extension of it can be categorical.

[24]Moreover, Gödel's second incompleteness theorem also applies to $Basax \cup \{\mathbf{Axb}\}$.

Summing up:

- *Basax* is consistent.
- *Basax* has many non-elementarily equivalent models.
- We will give a classification of the class of models of *Basax* (see subsection 3.6).
- The first-order theory of *Basax* is undecidable (hence non-categorical).
- Adding an extra axiom can make *Basax* categorical (hence decidable, since *Basax* is finite).
- Adding a different extra axiom can make *Basax* hereditarily undecidable (hence hereditarily non-categorical).

2.4 Models for *Basax* in dimension 2

In this subsection we show that $Basax(\mathbf{2})$ is consistent, via defining a frame model $\mathfrak{M}$ and showing that $\mathfrak{M} \models Basax(\mathbf{2})$. We will also give a model of $Basax(\mathbf{2})$, in which there are faster than light observers.

Let P be a function that to each $\ell \in \mathsf{Eucl}(\mathbf{2}, \mathfrak{R})$ associates a pair of two distinct points lying on ℓ. We will denote $P(\ell)$ by $\langle o_\ell, t_\ell \rangle$. To each such function P, we will define two frame models, $\mathfrak{M}_0^P$ and $\mathfrak{M}_1^P$. These two frame models will be very similar in spirit, but in $\mathfrak{M}_0^P$ we have as few observers as possible, while in $\mathfrak{M}_1^P$ there will be an observer on each line (with angle $\neq 1$).

First we define $\mathfrak{M} \stackrel{\text{def}}{=} \mathfrak{M}_0^P \stackrel{\text{def}}{=} \langle B, F, G; Obs, Ph, Ib, +, \cdot, \leq, E, W \rangle$, where

$\langle F, +, \cdot, \leq \rangle \stackrel{\text{def}}{=} \mathfrak{R}$, the ordered field of real numbers,

$\langle G, E \rangle \stackrel{\text{def}}{=} \langle \mathsf{Eucl}(\mathbf{2}, \mathfrak{R}), \epsilon \rangle$, the set of straight lines over $\mathfrak{R}$,

$Obs \stackrel{\text{def}}{=} \{\ell \in \mathsf{Eucl}(\mathbf{2}, \mathfrak{R}) : ang^2(\ell) < 1\}$,

$Ph \stackrel{\text{def}}{=} \{\ell \in \mathsf{Eucl}(\mathbf{2}, \mathfrak{R}) : ang^2(\ell) = 1\}$,

$B \stackrel{\text{def}}{=} Ib \stackrel{\text{def}}{=} Obs \cup Ph = \{\ell \in \mathsf{Eucl}(\mathbf{2}, \mathfrak{R}) : ang^2(\ell) \leq 1\}$.

By the above, **Ax1** and **Ax2** are true in $\mathfrak{M}$. It remains to define W. Let

$$m_0 \stackrel{\text{def}}{=} \bar{t} \stackrel{\text{def}}{=} \mathrm{R} \times \{0\}.$$

First we will define $w_{m_0} : {}^2\mathrm{R} \longrightarrow \mathcal{P}(B)$ and $\mathsf{f}_{km_0} : {}^2\mathrm{R} \longrightarrow {}^2\mathrm{R}$ for all $k \in \mathsf{Eucl}(\mathbf{2}, \mathfrak{R}), ang^2(k) \neq 1, k \neq m_0$. To define w_{m_0}, let $p \in {}^2\mathrm{R}$. Then

$$w_{m_0}(p) \stackrel{\text{def}}{=} \{\ell \in B : p \in \ell\}.$$

By this we have that for all $\ell \in B$,

$$tr_{m_0}(\ell) = \ell,$$

in particular, $tr_{m_0}(m_0) = m_0$. Thus **Ax3**, **Ax4**, **Ax5**, **AxE** are satisfied when m is replaced in them with m_0. See Figure 10.

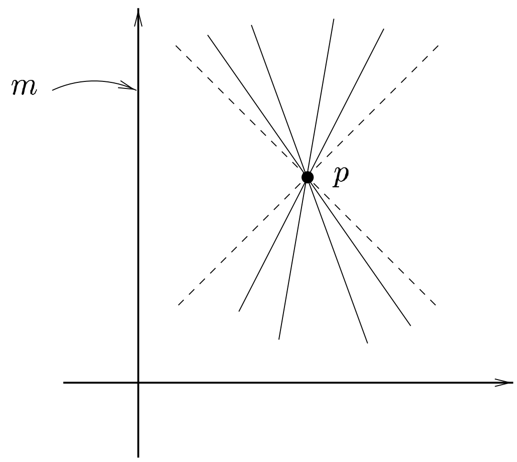


Figure 10: $w_{m_0}(p)$ in $\mathfrak{M}_0^P$.

Let $k \in \mathsf{Eucl}(\mathbf{2}, \mathfrak{R}), k \neq m_0, ang^2(k) \neq 1$ be arbitrary. We are going to define f_{km_0}. In the following, we will write f for f_{km_0}.

For any two distinct points $p, q \in {}^nF$, $\overline{pq}$ denotes the Euclidean line containing both p and q.

Recall that two distinct points, o_k and t_k are given to us by the parameter P of the model $\mathfrak{M} \stackrel{\text{def}}{=} \mathfrak{M}_0^P$. First we define the point s_k as the mirror image of t_k w.r.t. the line ℓ_k such that $o_k \in \ell_k$ and ℓ_k is parallel to the line $\overline{(0,0)(1,1)}$, See Figure 11.

In more detail: Let $o_k = (o_0, o_1), t_k = (t_0, t_1)$. We define

$$s_k \stackrel{\text{def}}{=} (o_0 + (t_1 - o_1), o_1 + (t_0 - o_0)).$$

By $ang^2(k) \neq 1$ we have that $s_k \neq t_k$, moreover, $s_k \neq a \cdot t_k$ for all $a \in \mathrm{R}$.

We will define $\mathsf{f} \stackrel{\text{def}}{=} \mathsf{f}_{km_0} : {}^2\mathrm{R} \longrightarrow {}^2\mathrm{R}$ to be the affine transformation[25] that takes $(0,0), (1,0), (0,1)$ to o_k, t_k, s_k respectively. See Figure 12.

[25] For the definition of an affine transformation see section 3.2. We will not need the definition here.

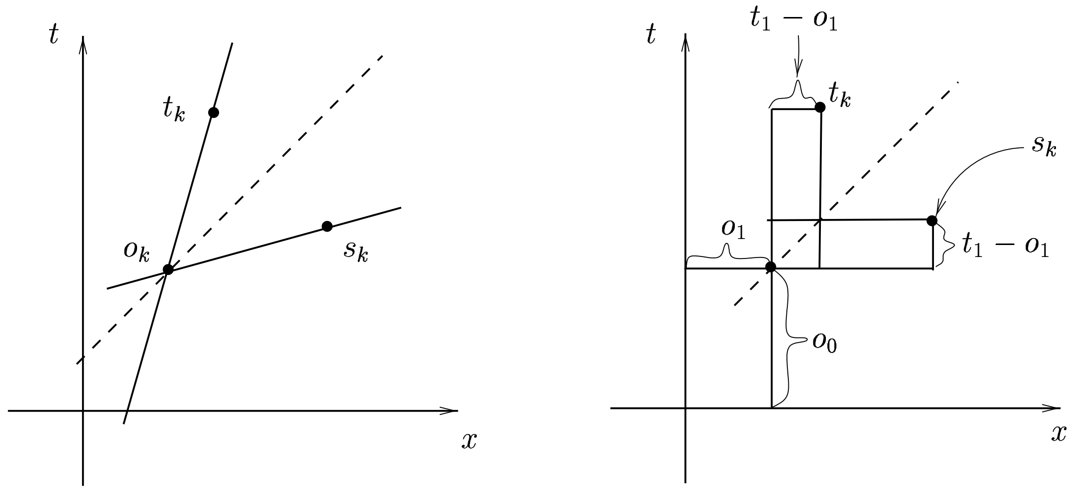


Figure 11: The definition of the point s_k.

In more detail,

$$\mathsf{f}_{km_0}(a,b) \stackrel{\text{def}}{=} a \cdot (t_k - o_k) + b \cdot (s_k - o_k) + o_k.$$

(Here we used that t_k, s_k, o_k are also vectors.) See Figure 13.

Intuitively, take a point $p = (a,b)$ in ${}^2\mathrm{R}$, and let $\mathsf{f}_{km_0}(a,b) = (a', b')$. Then a', b' are the co-ordinates of p in the co-ordinate system with basis $\{(1,0),(0,1)\}$, while a, b are the co-ordinates of p in the co-ordinate system with basis $\{(t_k - o_k), (s_k - o_k)\}$, see Figure 13.

By this, f_{km_0} is defined for all $k \in \mathsf{Eucl}(\mathbf{2}, \mathfrak{R}), k \neq m_0, ang^2(k) \neq 1$. We now define

$w_k \stackrel{\text{def}}{=} \mathsf{f}_{km_0} \circ w_{m_0}$ for all $k \in Obs \setminus \{m_0\}$, and

$W \stackrel{\text{def}}{=} \{\langle m, p, h\rangle : m \in Obs, h \in w_m(p)\}$.

By this, the model $\mathfrak{M} \stackrel{\text{def}}{=} \mathfrak{M}_0^P \stackrel{\text{def}}{=} \langle B, \ldots, W\rangle$ has been defined. $\mathfrak{M}_0^P$ is a frame model.

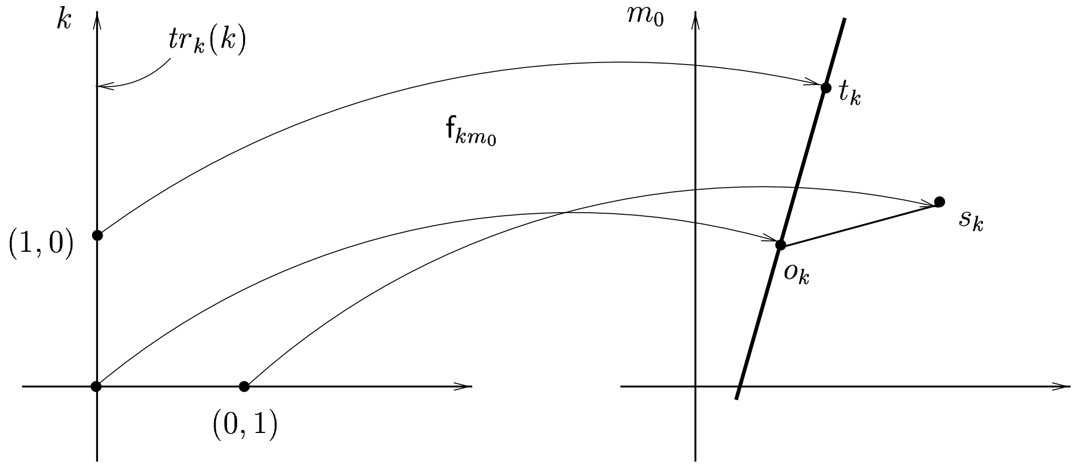


Figure 12: The world view transformation f_{km_0}.

THEOREM 2.12 $\mathfrak{M}_0^P \models Basax(\mathbf{2})$.

Proof. Let $\mathfrak{M} \stackrel{\text{def}}{=} \mathfrak{M}_0^P$. We have already observed that $\mathfrak{M} \models \mathbf{Ax1}, \mathbf{Ax2}$, and that $\mathbf{Ax3} - \mathbf{Ax5}, \mathbf{AxE}$ hold for the fixed observer $m_0 \in Obs$. Let $k \in Obs \setminus \{m_0\}$ be arbitrary. Denote

$\mathsf{f} \stackrel{\text{def}}{=} \mathsf{f}_{km_0}$,

$\mathsf{Eucl} \stackrel{\text{def}}{=} \mathsf{Eucl}(\mathbf{2}, \mathfrak{R})$.

We will check that the following $(i) - (v)$ hold:

(i) $\mathsf{f} : {}^2\mathrm{R} \longrightarrow {}^2\mathrm{R}$ is a bijection.

(ii) f takes lines to lines, i.e. $\mathsf{f}[\ell] \in \mathsf{Eucl}$ for all $\ell \in \mathsf{Eucl}$.

(iii) f takes $\bar{t}$ to k, i.e. $\mathsf{f}[\bar{t}] = k$.

(iv) f maps a slow line to a slow line, i.e. $(\forall \ell \in Obs)\mathsf{f}[\ell] \in Obs$.

(v) f maps "photon-lines" onto "photon-lines", i.e. $\forall \ell(\ell \in Ph) \Leftrightarrow \mathsf{f}[\ell] \in Ph)$.

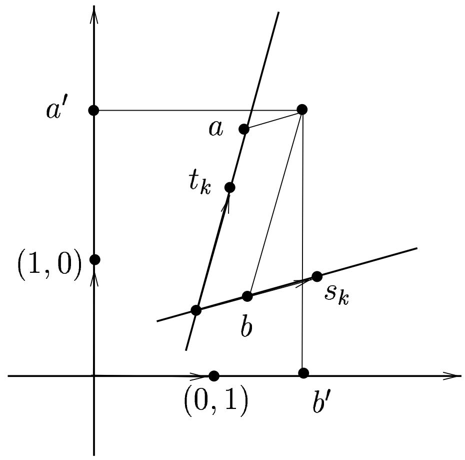


Figure 13: The world view transformation f_{km_0}.

Indeed, (i)-(ii) hold because f is a so called affine transformation[26]. (iii) holds then because $o_k, t_k \in k$, and $\mathsf{f}(0,0) = o_k$, $\mathsf{f}(1,0) = t_k$, $m_0 = \overline{(0,0)(0,1)}$, $k = \overline{o_k t_k}$. Since we defined s_k to be the mirror image of t_k w.r.t. ℓ_k, we have that $\mathsf{f}(1,1) = (t_k - o_k) + (s_k - o_k) + o_k$ lies on the line ℓ_k. Thus $\mathsf{f}[\overline{(0,0)(1,1)}] = \ell_k$. Similarly, $ang^2(\mathsf{f}[\overline{(0,0)(1,-1)}]) = 1$. See Figure 14. In dimension 2 (i.e. in $\mathsf{Eucl}(\mathbf{2},\mathbf{F})$), there are exactly two photon-lines going through each point, and we have seen that f takes the two photon-lines going through $(0,0)$ to the two photon-lines going through $\mathsf{f}(0,0)$.

By (i),(ii) we have that f takes parallel lines to parallel ones. This proves (v). To see that (iv) holds, use a similar argument, and use Figure 15.

We have checked that (i)-(v) hold. Now, in $\mathfrak{M}$ we have that for all $\ell \in B$

$$\ell = tr_k(\mathsf{f}[\ell]). \tag{1}$$

Indeed, let $\ell \in B$. Then $\mathsf{f}[\ell] \in B$, and $\mathsf{f}[\ell] = tr_{m_0}\mathsf{f}[\ell]$. Thus $\ell = \mathsf{f}_{m_0k}[\mathsf{f}[\ell]] = \mathsf{f}_{m_0k}[tr_{m_0}\mathsf{f}[\ell]] = tr_k[\mathsf{f}[\ell]]$, by Proposition2.6(vii). Since f is a bijection, by (ii) we have that both f and f^{-1} preserve Eucl. Using this, together with (ii)-(v), 1 and the fact that $\mathbf{Ax3} - \mathbf{Ax5}, \mathbf{AxE}$ hold for m_0, we get that $\mathbf{Ax3} - \mathbf{Ax5}, \mathbf{AxE}$ hold for k,

[26]This is so because $t_k \neq a \cdot s_k$ for all $a \in \mathrm{R}$, since $ang^2(k) \neq 1$.

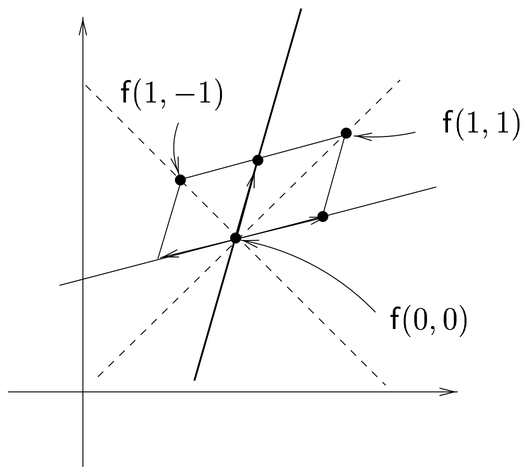


Figure 14: f takes "photon-lines" to "photon-lines".

too. From (i) we get that $Rng(w_k) = Rng(w_{m_0})$. Since k was arbitrary, this proves $\mathfrak{M} \models Basax(\mathbf{2})$. ∎

Now we define the other model $\mathfrak{M}_1^P$. The definition of $\mathfrak{M}_1^P$ is completely analogous to that of $\mathfrak{M}_0^P$, the only difference is that we allow all lines (with angle $\neq 1$) to be observers. In detail: let

$$Obs_1 \stackrel{\text{def}}{=} \{\ell \in \mathsf{Eucl}(\mathbf{2}, \mathfrak{R}) : ang^2(\ell) \neq 1\},$$

$$B_1 \stackrel{\text{def}}{=} Ib_1 \stackrel{\text{def}}{=} Obs_1 \cup Ph = \mathsf{Eucl}(\mathbf{2}, \mathfrak{R}).$$

Then $m_0 \in Obs \subseteq Obs_1$. We define

$$w'_{m_0}(p) \stackrel{\text{def}}{=} \{\ell \in B_1 : p \in \ell\},$$

$$w'_k \stackrel{\text{def}}{=} \mathsf{f}_{km_0} \circ w'_{m_0},$$

$$W' \stackrel{\text{def}}{=} \{\langle m, p, h\rangle : h \in w'_m(p)\},$$

$$\mathfrak{M}_1^P \stackrel{\text{def}}{=} \langle (B_1, Obs_1, Ph_1, Ib_1), \mathfrak{R}, G, E, W'\rangle.$$

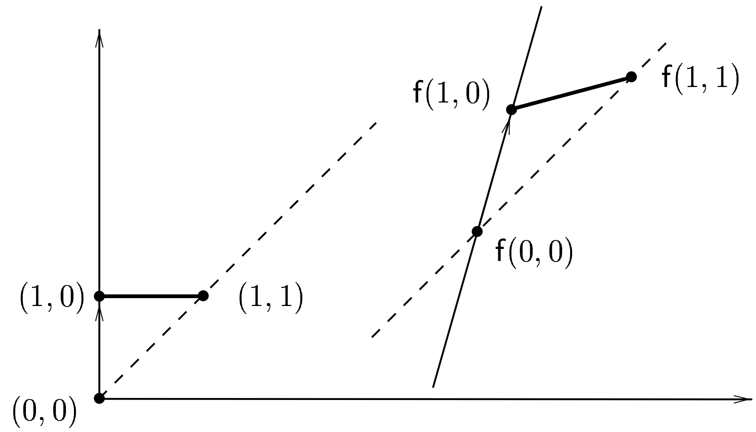


Figure 15: f takes slow lines to slow lines.

THEOREM 2.13 $\mathfrak{M}_1^P \models Basax(\mathbf{2})$.

The proof is analogous to that of Theorem 2.12, we omit it.

3 Special Relativity continued

3.1 Notation and definitions for section 3

Throughout, $n \geq 2$ is an arbitrary integer and $\mathfrak{F}$ is an arbitrary ordered field, unless otherwise specified. $\mathbf{F}$ and F denote the field reduct of $\mathfrak{F}$ and the universe of $\mathfrak{F}$, respectively. Notation depending on n, $\mathfrak{F}$ like $\mathsf{Eucl}(\mathbf{n},\mathbf{F})$ will be used without explicitly indicating n, $\mathfrak{F}$, so we will write e.g. Eucl for $\mathsf{Eucl}(\mathbf{n},\mathbf{F})$ (cf. Convention 2.2).

Notation 3.1

1. Throughout section 3 we denote $\models^{\mathrm{OFG}}$ simply by $\models$ (where $\models^{\mathrm{OFG}}$ was defined at the end of Definition 2.1 in §2.1), and we will never use $\models$ in its original purely logical sense (to avoid misunderstanding).

2. Let Σ be a set of formulas in the frame language of relativity theory. Then we define $\mathsf{Mod}_{\mathfrak{F}}(\Sigma)$ to be the class of all those frame models which are models of Σ and the field structure of which coincides with $\mathfrak{F}$, i.e.

$$\mathsf{Mod}_{\mathfrak{F}}(\Sigma) \stackrel{\text{def}}{=} \{\mathfrak{M} \in \mathsf{FM}(\mathbf{n},\mathfrak{F}) \ : \ \mathfrak{M} \models \Sigma\}\,.$$

3.
$$\begin{aligned}
\mathsf{Eucl} &\stackrel{\text{def}}{=} \mathsf{Eucl}(\mathbf{n},\mathbf{F})\,,\\
\mathsf{SlowEucl} &\stackrel{\text{def}}{=} \{\ell \in \mathsf{Eucl} \ : \ ang^2(\ell) < 1\}\,,\\
\mathsf{PhtEucl} &\stackrel{\text{def}}{=} \{\ell \in \mathsf{Eucl} \ : \ ang^2(\ell) = 1\}\,.
\end{aligned}$$

4. According to Convention 2.2(i) (§2.1) we denote the 0-th, first, second, and third axis of ${}^nF$ by $\bar{t}, \bar{x}, \bar{y}$ and $\bar{z}$, respectively, i.e.

$$\begin{aligned}
\bar{t} &\stackrel{\text{def}}{=} F \times {}^{n-1}\{0\}\,,\\
\bar{x} &\stackrel{\text{def}}{=} \{0\} \times F \times {}^{n-2}\{0\}\,,\\
\bar{y} &\stackrel{\text{def}}{=} \{0\} \times \{0\} \times F \times {}^{n-3}\{0\}\,, \text{ and}\\
\bar{z} &\stackrel{\text{def}}{=} \{0\} \times \{0\} \times \{0\} \times F \times {}^{n-4}\{0\}\,.
\end{aligned}$$

   According to Convention 2.2(i) we will sometime refer to $\bar{t}$ as <u>*time axis*</u>. If $n = 2$ then we use only $\bar{t}, \bar{x}$, if $n = 3$ we use $\bar{t}, \bar{x}, \bar{y}$.

5. We let S denote the "space" part of space-time ${}^nF$, i.e.

$$S \stackrel{\text{def}}{=} \{0\} \times {}^{n-1}F\,.$$

6. $1_t \stackrel{\text{def}}{=} e_0 \stackrel{\text{def}}{=} \langle 1, 0, \dots, 0\rangle$, $1_x \stackrel{\text{def}}{=} e_1 \stackrel{\text{def}}{=} \langle 0, 1, \dots, 0\rangle$, …, $e_{n-1} \stackrel{\text{def}}{=} \langle 0, 0, \dots, 1\rangle \in {}^nF$. If $n = 4$ then $1_t, 1_x, 1_y, 1_z$ denote, respectively, e_0, e_1, e_2, e_3. (If $n = 3$ then we use t, x, y only.)

7. Let $p, q \in {}^nF$ such that $p \neq q$. By $\overline{pq}$ we denote the Euclidean line which contains p and q.

   Whenever we write $\overline{pq}$, we assume that $p, q \in {}^nF$ and $p \neq q$.

8. Let $p \in {}^nF$. Then $|p| \stackrel{\text{def}}{=} p_0^2 + p_1^2 + \ldots + p_{n-1}^2$.

9. Let $x \in F$. Then $|x| \stackrel{\text{def}}{=} \max\{x, -x\}$.

10. ${}^+F \stackrel{\text{def}}{=} \{x \in F \;:\; x > 0\}\,.$

11. Let $\mathfrak{A}$ be an algebraic structure. Then $Aut(\mathfrak{A})$ denotes the set of all automorphisms of $\mathfrak{A}$.

12. Let $g : F \longrightarrow F$ be a function. Then $\tilde{g} : {}^nF \longrightarrow {}^nF$ denotes the function defined as follows. $(\forall p \in {}^nF)\tilde{g}(p) \stackrel{\text{def}}{=} \langle g(p_0), g(p_1), \dots, g(p_{n-1})\rangle$.

13. Let $p \in {}^nF$. Then $\tau_p : {}^nF \longrightarrow {}^nF$ denotes the translation by vector p, i.e.

$$(\forall q \in {}^nF)\ \tau_p(q) \stackrel{\text{def}}{=} p + q\,.$$

14. Id denotes the identical transformation of ${}^nF$ taking p to p.

15. Let $\mathfrak{N}$, $\mathfrak{M}$ be frame models. Then $\mathfrak{N} \subseteq \mathfrak{M}$ means that $\mathfrak{N}$ is a strong sub-model of $\mathfrak{M}$, i.e. all relations and sorts of $\mathfrak{M}$ are restricted to the universe of $\mathfrak{N}$ (cf. Chang-Keisler [7]). $\lhd$

Definition 3.2

1. A function $g : {}^nF \longrightarrow {}^nF$ is called an <u>*affine transformation*</u> iff there is an invertible linear transformation[27] h of the vector space ${}^n\mathbf{F}$ and a vector $q \in {}^nF$ such that $g = h \circ \tau_q$, i.e. g is an affine transformation iff it is an invertible linear transformation $p \mapsto h(p)$ composed with a translation $p \mapsto p + q$.

   The set of affine transformations is denoted by $Aftr = Aftr(\mathbf{n}, \mathbf{F})$.

[27] For the notion of linear transformation see e.g. Halmos [21]. For completeness we note that $f : {}^nF \longrightarrow {}^nF$ is a linear transformation iff f is a homomorphism of the vector space ${}^n\mathbf{F}$ into itself.

2. Let $j \leq n$. We say that P is a *j-dimensional plane* iff there is a j-dimensional subspace $\mathbf{W}$ of ${}^n\mathbf{F}$ and a vector $p \in {}^nF$ such that $P = W + p$, where $W + p \stackrel{\text{def}}{=} \{w + p : w \in W\}$.[28]
   By a *plane* we understand a 2-dimensional plane.

3. Let $\ell_1, \ell_2 \in \mathsf{Eucl}$.

   (i) We say that ℓ_1 and ℓ_2 are in the same plane if there is a 2-dimensional plane P such that $\ell_1, \ell_2 \subseteq P$.

   (ii) If there is a unique 2–dimensional plane P such that $\ell_1, \ell_2 \subseteq P$, then we denote this unique P by
   $$\mathsf{Plane}(\ell_1, \ell_2)\,.$$
   E.g. $\mathsf{Plane}(\bar{t}, \bar{x}) = F \times F \times {}^{n-2}\{0\}$ and
   $\mathsf{Plane}(\bar{t}, \bar{y}) = F \times \{0\} \times F \times {}^{n-3}\{0\}$.

   (iii) We say that ℓ_1 and ℓ_2 are *parallel*, in symbols $\ell_1 \parallel \ell_2$, iff ℓ_1 and ℓ_2 are in the same plane and $\ell_1 \cap \ell_2 = \emptyset$ or $\ell_1 = \ell_2$.

   (iv) Whenever we write $\mu \parallel \nu$, this means that $\mu, \nu \in \mathsf{Eucl}$ and $\mu \parallel \nu$.

4. (i) Assume $p, q \in {}^nF$. Then we say that *p is orthogonal to q*, in symbols $p \perp q$, iff $p_0q_0 + p_1q_1 + \cdots + p_{n-1}q_{n-1} = 0$.[29]

   (ii) Assume $\ell_1 = \{r_1 + a \cdot s_1 : a \in F\}$, $\ell_2 = \{r_2 + a \cdot s_2 : a \in F\} \in \mathsf{Eucl}$, for some $r_1, r_2 \in {}^nF$ and $s_1, s_2 \in {}^nF \setminus \{\bar{0}\}$. Then we say that ℓ_1 is orthogonal to ℓ_2, in symbols $\ell_1 \perp \ell_2$, iff $s_1 \perp s_2$. $\lhd$

[28] We use the universal algebraic convention that $\mathbf{W}$ denotes the algebra (vector space) and W denotes its universe. (We also note that by a plane one understands a set of form $W + p$.)

[29] For completeness we note that this is Euclidean orthogonality which is different from Minkowski orthogonality.

3.2 Some properties of the world view transformation

In this subsection we prove that *Basax* implies that the world view transformation f_{mk} takes all straight lines to straight lines. (Such transformations are called collineations in geometry.) We will do this in Thm.3.3 below. In this we will use the notation Eucl introduced in item 3 of Notation 3.1.

THEOREM 3.3

(i)	$Basax$	$\models$	$(\forall m,k \in Obs)(\forall \ell \in \mathsf{Eucl})\ \mathsf{f}_{mk}[\ell] \in \mathsf{Eucl}$.
(ii)	$\{\mathbf{Ax1}, \mathbf{Ax2}, \mathbf{Ax3}, \mathbf{Ax5}, \mathbf{Ax6}\}$	$\models$	$(\forall m,k \in Obs)(\forall \ell \in \mathsf{Eucl})\ \mathsf{f}_{mk}[\ell] \in \mathsf{Eucl}$.

We will give the **proof** of Thm.3.3 after Prop.3.10 below.

In the proof of Thm.3.3 we will use Prop.2.6(iv),(viii) in §2.3 and two elementary propositions from Euclidean geometry (cf. Prop.3.9, Prop.3.10 below) and Lemmas 3.6, 3.7 below. For completeness let us recall from §2.3 that Prop.2.6(iv) states that $\mathsf{f}_{mk} : {}^nF \longrightarrow {}^nF$ is a bijection, and Prop.2.6(viii) states that f_{mk} takes slow lines to lines, i.e. if $\ell \in \mathsf{SlowEucl}$ then $\mathsf{f}_{mk}[\ell] \in \mathsf{Eucl}$, where the notation $\mathsf{SlowEucl}$ was introduced in item 3 of Notation 3.1.

Before proving Thm.3.3 we state one of its corollaries in Thm.3.4 below. We will use notation/definitions introduced in items 11, 12 of Notation 3.1 and in item 2 of Def.3.2. (E.g. *Aftr* is the set of affine transformations.)

THEOREM 3.4
$Basax \models (\forall m,k \in Obs)\Big(\mathsf{f}_{mk} = \widetilde{\varphi} \circ g,\ \textit{for some } g \in Aftr \textit{ and } \varphi \in Aut(\mathbf{F})\Big).$

Proof: This is a corollary of Thm.3.3 and Lemma 3.5 below. ∎

The following lemma is known from algebra. It says that a function $f : {}^nF \to {}^nF$ is a bijection taking lines to lines iff it is an affine transformation composed with a field automorphism.

LEMMA 3.5 *Assume $f : {}^nF \longrightarrow {}^nF$ is a function. For (i), (ii) below we claim, (i)⇔(ii).*

(i) *f is a bijection with $(\forall \ell \in \mathsf{Eucl}) f[\ell] \in \mathsf{Eucl}$.*

(ii) *$f = \widetilde{\varphi} \circ g$, for some $g \in Aftr$ and $\varphi \in Aut(\mathbf{F})$.*

We omit the **proof** since it is available in the literature.

Now, we turn to proving Thm.3.3.

The following lemma states that *Basax* implies that f_{mk} takes slow parallel lines to parallel lines. We will use $\ell_1 \parallel \ell_2$ introduced in item 3 of Def.3.2.

LEMMA 3.6

$$Basax \models (\forall m, k \in Obs)(\forall \ell_1, \ell_2 \in \mathsf{SlowEucl})(\ell_1 \parallel \ell_2 \;\Rightarrow\; \mathsf{f}_{mk}[\ell_1] \parallel \mathsf{f}_{mk}[\ell_2]).$$

Proof: Let $\ell_1, \ell_2 \in \mathsf{SlowEucl}$ with $\ell_1 \parallel \ell_2$. We have to prove that $\mathsf{f}_{mk}[\ell_1] \parallel \mathsf{f}_{mk}[\ell_2]$. We may assume that $\ell_1 \neq \ell_2$. Let $\ell_3, \ell_4 \in \mathsf{SlowEucl}$ such that $\ell_1 \cap \ell_3 = \{p\}$, $\ell_1 \cap \ell_4 = \{q\}$, $\ell_2 \cap \ell_4 = \{r\}$, $\ell_2 \cap \ell_3 = \{s\}$, $\ell_3 \cap \ell_4 = \{t\}$, for some distinct $p, q, r, s, t \in {}^nF$ (see Figure 16). Let such p, q, r, s, t be fixed.

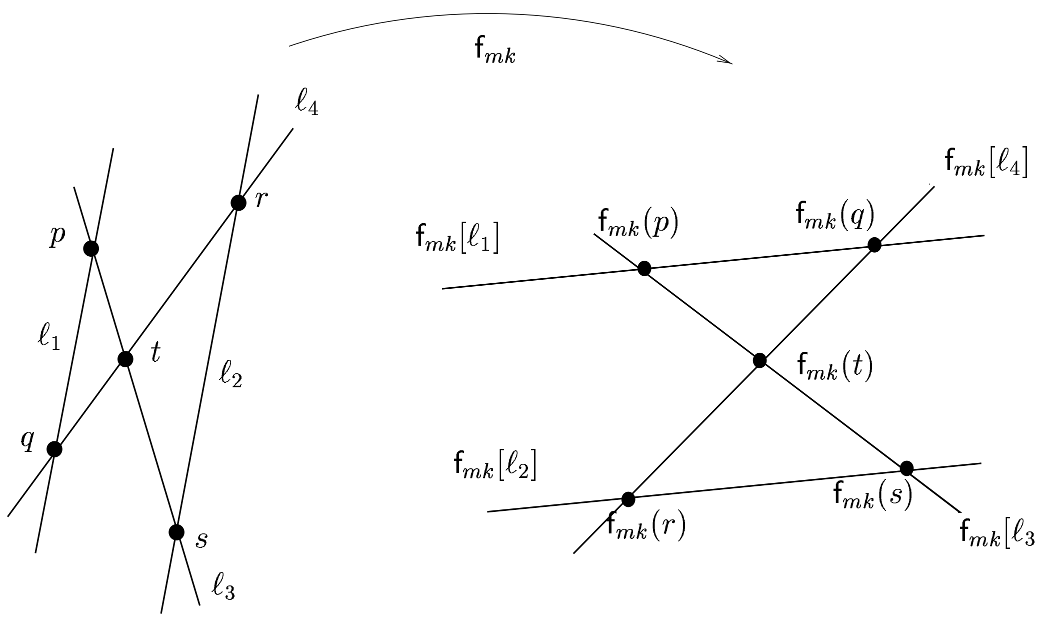


Figure 16: Illustration for the proof of Lemma 3.6.

Obviously such ℓ_3 and ℓ_4 exist. Now by the above construction and by f_{mk} being a bijection taking slow lines to lines (cf. Prop.2.6(iv),(viii)), we have that (2)–(5) below hold (see Figure 16).

$$\begin{aligned}
\mathsf{f}_{mk}[\ell_1] \cap \mathsf{f}_{mk}[\ell_3] &= \{\mathsf{f}_{mk}(p)\}\,, \\
\mathsf{f}_{mk}[\ell_1] \cap \mathsf{f}_{mk}[\ell_4] &= \{\mathsf{f}_{mk}(q)\}\,,
\end{aligned}$$

(2) $$\mathsf{f}_{mk}[\ell_2] \cap \mathsf{f}_{mk}[\ell_4] = \{\mathsf{f}_{mk}(r)\}\,,$$
$$\mathsf{f}_{mk}[\ell_2] \cap \mathsf{f}_{mk}[\ell_3] = \{\mathsf{f}_{mk}(s)\}\,, \text{ and}$$
$$\mathsf{f}_{mk}[\ell_3] \cap \mathsf{f}_{mk}[\ell_4] = \{\mathsf{f}_{mk}(t)\}\,.$$

(3) $$\mathsf{f}_{mk}(p), \mathsf{f}_{mk}(q), \mathsf{f}_{mk}(r), \mathsf{f}_{mk}(s), \mathsf{f}_{mk}(t) \text{ are distinct points.}$$

(4) $$\mathsf{f}_{mk}[\ell_1], \mathsf{f}_{mk}[\ell_2], \mathsf{f}_{mk}[\ell_3], \mathsf{f}_{mk}[\ell_4] \in \mathsf{Eucl}\,.$$

(5) $$\mathsf{f}_{mk}[\ell_1] \cap \mathsf{f}_{mk}[\ell_2] = \emptyset\,.$$

By (2)–(4), we have that $\mathsf{f}_{mk}[\ell_1]$ and $\mathsf{f}_{mk}[\ell_2]$ are in the same plane. This and (5) imply that $\mathsf{f}_{mk}[\ell_1] \parallel \mathsf{f}_{mk}[\ell_2]$. ∎

The following lemma states that if $\overline{pq}$ is a slow line then f_{mk} takes the midpoint of segment pq to the midpoint of segment $\mathsf{f}_{mk}(p)\mathsf{f}_{mk}(q)$, where the notation $\overline{pq}$ was introduced in item 7 of Notation 3.1.

LEMMA 3.7 *Basax* $\models (\forall m, k \in Obs)(\forall p, q \in {}^nF)\Big(\overline{pq} \in \mathsf{SlowEucl} \quad \Rightarrow$

$$\mathsf{f}_{mk}\left(\frac{p+q}{2}\right) = \frac{\mathsf{f}_{mk}(p) + \mathsf{f}_{mk}(q)}{2}\Big).$$ [30]

(See Figure 17.)

We will give the **proof** after the proof of Prop.3.9 below.

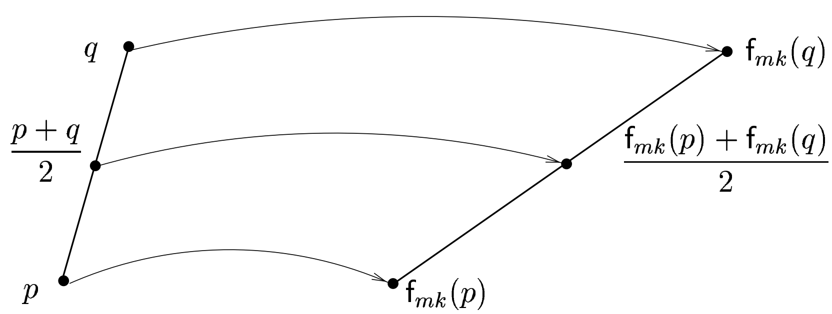


Figure 17: Illustration for Lemma 3.7.

For the proof of Lemma 3.7 we will use Proposition 3.9 below which is a proposition in elementary Euclidean geometry and it says that the diagonals of a parallelogram bisect each other.

[30] Since p is a vector and $\frac{1}{2} \in F$ $\;\frac{1}{2} \cdot p = \frac{p}{2}$ is defined. (Similarly for $p+q$ in place of p.)

Definition 3.8 Let $q, r, p, s \in {}^nF$. We say that $\langle q, r, p, s\rangle$ is a <u>*parallelogram*</u> iff the following hold. No three of q, r, p, s is on the same line (i.e. collinear), $\overline{qr} \parallel \overline{sp}$, and $\overline{qs} \parallel \overline{rp}$. We write $qrps$ for $\langle q, r, p, s\rangle$. $\lhd$

PROPOSITION 3.9 *Assume $q, r, p, s \in {}^nF$ such that $qrps$ is a parallelogram. Then the diagonals of parallelogram $qrps$ bisect each other, that is $\overline{pq} \cap \overline{rs} = \{\frac{p+q}{2}\} = \{\frac{r+s}{2}\}$ (see Figure 18).*

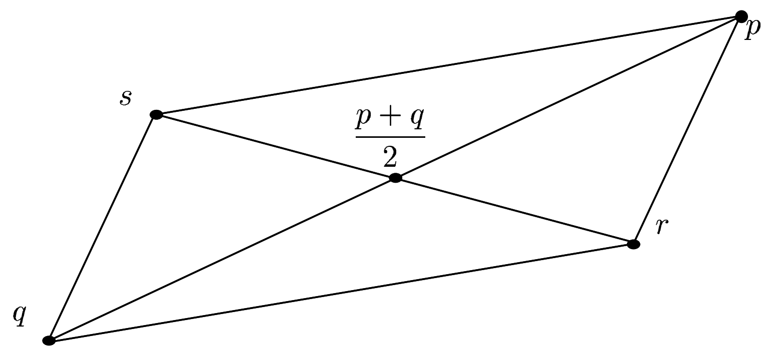


Figure 18: The diagonals of a parallelogram bisect each other.

Proof: The proof known from geometry uses only those axioms of geometry which are true in $\mathsf{Eucl}(\mathbf{n}, \mathbf{F})$ for any field $\mathbf{F}$. The reader is invited to check the details. ∎

Proof of Lemma 3.7: Let $p, q \in {}^nF$ with $\overline{pq} \in \mathsf{SlowEucl}$. We will show that $\mathsf{f}_{mk}(\frac{p+q}{2}) = \frac{\mathsf{f}_{mk}(p)+\mathsf{f}_{mk}(q)}{2}$. To prove this let $r, s \in {}^nF$ such that $qrps$ is a parallelogram and $\overline{rs}, \overline{qr}, \overline{sp}, \overline{qs}, \overline{rp} \in \mathsf{SlowEucl}$ (see Figure 20). (It is easy to see that such r and s exist because of the following. Choose $\ell \in \mathsf{SlowEucl}$ such that $\frac{p+q}{2} \in \ell$. Choose $r, s \in \ell$ such that r and s are "near to" $\frac{p+q}{2}$, then the lines $\overline{rs}, \overline{qr}, \overline{sp}, \overline{qs}, \overline{rp} \in \mathsf{SlowEucl}$, since $\overline{pq} \in \mathsf{SlowEucl}$.) Then by applying Lemma 3.6, since $qrsp$ is a parallelogram, $\mathsf{f}_{mk}(q)\mathsf{f}_{mk}(r)\mathsf{f}_{mk}(p)\mathsf{f}_{mk}(s)$ will be a parallelogram (see Figure 19).

By Prop.3.9 the intersection of the diagonals of parallelograms $qrps$ and $\mathsf{f}_{mk}(q)\mathsf{f}_{mk}(r)\mathsf{f}_{mk}(p)\mathsf{f}_{mk}(s)$ are $\frac{p+q}{2}$, $\frac{\mathsf{f}_{mk}(p)+\mathsf{f}_{mk}(q)}{2}$, respectively. Since f_{mk} is a bijection taking slow lines to lines (and since $\overline{pq}, \overline{rs}, \overline{qr}, \overline{ps}, \overline{qs}, \overline{rp} \in \mathsf{SlowEucl}$) f_{mk} will take the intersection $(\frac{p+q}{2})$ of the diagonals of the parallelogram $qrps$ to the intersection $\left(\frac{\mathsf{f}_{mk}(p)+\mathsf{f}_{mk}(q)}{2}\right)$ of the diagonals of parallelogram $\mathsf{f}_{mk}(q)\mathsf{f}_{mk}(r)\mathsf{f}_{mk}(p)\mathsf{f}_{mk}(s)$. Hence

$$\mathsf{f}_{mk}\left(\frac{p+q}{2}\right) = \frac{\mathsf{f}_{mk}(p)+\mathsf{f}_{mk}(q)}{2}. \quad ∎$$

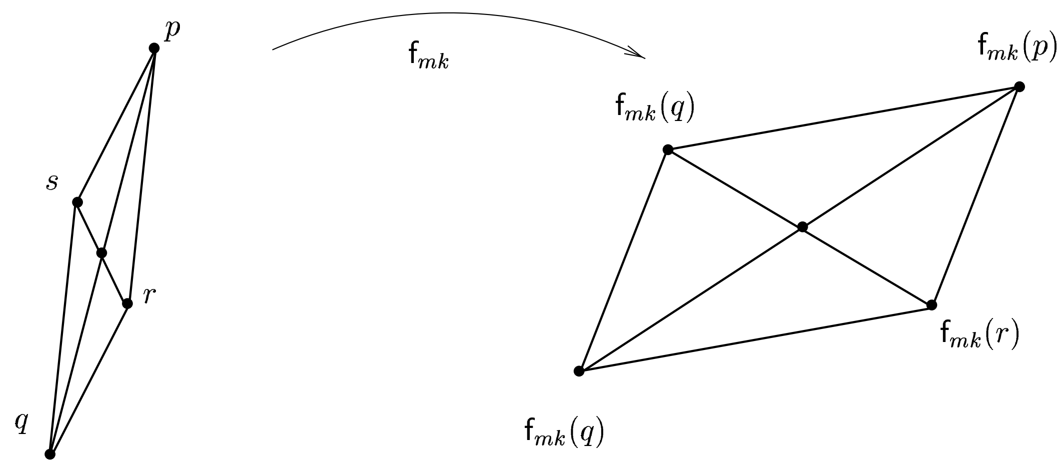


Figure 19: Illustration for the proof of Lemma 3.7.

The following proposition is a proposition of elementary Euclidean geometry.

PROPOSITION 3.10 *Assume $p,q,r,s,t,u,v \in {}^nF$ are distinct points such that $\overline{ps} \neq \overline{pt}$, $u \in \overline{ps}$, $v \in \overline{pt}$, $r \in \overline{uv}$, $\overline{st} \parallel \overline{uv}$, and q is the midpoint of segment st (i.e. $q = \frac{s+t}{2}$). Then*

(r is the midpoint of segment uv)[31] $\iff$ *(p,q,r are collinear)*[32].
(See Figure 20).

Proof: The proof known from geometry uses only those axioms of geometry which are true in $\mathsf{Eucl}(\mathbf{n},\mathbf{F})$ for any field $\mathbf{F}$. The reader is invited to check the details. ∎

Proof of Thm.3.3: We will present the proof only for (i) but it will be clear that we never use **Ax4** and **AxE**.

Let $p,q,r \in {}^nF$ be collinear and distinct. To prove Thm.3.3, since f_{mk} is a bijection, it is enough to prove that $\mathsf{f}_{mk}(p), \mathsf{f}_{mk}(q), \mathsf{f}_{mk}(r)$ are collinear. Let $s,t,u,v \in {}^nF$ such that p,q,r,s,t,u,v satisfy the conditions of Prop.3.10 (i.e. p,q,r,s,t,u,v are distinct, $\overline{ps} \neq \overline{pt}$, $u \in \overline{ps}$, $v \in \overline{pt}$, $r \in \overline{uv}$, $\overline{st} \parallel \overline{uv}$, and $q = \frac{s+t}{2}$) and they satisfy an extra condition which is $\overline{ps}, \overline{pt}, \overline{st}, \overline{uv} \in \mathsf{SlowEucl}$ (see Figure 20). (It is easy to see that such s,t,u,v exist because of the following. Choose $\overline{st}, \overline{uv}$ to be parallel with the time axis, and choose s and t "very far" from q. Then clearly $\overline{st}, \overline{uv}, \overline{ps}, \overline{st}$ will be slow lines). Then by direction "⇐" of Prop.3.10, we have that r is the midpoint

[31] I.e. $r = \frac{u+v}{2}$
[32] I.e. there is a line containing p,q,r.

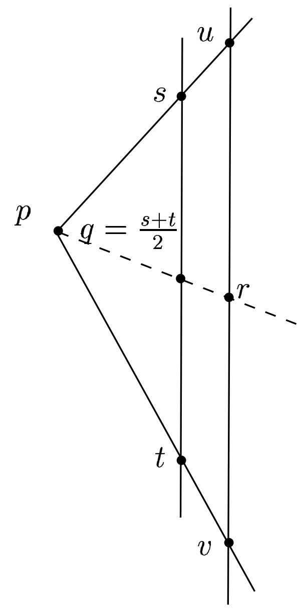


Figure 20: Illustration for Prop.3.10 and for the proof of Thm.3.3.

of segment uv. Since $\overline{st}, \overline{uv} \in \mathsf{SlowEucl}$ and q and r are the midpoints of segments st and uv, respectively, by Lemma 3.7, we have that (6) and (7) below hold.

(6) $\quad \mathsf{f}_{mk}(q)$ is the midpoint of segment $\mathsf{f}_{mk}(s)\mathsf{f}_{mk}(t)$.

(7) $\quad \mathsf{f}_{mk}(r)$ is the midpoint of segment $\mathsf{f}_{mk}(u)\mathsf{f}_{mk}(v)$.

Now by the above construction, by f_{mk} being a bijection taking slow lines to lines, and by (6), we have that $\mathsf{f}_{mk}(p), \mathsf{f}_{mk}(q), \mathsf{f}_{mk}(r), \mathsf{f}_{mk}(s), \mathsf{f}_{mk}(t), \mathsf{f}_{mk}(u), \mathsf{f}_{mk}(v)$ satisfy the conditions of Prop.3.10 (see Figure 21).

But $\mathsf{f}_{mk}(r)$ is the midpoint of segment $\mathsf{f}_{mk}(u)\mathsf{f}_{mk}(v)$ by (7), hence by Prop.3.10, we have that $\mathsf{f}_{mk}(p), \mathsf{f}_{mk}(q), \mathsf{f}_{mk}(r)$ are collinear. ∎

COROLLARY 3.11 Assume $f : {}^nF \longrightarrow {}^nF$ is a bijection such that $(\forall \ell \in \mathsf{SlowEucl}) f[\ell] \in \mathsf{Eucl}$. Then $(\forall \ell \in \mathsf{Eucl}) f[\ell] \in \mathsf{Eucl}$.

Proof: This is a corollary of the proof of Thm.3.3. (It can be checked that we did not use more than the present conditions). ∎

In the following corollary we will use the notion of j–dimensional plane introduced in item 2 of Def.3.2.

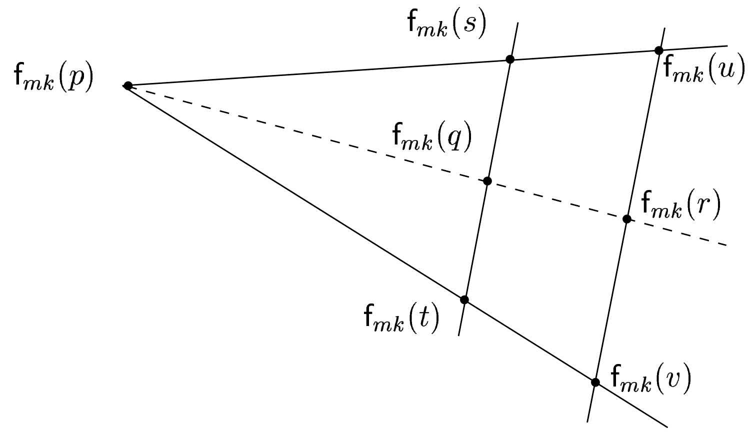


Figure 21: Illustration for the proof of Thm.3.3.

COROLLARY 3.12 Assume $j \leq n$, and assume P is a j-dimensional plane. Then

$$Basax \models (\forall m, k \in Obs)(\mathsf{f}_{mk}[P] \text{ is a } j\text{-dimensional plane}).$$

Proof: This is a corollary of Thm.3.3. ∎

Let us recall from item 3 of Notation 3.1 that $\mathsf{PhtEucl} \stackrel{\text{def}}{=} \{\ell \in \mathsf{Eucl} : ang^2(\ell) = 1\}$.

PROPOSITION 3.13
$Basax \models (\forall m, k \in Obs)(\forall \ell \in \mathsf{Eucl})(\ell \in \mathsf{PhtEucl} \ \Leftrightarrow\ \mathsf{f}_{mk}[\ell] \in \mathsf{PhtEucl})$.

Proof: The proof is straightforward, but for completeness we mention that the proposition follows by **Ax5** and **AxE**. ∎

For Prop.3.15 below we will need the notion of rhombus. This comes next. For this definition we will use the definition/notation of orthogonality ($\perp$) introduced in item 4 of Def.3.2.

Definition 3.14 Assume $q, r, p, s \in {}^nF$ and assume that $qrps$ is a parallelogram. We say that $qrps$ is a *rhombus* iff $\overline{qp} \perp \overline{rs}$. See Figure 22. $\lhd$

In the following proposition we will use notation $\bar{t}$, $\bar{x}$, 1_t, 1_x introduced in items 4, 6 of Notation 3.1. Let us recall that for $n = 2$ $\quad \bar{t} \stackrel{\text{def}}{=} F \times \{0\}$, $\bar{x} \stackrel{\text{def}}{=} \{0\} \times F$, $1_t \stackrel{\text{def}}{=} \langle 1, 0\rangle$ and $1_x \stackrel{\text{def}}{=} \langle 0, 1\rangle$.

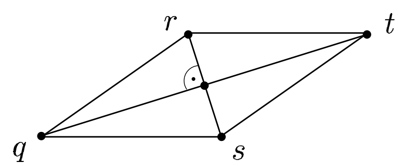


Figure 22: Illustration for Def.3.14.

PROPOSITION 3.15 *Assume* $\mathfrak{M} \in \mathsf{Mod}_{\text{OFG}}(Basax(\mathbf{2}))$*. Let* $m, k \in Obs$ *and let* $\mathsf{f} \stackrel{\text{def}}{=} \mathsf{f}_{mk}$*. Then (i)–(iii) below hold.*

(i) $\mathsf{f}(1_t)$ *and* $\mathsf{f}(1_x)$ *are mirror images of each other w.r.t. a line* ℓ *with* $\ell \in \mathsf{PhtEucl}$ *and* $\ell \ni \mathsf{f}(\bar{0})$*.*

(ii) $\mathsf{f}[\bar{t}\,]$ *and* $\mathsf{f}[\bar{x}]$ *are mirror images of each other w.r.t. the lines* $\ell_1, \ell_2 \in \mathsf{PhtEucl}$ *with* $\ell_1, \ell_2 \ni \mathsf{f}(\bar{0})$ *and* $\ell_1 \neq \ell_2$*.*

(iii) $\mathsf{f}(\bar{0})\mathsf{f}(1_t)\mathsf{f}(1,1)\mathsf{f}(1_x)$ *is a rhombus such that* $\overline{\mathsf{f}(\bar{0})\mathsf{f}(1,1)},\ \overline{\mathsf{f}(1_t)\mathsf{f}(1_x)} \in \mathsf{PhtEucl}$*.*

See Figure 23.

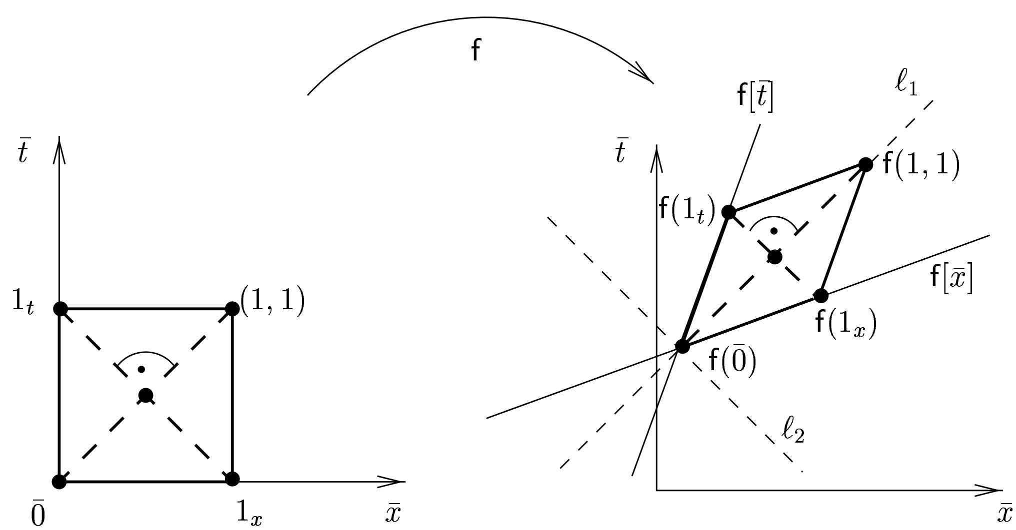


Figure 23: Illustration for Prop.3.15.

Proof of Prop.3.15: Let $\mathfrak{M} \in \mathsf{Mod}_{\text{OFG}}(Basax(\mathbf{2}))$, $m, k \in Obs$ and $\mathsf{f} \stackrel{\text{def}}{=} \mathsf{f}_{mk}$. Throughout the proof the reader is asked to consult Figure 23.

Throughout the proof we will use that f is a bijection (cf. Prop.2.6(iv)).

By Thm.3.3 and Prop.3.13 we have that (8) and (9) below hold.

(8) $(\forall \ell \in \mathsf{Eucl}) \quad \mathsf{f}[\ell] \in \mathsf{Eucl}\,.$

(9) $(\forall \ell \in \mathsf{Eucl}) \quad (\ell \in \mathsf{PhtEucl} \;\Leftrightarrow\; \mathsf{f}[\ell] \in \mathsf{PhtEucl})\,.$

Now

(10) $\mathsf{f}(\bar{0})\mathsf{f}(1_t)\mathsf{f}(1,1)\mathsf{f}(1_x)$ is a parallelogram,

since f is a bijection satisfying (8) above and since "$\bar{0}1_t\langle 1,1\rangle 1_x$" is a parallelogram.

We have

(11) $\overline{\mathsf{f}(\bar{0})\mathsf{f}(1,1)}\,,\ \overline{\mathsf{f}(1_t)\mathsf{f}(1_x)} \in \mathsf{PhtEucl}\,,$

since $\overline{\bar{0}\langle 1,1\rangle}\,,\ \overline{1_t 1_x} \in \mathsf{PhtEucl}$ and since f is a bijection satisfying (9) above.

Now by (10), we have that $\overline{\mathsf{f}(\bar{0})\mathsf{f}(1,1)} \neq \overline{\mathsf{f}(1_t)\mathsf{f}(1_x)}$ and $\overline{\mathsf{f}(\bar{0})\mathsf{f}(1,1)} \cap \overline{\mathsf{f}(1_t)\mathsf{f}(1_x)} \neq \emptyset$. By this, by $n = 2$ and by (11), we have

(12) $\overline{\mathsf{f}(\bar{0})\mathsf{f}(1,1)} \perp \overline{\mathsf{f}(1_t)\mathsf{f}(1_x)}\,.$

(10), (11) and (12) completes the proof of item (iii) of Prop.3.15.

Item (i) of Prop.3.15 follows from item (iii) for the choice $\ell = \overline{\mathsf{f}(\bar{0})\mathsf{f}(1,1)}$.

Item (ii) of Prop.3.15 follows from item (iii) of Prop.3.15 and from (8) above for the choice $\ell_1 = \overline{\mathsf{f}(\bar{0})\mathsf{f}(1,1)}$ and $\ell_2 \in \mathsf{PhtEucl}$ with $\ell_2 \ni \mathsf{f}(\bar{0})$ and $\ell_2 \neq \ell_1$. Clearly such an ℓ_2 exists and is unique by $n = 2$.

This completes the proof of Prop.3.15. ∎

COROLLARY 3.16 Assume $\mathsf{f} : {}^2F \longrightarrow {}^2F$ is a bijection such that

$$(\forall \ell \in \mathsf{Eucl})\Big(\mathsf{f}[\ell] \in \mathsf{Eucl} \;\wedge\; (\mathsf{f}[\ell] \in \mathsf{PhtEucl} \;\Leftrightarrow\; \ell \in \mathsf{PhtEucl})\Big)\,.$$

Then $\mathsf{f}[\bar{t}\,]$ and $\mathsf{f}[\bar{x}]$ are mirror images of each other w.r.t. the lines $\ell_1, \ell_2 \in \mathsf{PhtEucl}$ with $\ell_1, \ell_2 \ni \mathsf{f}(\bar{0})$ and $\ell_1 \neq \ell_2$.

Proof: This is a corollary of the proof of Prop.3.15(ii). (It can be checked that we did not use more than the present conditions.) ∎

3.3 The refined theory *Newbasax*

In this subsection we introduce the axiom system *Newbasax*. We show that *Newbasax* is strictly weaker than *Basax* and we characterize the models of *Newbasax* in terms of models of *Basax*: the models of *Newbasax* are, roughly speaking, unions of models of *Basax*.

Let $p \in {}^nF$ and $\varepsilon \in {}^+F$ (where ${}^+F$ was introduced in item 10 of Notation 3.1). Then by $S(p,\varepsilon)$ we denote the $\underline{\varepsilon\text{-neighborhood of } p}$ defined as follows. Recall that $|p|$ was defined in item 8 of Notation 3.1.

$$S(p,\varepsilon) \stackrel{\text{def}}{=} \{q \in {}^nF \,:\, |q-p| < \varepsilon\}\,.$$

Let $H \subseteq {}^nF$. We say that H is an *open set* iff

$$(\forall q \in H)(\exists \varepsilon \in {}^+F)\; S(q,\varepsilon) \subseteq H\,.$$

The set of open subsets of ${}^nF$ is denoted by $Open = Open(\mathbf{n}, \mathfrak{F})$.

As it is indicated at the beginning of §2.2, we introduce a more refined system of axioms, *Newbasax*. This comes next. Recall from subsection 2.2 (Definition 2.4) the axioms **Ax1**–**AxE** and the set *Basax*.

Below we postulate axioms $\mathbf{Ax6_{00}}$, $\mathbf{Ax6_{01}}$, $\mathbf{Ax3_0}$, $\mathbf{AxE_0}$.

$\mathbf{Ax6_{00}}$ $(\forall m,k \in Obs)\; w_m[tr_m(k)] \subseteq Rng(w_k)$.

Intuitively, observer k sees all those events which are seen by m on k's life line.

$\mathbf{Ax6_{01}}$ $(\forall m,k \in Obs)\; Dom(\mathsf{f}_{mk}) \in Open$.

Intuitively, if observers m and k see event $H \subseteq B$ then k sees all those events which m sees "close" to H.

$\mathbf{Ax3_0}$ $(\forall h \in Ib)\; (tr_m(h) \in G \cup \{\emptyset\} \quad \wedge \quad (\exists k \in Obs) tr_k(h) \neq \emptyset)$.

That is, the life line of any inertial body h as seen by any observer m must be a geodesics or the empty-set, and there is an observer k such that the life line of h for k is not the empty-set. $\mathbf{Ax3_0}$ differs from **Ax3** in that the life line of an inertial body seen by an observer can be the empty-set.

$\mathbf{AxE_0}$ $(\forall m \in Obs)(\forall ph \in Ph)(tr_m(ph) \neq \emptyset \quad \Rightarrow \quad v_m(ph) = 1)$.

That is, if the life line of photon ph is not the empty-set for observer m, then the speed of ph for observer m is 1.

Definition 3.17 (*Newbasax*)
We define

$Newbasax \stackrel{\text{def}}{=} Basax \setminus \{\mathbf{Ax6}, \mathbf{Ax3}, \mathbf{AxE}\} \cup \{\mathbf{Ax6_{00}}, \mathbf{Ax6_{01}}, \mathbf{Ax3_0}, \mathbf{AxE_0}\}$,

where the axioms $\mathbf{Ax6_{00}}$, $\mathbf{Ax6_{01}}$, $\mathbf{Ax3_0}$, $\mathbf{AxE_0}$ were defined above. $\lhd$

Remark 3.18 We note that $Basax \models Newbasax$. $\lhd$

In definition below we will define a binary relation $\stackrel{\odot}{\to}$ between observers and bodies. The intuitive meaning of $m \stackrel{\odot}{\to} b$ is that observer m "sees" body b, that is the life line of b seen by m is not the empty-set.

Definition 3.19 Let $\mathfrak{M} \in \mathsf{FM}$. We define the binary relation $\stackrel{\odot}{\to} \subseteq Obs \times B$ as follows.

$$(\forall m \in Obs)(\forall b \in B)(m \stackrel{\odot}{\to} b \stackrel{\text{def}}{\iff} tr_m(b) \neq \emptyset)\,. \qquad \lhd$$

Thm.3.20 below states that *Newbasax* together with **Ax3** is equivalent with *Basax*.

THEOREM 3.20 *(i) and (ii) below hold.*

(i) $Newbasax \cup \{\mathbf{Ax3}\} \mathrel{=\!\!|\!\!\models} Basax$.

(ii) $Newbasax \cup \{(\forall m, k \in Obs)\ m \stackrel{\odot}{\to} k\} \mathrel{=\!\!|\!\!\models} Basax$.

We will give the **proof** after Thm.3.23 below.

The following theorem states that two observers either see the same events or they see completely different events.

THEOREM 3.21

$Newbasax \models (\forall m, k \in Obs)(Rng(w_m) = Rng(w_k) \quad \lor \quad Rng(w_m) \cap Rng(w_k) = \emptyset)\,.$

We will give the **proof** after the proof of Lemma 3.27 way below.

THEOREM 3.22

$$Newbasax \models (\forall m, k \in Obs)(m \stackrel{\odot}{\to} k \;\Leftrightarrow\; Rng(w_m) = Rng(w_k))\,.$$

Proof: Let $\mathfrak{M} \in \mathsf{Mod}_{\rm OFG}(Newbasax)$. Let $m, k \in Obs$. Then $(Rng(w_m) = Rng(w_k) \Rightarrow m \overset{\odot}{\to} k)$ is obvious, hence we will concentrate on the other direction. To prove $(m \overset{\odot}{\to} k \Rightarrow Rng(w_m) = Rng(w_k))$ assume that $m \overset{\odot}{\to} k$. $m \overset{\odot}{\to} k$ and $\mathbf{Ax6_{00}}$ imply that $Rng(w_m) \cap Rng(w_k) \neq \emptyset$. By this and by Thm.3.21, we have that $Rng(w_m) = Rng(w_k)$. ∎

THEOREM 3.23
Newbasax $\models$ "$\overset{\odot}{\to}$ *is an equivalence relation when restricted to Obs*".

Proof: The proof easily follows from Thm.3.22.

Poof of Thm.3.20: Direction $=\!|$ is obvious. Hence it remains to prove direction $\models$.

Let us notice that $\{\mathbf{Ax2}, \mathbf{Ax3}\} \models (\forall m, k \in Obs) m \overset{\odot}{\to} k$. Hence it is enough to prove $Newbasax \cup \{(\forall m, k \in Obs)\ m \overset{\odot}{\to} k\} \models Basax$. Next we turn to prove this. By Thm.3.22 we have $Newbasax \cup \{(\forall m, k \in Obs)\ m \overset{\odot}{\to} k\} \models \mathbf{Ax6}$. Now it is easy to check that $Newbasax \cup \{\mathbf{Ax6}\} \models \{\mathbf{Ax3}, \mathbf{AxE}\}$. ∎

Thm.3.24 below says that $\mathfrak{M}$ is a model of *Newbasax* iff $\mathfrak{M}$ can be obtained by taking a kind of a "disjoint union" of a set of models of *Basax*. Thus, a study of models of *Basax* will also give a study of models of *Newbasax*, cf. §3.5, §3.6. We will use the notation $\mathsf{Mod}_{\mathfrak{F}}(\Sigma)$ introduced in item 2 of Notation 3.1.

THEOREM 3.24

$$\mathfrak{M} \in \mathsf{Mod}_{\mathfrak{F}}(Newbasax)$$

$$\Longleftrightarrow$$

$$(\exists \mathsf{K} \subseteq \mathsf{Mod}_{\mathfrak{F}}(Basax))\Big((\forall \mathfrak{N}_0, \mathfrak{N}_1 \in \mathsf{K})(\mathfrak{N}_0 \neq \mathfrak{N}_1 \quad \Rightarrow$$

$$(Obs^{\mathfrak{N}_0} \cap B^{\mathfrak{N}_1} = \emptyset \quad \& \quad Ph^{\mathfrak{N}_0} \cap B^{\mathfrak{N}_1} \subseteq Ph^{\mathfrak{N}_1} \quad \& \quad Ib^{\mathfrak{N}_0} \cap B^{\mathfrak{N}_1} \subseteq Ib^{\mathfrak{N}_1})) \textit{ and}$$

$$\mathfrak{M} = \langle (B^{\mathfrak{M}}, Obs^{\mathfrak{M}}, Ph^{\mathfrak{M}}, Ib^{\mathfrak{M}}), \mathfrak{F}, G; \ \mathrm{E}\ , W^{\mathfrak{M}}\rangle, \textit{ where}$$

$$B^{\mathfrak{M}} \overset{\text{def}}{=} \bigcup_{\mathfrak{N} \in \mathsf{K}} B^{\mathfrak{N}},$$

$$Obs^{\mathfrak{M}} \overset{\text{def}}{=} \bigcup_{\mathfrak{N} \in \mathsf{K}} Obs^{\mathfrak{N}},$$

$$Ib^{\mathfrak{M}} \stackrel{\text{def}}{=} \bigcup_{\mathfrak{N}\in\mathsf{K}} Ib^{\mathfrak{N}},$$
$$Ph^{\mathfrak{M}} \stackrel{\text{def}}{=} \bigcup_{\mathfrak{N}\in\mathsf{K}} Ph^{\mathfrak{N}}, \; \textit{and}$$
$$W^{\mathfrak{M}} \stackrel{\text{def}}{=} \bigcup_{\mathfrak{N}\in\mathsf{K}} W^{\mathfrak{N}}\Big).$$

Proof:
(i) Direction $\Longleftarrow$ goes by checking the axioms.

(ii) Direction $\Longrightarrow$ follows from Theorems 3.21, 3.22, 3.23 as follows. Assume $\mathfrak{M} = \langle B, Obs, \ldots\rangle \models Newbasax$. By Thm.3.23 there is an equivalence relation $\equiv\; \subseteq Obs \times Obs$ (induced by $\stackrel{\odot}{\to}$). For simplicity assume $Obs/\equiv\; = \{Obs_0, Obs_1\}$. For each Obs_i ($i < 2$) we construct a *Basax* model $\mathfrak{N}_i$. Let $\mathfrak{N}_i = \langle (B_i, Obs_i, Ph_i, Ib_i), \mathfrak{F}, G; \;\mathrm{E}\;, W_i\rangle$, where

$$Ph_i \stackrel{\text{def}}{=} \{ph \in Ph \;:\; (\exists m \in Obs_i)\; m \stackrel{\odot}{\to} ph\},$$
$$Ib_i \stackrel{\text{def}}{=} \{h \in Ib \;:\; (\exists m \in Obs_i)\; m \stackrel{\odot}{\to} h\},$$
$$B_i \stackrel{\text{def}}{=} B \setminus Ib_j \cup Ib_i \quad (\{i,j\} = \{0,1\}), \text{ and}$$
$$W_i \stackrel{\text{def}}{=} W\lceil(B_i \times {}^nF \times B_i).$$

Now one can check that $\mathfrak{N}_i \models Basax$ and $\mathfrak{M}$ is obtained from $\mathfrak{N}_0$, $\mathfrak{N}_1$ as described in the theorem. ∎

THEOREM 3.25 *Newbasax* $\not\models$ *Basax.*

Proof: Let $\mathfrak{N}_0, \mathfrak{N}_1 \in \mathsf{Mod}_{\mathrm{OFG}}(Basax)$ with $B^{\mathfrak{N}_0} \cap B^{\mathfrak{N}_1} = \emptyset$. Let $\mathfrak{M}$ be the model which is obtained by $\mathsf{K} = \{\mathfrak{N}_0, \mathfrak{N}_1\}$ as described in Thm.3.24. Then $\mathfrak{M} \models Newbasax$ by Thm.3.24.

It is easy to check that if $m_0 \in Obs^{\mathfrak{N}_0}$ and $m_1 \in Obs^{\mathfrak{N}_1}$ then $tr_{m_0}(m_1) = \emptyset$. Hence $\mathfrak{M} \not\models$ **Ax3**. ∎

Now we turn to the proof of Thm.3.21. Lemmas 3.26, 3.27 are needed for the proof of Thm.3.21. Lemma 3.26 below is an analogon of Claim 2.8.

LEMMA 3.26

$$Newbasax \models (\forall m, k \in Obs)(m \stackrel{\odot}{\to} k \;\Rightarrow\; v_m(k) \neq 1).$$

Proof: The proof goes by contradiction. Let $\mathfrak{M} \in \mathsf{Mod}_{\rm OFG}(Newbasax)$. Let $m, k \in Obs$ such that $m \stackrel{\odot}{\to} k$ and $v_m(k) = 1$. By **Ax5** we have that there is $ph \in Ph$ such that $tr_m(ph) = tr_m(k)$. Let such a ph be fixed. Let $p, q \in tr_m(k)$ such that $p \neq q$. Now $ph, k \in w_m(p) \cap w_m(q)$ and **Ax6$_{00}$** implies that $w_k(p') = w_m(p)$ and $w_k(q') = w_m(q)$, and $ph, k \in w_k(p') \cap w_k(q')$, for some $p', q' \in {}^nF$. Let such p' and q' be fixed. By $p \neq q$ and Claim 2.10(ii), we have that $p' \neq q'$. By $k \in w_k(p') \cap w_k(q')$ and **Ax4** we have $\overline{p'q'} = \bar{t} \stackrel{\rm def}{=} F \times {}^{n-1}\{0\}$. Now $\overline{p'q'} = \bar{t}$ and $ph \in w_k(p') \cap w_k(q')$ contradicts to **AxE$_0$**. ∎

LEMMA 3.27 *Newbasax* $\models (\forall m, k \in Obs)(\forall p, q \in {}^nF)$

$$\Big((ang^2(\overline{pq}) = 1 \ \wedge \ p \in Dom(\mathsf{f}_{mk})) \ \Rightarrow \ q \in Dom(\mathsf{f}_{mk}) \Big) .$$

Proof: Let $\mathfrak{M} \in \mathsf{Mod}_{\rm OFG}(Newbasax)$. Let $m, k \in Obs$ and $p, q \in {}^nF$ with $ang^2(\overline{pq}) = 1$ and $p \in Dom(\mathsf{f}_{mk})$. We have to prove that $q \in Dom(\mathsf{f}_{mk})$.

By $ang^2(\overline{pq}) = 1$ and **Ax5** we have $tr_m(ph) = \overline{pq}$, for some $ph \in Ph$. Let this ph be fixed. By **Ax6$_{01}$** and $p \in Dom(\mathsf{f}_{mk})$, we have that $S(p, \varepsilon) \subseteq Dom(\mathsf{f}_{mk})$, for some $\varepsilon \in {}^+F$. Let this ε be fixed.

Then it is easy to see that there are $r, s, u, v \in S(p, \varepsilon)$ such that p, q, r, s, u, v are different, $q \in \overline{rs}$, $q \in \overline{pu}$, $\overline{pr} \cap \overline{su} = \{v\}$, and $ang^2(\overline{rs}), ang^2(\overline{pr}), ang^2(\overline{su}) < 1$ (see Figure 24). Let such r, s, u, v be fixed.

By $ang^2(\overline{rs}), ang^2(\overline{pr}), ang^2(\overline{su}) < 1$ and **Ax5** we have that $tr_m(m_1) = \overline{rs}$, $tr_m(m_2) = \overline{su}$, and $tr_m(m_3) = \overline{pr}$, for some $m_1, m_2, m_3 \in Obs$. Let such m_1, m_2, m_3 be fixed. By the above construction we have that

$$\begin{aligned} tr_m(ph) \cap tr_m(m_3) &= \{p\}, \\ tr_m(m_1) \cap tr_m(m_3) &= \{r\}, \\ tr_m(m_1) \cap tr_m(m_2) &= \{s\}, \\ tr_m(ph) \cap tr_m(m_2) &= \{u\}, \\ tr_m(m_2) \cap tr_m(m_3) &= \{v\}. \end{aligned} \tag{13}$$

Now by $p, r, s, u, v \in S(p, \varepsilon) \subseteq Dom(\mathsf{f}_{mk})$ by **Ax3$_0$**, by Claim 2.10(ii) and by (13), we have

$$\begin{aligned} tr_k(ph) \cap tr_k(m_3) &= \{\mathsf{f}_{mk}(p)\}, \\ tr_k(m_1) \cap tr_k(m_3) &= \{\mathsf{f}_{mk}(r)\}, \\ tr_k(m_1) \cap tr_k(m_2) &= \{\mathsf{f}_{mk}(s)\}, \\ tr_k(ph) \cap tr_k(m_2) &= \{\mathsf{f}_{mk}(u)\}, \\ tr_k(m_2) \cap tr_k(m_3) &= \{\mathsf{f}_{mk}(v)\}. \end{aligned} \tag{14}$$

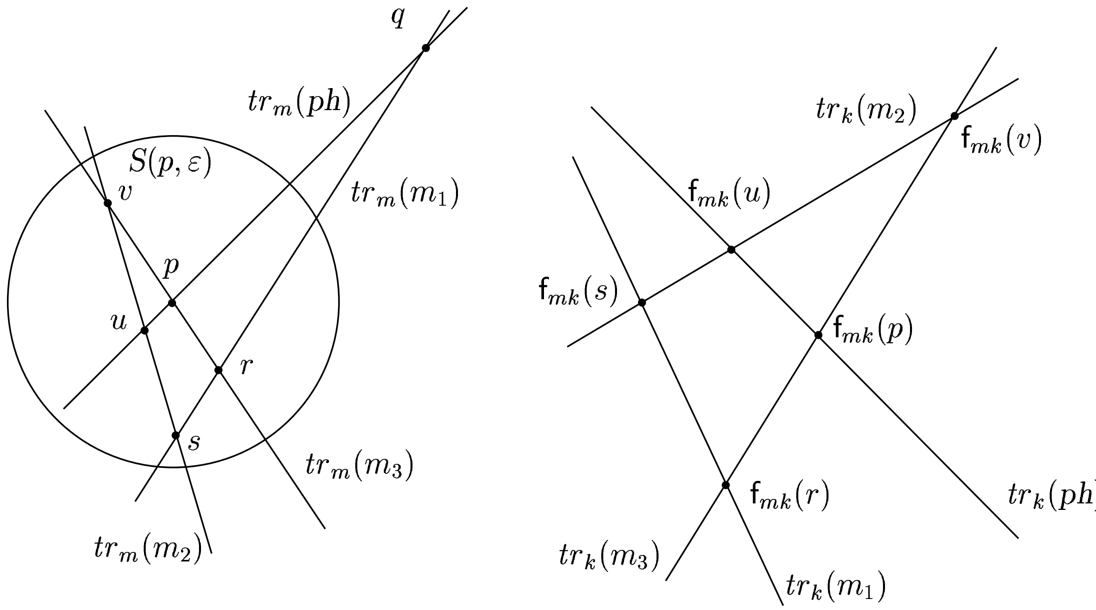


Figure 24: Illustration for the proof of Lemma 3.27.

$\mathsf{f}_{mk}(p), \mathsf{f}_{mk}(r), \mathsf{f}_{mk}(s), \mathsf{f}_{mk}(u), \mathsf{f}_{mk}(v)$ are different points by Claim 2.10(ii) because p, r, s, u, v were different. Hence by (14) and $\mathbf{Ax3_0}$, we have that $tr_k(m_1)$ and $tr_k(ph)$ are in the same plane. Now by this and by Lemma 3.26, we have $tr_k(m_1) \cap tr_k(ph) = \{w\}$, for some $w \in {}^nF$. Let this w be fixed. Now

(15) $$m_1, ph \in w_m(q) \cap w_k(w)\,.$$

By (15) and by $\mathbf{Ax6_{00}}$, we have that there are $q', w' \in {}^nF$ such that

(16) $$w_{m_1}(q') = w_m(q) \quad \text{and} \quad w_{m_1}(w') = w_k(w)\,.$$

Let such q' and w' be fixed. By (15) and by (16), we have that $m_1, ph \in w_{m_1}(q') \cap w_{m_1}(w')$. By this, by $\mathbf{Ax4}$ and by $\mathbf{AxE_0}$, we have $q' = w'$. By $q' = w'$ and by (16), we have $w_m(q) = w_k(w)$. Hence $q \in Dom(\mathsf{f}_{mk})$. ∎

Proof of Thm.3.21: The proof follows by Lemma 3.27 as follows. Let $\mathfrak{M} \in \mathsf{Mod}_{\text{OFG}}(Newbasax)$. Let $m, k \in Obs$ with $Rng(w_m) \cap Rng(w_k) \neq \emptyset$. We have to prove that $Rng(w_m) = Rng(w_k)$. To prove this it is enough to prove that $Dom(\mathsf{f}_{mk}) = {}^nF$ and $Dom(\mathsf{f}_{km}) = {}^nF$. By $Rng(w_m) \cap Rng(w_k) \neq \emptyset$, we have that there is $p \in Dom(\mathsf{f}_{mk})$. Let $q \in {}^nF$. We will prove that $q \in Dom(\mathsf{f}_{mk})$. It is easy to see that

(17) $$(\exists j \in \omega)(\exists r_0, r_1, \ldots, r_j \in {}^nF)\Big((\forall i < j)\,ang^2(\overline{r_i r_{i+1}}) = 1 \;\&\; r_0 = p \;\&\; r_j = q\Big)\,.$$

Let such j and $r_0, \ldots, r_j$ be fixed. Now by applying Lemma 3.27 j times, by (17), we get that $q \in Dom(\mathsf{f}_{mk})$. Hence $Dom(\mathsf{f}_{mk}) = {}^nF$. Analogously $Dom(\mathsf{f}_{km}) = {}^nF$. ∎

3.4 Faster than light observers

In the first sub-section (§3.4.1) of the present section, we show that our relativity theories (*Basax*, *Newbasax*) introduced so far imply the nonexistence of FTL observers, for $n > 2$. (Let us recall that in §2 we saw that FTL observers are permitted when $n = 2$.) At this point, we would like to emphasize that the main theorem of §3.4.1 is a theorem of purely logical nature and does not involve concepts like "mass", "force" or "energy". The theorem (and its improvement in §3.4.2) says that a very small number of weak and natural assumptions already implies "logical" impossibility of say "tachyons being observers".

In §3.4.2 we investigate the reason for our "no FTL observer" result, by weakening the axioms of *Newbasax*. We arrive at various rather weak systems, some of which are called *Bax*, Bax^-, *Relphax*, and *Reich*(*Bax*). We will see that most (but not all) of these systems still prove the nonexistence of FTL observers (assuming $n > 2$, of course). On the other hand, e.g. *Relphax* permits FTL observers for arbitrary n. The interesting aspect of this is that *Relphax* is not too weak to be considered as a possible (special) relativity theory.

These weak axiom systems (*Bax*, Bax^- etc.) lead up to the second purpose of the present section which was not mentioned in the title. This second purpose is motivated by the literature and is twofold: (i) Friedman [13] started a kind of conceptual analysis of relativity theory which is taken up and is further elaborated in §§3.4.2–3.4.3. (ii) Reichenbach, Grünbaum and others (cf. e.g. Szabó [42]) initiated a variant of relativity theory which differs from Einstein's one in the treatment of simultaneity. In a future version of the present work, we plan to formalize the Reichenbachian versions (*Reich*(*Bax*), *Reich*(*Basax*) etc.) of our relativity theories (*Bax*, *Basax*, etc. respectively), in first order logic.[33] Of course, besides formalizing these Reichenbachian versions, we want to study them, among others in order to answer some problems raised in e.g. Friedman [13]. Sub-sections 3.4.2–3.4.3 contain preparations for this plan. E.g. the theory Bax^- is flexible enough to serve as a

[33]These sections already exist as hand-written manuscripts, cf. Madarász-Németi [29].

basis (or starting point) for developing the Reichenbachian theory $Reich(Bax)$.

3.4.1 Main stream investigations

In this sub-section we prove that *Basax* implies that there is no FTL observer if we assume that $n \geq 3$. We show that the assumption $n \geq 3$ is necessary.

The following theorem says that if $n \geq 3$, then $Basax(\mathbf{n})$ implies that there is no FTL observer, while $Basax(\mathbf{2})$ allows the existence of FTL observers. We will use the notation $\mathsf{Mod}_{\mathfrak{F}}(\Sigma)$ introduced in item 2 of Notation 3.1.

THEOREM 3.28 *Let $n \geq 3$. Then (i)–(iii) below hold.*

(i) $Basax(\mathbf{n}) \models (\forall m, k \in Obs)\ v_m(k) < 1.$

(ii) $Basax(\mathbf{2}) \not\models (\forall m, k \in Obs)\ v_m(k) < 1.$

(iii) *Assume $\mathfrak{F}$ is an arbitrary ordered field. Then*
$\mathsf{Mod}_{\mathfrak{F}}(Basax(\mathbf{2})) \not\models (\forall m, k \in Obs)\ v_m(k) < 1.$

On the **proof**: We will give the proof later in this sub-section, after Lemma 3.31.

Thm.3.29 below is a corollary of Thm.3.28 and Thm.3.24 (§3.3). Recall from §3.3 (Def.3.17) the axiom system *Newbasax* which is a refined version of *Basax*.

THEOREM 3.29 *Let $n \geq 3$. Then*

$$Newbasax(\mathbf{n}) \models (\forall m, k \in Obs)\ v_m(k) < 1.$$

Proof: The theorem follows directly from Thm.3.28 and Thm.3.24 (§3.3). ∎

Our next axiom, $\mathbf{Ax}(\sqrt{\ })$, is of a technical nature. Namely, sometimes we will need to assume that square roots of positive numbers exist in $\mathfrak{F}$.

$\mathbf{Ax}(\sqrt{\ })$ $\quad (\forall 0 < x \in F)(\exists y \in F) y^2 = x.$

If $\mathfrak{F} \models \mathbf{Ax}(\sqrt{\ })$ then we say that $\mathfrak{F}$ is <u>*Euclidean*</u>. Clearly, $\mathfrak{R} \models \mathbf{Ax}(\sqrt{\ })$. For any $0 < x \in F$, $\sqrt{x}$ denotes that positive y for which $y^2 = x$.

For the proof of Thm.3.28 we need Lemmas 3.30, 3.31 below. First we will state the lemmas, after that we will give the proof of Thm.3.28. After the proof of Thm.3.28 we will give the proof of the lemmas.

LEMMA 3.30 *Assume $n \geq 3$ and $\mathfrak{F}$ is Euclidean. Assume $\mathsf{f} : {}^nF \to {}^nF$ is a bijection such that*

$$(\star) \qquad (\forall \ell \in \mathsf{Eucl})\Big(\mathsf{f}[\ell] \in \mathsf{Eucl} \ \wedge \ (\mathsf{f}[\ell] \in \mathsf{PhtEucl} \ \Leftrightarrow \ \ell \in \mathsf{PhtEucl})\Big).$$

Then $\mathsf{f}[\bar{t}\,] \in \mathsf{SlowEucl}$.

On the **proof**: We will give the proof of Lemma 3.30 after Remark 3.33 below.

LEMMA 3.31 *Assume $\mathsf{f} \in Aftr(\mathbf{n}, \mathbf{F})$ satisfying $(\star)$ in Lemma 3.30. Assume $\mathfrak{F}_* = \langle \mathbf{F}_*, \leq \rangle$ is an ordered field such that $\mathfrak{F} \subseteq \mathfrak{F}_*$.*[34] *Let $\mathsf{f}_* \in Aftr(\mathbf{n}, \mathbf{F}_*)$ such that $\mathsf{f}_* \lceil {}^nF = \mathsf{f}$. Let us notice that such an f_* exists and is unique. Then f_* satisfies $(\star)$ in Lemma 3.30 when f_* and $\mathfrak{F}_*$ are substituted in place of f and $\mathfrak{F}$, respectively.*

On the **proof**: We will give the proof of Lemma 3.31 after the proof of Lemma 3.30.

Proof of Thm.3.28(i): Let $n \geq 3$ and $\mathfrak{M} \in \mathsf{Mod}_{\mathrm{OFG}}(Basax(\mathbf{n}))$. Let $m, k \in Obs$. We have to prove that $v_m(k) < 1$.

<u>*Intuitive idea of the proof:*</u> We want to prove that $tr_m(k)$ is "slow". We know that $\mathsf{f}_{km} = \widetilde{\varphi} \circ \mathsf{f}$, where f is an affine transformation satisfying $(\star)$ of 3.30. By 3.31 f will continue satisfying $(\star)$ in a larger field $\mathfrak{F}_*$, which, in turn will be Euclidean. Looking at it from $\mathfrak{F}_*$, $\mathsf{f}[\bar{t}\,]$ must be "slow" by 3.30. Therefore $\mathsf{f}[\bar{t}\,]$ must be slow in $\mathfrak{F}$, too, and then $tr_m(k) = \mathsf{f}[\bar{t}]$ will complete the proof.

<u>*Formally:*</u> By Theorems 3.3, 3.4 and Proposition 3.13, we have that $\mathsf{f}_{km} = \widetilde{\varphi} \circ \mathsf{f}$, for some $\varphi \in Aut(\mathbf{F})$ and for some $\mathsf{f} \in Aftr(\mathbf{n}, \mathbf{F})$ satisfying $(\star)$ in Lemma 3.30. Let such φ and f be fixed. By $\mathsf{f}_{km} = \widetilde{\varphi} \circ \mathsf{f}$, by $\widetilde{\varphi}[\bar{t}\,] = \bar{t}$, by $tr_k(k) = \bar{t}$ and by $\mathsf{f}_{km}[tr_k(k)] = tr_m(k)$, we have

$$(19) \qquad \mathsf{f}[\bar{t}\,] = tr_m(k)\,.$$

[34] $\mathfrak{F} \subseteq \mathfrak{F}_*$ means that $\mathfrak{F}$ is a strong sub-model of $\mathfrak{F}_*$ in the sense of Chang-Keisler [7].

Let $\mathfrak{F}_* = \langle \mathbf{F}_*, \leq \rangle$ be such that $\mathfrak{F}_*$ is Euclidean and $\mathfrak{F} \subseteq \mathfrak{F}_*$. Such an $\mathfrak{F}_*$ exists, e.g. the real closure[35] of $\mathfrak{F}$ is such. Let $\mathsf{f}_* \in Aftr(\mathbf{n}, \mathbf{F}_*)$ such that $\mathsf{f}_* \lceil {}^nF = \mathsf{f}$. Then by Lemma 3.31, f_* satisfies $(\star)$ in Lemma 3.30 when f_* and $\mathfrak{F}_*$ are substituted in place of f and $\mathfrak{F}$, respectively. Let $\bar{t}_* := F_* \times {}^{n-1}\{0\}$. Then $\mathsf{f}_*[\bar{t}_*] \in \mathsf{SlowEucl}(\mathbf{n}, \mathfrak{F}_*)$ by Lemma 3.30. But $\mathsf{f}[\bar{t}\,] \subseteq \mathsf{f}_*[\bar{t}_*]$ by $\mathsf{f}_* \lceil {}^nF = \mathsf{f}$ and $\bar{t} \subseteq \bar{t}_*$. Hence $\mathsf{f}[\bar{t}\,] \in \mathsf{SlowEucl}$. By this and by (19), we have $v_m(k) < 1$. ∎

Proof of Thm.3.28(iii): Let $\mathfrak{F}$ be arbitrary. Let P be a choice function that to each $\ell \in \mathsf{Eucl}(\mathbf{2}, \mathbf{F})$ associates two distinct points lying on ℓ. Let the model $\mathfrak{M}_1^P$ be defined for $\mathfrak{F}$ as it was defined in §2.4 for the case when $\mathfrak{F}$ was $\mathfrak{R}$ the ordered field of real numbers. There are FTL observers in the frame model $\mathfrak{M}_1^P$. Analogously to Thm.2.13 in §2.4 one can prove that $\mathfrak{M}_1^P \models Basax(\mathbf{2})$.

Proof of Thm.3.28(ii): Item (ii) directly follows from item (iii) of Thm.3.28. ∎

Now we turn to the proof of Lemma 3.30. For the proof of Lemma 3.30 we need the definition of the light-cone of $p \in {}^nF$. This comes next.

Definition 3.32

(i) We define the <u>*light-cone*</u> of $\bar{0} \in {}^nF$ as follows:

$$\mathsf{LightCone}(\bar{0}) \stackrel{\text{def}}{=} \{q \in {}^nF \ : \ q_0^2 = q_1^2 + q_2^2 + \ldots + q_{n-1}^2\}\,.$$

(ii) We define the light-cone of $p \in {}^nF$ as follows:

$$\mathsf{LightCone}(p) \stackrel{\text{def}}{=} \mathsf{LightCone}(\bar{0}) + p \stackrel{\text{def}}{=} \{q + p \ : \ q \in \mathsf{LightCone}(\bar{0})\}\,. \qquad \lhd$$

Remark 3.33 Let us notice that for every $p \in {}^nF$

$$\mathsf{LightCone}(p) = \bigcup\{\ell \in \mathsf{PhtEucl} \ : \ p \in \ell\}\,.$$

That is, in a model of *Basax*, $\mathsf{LightCone}(p)$ is the union of traces of photons going through p. This is where the name "light-cone" comes from.

[35] The notion of the real closure of an ordered field can be found e.g. in [14].

Proof of Lemma 3.30: In the proof we will use notation/definitions introduced in items 3, 4, 5, 7, 9, 13 of Notation 3.1 and in items 2, 3 of Def.3.2. The proof goes by contradiction. Let $\mathsf{f} : {}^nF \to {}^nF$ be a bijection satisfying $(\star)$. Assume that $\mathsf{f}[\bar{t}\,] \notin \mathsf{SlowEucl}$.

<u>*Intuitive idea of the proof:*</u> We will see that $\mathsf{f}[\bar{t}\,] \notin \mathsf{SlowEucl}$ will imply that $\mathsf{f}[S]$ (S denotes the space part of space-time) will contain a line "inside" a light-cone. But then the intersection of $\mathsf{f}[S]$ and this light-cone will contain a line. Clearly this line will be a "photon-line" and its inverse image by f will be contained in S. See Figure 26. This will contradict to $(\star)$, i.e. to the fact that the inverse images of "photon-lines" by f must be "photon-lines".

<u>*Formally:*</u> By $(\star)$, we have $\mathsf{f}[\bar{t}\,] \notin \mathsf{PhtEucl}$. Therefore $ang^2(\mathsf{f}[\bar{t}\,]) > 1$. Without loss of generality we can assume (20) and (21) below.

$$\mathsf{f}[\mathsf{Plane}(\bar{t},\bar{x})] = \mathsf{Plane}(\bar{t},\bar{x}) \quad \textit{and} \tag{20}$$
$$\mathsf{f}(\bar{0}) = \bar{0}\,. \tag{21}$$

We will explain at the end of the proof why we can assume (20), (21) above.

By f being a bijection taking lines to lines and by (21), we have

$$\mathsf{f}[S] \text{ is an } (n-1)\text{–dimensional subspace of } {}^n\mathbf{F}\,.^{36} \tag{22}$$

Now by the assumption $ang^2(\mathsf{f}[\bar{t}\,]) > 1$ and by (20), (21), we have

$$\mathsf{f}[\bar{t}\,] = \{x \cdot \langle a, 1, 0, \ldots, 0\rangle \ : \ x \in F\}, \text{ for some } a \in F \text{ with } |a| < 1\,. \tag{23}$$

Let this a be fixed.

Let ℓ be the mirror image of $\mathsf{f}[\bar{t}\,]$ w.r.t. either one of the lines in $\mathsf{PhtEucl} \cap \mathsf{Plane}(\bar{t},\bar{x})$ going through $\bar{0}$, i.e. let

$$\ell \stackrel{\text{def}}{=} \{x \cdot \langle 1, a, 0, \ldots, 0\rangle \ : \ x \in F\}\,.$$

Let us notice that $\ell \in \mathsf{SlowEucl}$ by $|a| < 1$.

Claim 3.34 *$\ell \subseteq \mathsf{f}[S]$. Actually, $\ell = \mathsf{f}[\bar{x}]$.*

<u>*Proof of Claim 3.34:*</u> Throughout the proof the reader is asked to consult Figure 25. By (20) we can define $\mathsf{f}_0 : {}^2F \longrightarrow {}^2F$ as follows:

$$(\forall p \in {}^2F)\ \mathsf{f}_0(p_0,p_1) \stackrel{\text{def}}{=} \mathsf{f}(p_0,p_1,0,\ldots,0)\,.$$

[36] For the visually oriented reader we note that $\mathsf{f}[S]$ is a, so called, hyperplane.

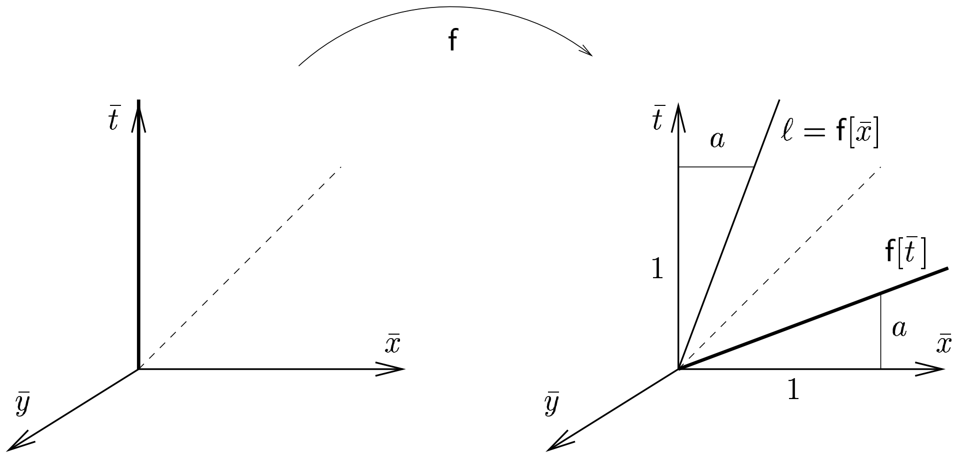


Figure 25: Illustration for Claim 3.34.

Then $\mathsf{f}_0 : {}^2F \longrightarrow {}^2F$ is a bijection such that

$$(\forall \ell \in \mathsf{Eucl})\Big(\mathsf{f}_0[\ell] \in \mathsf{Eucl} \quad \wedge \quad (\mathsf{f}_0[\ell] \in \mathsf{PhtEucl} \ \Leftrightarrow \ \ell \in \mathsf{PhtEucl})\Big),$$

because f is a bijection satisfying ($\star$) in Lemma 3.30. By Corollary 3.16 (§3.2), and by $\mathsf{f}(\bar{0}) = \bar{0}$ (cf. (21)), we have that $\mathsf{f}_0[\bar{t}\,]$ and $\mathsf{f}_0[\bar{x}]$ are mirror images of each other w.r.t. either one of the lines in $\mathsf{PhtEucl}(\mathbf{2})$ containing $\bar{0}$.[37] Using this we conclude $\mathsf{f}[\bar{x}] = \ell$ by the definitions of ℓ and f.

QED (Claim 3.34)

Claim 3.35 *There is $p \in {}^nF$ such that $\bar{0} \neq p \in \mathsf{LightCone}(\bar{0}) \cap \mathsf{f}[S]$. (See Figure 26.)*

We will give the *detailed proof* of Claim 3.35 very soon. Claim 3.35 is true in the case $n = 3$ and $\mathfrak{F} = \mathfrak{R}$, because the plane $\mathsf{f}[S]$ has a point inside the cone $\mathsf{LightCone}(\bar{0})$, namely $\ell \subseteq \mathsf{f}[S]$ and ℓ is inside the cone by $\ell \in \mathsf{SlowEucl}$. Thus the plane $\mathsf{f}[S]$ intersects the cone in a point different from $\bar{0}$. (See Figure 26.) We will give the detailed proof for the general case ($n \geq 3$ and $\mathfrak{F}$ is an arbitrary Euclidean field) very soon. We note that if $\mathfrak{F}$ is the ordered field of rational numbers, (in which $\sqrt{2}$ does not exist), then the intersection of the plane $\mathsf{Plane}(\bar{t}, \overline{0}\langle 0, 1, 1\rangle)$ with the cone $\mathsf{LightCone}(\bar{0})$ is the singleton of $\bar{0}$, while the plane clearly contains points that are inside the cone (e.g. $\bar{t}$ is inside the cone). This motivates the use of $\mathbf{Ax}(\sqrt{\ })$ in the proof of Claim 3.35.

Now we return to the proof of Lemma 3.30. Throughout the proof the reader is asked to consult Figure 26. By Claim 3.35, there is $\bar{0} \neq p \in \mathsf{LightCone}(\bar{0}) \cap \mathsf{f}[S]$.

[37] According to our convention $\mathsf{PhtEucl}(\mathbf{2})$ denotes $\mathsf{PhtEucl}(\mathbf{2}, \mathbf{F})$.

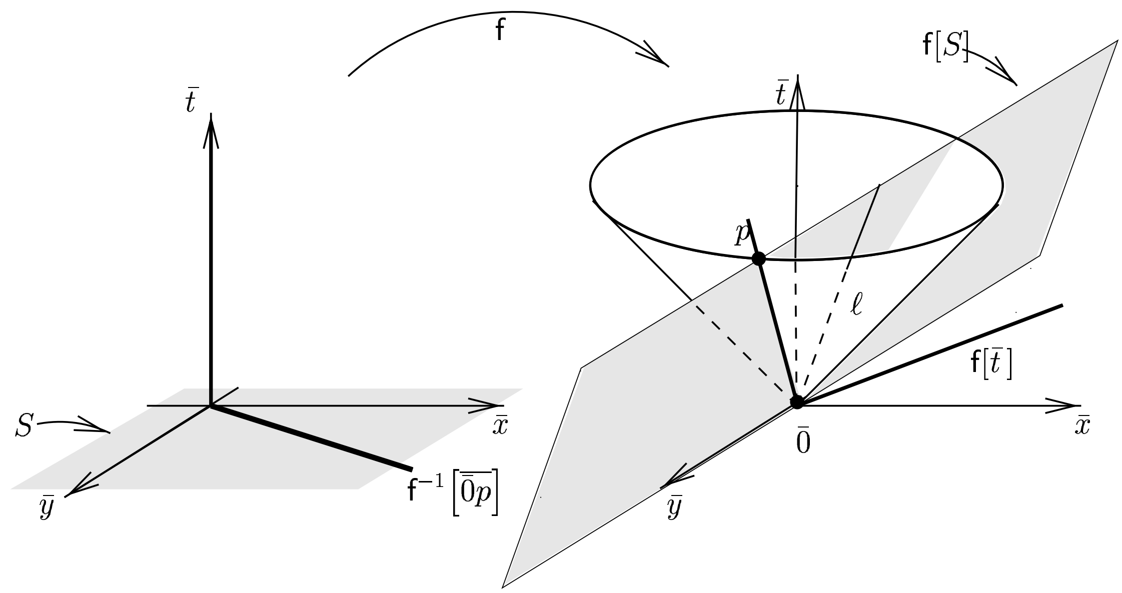


Figure 26: Illustration for the proof of Lemma 3.30 and for Claim 3.35.

Let such a p be fixed. $p \in \mathsf{LightCone}(\bar{0}) \setminus \{\bar{0}\}$ implies that $\overline{\bar{0}p} \in \mathsf{PhtEucl}$. Now we conclude $\overline{\bar{0}p} \subseteq \mathsf{f}[S]$ by $p \in \mathsf{f}[S]$ and by $\mathsf{f}[S]$ being a subspace of ${}^n\mathbf{F}$ (cf. (22)). By this we have $\mathsf{f}^{-1}\left[\overline{\bar{0}p}\right] \in S$, hence $\mathsf{f}^{-1}\left[\overline{\bar{0}p}\right] \notin \mathsf{PhtEucl}$. $\overline{\bar{0}p} \in \mathsf{PhtEucl}$ and $\mathsf{f}^{-1}\left[\overline{\bar{0}p}\right] \notin \mathsf{PhtEucl}$ contradicts to $(\star)$ in the formulation of Lemma 3.30. Hence $\mathsf{f}[\bar{t}\,] \in \mathsf{SlowEucl}$.

Lemma 3.30 is proved modulo Claim 3.35, and the explanation of why we can assume (20) and (21) above. Now we turn to prove these.

Proof of Claim 3.35: Throughout the proof the reader is asked to consult Figure 27. Let ℓ be as defined above Claim 3.34, i.e. $\ell \stackrel{\text{def}}{=} \{x \cdot \langle 1, a, 0, \dots, 0\rangle \;:\; x \in F\}$. Let us recall that $a \in F$ was fixed below (23) and $|a| < 1$. By Claim 3.34, we have that $\ell \subseteq \mathsf{f}[S]$. Let P be the plane parallel with $\mathsf{Plane}(\bar{x}, \bar{y})$ and with height 1, i.e. let

$$P \stackrel{\text{def}}{=} \{\langle 1, x, y, 0, \dots, 0\rangle \;:\; x, y \in F\}\,.$$

Let $q \stackrel{\text{def}}{=} \langle 1, a, 0, \dots, 0\rangle$. Then $q \in \mathsf{f}[S] \cap P$ by $q \in \ell \subseteq \mathsf{f}[S]$. Since $\mathsf{f}[S]$ is an $(n-1)$–dimensional subspace, $n \geq 3$, and $q \in \mathsf{f}[S] \cap P$, $\mathsf{f}[S] \cap P$ contains a line ℓ_0 such that $q \in \ell_0$.[38]

[38]This is known from linear algebra; intersection of a (≥ 2)–dimensional subspace with a plane contains a line through each point in the intersection.

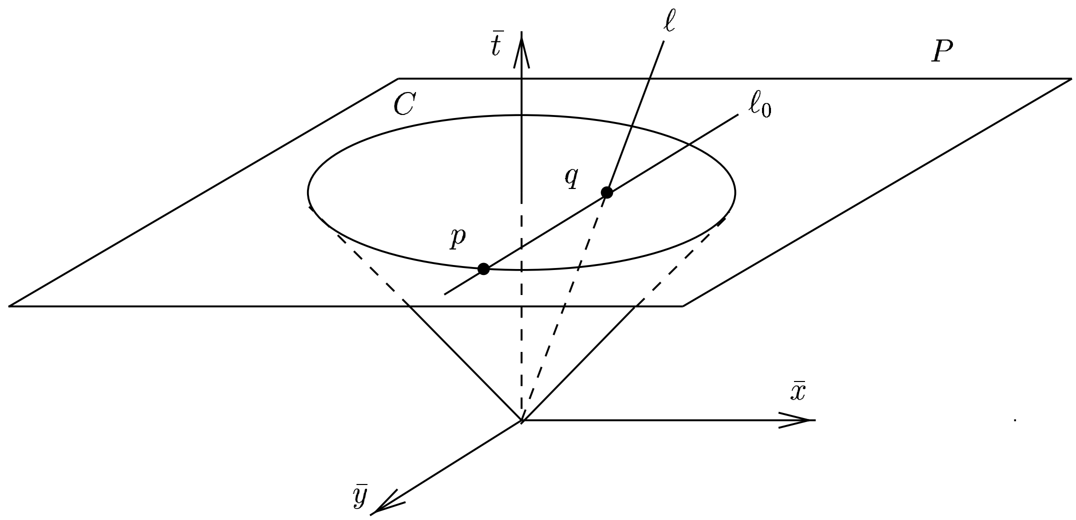


Figure 27: Illustration for the proof of Claim 3.35.

The intersection $\mathsf{LightCone}(\bar{0}) \cap P$ is a circle, i.e.

$$C \stackrel{\text{def}}{=} \mathsf{LightCone}(\bar{0}) \cap P = \{\langle 1, x, y, 0, \ldots, 0\rangle \ : \ x^2 + y^2 = 1\}\,.$$

The point q is inside this circle by $|a| < 1$. We will show that

$$\ell_0 \cap C \neq \emptyset\,. \tag{24}$$

This is true, because it is known from geometry that if $\mathfrak{F}$ is Euclidean then inside every plane, whenever a line has a point inside a circle, it intersects the circle.[39] For completeness, we give here a direct proof of (24), too.

Since $q \in \ell_0 \subseteq P$, we have that, for some $b \in F$,

$$\ell_0 = \{\langle 1, x, y, 0, \ldots, 0\rangle \ : \ x = by + a\}\,.$$

Let this b be fixed. Then $\langle 1, x, y, 0, \ldots, 0\rangle \in \ell_0 \cap C$ iff

$$x^2 + y^2 = 1 \quad \text{and} \quad x = by + a\,.$$

By replacing x with $by + a$ in the first equation we get

$$(b^2 + 1)y^2 + 2bay + (a^2 - 1) = 0\,.$$

[39] Cf. Jones-Morris-Pearson[24] chapter 5 or Szmielew [43]. Actually, (i) and (ii) below are equivalent in every two-dimensional geometry over an ordered field $\mathfrak{F}$:
(i) If a line ℓ has a point inside a circle, then ℓ intersects the circle.
(ii) The square root of any $x > 0$ exists.

This quadratic equation, by $\mathfrak{F}$ being Euclidean, has a solution iff

$$(2ba)^2 - 4(b^2+1)(a^2-1) \geq 0\,.$$[40]

Carrying out some simplifications, this is equivalent to

$$b^2 + (1-a^2) \geq 0\,.$$

By $|a| < 1$ this always holds. (24) is proved.
Now, let $p \in \ell_0 \cap C$. Then $p \in \mathsf{f}[S] \cap \mathsf{LightCone}(\bar{0})$ by $\ell_0 \subseteq \mathsf{f}[S]$, $C \subseteq \mathsf{LightCone}(\bar{0})$. Also, $p \neq \bar{0}$ because $p \in P$ while $\bar{0} \notin P$.

QED (Claim 3.35)

Explanation of why we can assume (20) and (21) above: Throughout this explanation the reader is asked to consult Figure 28. Let us recall from the beginning of the proof of Lemma 3.30 that f is a bijection satisfying $(\star)$ in the formulation of Lemma 3.30 and $\mathsf{f}[\bar{t}\,] \notin \mathsf{SlowEucl}$. We also recall assumptions (20) and (21):

$$\begin{aligned} \mathsf{f}[\mathsf{Plane}(\bar{t},\bar{x})] &= \mathsf{Plane}(\bar{t},\bar{x}) \quad \textit{and} \\ \mathsf{f}(\bar{0}) &= \bar{0}\,. \end{aligned}$$

Now we will explain why these assumptions can be made.

Let $s := \mathsf{f}(\bar{0})$. Let us recall that τ_{-s} denotes the translation by vector $-s$. Now let $\mathsf{f}_s := \mathsf{f} \circ \tau_{-s}$. We have $\mathsf{f}_s(\bar{0}) = \bar{0}$ and $\mathsf{f}_s[\bar{t}\,] \notin \mathsf{SlowEucl}$ by $\mathsf{f}(\bar{0}) = s$ and $\mathsf{f}[\bar{t}\,] \notin \mathsf{SlowEucl}$.

Let ϱ_1 and ϱ_2 be linear transformations such that ϱ_1 and ϱ_2 satisfy 1-3 below.

1. ϱ_1 and ϱ_2 leave $\bar{t}$ (the time axis) point-wise fixed.

2. ϱ_1 and ϱ_2 are distance preserving transformations (i.e. congruence transformations).

3. $\varrho_1[\mathsf{Plane}(\bar{t},\bar{x})] = \mathsf{Plane}(\bar{t},\mathsf{f}_s^{-1}[\bar{t}\,])$ and $\varrho_2[\mathsf{Plane}(\bar{t},\mathsf{f}_s[\bar{t}\,]) = \mathsf{Plane}(\bar{t},\bar{x})$.

By $\mathfrak{F}$ being Euclidean, it is easy to see that such ϱ_1 and ϱ_2 exist, e.g. rotations around time axis $\bar{t}$ which take $\mathsf{Plane}(\bar{t},\bar{x})$ and $\mathsf{Plane}(\bar{t},\mathsf{f}_s[\bar{t}\,])$ to $\mathsf{Plane}(\bar{t},\mathsf{f}_s^{-1}[\bar{t}\,])$ and $\mathsf{Plane}(\bar{t},\bar{x})$, respectively, are such.[41] See Figure 28.

It is easy to check that $\varrho_1 \circ \mathsf{f}_s \circ \varrho_2$ is a bijection satisfying $(\star)$ in Lemma 3.30 when $\varrho_1 \circ \mathsf{f}_s \circ \varrho_2$ is substituted in place of f, $\varrho_1 \circ \mathsf{f}_s \circ \varrho_2[\bar{t}\,] \notin \mathsf{SlowEucl}$, and $\varrho_1 \circ \mathsf{f}_s \circ \varrho_2$ satisfy (20), (21). By this it is clear that assumptions (20), (21) can be made. ∎

[40] We note that this means that the so called discriminant of the above quadratic equation is non-negative.

[41] For a detailed proof cf. the proof of Lemma 3.96 in §3.5.

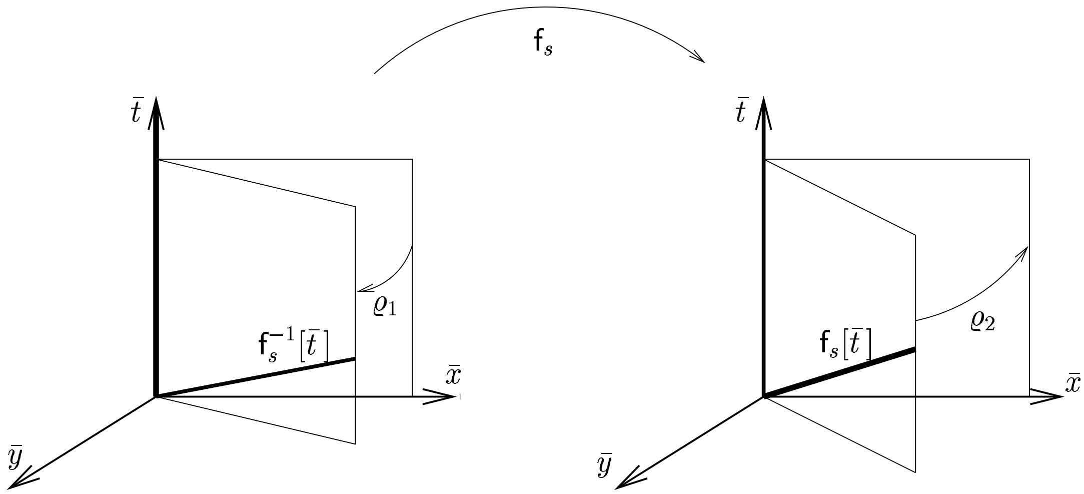


Figure 28: Illustration for explanation of assumptions (20), (21).

Proof of Lemma 3.31: A detailed proof will be given only for $n = 3$. After that we will outline how to modify the proof for $n = 3$ to obtain a proof for $n = 4$. For the case $n = 4$, Lemma 3.91 in §3.4.4 is a generalization of Lemma 3.31. In §3.4.4 we will give a detailed proof of Lemma 3.91, therefore we will not give a detailed proof of Lemma 3.31 for $n = 4$.

Claim 3.36 *Let $\mathfrak{F}_*, \mathsf{f}, \mathsf{f}_*$ be as in the formulation of Lemma 3.31. Then*

$$(\star\star) \qquad (\forall \ell \in \mathsf{PhtEucl}(\mathbf{n}, \mathbf{F}_*))\ \mathsf{f}_*[\ell] \in \mathsf{PhtEucl}(\mathbf{n}, \mathbf{F}_*).$$

We will give the *proof* of Claim 3.36 very soon. Lemma 3.31 follows from Claim 3.36 because of the following. Intuitively: If f satisfies $(\star)$ in Lemma 3.30 then also f^{-1} satisfies $(\star)$. Then applying Claim 3.36 to f and f^{-1}, respectively, we obtain Lemma 3.31. More formally: Let $\mathfrak{F}_*, \mathsf{f}, \mathsf{f}_*$ be as in the formulation of Lemma 3.31. Let $(\mathsf{f}^{-1})_* \in Aftr(\mathbf{n}, \mathbf{F}_*)$ such that $(\mathsf{f}^{-1})_* \lceil {}^n F = \mathsf{f}^{-1}$. Obviously $(\mathsf{f}^{-1})_* = (\mathsf{f}_*)^{-1}$. Now applying Claim 3.36 to $\mathfrak{F}_*, \mathsf{f}, \mathsf{f}_*$ and $\mathfrak{F}_*, \mathsf{f}^{-1}, (\mathsf{f}^{-1})_*$, respectively, we get that

$$(\forall \ell \in \mathsf{PhtEucl}(\mathbf{n}, \mathbf{F}_*))\ (\mathsf{f}_*[\ell] \in \mathsf{PhtEucl}(\mathbf{n}, \mathbf{F}_*)\ \wedge\ (\mathsf{f}_*)^{-1}[\ell] \in \mathsf{PhtEucl}(\mathbf{n}, \mathbf{F}_*)).$$

Hence f_* satisfies $(\star)$ in Lemma 3.30 when f_* and $\mathfrak{F}_*$ are substituted in place of f and $\mathfrak{F}$, respectively.

Proof of Claim 3.36: Let $n = 3$ and $\mathfrak{F}_*, \mathsf{f}, \mathsf{f}_*$ be as in formulation of Lemma 3.31. We have to prove that f_* satisfies $(\star\star)$ in Claim 3.36. Without loss of generality we may assume that $\mathsf{f}(\bar{0}) = \bar{0}$.

On the structure of the proof: Item (27) below is a reformulation of saying that f satisfies $(\star\star)$ of 3.36. Item (31) says the same for f_*. Therefore our task is to prove (31) from (27). This is done by the linear algebraic considerations given below.

By our assumption that f is a linear transformation we have that

$$(\forall p \in {}^3F)\ \mathsf{f}(p) = \langle p_0a_{00} + p_1a_{10} + p_2a_{20}, p_0a_{01} + p_1a_{11} + p_2a_{21}, p_0a_{02} + p_1a_{12} + p_2a_{22}\rangle\,,$$

for some $a_{ij} \in F$, where $i, j \in 3$. Let these a_{ij}'s be fixed. By $\mathsf{f}_* \in Aftr(\mathbf{3}, \mathbf{F}_*)$ and $\mathsf{f}_*\lceil {}^3F = \mathsf{f}$, we have

(26) $$(\forall p \in {}^3F_*)\mathsf{f}(p) = \langle p_0a_{00} + p_1a_{10} + p_2a_{20}, p_0a_{01} + p_1a_{11} + p_2a_{21}, p_0a_{02} + p_1a_{12} + p_2a_{22}\rangle\,.$$

By f satisfying $(\star)$, we have

(27) $$(\forall p \in {}^3F)\ \Big(p_0^2 = p_1^2 + p_2^2 \ \Rightarrow\ (p_0a_{00} + p_1a_{10} + p_2a_{20})^2 = (p_0a_{01} + p_1a_{11} + p_2a_{21})^2 + (p_0a_{02} + p_1a_{12} + p_2a_{22})^2\Big)\,.$$

Let $b_0 := a_{00}^2 - a_{01}^2 - a_{02}^2$, $b_1 := a_{10}^2 - a_{11}^2 - a_{12}^2$, $b_2 := a_{20}^2 - a_{21}^2 - a_{22}^2$, $b_3 := 2a_{00}a_{10} - 2a_{01}a_{11} - 2a_{02}a_{12}$, $b_4 := 2a_{00}a_{20} - 2a_{01}a_{21} - 2a_{02}a_{22}$, $b_5 := 2a_{10}a_{20} - 2a_{11}a_{21} - 2a_{12}a_{22}$. By this and by (27), we get

(28) $$(\forall p \in {}^3F)\ \Big(p_0^2 = p_1^2 + p_2^2 \ \Rightarrow\ p_0^2b_0 + p_1^2b_1 + p_2^2b_2 + p_0p_1b_3 + p_0p_2b_4 + p_1p_2b_5 = 0\Big)\,.$$

But (28) is equivalent with (29) below.

(29) $\langle b_0, b_1, b_2, b_3, b_4, b_5\rangle$ is a solution for the system of linear equations
$$E := \{p_0^2x_0 + p_1^2x_1 + p_2^2x_2 + p_0p_1x_3 + p_0p_2x_4 + p_1p_2x_5 \ :\ p \in {}^3F \ \&\ p_0^2 = p_1^2 + p_2^2\}.$$

To prove that f_* satisfies $(\star\star)$ it is enough to prove that

(30) $\langle b_0, b_1, b_2, b_3, b_4, b_5\rangle$ is a solution for the system of linear equations
$$E_* := \{p_0^2x_0 + p_1^2x_1 + p_2^2x_2 + p_0p_1x_3 + p_0p_2x_4 + p_1p_2x_5 \ :\ p \in {}^3F_* \ \&\ p_0^2 = p_1^2 + p_2^2\},$$

because (30) is equivalent with

(31) $$(\forall p \in {}^3F_*)\ \Big(p_0^2 = p_1^2 + p_2^2 \ \Rightarrow\ (p_0a_{00} + p_1a_{10} + p_2a_{20})^2 = (p_0a_{01} + p_1a_{11} + p_2a_{21})^2 + (p_0a_{02} + p_1a_{12} + p_2a_{22})^2\Big)\,,$$

and (26) and (31) implies that f_* satisfies ($\star\star$). To complete the proof it remains to prove (30) above. It is easy to check that the vectors in

$$B := \Big\{ \langle p_0^2, p_1^2, p_2^2, p_0p_1, p_0p_2, p_1p_2 \rangle \ : \\ p \in \{\langle 1,1,0\rangle, \langle 1,-1,0\rangle, \langle 1,0,1\rangle, \langle 1,0,-1\rangle, \langle 5,2,3\rangle\} \Big\}$$

are linearly independent. Let $\mathbf{W}_*$ denote the subspace of ${}^6\mathbf{F}_*$ generated by

$$A_* := \{ \langle p_0^2, p_1^2, p_2^2, p_0p_1, p_0p_2, p_1p_2 \rangle \ : \ p \in {}^3F_* \ \& \ p_0^2 = p_1^2 + p_2^2 \}\,.$$

$\mathbf{W}_*$ is at most 5-dimensional because of the "condition" $p_0^2 = p_1^2 + p_2^2$ in the definition of A_*. Hence B is a basis of $\mathbf{W}_*$ by $B \subseteq A_*$. By this, we have that each equation in E_* (cf. (30)) is a linear combination of equations in

$$J := \Big\{ p_0^2x_0 + p_1^2x_1 + p_2^2x_2 + p_0p_1x_3 + p_0p_2x_4 + p_1p_2x_5 \ : \\ p \in \{\langle 1,1,0\rangle, \langle 1,-1,0\rangle, \langle 1,0,1\rangle, \langle 1,0,-1\rangle, \langle 5,2,3\rangle\} \Big\}.$$

By this, we have that (30) holds because $\langle b_0, b_1, b_2, b_3, b_4, b_5 \rangle$ is a solution for the system of equations J by (29) and $J \subseteq E$. This completes the proof for $n = 3$. For $n = 4$ the proof is similar. Analogously to the proof for $n = 3$ it has to be shown that in the set

$$\{ \langle p_0^2, p_1^2, p_2^2, p_3^2, p_0p_1, p_0p_2, p_0p_3, p_1p_2, p_1p_3, p_2p_3 \rangle \ : \ p \in {}^4\mathrm{Z} \ \& \ p_0^2 = p_1^2 + p_2^2 + p_3^2 \}$$

can be found 9 linearly independent vectors. ∎

We will return to the possibility of (strongly) improving the result (Thm.3.28) "there are no FTL observers" after Thm.3.56 at the end of §3.4.2 below.

3.4.2 Weakening the axioms (FTL observers)

To our minds, Theorems 3.28(i) and 3.29 are important and strong theorems. They say that from our (deliberately weak) axiom systems of relativity theory it already follows that no observer can move faster than light. Hence it is worth to discuss which axiom is responsible for this. Therefore we have started a series of investigations such that we weaken our axiom system (in several ways), and then we check whether the new, weaker axiom systems allow FTL observers. These investigations also make it possible to understand better why Theorems 3.28, 3.29 are true.[42]

In this sub-section we present several examples of these investigations. First we present a very weak version (*Bax*) of *Newbasax* and show that this weak version still excludes FTL observers. We also look at even weaker versions of *Bax*. Then we present another weakened axiom system (*Relphax*), and show that it allows FTL observers. In all these cases we weaken Einstein's axiom, **AxE**. Clearly, if we omitted **AxE** from *Basax*, then the new axiom system would allow FTL observers. Besides the present FTL motivation, we will have other kinds of motivation for investigating weak systems like *Bax* (or even weaker ones cf. e.g. *Reich*(*Bax*), Bax^-, *Bax*(*nop*) way below [*Reich*(*Bax*) and *Bax*(*nop*) will be included in a later version of this work, cf. Madarász-Németi [29]]). Some of these other motivations will be dicussed in Remarks 3.48, 3.53 way below.

In the case of *Bax* we relax the condition that the speed of light is the same for all observers (but we retain the condition that each observer sees photons moving in all directions with the same speed). In the case of *Relphax* we relax the condition (that now is built into the language), that all observers perceive the same bodies as photons, i.e. being a photon becomes "relative" to observers.

The axiom system *Bax* (speed of light is observer-dependent) and connection to the literature

Besides the purposes outlined above, investigating *Bax* also serves other purposes e.g. a kind of continuation of the conceptual analysis started by Friedman [13]. We

[42]This kind of research is also about how the universe could look like from the logical point of view, and this question is more elaborated in section §5.1. These investigations are also motivated by the existence of Tachyon–Theory. About tachyon–theory we note that tachyons are hypothetical particles which move faster than the speed of light, cf. e.g. [10], [6].

will write more about this beginning with Remark 3.48 way below.

Let us recall that Thm.3.29 says that *Newbasax* implies that there is no FTL observer, and let us recall from §3.3 that *Newbasax* is a refined version of our basic axiom system *Basax*. We now introduce a new axiom system *Bax* which will be a refined version of *Newbasax*. As we said, we will fine-tune $\mathbf{AxE_0}$ of *Newbasax* in that the speed of light will not be the same for every observer, but for each observer photons moving in different directions will have the same speed. We will change **Ax5** only because $\mathbf{AxE_0}$ will be changed. After that we will state a theorem which says that *Bax* still implies that there is no FTL observer, if $n \geq 3$. More on this subject is in Madarász-Németi [28] and Madarász [26].

We will discuss the connections between *Bax* and the Kennedy-Thorndike experiment (cf. Taylor-Wheeler [44, pp.86–88] for the latter) in Remark 3.59 at the end of the present sub-section.

Below we postulate axioms $\mathbf{AxE_{00}}$, $\mathbf{AxE_{01}}$, $\mathbf{Ax5^{Obs}}$, and $\mathbf{Ax5^{Ph}}$. Recall from §3.3 the definition of relation $\overset{\odot}{\rightarrow}$ (Def.3.19) and the definition of *Newbasax* (Def.3.17).

$\mathbf{AxE_{00}}$ $(\forall m \in Obs)(\forall ph_1, ph_2 \in Ph)$
$$\Big((m \overset{\odot}{\rightarrow} ph_1 \ \land \ m \overset{\odot}{\rightarrow} ph_2) \ \Rightarrow \ v_m(ph_1) = v_m(ph_2)\Big).$$

That is, if observer m sees photons ph_1, ph_2 then the speed of ph_1 and ph_2 is same for m.

$\mathbf{AxE_{01}}$ $(\forall m \in Obs)(\forall ph \in Ph)(m \overset{\odot}{\rightarrow} ph \ \Rightarrow \ v_m(ph) \neq 0)$.

That is, there is no photon at rest.

$\mathbf{Ax5^{Obs}}$ $(\forall m \in Obs)(\exists ph \in Ph)(\forall \ell \in G)$
$$\Big(m \overset{\odot}{\rightarrow} ph \quad \land \quad [ang^2(\ell) < v_m(ph) \ \Rightarrow \ (\exists k \in Obs) tr_m(k) = \ell]\Big).$$

That is, every observer m sees some photon ph such that on every line slower than this photon there is an observer.

$\mathbf{Ax5^{Ph}}$ $(\forall m \in Obs)(\forall ph \in Ph)(\forall \ell \in G)$
$$(ang^2(\ell) = v_m(ph) \ \Rightarrow \ (\exists ph \in Ph) tr_m(ph) = \ell).$$

The intuitive content of $\mathbf{Ax5^{Ph}}$ will be quite important for us: Assume that observer m sees a photon with speed v. Now if in some other direction speed v can be realized by a line ℓ, then in that direction too there is a photon moving with the same speed v. We will call $\mathbf{Ax5^{Ph}}$ a *Weak Principle of Isotropy* (WPI) because it can be interpred as follows: (i) all directions are alike as far as speed of light is concerned, i.e. speed of light behaves the same way in all directions; (ii) more carefully: Assume $\mathbf{Ax}(\sqrt{\ })$. Now $\mathbf{Ax5^{Ph}}$ says that if observer m sees a photon ph with speed v in some direction, then in every other direction m will see a photon with the same speed v.

Definition 3.37
We define

$$Bax \stackrel{\text{def}}{=} \Big(Newbasax \setminus \{\mathbf{Ax5}, \mathbf{AxE_0}\}\Big) \cup \{\mathbf{Ax5^{Obs}}, \mathbf{Ax5^{Ph}}, \mathbf{AxE_{00}}, \mathbf{AxE_{01}}\},$$

where $\mathbf{Ax5^{Obs}}$, $\mathbf{Ax5^{Ph}}$, $\mathbf{AxE_{00}}$, $\mathbf{AxE_{01}}$ are defined above. Therefore

$$Bax = \{\mathbf{Ax1}, \mathbf{Ax2}, \mathbf{Ax3_0}, \mathbf{Ax4}, \mathbf{Ax5^{Obs}}, \mathbf{Ax5^{Ph}}, \mathbf{Ax6_{00}}, \mathbf{Ax6_{01}}, \mathbf{AxE_{00}}, \mathbf{AxE_{01}}\},$$

where for completeness we summarize below the axioms.

Ax1 $G = \mathsf{Eucl}(\mathbf{n}, \mathbf{F})$.

Ax2 $Obs \cup Ph \subseteq Ib$.

$\mathbf{Ax3_0}$ $tr_m(h) \in G \cup \{\emptyset\} \ \wedge \ (\exists k) tr_k(h) \neq \emptyset$.

Ax4 $tr_m(m) = \bar{t}$.

$\mathbf{Ax5^{Obs}}$ $(\exists ph)(\forall \ell)[m \overset{\odot}{\to} ph \ \wedge \ (ang^2(\ell) < v_m(ph) \ \Rightarrow \ (\exists k)\ell = tr_m(k))]$.

$\mathbf{Ax5^{Ph}}$ $ang^2(\ell) = v_m(ph) \ \Rightarrow \ (\exists ph)\ell = tr_m(ph)$.

$\mathbf{Ax6_{00}}$ $w_m[tr_m(k)] \subseteq Rng(w_k)$.

$\mathbf{Ax6_{01}}$ $Dom(\mathsf{f}_{mk}) \in Open$.

$\mathbf{AxE_{00}}$ $(m \overset{\odot}{\to} ph_1, ph_2) \ \Rightarrow \ v_m(ph_1) = v_m(ph_2)$.

$\mathbf{AxE_{01}}$ $v_m(ph) \neq \emptyset$.

$\triangleleft$

Remark 3.38 We will see later in Prop.3.43 that in *Bax* we can replace $\mathbf{AxE_{00}}$, saying that photons move with the same speed in each direction, with a weaker axiom **AxP1** saying that in each direction photons can move with only a unique speed (which speed may depend on the direction). This will lead us to investigating connections with well-studied directions in the literature. $\triangleleft$

Definition 3.39 Let $\mathfrak{M} \in \mathsf{Mod}_{\text{OFG}}(Bax \setminus \{\mathbf{AxE_{01}}\})$. For every $m \in Obs$ we will define $c_m \in F \cup \{\infty\}$ as follows. By $\mathbf{Ax5^{Obs}}$, $m \xrightarrow{\odot} ph$, for some $ph \in Ph$. Let this ph be fixed. We define

$$c_m \stackrel{\text{def}}{=} v_m(ph)\,.$$

The definition of c_m is unambiguous by $\mathbf{AxE_{00}}$. Intuitively, c_m is the speed of light for observer m. $\mathbf{AxE_{01}}$ then implies that $c_m \neq 0$. $\lhd$

Recall (from Thm.3.24 in §3.3) that the models of *Newbasax* are, roughly speaking, unions of models of *Basax*. Now, the models of *Bax* are like those of *Newbasax*, except that the speed of light can vary from world-view to world-view in the model. *Newbasax* is equivalent then to $(Bax \cup \{c_m = 1\})$. *Bax* is consistent, and is still weaker than *Newbasax*, i.e. $Bax \not\models Newbasax$. (A model for *Bax* which is not a model for *Newbasax* is given in Madarász [26].)

Next we state an FTL-type theorem, i.e. that *Bax* implies that there is no FTL observer. Clearly $Basax \models Newbasax \models Bax$. Therefore Thm.3.40(i) below implies Theorems 3.28(i), 3.29.

THEOREM 3.40 *Assume $n \geq 3$. Then (i) and (ii) below hold.*

(i) $Bax \models (\forall m, k \in Obs)(m \xrightarrow{\odot} k \Rightarrow v_m(k) < c_m)$.

(ii) $Bax \setminus \{\mathbf{AxE_{01}}\} \models (\forall m, k \in Obs)\big(m \xrightarrow{\odot} k \Rightarrow (v_m(k) \leq c_m \wedge (c_m \neq 0 \Rightarrow v_m(k) < c_m))\big)$.

On the **proof**: We will give the proof for $n = 3$ in §3.4.4 after Lemma 3.78, and for $n = 4$ in §3.4.4 after Lemma 3.92.

For $n = 3$ the proof will be given in the following way. To every model $\mathfrak{M}$ of *Bax* a model $\mathfrak{N}$ of *Newbasax* will be associated, roughly speaking in such a way that $v_m(k) < c_m$ holds in $\mathfrak{M}$ iff $v_m(k) < 1$ holds in $\mathfrak{N}$. Then using Thm.3.29, which says that *Newbasax* does not allow FTL observers, we will conclude that *Bax* does not allow FTL observers. The proof for $n = 4$ will be carried out analogously to the proof of Thm.3.28(i), recall that the latter theorem says that *Basax* does not allow FTL observers. To do this we will have to formulate and prove analogous counterparts of theorems of §3.2 and §3.3 for *Bax*.

It remains an open question whether the proof for $n = 3$ can be generalized to a proof for $n = 4$.

Before continuing the study of FTL observers, we introduce a weakened version of *Bax* in order to study connections with the literature.

In what follows we will introduce a weaker (than $\mathbf{AxE_{00}}$) axiom **AxP1** mentioned in Remark 3.38. **AxP1** allows the speed of light vary in each direction. This will be important e.g. in the Reichenbachian versions of relativity, because the main idea there is that we only want to make restrictions on the two-way speed of photons, without making any assumption about the one-way speed of photons. In Friedman [13], variants of Einstein's axiom **AxE** are recalled from the literature and are systematized into three principles (P1)–(P3). (We will recall (P1)–(P3) in Remark 3.48.) **AxP1** is a formalized version of Friedman's (P1), and we will elaborate on the connections (between our axiom systems and Friedman's principles) in Remark 3.48. We will return to discussing the question of how careful or how faithful our formalization of Friedman's principle (P1) is, in §3.4.3. To make **AxP1** shorter we use Definitions 3.41, 3.42 below.

Definition 3.41 We define functions $time : {}^nF \to F$ and $space : {}^nF \to {}^{n-1}F$ as follows.

$$(\forall p \in {}^nF)(time(p) :\stackrel{\text{def}}{=} p_0 \quad \wedge \quad space(p) :\stackrel{\text{def}}{=} \langle p_1, p_2, \ldots, p_{n-1}\rangle).$$

Sometimes, for brevity we denote $time(p)$ with p_t.

$\triangleleft$

Definition 3.42

(i) By a <u>*spatial direction*</u> or simply by a <u>*direction*</u> we understand a <u>*space*</u>-vector $d \in {}^{n-1}F$, with $d \neq \bar{0}$.

(ii) directions $:\stackrel{\text{def}}{=} \{d \in {}^{n-1}F \ : \ d \neq \bar{0}\}$.

(iii) Let $\mathfrak{M}$ be a frame model. Then body b is said to <u>*move in direction*</u> d (as seen by observer m) iff

$$(\forall p, q \in tr_m(b))(\exists r \in F)(space(q) - space(p) = r \cdot d).$$

Body b is said to move <u>*forward*</u> in direction d (as seen by observer m) iff

$$(\forall p, q \in tr_m(b))(\exists r \in {}^+F \cup \{0\})$$
$$(time(p) \leq time(q) \ \Rightarrow \ space(q) - space(p) = r \cdot d),$$ see Figure 29.

Body b is said to move <u>*backwards*</u> in direction d (as seen by observer m) iff

$$(\forall p, q \in tr_m(b))(\exists r \in {}^+F \cup \{0\})$$
$$(time(p) \geq time(q) \ \Rightarrow \ space(q) - space(p) = r \cdot d),$$ see Figure 29.

$\triangleleft$

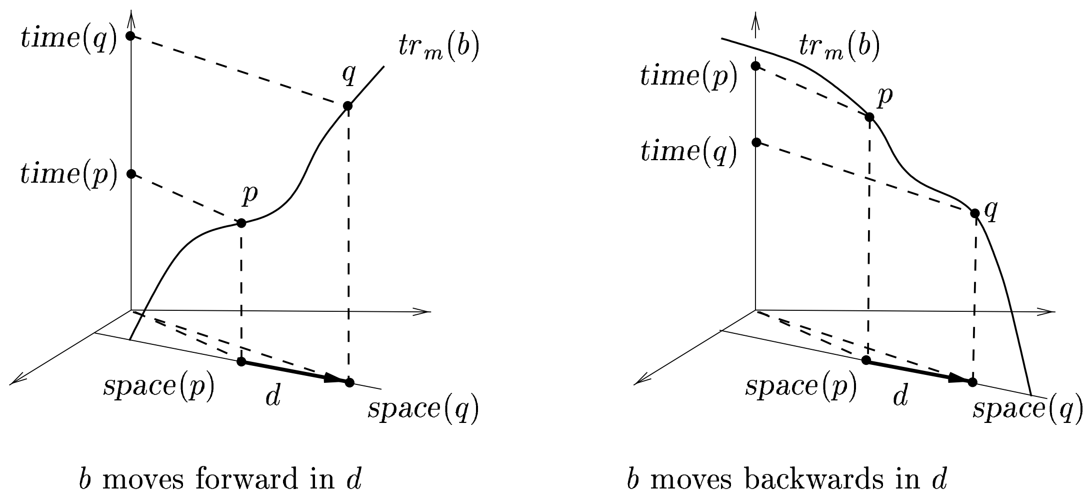


Figure 29: Illustration for Def.3.42.

Up to this point, when we said that body b moves in a certain direction, then we meant to say that b moves <u>*forward*</u> in that direction. From the present point on we will indicate whether b moves forward or backwards in a certain direction, except when there is no danger of confusion.

AxP1 $(\forall m \in Obs)(\forall ph_1, ph_2 \in Ph)(\forall d \in \text{directions})\Big((ph_1$ and ph_2 is moving forward in direction d as seen by m and $tr_m(ph_1) \cap tr_m(ph_2) \neq \emptyset) \Rightarrow$

$$tr_m(ph_1) = tr_m(ph_2)\Big).$$

Intuitively, photons going out from a point of space-time in the same direction (forward) have the same speed. In other words: Starting out from one point p of space-time, in every direction there is at most one "speed of light".

The following proposition states that the theory of *Bax* remains the same if we replace $\mathbf{AxE_{00}}$ with **AxP1**.

PROPOSITION 3.43

(i) *Bax* $\dashv\models (Bax \setminus \{\mathbf{AxE_{00}}\}) \cup \{\mathbf{AxP1}\}$.[43]

(ii) *Assume* **Ax1**-$\mathbf{Ax3_0}$, $\mathbf{Ax5^{Ph}}$, $\mathbf{Ax}(\sqrt{\ })$. *Then* $\mathbf{AxE_{00}} \dashv\models \mathbf{AxP1}$.

[43] We think that this is also true without $\mathbf{AxE_{01}}$.

Proof:

Proof of (ii): The proof of "direction $\models$" is obvious. To prove "direction $\Dashv$" let $\mathfrak{M} \models \{\mathbf{Ax1}, \mathbf{Ax2}, \mathbf{Ax3_0}, \mathbf{AxP1}, \mathbf{Ax5^{Ph}}, \mathbf{Ax}(\sqrt{\ })\}$. We have to prove that $\mathfrak{M} \models \mathbf{AxE_{00}}$. To see this let $m \in Obs$, $d \in$ directions, and $ph_1, ph_2 \in Ph$ such that $m \overset{\odot}{\to} ph_1$ and ph_2 moves forward in direction d as seen by m. Let $ph \in Ph$ such that ph moves forward in direction d as seen by m, $tr_m(ph) \cap tr_m(ph_2) \neq \emptyset$, and $v_m(ph) = v_m(ph_1)$. Such a ph exists by $\mathbf{Ax5^{Ph}}$ and $\mathbf{Ax}(\sqrt{\ })$. Since ph and ph_2 move forward in direction d as seen by m and $tr_m(ph) \cap tr_m(ph_2) \neq \emptyset$, we have $tr_m(ph) = tr_m(ph_2)$ by $\mathbf{AxP1}$. But this implies $v_m(ph_2) = v_m(ph) = v_m(ph_1)$, and this completes the proof of (ii).

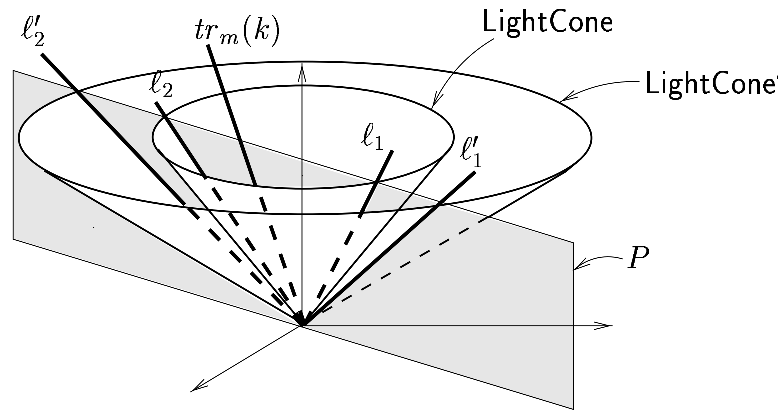


Figure 30: Illustration for the proof of Prop.3.43(i).

Outline of the proof of (i): Throughout the proof of (i) the reader is asked to consult Figure 30. We will prove the "direction $\Dashv$", because the proof of the other direction is straightforward. To prove direction $\Dashv$ let $\mathfrak{M}$ be a frame model of $(Bax \setminus \{\mathbf{AxE_{00}}\}) \cup \mathbf{AxP1}$. We have to prove $\mathfrak{M} \models \mathbf{AxE_{00}}$. In the proof of (ii) we have seen that $\mathfrak{M} \models \mathbf{AxE_{00}}$ holds under assuming $\mathbf{Ax}(\sqrt{\ })$. Now without assuming $\mathbf{Ax}(\sqrt{\ })$ the proof is not so obvious. Let $m \in Obs$. Let $ph \in Ph$ such that

(32) $\quad m \overset{\odot}{\to} ph \ \land\ (\forall \ell \in \mathsf{Eucl})(ang^2(\ell) < v_m(ph) \ \Rightarrow\ (\exists k \in Obs) tr_m(k) = \ell).$

Such a ph exists by $\mathbf{Ax5^{Obs}}$. Let $ph' \in Ph$ be arbitrary. To prove $\mathfrak{M} \models \mathbf{AxE_{00}}$ it is enough to prove that $v_m(ph) = v_m(ph')$. Let

$$\mathsf{LightCone} \stackrel{\text{def}}{=} \bigcup\{\ell \in \mathsf{Eucl} : \bar{0} \in \ell \ \& \ ang^2(\ell) = v_m(ph)\},$$
$$\mathsf{LightCone}' \stackrel{\text{def}}{=} \bigcup\{\ell \in \mathsf{Eucl} : \bar{0} \in \ell \ \& \ ang^2(\ell) = v_m(ph')\}.$$

To prove $v_m(ph) = v_m(ph')$ it is enough to prove that $\mathsf{LightCone} = \mathsf{LightCone}'$. The proof of this goes by contradiction. Assume $\mathsf{LightCone} \neq \mathsf{LightCone}'$. It is not hard to check that there is a plane P such that $\bar{0} \in P$, $P \cap \mathsf{LightCone} = \ell_1 \cup \ell_2$ and $P \cap \mathsf{LightCone}' = \ell_1' \cup \ell_2'$, for some pairwise different $\ell_1, \ell_2, \ell_1', \ell_2' \in \mathsf{Eucl}$. Let such $P, \ell_1, \ell_2, \ell_1', \ell_2'$ be fixed. Now by $\mathbf{Ax5^{Ph}}$ there are $ph_1, ph_2, ph_1', ph_2' \in Ph$ such that $tr_m(ph_1) = \ell_1$, $tr_m(ph_2) = \ell_2$, $tr_m(ph_1') = \ell_1'$ and $tr_m(ph_2') = \ell_2'$. Now by (32), there is $k \in Obs$ with $\bar{0} \in tr_m(k) \subseteq P$. Checking the details is left to the reader.
$\mathsf{f}_{mk} : {}^nF \to {}^nF$ is a bijection taking lines to lines by Thm.3.45 way below. But then

$tr_k(ph_1) \cap tr_k(ph_2) \cap tr_k(ph_1') \cap tr_k(ph_2') \neq \emptyset$,
$(\exists d \in \text{directions})$ ph_1, ph_2, ph_1', ph_2' move in direction d as seen by k, and
$tr_k(ph_1)$, $tr_k(ph_2)$, $tr_k(ph_1')$, $tr_k(ph_2')$ are pairwise different.

But this contradicts to **AxP1**. Hence $\mathsf{LightCone} = \mathsf{LightCone}'$ and $v_m(ph) = v_m(ph')$. This completes the proof. ∎

The reason why the above proposition is true is that the Weak Principle of Isotropy ($\mathbf{Ax5^{Ph}}$) is so strong that it makes $\mathbf{AxE_{00}}$ equivalent with **AxP1** (under assuming $\mathbf{Ax}(\sqrt{\ })$ and that the traces of photons are straight lines [or empty]).

As we said, **AxP1** is a formalized version of Friedman's principle (P1). In what follows, we aim at weakening Bax to Bax^- so that Bax^- could be considered as a "complete" formalization of (P1). For this purpose (i) we will replace $\mathbf{AxE_{00}}$ with **AxP1**. Further (ii) by looking at Prop.3.43(ii) above, we notice that we have to weaken $\mathbf{Ax5^{Ph}}$ because otherwise our theory will be strictly stronger than (P1) (i.e. it will imply $\mathbf{AxE_{00}}$). This weakened version of $\mathbf{Ax5^{Ph}}$ is $\mathbf{Ax5_{Ph}}$ below. We will also change $\mathbf{Ax5^{Obs}}$ to $\mathbf{Ax5_{Obs}}$ below, because $\mathbf{Ax5^{Obs}}$ does not fit the "philosophy" or paradigm of (P1).

$\mathbf{Ax5_{Ph}}$ $(\forall m \in Obs)(\forall p \in {}^nF)(\forall d \in \text{directions})(\exists ph \in Ph)$
(ph is moving forward in direction d as seen by m). See Figure 31.

Intuitively, from any point p of space-time in any direction there is a photon moving forward in that direction.

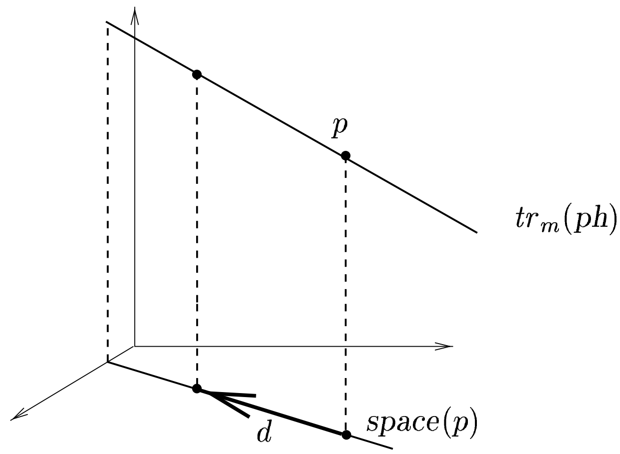


Figure 31: Illustration for $\mathbf{Ax5_{Ph}}$.

$\mathbf{Ax5_{Obs}}$ $(\forall m \in Obs)(\forall p \in {}^nF)(\forall d \in \text{directions})$
$\Big(\big[(\exists ph \in Ph)(p \in tr_m(ph) \ \wedge \ (ph \text{ is moving forward in } d \text{ as seen by } m)\big] \ \Rightarrow$
$\big[(\exists ph \in Ph)\big(p \in tr_m(ph) \ \wedge \ (ph \text{ is moving forward in } d \text{ as seen by } m) \wedge$
$(\forall v \in F)(0 \leq v < v_m(ph) \ \Rightarrow \ (\exists k \in Obs)(p \in tr_m(k) \ \wedge \ v_m(k) = v \wedge$
$(k \text{ is moving forward in } d \text{ as seen by } m))\big)\big]\Big)$.[44] See Figure 32.

Intuitively: Let us fix an observer m, a direction d, and a point p of space-time. We will speak about things moving forward in direction d through point p as seen by m (without mentioning all this data). Assume there is a photon moving in direction d. Then there is a photon in the same direction which is limiting in the following sense: For all speeds slower than this limiting photon, there is an observer moving with this speed. See Figure 32.

Definition 3.44 We define

$$Bax^- \stackrel{\text{def}}{=} \Big(Bax \setminus \{\mathbf{Ax5^{Ph}}, \mathbf{Ax5^{Obs}}, \mathbf{AxE_{00}}\}\Big) \cup \{\mathbf{Ax5_{Obs}}, \mathbf{Ax5_{Ph}}, \mathbf{AxP1}\},$$

where $\mathbf{Ax5_{Obs}}$, $\mathbf{Ax5_{Ph}}$ and $\mathbf{AxP1}$ are defined above. Therefore

$$Bax^- = \{\mathbf{Ax1}, \mathbf{Ax2}, \mathbf{Ax3_0}, \mathbf{Ax4}, \mathbf{Ax5_{Ph}}, \mathbf{Ax5_{Obs}}, \mathbf{Ax6_{00}}, \mathbf{Ax6_{01}}, \mathbf{AxE_{01}}, \mathbf{AxP1}\}.$$

$\lhd$

[44] We are under the impression that $\mathbf{Ax5_{Obs}}$ will imply $\mathbf{Ax}(\sqrt{\ })$. It is a task of future research to find natural (and short) versions of $\mathbf{Ax5_{Obs}}$ and $\mathbf{Ax5_{Ph}}$ which are provable from Bax.

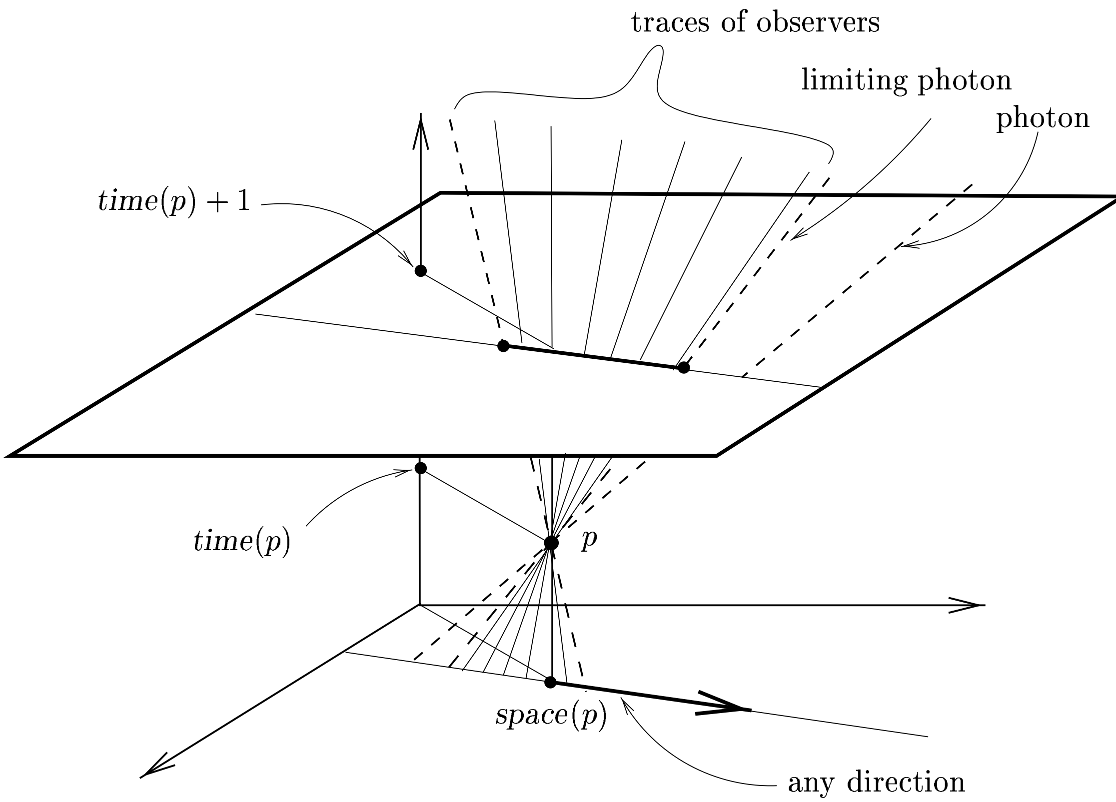


Figure 32: Illustration for $\mathbf{Ax5_{Obs}}$.

The following theorem is an analogon of Thm.3.3 (§3.2).

THEOREM 3.45 *$Bax^- \models (\forall m, k \in Obs)\Big(m \xrightarrow{\odot} k \ \Rightarrow\ (\mathsf{f}_{mk} : {}^nF \to {}^nF$ is a bijection taking lines to lines$)\Big)$.*

On the proof: Using the methods of the proof of Thm.3.3 (§3.2) and the proof of Lemma 3.27 (§3.3) one can prove Thm.3.45. ∎

PROPOSITION 3.46

(i) $Bax \cup \{\mathbf{Ax}(\sqrt{\ })\} \models Bax^- \cup \{\mathbf{Ax}(\sqrt{\ })\} \not\models Bax$.

(ii) *Assume $n \leq 3$. Then $Bax \models Bax^-$.*

Proof:
(i) We omit the proof of (i).
(ii) For $n < 3$, the proof is easy. Assume $n = 3$. In the proof of Thm.3.40 for $n = 3$ beginning with p.107, we transform each $Bax(\mathbf{3})$-model $\mathfrak{M}$ to a $Newbasax(\mathbf{3})$ model. By Thm.3.114 (in §3.6 way below), we conclude that $\mathbf{Ax}(\sqrt{\ })$ is true in these *Newbasax* models. Since the construction did not change the field $\mathfrak{F}^{\mathfrak{M}}$, we conclude that

(33) $$Bax(\mathbf{3}) \models \mathbf{Ax}(\sqrt{\ }).\ \blacksquare$$

Remark 3.47 One might wonder why $\mathbf{Ax}(\sqrt{\ })$ is needed in (i) of the above proposition. The reason for this is that in *Bax* we did not assume $\mathbf{Ax}(\sqrt{\ })$ and therefore in certain models of *Bax* there may exist (spatial) directions, say d, in which no photon moves. This is consistent with **Ax5**, because in direction d there need not exist a line ℓ with $ang^2(\ell) = 1$. On the other hand Bax^- does postulate that in every (spatial) direction there is a photon moving. Hence we do not know whether there is $n > 3$ with $Bax(\mathbf{n}) \not\models Bax^-(\mathbf{n})$.

In the present work when comparing refinements of *Bax* we will almost always assume $\mathbf{Ax}(\sqrt{\ })$. E.g. if we say that Bax^- is weaker than *Bax* then we really mean to say that $Bax^- + \mathbf{Ax}(\sqrt{\ })$ is weaker than $Bax + \mathbf{Ax}(\sqrt{\ })$. $\triangleleft$

Remark 3.48 (On connections with the literature) Our introduction (and study) of *Bax* can be viewed as a continuation of the conceptual analysis of relativity started in Friedman [13] p.159 §IV.6. Namely Friedman [13, p.159] introduces informal axioms (P1), (P2) and (P3) for the purposes of conceptual analysis of the usual axioms about the speed of light, showing up in various versions of relativity theory. Let us recall Friedman's principles (P1), (P2) and (P3) concerning the speed of light.

(P1) The constancy of the velocity[45] of light: Light is propagated with a constant velocity c independent of the velocity of its source.

(P2) The invariance of the velocity of light: Light has the same constant velocity c in all inertial reference frames.

(P3) The limiting character of the velocity of light: No "causal" signal can propagate with velocity greater than that of light.

[45] Friedman uses the word "velocity". In certain contexts we will use the word "speed" instead (cf. Gardner [15, p.7]).

(P1) says that photons move rather like sound moves and not like bullets (emitted from guns) move: the velocity of the gun from which a bullet is emitted adds to the velocity of bullet, while the velocity of sound depends only on the medium in which it is propagated. When saying that the velocity of light is independent of the velocity of its source, we mean that photons emitted at a point p of space-time, in direction d by various sources of light, like e.g. by a moving light-bulb and another non-moving light-bulb, have the same speed. By saying that this speed is "constant", we mean that a particular light-bulb at a particular point of space-time in a particular direction emits photons with one speed only.[46] Since we do not talk in our language about "sources" of light, or "emission" of light, it seems that it is a good formalization of (P1) that at any point p of space-time in any direction d, there is at most one photon trace. This is what **AxP1** says. So, the velocity of light-particles depends only on two data: (i) the point p of space-time where the light-particle is emitted, and (ii) the direction d in which the light-particle goes (and this velocity does not depend on other things, e.g. not on which light-bulb emitted the photon). Later in §3.4.3 we will denote the speed of this photon, as seen by an observer m, with $c_m(p, d)$. Then (P2) says that $c_m(p, d)$ does not depend on m, p or d. We consider Bax^- (cf. Def.3.44) as the completely formalized[47] counterpart of (P1). We consider Bax as the formalized counterpart of (P1 + Weak Principle of Isotropy), where Weak Principle of Isotropy (WPI) is formalized as $\mathbf{Ax5^{Ph}}$ above 3.37. Further we consider *Newbasax* as the formalized counterpart of (P2). We note that (P1+WPI) $\not\models$ (P2). When one uses a principle like (P1), usually one takes as granted some background axioms. In special relativity such a background axiom is e.g. that the traces of inertial bodies are straight lines. We will take as background axioms $\mathbf{Ax1}$, $\mathbf{Ax2}$, $\mathbf{Ax3_0}$, $\mathbf{Ax4}$, $\mathbf{Ax5_{Ph}}$, $\mathbf{Ax5_{Obs}}$, $\mathbf{Ax6_{00}}$, $\mathbf{Ax6_{01}}$, $\mathbf{AxE_{01}}$ which seem to be implicitly assumed in all versions of relativity in Friedman [13]. We will call the collection of these trivial axioms SPR_0, where the abbreviation SPR_0 refers to "the trivial part" of the Special Principle of Relativity (SPR) in the sense of Friedman [13, p.149, principle (R)]. Now the formalized version of (P1+SPR_0) is Bax^-, the formalized version of (P1+WPI+SPR_0) is Bax, and *Newbasax* will turn out to be the formalized version of (P2+SPR_0) [cf. Propositions 3.49, 3.50].[48] For completeness, we note that (P2+SPR_0) $\models$ WPI. We admit that Bax is only one of the possible formalizations of (P1+WPI+SPR_0). Another possible formalization of

[46]This is in accordance with [13, p.160], where it is said that one of the most important consequences of (P1) is that we have so-called light-cone (in each point). We will elaborate on the light-cone aspect more in §3.4.3.

[47]in first order logic

[48]Formally, the precise counterpart of (P2) would say that $(\exists c \in F)(\forall m \in Obs)(m$ sees the speed of light to be $c)$. But this means that $(\forall$ model $\mathfrak{M})(\exists c \in F)$ etc. Then without essential loss of generality we may assume $c = 1$.

(P1+WPI+SPR_0) is $Reich(Bax)$ in §**??** way below (this section will be included at a future stage of development of this work, cf. Madarász-Németi [29]). We sum this up in the following table:

Bax^-	is the formalization of	(P1)	+ SPR_0.
Bax	is the formalization of	(P1) + (WPI)	+ SPR_0.
$Newbasax$	is the formalization of	(P2)	+ SPR_0.

For completeness, we note that we never assume Friedman's (P3) as an axiom for the following reason: Part of (P3) turns out to be a theorem of our *Newbasax* (and *Bax*) (hence of Friedman's (P2) + Special Principle of Relativity, too) [cf. Theorems 3.29, 3.40 herein], while the other part of (P3) concerning bodies which are not observers does not seem to be needed in any part of developing the theory. Actually, we do have some philosophical reasons for not assuming this second part of (P3).

We should emphasize that our principle SPR_0 is strictly weaker than Special Principle of Relativity in Einstein's 1905 paper. Therefore we do not call our SPR_0 "Special Principle of Relativity" outside this remark. Our reason for not using the original Special Principle of Relativity is that (it is so strong that) it would blur the distinction between (P1) and (P2) (as indeed is pointed out in Friedman [13, p.160]).[49]

As we said, we will return to more careful (or thorough) considerations concerning the first-order formalization(s) of Friedman's principle (P1) in §3.4.3 way below.

$\lhd$

Propositions 3.49, 3.50 below serve to illuminate parts of Remark 3.48. Principle SPR_0 ($= Bax^- \setminus \{\mathbf{AxP1}\}$) was introduced in that remark.

Whenever we state a proposition beginning with "assume $\mathbf{Ax}(\sqrt{\ })$" this means that $Th_1 \models Th_2$ abbreviates the longer statement

$$Th_1 \cup \{\mathbf{Ax}(\sqrt{\ })\} \models Th_2 \cup \{\mathbf{Ax}(\sqrt{\ })\}.$$

PROPOSITION 3.49 *Assume* $\mathbf{Ax}(\sqrt{\ })$. *Then (i) and (ii) below hold.*

(i) $\mathrm{SPR}_0 \cup \{\mathbf{AxE_0}\} \mathrel{=\!\!|\!\models} Newbasax$.

(ii) $\mathrm{SPR}_0 \cup \{\mathbf{Ax5^{Ph}}, \mathbf{AxP1}\} \mathrel{=\!\!|\!\models} Bax$.

[49] Actually Gyula Dávid announced that he has a proof of Lorentz transform from Special Principle of Relativity without using (P1). This is an extra reason for believing that Special Principle of Relativity is too strong for our purposes (of conceptual analysis) here.

On the proof: The proof of item (i) is straightforward. Item (ii) follows by Prop.3.43(ii). ∎

PROPOSITION 3.50

(i) $(\mathrm{SPR}_0 \setminus \{\mathbf{AxE_{01}}\}) \cup \mathbf{AxE_0} \models \mathit{Newbasax}$.

(ii) $\mathrm{SPR}_0 \cup \{\mathbf{Ax5^{Ph}}, \mathbf{AxP1}\} \models \mathit{Bax}$.

On the proof: The proof of item (i) is straightforward. The proof of item (ii) is similar to the proof of Prop.3.43. ∎

QUESTION 3.51 *Does Bax^- imply that there are no FTL observers?*

◁

QUESTION 3.52 *If the answer to the above question turns out to be "NO", then is there a model $\mathfrak{M}$ of Bax^- with FTL observers such that*

$$\mathfrak{M} \models (\forall m \in Obs)(\forall ph \in Ph)\, v_m(ph) \neq \infty?$$

◁

Remark 3.53 (photon-free relativity and Reichenbachian relativity) In developing the theory of *Bax* we started weakening relativity in the direction of eliminating Einstein's axiom about the speed of light.

(i) A further refinement in this direction is the axiomatic theory *Reich*(*Bax*), going back to Reichenbach, Grünbaum and others (cf. Friedman [13], Reichenbach [34], Grünbaum [19], Winnie [47]). *Reich*(*Bax*) will be found in the present §3.

(ii) In passing we note that the proof-theoretic powers (under assuming $\mathbf{Ax}(\sqrt{\ })$) of our theories are arranged as follows:

$$Bax^- < Reich(Bax) < Bax < Newbasax < Basax,$$

where $Th_1 < Th_2$ means that $Th_2 \models Th_1 \not\models Th_2$.

(iii) A further continuation of this direction ($Newbasax \mapsto Bax \mapsto Reich(Bax) \mapsto Bax^-$) would be an axiomatic theory containing **Ax1**-**Ax4** and not mentioning photons at all. However, for this, we need to reformulate **Ax5** (or its version e.g. in *Bax*) because of the following. We do need the existence of observers, even if we drop the existence of photons. But in the present form of **Ax5** we state the existence of observers via photons. To eliminate photons from the observer-part of **Ax5**, we will need the tools developed in §3.7 "Symmetry axioms". Therefore we will return to the now outlined project of eliminating photons from *Bax* at the end of §3.7 in sub-section "Speed-of-light free part of our axiom systems" (§3.7.∞) of this work (§3.7.∞ will be included at a later stage of development, cf. Madarász-Németi [29]).

$\triangleleft$

Let us return to systematic study of FTL observers. As we can see in Thm.3.40 way above, FTL observers cannot exist even if we assume the very weak axiom system *Bax*.

The axiom system *Relphax* (being a photon is observer-dependent)[50]

In what follows we will introduce a new, refined version *Relphax* of *Basax*. We will again fine-tune **AxE**, but in a different way as we did in the case of *Bax*. In the new axiom system *Relphax*, being a photon will be relative, and it will depend on the observer.

To refine (or weaken) **AxE**, we will introduce a new variant L^+ for our language of relativity theory.

Definition 3.54 The new language L^+ for relativity theory is the same as the old one except that *Ph* is no more a unary relation. *Ph* is a binary relation of sort $\langle B, B \rangle$.

[50] For related investigations concerning the case of $n = 2$ we refer to Dávid [9].

More precisely B, Q and G are the same sorts as in the old definition of our language, i.e. Def.2.1 of §2.1, and $\mathfrak{M}$ is a *model of language L^+* iff

$$\mathfrak{M} = \langle B, F, G; Obs, Ph, Ib, +, \cdot, \leq, \mathrm{E}, W \rangle, \text{ where}$$

B, $\langle F, +, \cdot, \leq \rangle$, G, Obs, Ib, E, and W are as in Def.2.1 of §2.1, and

- Ph is a binary relation of sort $\langle \mathrm{B}, \mathrm{B} \rangle$.

Now, similarly as in Def.2.1 a model $\mathfrak{M}$ of L^+ is a *frame model of L^+* iff

$$\mathfrak{M} \models \mathbf{Ax}_{\mathrm{OF}} \cup \{\mathbf{Ax}_{\mathrm{G}}\} \cup \{W(m,p,h) \Rightarrow Obs(m)\}, \text{ where}$$

$\mathbf{Ax}_{\mathrm{OF}}$ and $\mathbf{Ax}_{\mathrm{G}}$ were defined in Def.2.1. Now $\models^{\mathrm{OFG}}$ and $\mathsf{Mod}_{\mathrm{OFG}}(\Sigma)$ are defined as at the end of Def.2.1. According to Notation 3.1, we use $\models$ for $\models^{\mathrm{OFG}}$.

Let $\mathfrak{M}$ be a frame model of L^+. If $m \in Obs$ and $b \in B$ and $\langle m, b \rangle \in Ph$ then we will write $Ph_m(b)$. Intuitively, this means that for observer m body b is a photon. Further

$$Ph_m \stackrel{\text{def}}{=} \{b \in B \,:\, Ph(m,b)\}. \qquad \triangleleft$$

Now $\mathbf{AxE_1}$ and $\mathbf{AxE_2}$ below constitute a weaker version of $\mathbf{AxE}$. We will change $\mathbf{Ax5}$ and $\mathbf{Ax2}$ to $\mathbf{Ax5_1}$ and $\mathbf{Ax2_1}$, respectively, only to fit our axioms to the new language L^+.

Below we postulate axioms $\mathbf{AxE_1}$, $\mathbf{AxE_2}$, $\mathbf{Ax2_1}$, $\mathbf{Ax5_1}$.

$\mathbf{AxE_1}$ $(\forall m \in Obs)(\forall b \in Ph_m)\ v_m(b) = 1.$

Intuitively, for each observer m the speed of light is 1.

$\mathbf{AxE_2}$ $(\forall m, k \in Obs)\Big(v_m(k) < 1 \ \Rightarrow\ (\forall b \in B)(Ph_m(b) \ \Leftrightarrow Ph_k(b))\Big).$

Intuitively, if observer m sees observer k moving slower than the speed of light, then body b is a photon for m iff it is a photon for k.

$\mathbf{Ax2_1}$ $Obs \subseteq Ib \ \wedge\ (\forall m \in Obs)\ Ph_m \subseteq Ib.$

$\mathbf{Ax5_1}$ $(\forall m \in Obs)(\forall \ell \in G)\Big((ang^2(\ell) < 1 \ \Rightarrow\ (\exists k \in Obs)\ \ell = tr_m(k)) \ \wedge$
$(ang^2(\ell) = 1 \ \Rightarrow\ (\exists ph \in Ph_m)\ \ell = tr_m(ph))\Big).$

Definition 3.55
We define

$$Relphax \stackrel{\text{def}}{=} \Big(Basax \setminus \{\mathbf{Ax2}, \mathbf{Ax5}, \mathbf{AxE}\}\Big) \cup \{\mathbf{Ax2_1}, \mathbf{Ax5_1}, \mathbf{AxE_1}, \mathbf{AxE_2}\}. \qquad \triangleleft$$

The following theorem says that in *Relphax* FTL observers are consistently possible.

THEOREM 3.56 *Assume $n \geq 2$. Then (i) and (ii) below hold.*

(i) $Relphax \not\models (\forall m, k \in Obs)\ v_m(k) < 1$.

(ii) *Assume $\mathfrak{F}$ is Euclidean or $\mathsf{Mod}_{\mathfrak{F}}(Basax) \neq \emptyset$. Then there is $\mathfrak{M} \in \mathsf{Mod}_{\mathfrak{F}}(Relphax)$ such that*

$$\mathfrak{M} \models (\exists m, k \in Obs)\ v_m(k) > 1\,.$$

The **proof** is in Madarász-Németi [28] and Madarász [26] (available from J. Madarász).

We suggest that the reader compare Theorems 3.28, 3.29, 3.40, and Theorem 3.56.

We note that one can consider *Relphax* to be a weakened version of *Basax*, namely $Basax \dashv\models (Relphax \cup \{Ph_m = Ph_k\})$.

We have $Basax(\mathbf{2}) \dashv\models Relphax(\mathbf{2})$. Investigations of FTL phenomena in $Basax(\mathbf{2})$ was done in §2 of an earlier version of the present work. Cf. also Dávid [9].

A brief return to *Bax* etc.

Now, we turn to the possibility of improving the "no FTL observers" theorem. Let us discuss briefly how much of our assumption *Basax* is needed for proving the nonexistence of FTL observers, in Thm.3.28(i). We will concentrate on the question of how much of **Ax5** was needed.

In the next conjecture we will use the notion of the angle between two lines ℓ, ℓ_1. Though we did not define this, we hope the reader will understand what we mean. Since we will need this notion only in two conjectures, we do not define it.

Conjecture 3.57 *Replace the photon part of* **Ax5** *by the following weaker axiom.*

(*) $(\forall \varepsilon \in {}^{+}F)(\forall \ell \in \mathsf{PhtEucl})(\forall p \in {}^{n}F)(\forall m \in Obs)(\exists ph \in Ph)$
$\Big($*the angle between* ℓ *and* $tr_m(ph)$ *is smaller than* ε, *and* $p \in tr_m(ph)\Big)$.

Let $Basax_0$ be obtained from Basax by replacing the photon part of **Ax5** *with (*) above. Then we conjecture that for $n \geq 3$*

$$Basax_0(\mathbf{n}) \models \textit{ there are no FTL observers.} \qquad \triangleleft$$

The axiom system *Bax* is introduced in Def.3.37.

Conjecture 3.58 *We conjecture that Conjecture 3.57 remains true if we replace Basax with Bax in it. I.e. we obtain Bax_0 by replacing $\mathbf{Ax5^{Ph}}$ with*
$(\forall\varepsilon \in {}^{+}F)(\forall\ell \in \mathsf{Eucl})(\forall p \in {}^{n}F)(\forall m \in Obs)(\exists ph \in Ph)\Big(ang^2(\ell) = c_m \Rightarrow$
(the angle between ℓ and $tr_m(ph)$ is smaller than ε, and $p \in tr_m(ph)$))$\Big)$.
Then we conjecture that for $n \geq 3$

$$Bax_0 \models \textit{there are no FTL observers.} \qquad \triangleleft$$

Remark 3.59 The Kennedy-Thorndike experiment (cf. Taylor-Wheeler [44, pp.86–88]) seems to suggest that the reality (of special relativity) may be closer to *Newbasax* than to *Bax*. All the same, we have motivation for studying *Bax* (and even weaker systems) summarized in (i)–(v) below:

(i) Conceptual analysis (like the one in Friedman [13], cf. also the introduction of this work).

(ii) We do not know what future experiments will say.

(iii) Applicability of the *"theory"* of *Bax* to accelerated observers and other general relativity situations where the flexibility of *Bax* renders it more applicable than *Newbasax*. (E.g. certain structures are "'locally *Bax*" but not "locally *Newbasax*".)

(iv) Friedman [13] p.159 line 6 bottom up – p.160 line 2, writes that Maxwell's (classical) electrodynamics predicts *Bax* but not *Newbasax* (or using Friedman's terminology it predicts (P1 + something) but not (P2)).[51]

[51] This is relevant because of the following. Consider the "dynamics of theories" as outlined e.g. in Andréka-Gergely- Németi-Sain [2], [1], [33] (of which a more accessible continuation is Jánossy et al. [23]). In this "paradigm" one can reconstruct the development of special relativity the following way. We take two pre-relativistic theories, (1) Newton's mechanics and (2) Maxwell's electrodynamics. Then we form the so called amalgamated coproduct of the two theories where the basis of amalgamation is the postulate saying that what is an electromagnetic wave in theory (2) is a particular case of inertial body in theory (1). For completeness, we note that the mathematical mechanism (in algebraic form) of forming such amalgamations is studied in Madarász [27], cf. also Andréka-Németi-Sain [4]. If we amalgamate these two pre-relativistic theories, (1) and (2) we will

(v) A further motivation for looking at *Bax* is that Friedman suggests to study the logical connections between (P1) and (P3). To do this in a logical framework, we look at a logical counterpart Bax^- of (P1) [as opposed to the counterpart *Newbasax* of (P2)] and study the (P3)-style aspects (or properties) of this theory e.g. in the form of the "no FTL observers" theorems of this section. We will return to further logical analysis of (P3) and e.g. its connections with (P1), and/or (P2) in a later section (at a later stage of development, cf. Madarász-Németi).

$\triangleleft$

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arrive at a theory say Th_3 which turns out to be inconsistent. Then we use the usual methodology of logic to weaken the axioms little-by-little until the so obtained version Th_3^- of Th_3 becomes consistent. Then this Th_3^- is called a possible version of special relativity.

Later we repeat this act of amalgamating theories by putting together special relativity and Newton's theory of gravitation. Again the amalgamated theory will be inconsistent, and then again we can try to apply the methods of logic to modify the axioms until it becomes consistent.

3.4.3 On the careful (or faithful) formalization of Friedman's principle (P1)

For completeness in this sub-section we contemplate the question of what we exactly mean by saying that the speed of light is independent of the velocity of its source. The reader not interested in this "exercise of mind" may safely skip the present sub-section.

In this sub-section we will be more "thorough" in formalizing principle (P1), arriving at axioms weaker than **AxP1**.[52] The main new idea is that, drawing from the analogy with speed of sound, we allow that a photon supposed to move forward in direction d, actually moves backwards in direction d (in the analogy with sound the reason may be either that there is a wind, or that our observer — e.g. a supersonic airplane — moves faster in direction d than sound).[53] In §3.4.2 we already had "tilted" light-cones, in the present sub-section we will allow light-cones tilt so much that the time-axis is not inside the cone. See Figure 33.

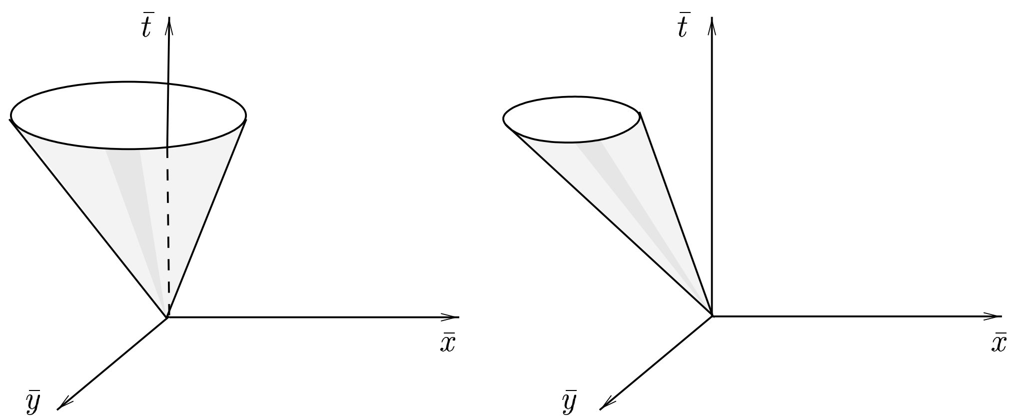


Figure 33: We will allow light-cones tilt so much that the time-axis is not inside the cone.

In our previous formalization Bax^-, for any observer m, and for any direction, say the direction marked by the vector 1_x, there is a photon moving forward in

[52](P1) was recalled in Remark 3.48 in §3.4.2.

[53]In the literature of relativity, this kind of hypothetical "wind" is often called ether-wind.

direction 1_x. If we keep the analogy with sound in mind, and if we do not want to exclude FTL observers (at least not a priori), then one might imagine that for some observer m a photon supposed to move forward in direction 1_x might seem to be moving (slowly) forward in the opposite direction -1_x. Then m would see two photons moving along the $\bar{x}$ axis, one moving forward with speed say 0.1 in direction -1_x while the other moving forward with speed say 1.1 also in direction -1_x.

In our first refined formalization of (P1), in **AxP1⁻**, we will require that from any point p of space-time, in any direction there are at most two photon-traces (moving forward or backwards in this direction). This way we will arrive at a very week system Bax^{--}. Then we will add various restrictions concerning the shapes of light-cones.

AxP1⁻ $(\forall m \in Obs)(\forall p \in {}^nF)(\forall d \in \text{directions})$
$\Big(\Big|\{tr_m(ph)\ :\ p \in tr_m(ph)\ \&\ ph \text{ is moving in direction}^{54}\ d\ \&\ ph \in Ph\}\Big| \le 2\Big).$

Intuitively: For any observer m and point p, in any *spatial direction* d, m will observe at most two kinds of photons "starting out from p", one supposed to move forward in this direction, while the other one supposed to move backwards. See Figure 34. Thinking further on the analogy of supersonic airplanes (and the velocity of sound), we conclude that in some directions there might be only one photon, and in still other directions there might be none. To illustrate this, consider Figure 34. Since the light-cone (or sound-cone for airplanes) only touches the plane $\mathsf{Plane}(\bar{t}, \bar{y})$ there will be only one photon in direction 1_y. This corresponds to the case when the supersonic airplane moves exactly with the speed of sound.[55] If the airplane goes a bit faster, then there will be no "photon" (or sound wave) in direction 1_y.

Our **AxP1⁻** is still a formalization of Friedman's (P1) which says that the speed of light does not depend on the velocity of its source. We turn to define the axiom system Bax^{--}, which could be considered as a careful (or almost finicky) formalization of Friedman's (P1). To do this we will replace **AxP1** by **AxP1⁻** in Bax^-. We note that $(Bax^- \setminus \{\mathbf{AxP1}\}) + \{\mathbf{AxP1^-}\}$ is equivalent with Bax^- under the assumption that there are no photons with infinite speed.[56] To obtain a weaker axiom system than Bax^- we have to replace $\mathbf{Ax5_{Ph}}$ and $\mathbf{Ax5_{Obs}}$ by $\mathbf{Ax5^-_{Ph}}$ and $\mathbf{Ax5^-_{Obs}}$ below, respectively.

[54] as seen by m

[55] Here we disregard the fact that this might destroy the plane.

[56] This is so because $\{\mathbf{Ax5_{Ph}}, \mathbf{AxP1^-}, \mathbf{AxE_{01}}\} \models \mathbf{AxP1}$ under the assumption that there are no photons with infinite speed.

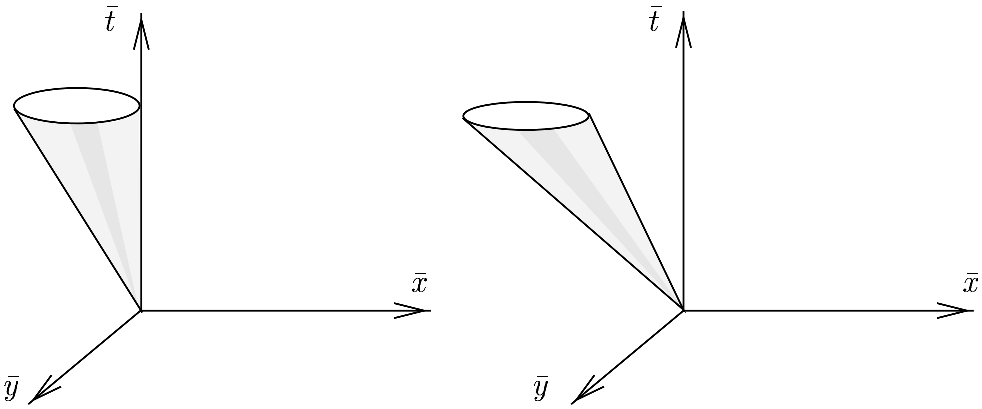


Figure 34: Illustration for $\mathbf{AxP1^-}$: In some directions there might be only one photon, and still other directions there might be none.

$\mathbf{Ax5^-_{Ph}}$ Assume observer m sees photons ph_1 and ph_2 moving forward in directions d_1 and d_2, respectively through point p of space-time. Then for any direction d_3 in "between" d_1 and d_2, m sees a third photon ph_3 going through point p and moving forward in direction d_3, where d_3 is between d_1 and d_2 iff there are $\mu, \lambda \in {}^+F$ such that $d_3 = \mu \cdot d_1 + \lambda \cdot d_2$. Every observer m at every point p of space-time sees at least two photons, which are moving in different directions.

To formulate $\mathbf{Ax5^-_{Obs}}$ we will use Def.3.60 below.

Definition 3.60 Let $\ell_1, \ell_2 \in \mathsf{Eucl}$ and $p \in {}^nF$ such that $\ell_1 \cap \ell_2 = \{p\}$. Let $\ell \in \mathsf{Eucl}$. Then ℓ is <u>*between*</u> ℓ_1 and ℓ_2 iff $p \in \ell$ and there is $\ell' \in \mathsf{Eucl}$ such that $\ell \parallel \ell'$ and there are $q \in \ell' \cap \ell_1$ and $r \in \ell' \cap \ell_2$ such that $q \neq p$ and $time(q) \leq time(p) \leq time(r)$, see Figure 35.

$\lhd$

$\mathbf{Ax5^-_{Obs}}$ Assume observer m sees photons ph_1 and ph_2 moving in direction d through point p of space-time and $tr_m(ph_1) \neq tr_m(ph_2)$. Assume $\ell \in \mathsf{Eucl}$ such that ℓ is between $tr_m(ph_1)$ and $tr_m(ph_2)$. Then there is an observer k such that $tr_m(k) = \ell$.

Definition 3.61

$Bax^{--} \stackrel{\text{def}}{=} (Bax^- \setminus \{\mathbf{AxE_{01}}, \mathbf{Ax5_{Ph}}, \mathbf{Ax5_{Obs}}, \mathbf{AxP1}\}) \cup \{\mathbf{Ax5^-_{Ph}}, \mathbf{Ax5^-_{Obs}}, \mathbf{AxP1^-}\}$. $\lhd$

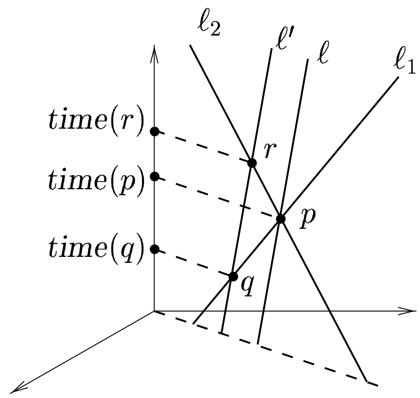


Figure 35: Illustration for Def.3.60: ℓ is between ℓ_1 and ℓ_2.

Notice that we omitted $\mathbf{AxE_{01}}$ which says that the speed of a photon is not 0.

Next, we consider two potential axioms $\mathbf{AxP1_1^-}$ and $\mathbf{AxP1_2^-}$ which could be added to Bax^{--} to make it a stronger version of Friedman's (P1).

$\mathbf{AxP1_1^-}$ $(\forall m \in Obs)(\forall p \in {}^nF)(\exists k \in Obs \cap w_m(p))(\forall d \in \text{directions})$ $(\exists ph \in Ph \cap w_m(p))(ph \text{ is moving forward in direction } d \text{ as seen by } k)$.

That is, for every event E there is an observer $k \in E$ such that k sees in any direction d a photon $ph \in E$ moving forward in direction d.

By $\mathsf{Planes} = \mathsf{Planes}(\mathbf{n}, \mathbf{F})$ we denote the set of all planes of ${}^n\mathbf{F}$.

$\mathbf{AxP1_2^-}$ $(\forall m \in Obs)(\forall p \in {}^nF)(\exists \ell \in \mathsf{Eucl})\Big((\forall P \in \mathsf{Planes})(\ell \subseteq P \;\Rightarrow$ $(\exists ph_1, ph_2 \in Ph)\big(tr_m(ph_1), tr_m(ph_2) \subseteq P \;\wedge\; tr_m(ph_1) \cap tr_m(ph_2) = \{p\} \;\wedge$ $(\ell \text{ is between } tr_m(ph_1) \text{ and } tr_m(ph_2))\big)\Big)$. See Figure 36.

The reader may ask, what is the role of the line ℓ in $\mathbf{AxP1_2^-}$? The answer is this: Let us think about the analogy with the speed of sound. If there is a wind, then "against the wind" sound goes slower, while "with the wind" it goes faster. Hence, in direction 1_x speed of light c_x may be small while in direction -1_x the speed c_{-x} might be very large. If we think of airplanes moving faster than the speed of sound, we realize that (in theory) it is reasonable to allow c_x to be a negative number. Imagine e.g. $c_x = -0.1$ and $c_{-x} = 1.1$. Then,

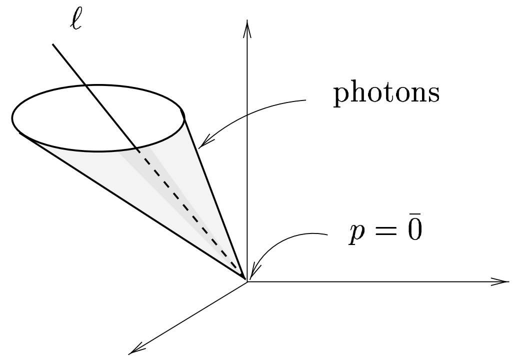


Figure 36: Illustration for $\mathbf{AxP1_2^-}$.

moving along the $\bar{x}$ axis we see two kinds of photons, one with speed 0.1, the other with 1.1 and <u>both</u> moving forward in direction -1_x. So, in principle, the photon ph_x moving in direction 1_x might have a negative speed (-0.1) and therefore ph_x might appear to the observer as if it was moving forward in the direction -1_x. All this is quite natural, if we think of sound in place of light and if our observer is a supersonic airplane. If we push these ideas further, we will arrive at the above formulation of $\mathbf{AxP1_2^-}$: Here, the role of ℓ is analogous with the role of observer k in $\mathbf{AxP1_1^-}$. We can think of ℓ as the life-line of a leaf drifting in the wind.

We note that $\mathbf{AxP1_2^-} \not\models \mathbf{AxP1^-}$. We turn to define an axiom system Bax_1^{--} and Bax_2^{--} by adding $\mathbf{AxP1_1^-}$ and $\mathbf{AxP1_2^-}$ to Bax^{--}, respectively.

Definition 3.62

$$Bax_1^{--} \stackrel{\text{def}}{=} Bax^{--} \cup \{\mathbf{AxP1_1^-}\}.$$
$$Bax_2^{--} \stackrel{\text{def}}{=} Bax^{--} \cup \{\mathbf{AxP1_2^-}\}.$$

◁

Next, we consider an axiom $\mathbf{AxP1_3^-}$ which could be added to Bax_1^{--}. The axiom system obtained in such a way will be called Bax_3^{--}. We note that in a certain sense Bax_3^{--} is motivated by Figure 16.10 on p.220 d'Inverno [10].

AxP1$_3^-$ $(\forall m \in Obs)(\forall p \in \bar{t}\,)(\forall d \in \text{directions})(\exists ph \in Ph)$
$(p \in tr_m(ph)$ and ph is moving forward in direction d as seen by $m)$.

That is, each observer m through any point p of its life-line in any direction d sees a photon moving forward in direction d.

Definition 3.63

$$Bax_3^{--} \stackrel{\text{def}}{=} Bax_1^{--} \cup \{\mathbf{AxP1_3^-}\}.$$

◁

Question for future research 3.64 *Investigate axiom systems Bax^{--}, Bax_1^{--}, Bax_2^{--}, Bax_3^{--} in the spirit as we investigated axiom systems e.g. Bax or Basax. Further, compare them with our weak axiom systems like Bax^{-} or Reich(Bax) or Bax(nop), where the latter two will be introduced later (at a later stage of development, cf. Madarász-Németi [29]).* ◁

Let us return to Bax^{--}. Now, Bax^{--} is a variant of relativity theory, in which the speed-of-light axiom is weakened to say that the speed of light does not depend on the speed of its source.

So in a certain sense, we could consider Bax^{--} as the formalized counterpart of Friedman's (P1). However, for the purposes of relativity theory, Bax^{--} is *extremely* weak. Therefore, we will strengthen Bax^{--} with extra assumptions. To formulate these extra assumptions, first we make it explicit that the velocity of a photon ph depends only on (i) the point p of space-time where ph was created and (ii) on the direction d in which (according to m) ph is moving. The formal version of this is $(*)$ below. For $(*)$ we need the following definition.

Definition 3.65 Let $\mathfrak{M}$ be a frame model satisfying **Ax1**, **Ax2**, **Ax3$_0$**, i.e. in which traces of inertial bodies are straight lines (or empty). Let $m \in Obs$, $b \in Ib$ and $d \in$ directions such that body b moves in direction d (as seen by m). Then the *speed* of body b in direction d (as seen by m) is $v_m(b)$ if body b moves forward in direction d and is $-v_m(b)$ otherwise. ◁

$(*)$ To each observer m, there is a partial binary function $c_m : {}^nF \times (\text{directions}) \to F$, that is $\langle p, d\rangle \mapsto c_m(p,d)$, such that m will observe that all photons emitted at space-time point p supposed to be moving forward in direction d have speed $c_m(p,d)$ in direction d as seen by m. Since in our language we cannot say "*emitted* at point p" instead we say going through point p (i.e. $p \in tr_m(ph)$).

Let us notice that $c_m(p,d)$ may be negative. Therefore observer m may see (or have the illusion, so to speak) that a photon ph which was expected to be moving forward in direction d is actually moving backwards in *direction* d. This is why in axiom **AxP1⁻** we said *only* that there are at most two photon velocities corresponding to a space-time point p and a direction d and did *not* require that these two photons should move in opposite directions. So, priciple $(*)$ is derived from Friedman's (P1), and axiom **AxP1⁻** is in turn derived from (or justifed by) $(*)$. This way, we obtained Bax^{--}. However, Bax^{--} is very weak because we did not say anything about how $c_m(p,d)$ depends on its arguments p and d.

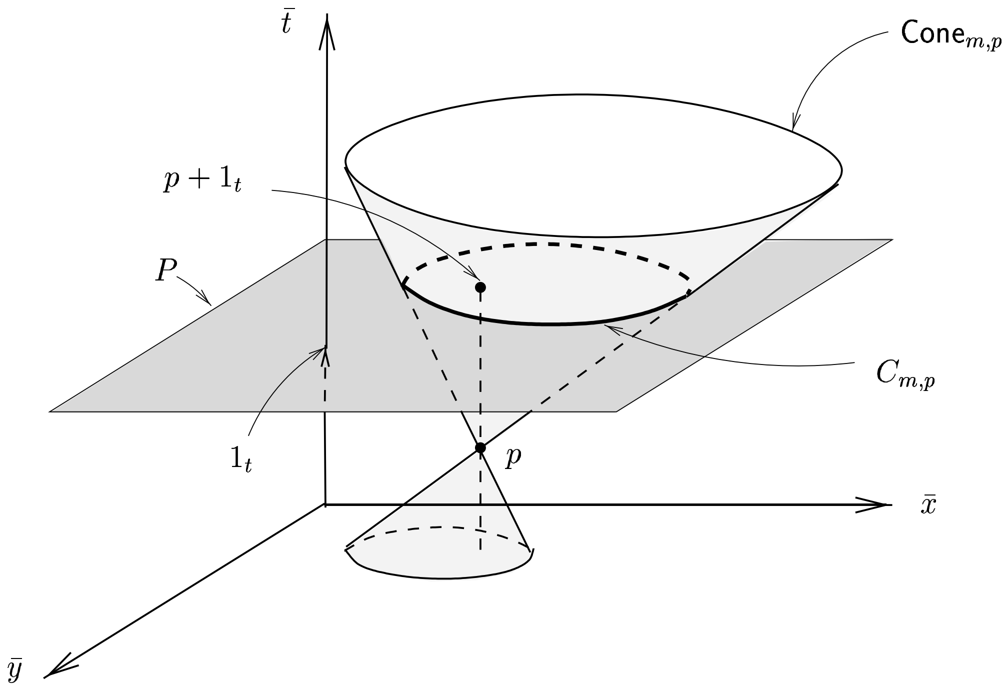


Figure 37: Illustration for Def.3.66.

Definition 3.66 Let

$$Bax^{--}_{+} \stackrel{\text{def}}{=} Bax^{--} \ + \ \text{the following postulates (i) and (ii).}$$

First we need a definition:

$$\mathsf{Cone}_{m,p} := \bigcup\{tr_m(ph) \ : \ ph \in Ph \ \& \ p \in tr_m(ph)\}.$$

(i) $c_m(p,d)$ is a <u>*continuous*</u> function of both its arguments (p and d).

(ii) First we formalize this condition for the case $n = 3$. Let P be the plane $P := \{q + (1_t + p) \ : \ q \in S\}$. Let $C_{m,p} = P \cap \mathsf{Cone}_{m,p}$. Now, we postulate that $C_{m,p}$ is homeomorphic with a circle in plane S. This homeomorphism is defined via the usual topology inherited from the space ${}^n\mathbf{F}$. Let us turn to the case of arbitrary n. Now, $C_{m,p}$ is defined as above and we postulate that $C_{m,p}$ is homeomorphic with the $(n-1)$-sphere $\{q \in S \ : \ |q| = 1\}$. See Figure 37.

$\lhd$

Question for future research 3.67 *Investigate Bax_{+}^{--} in the same spirit as we investigated e.g. Bax or Basax. Further, compare it with our weak axiom systems like Bax^{-} or Reich(Bax) or Bax(nop), where the latter two will be defined later (at a later sage of development, cf. Madarász-Németi [29]). For example, is $Bax_{+}^{--} \models$ (f_{mk} preserves Euclidean lines) true? In this direction we note the following:*

PROPOSITION 3.68 *(i) $Bax_{+}^{--} \not\models \nexists$ FTL observers.*

(ii) In Bax_{+}^{--} something like a "light-cone" already exists. Indeed $\mathsf{Cone}_{m,p}$ can be visualized like a cone-like surface e.g. like on Figure 38.

(iii) $Bax_{+}^{--} \models \{\mathsf{Cone}_{m,p}$ solidifies into something like a cone-like surface. For $n = 3$ the horizontal intersections of this "cone" are closed curves but need not be circles or even ellipses.$\}$
It is left to the reader to formalize this statement for $n > 3$.

We omit the **proof**.

Let us strengthen Bax_{+}^{--} a little more:

Definition 3.69

$$Bax_{++}^{--} \stackrel{\text{def}}{=} Bax_{+}^{--} + \Big\{c_m(p,d) \text{ does not depend on time, i.e. } (\forall \Delta t \in \bar{t})\, c_m(p,d) = c_m(p + \Delta t, d)\Big\}.$$

$\lhd$

Now Bax_{++}^{--} implies that the speed of light going in direction d is the same everywhere in the plane $\mathsf{Plane}(\bar{t}, d)$ determined by $\bar{t}$ and direction d.

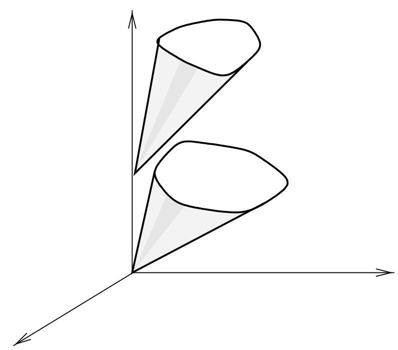


Figure 38: $\mathsf{Cone}_{m,p}$ can be visualized like a cone-like surface.

Definition 3.70 Let

$$Bax(\mathrm{P1}) \stackrel{\mathrm{def}}{=} Bax^{--}_{++} + \Big\{c_m(p,d) \text{ does not depend on } p, \text{ i.e. } c_m(p,d) = c_m(p',d) \text{ for all } p,p' \in {}^nF\Big\}.$$

◁

We note that

$$Bax^{--}_{++} < Bax(\mathrm{P1}) < Reich(Bax).$$

That is, our Bax^{--}_{++} is still compatible[57] with the Reichenbachian relativity (in the latter's full generality), where the latter will be introduced in a later section in a later version of this work (cf. Madarász-Németi [29], but cf. Friedman [13, pp.165-176].

Question for future research 3.71 *Investigate $Bax(\mathrm{P1})$ (in the spirit as we investigated e.g. Bax here), compare it with our present investigation of Bax^-, $Reich(Bax)$, $Bax(nop)$, where the latter two will be introduced later (at a later stage of development). E.g. does f_{mk} preserve straight lines, in $Bax(\mathrm{P1})$?*

◁

We note the following:

PROPOSITION 3.72

$$Bax(\mathrm{P1}) \not\models \nexists \textit{ FTL observer}.$$

[57] in the sense that $Reich(Bax)$ is a special case of Bax^{--}_{++}.

We omit the **proof**.

We consider $Bax(\mathrm{P1})$ and Bax^{--}_{++} as the very carefully formalized counterparts of Friedman's (P1), while we consider our Bax^{-} as a more "pragmatically" formalized version of (P1). It seems that Bax^{-} is an adequate formalization of Friedman's (P1) for the purposes of the present work, at least for a first systematization of the subject. For a future, second (or third) refined systematization, the more "finicky" formalizations $Bax(\mathrm{P1})$ and Bax^{--}_{++} will probably prove useful.

For the present investigation (for the time being) we will stick with Bax^{-} and we will treat Bax^{--}, $Bax(\mathrm{P1})$, Bax^{--}_{++} as potential future research subjects.

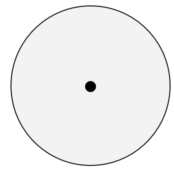
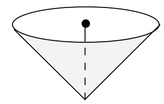
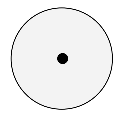
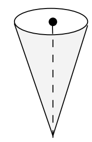
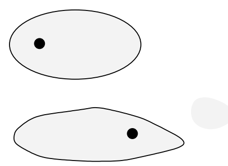
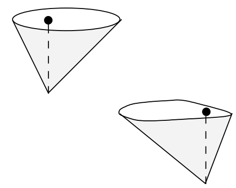
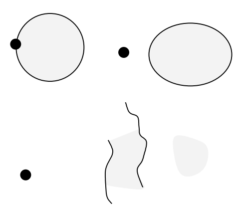
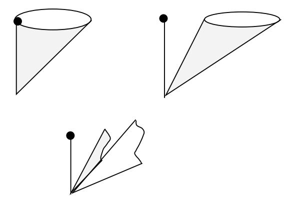
In the table below, we summarize our hierarchy

$$Bax^{--} < Bax^{-} < Bax < Newbasax$$

of theories from the point of view of the shape of light-cones and $c_m(p,d)$. Explanation for the table: Throughout the table $n = 3$. We are looking at a light-cone starting out from point p of space-time. I.e. we are looking at $\mathsf{Cone}_{m,p}$. In column 3, we see the intersection of the light-cone with the "simultaneity" i.e. plane $P = S + (p + 1_t)$, cf. Def.3.66 and Figure 37. The "circle" is the intersection of the cone with the plane P, i.e. it is $C_{m,p}$, in other words it is the set of intersections of photons (of finite speeds) starting from p with plane P. The shaded area is the set of intersection points of observers going through point p (with P). As one can see, in the first two rows, there is no shaded area outside the circles; this fact is caused by our "no FTL observers" theorems. The "fat" point represents the intersection (with P) of the coordinate-line parallel with $\bar{t}$ and going through p. In the 3-rd and 4-th rows (Bax^{-}, Bax^{--}) there are more than one figures some further up and some lower down. The upward figures represent the "ideal" cases we have in mind while the downward figures represent pathological cases which *might*[58] not be excluded by the axioms. E.g. consider the shaded area outside the "circle" in the Bax^{--} row.

We note that it can be proved that in Bax^{-} the "circle" (i.e. the intersection of the light-cone with the simultaneity P) is convex in the sense that any Euclidean line intersects it at most in two points.

[58]Perhaps some of these are excluded but we did not prove it yet.

Newbasax	$c_m(p,d) = 1$		
Bax	$c_m(p,d) = c_m$		
Bax^-	$c_m(p,d) > 0$		
Bax^{--} and versions	$c_m(p,d)$ (≤ 0 allowed)		

Shapes of light-cones in our theories

Group axioms for relativity

The set of Newtonian transformations *Newt* is defined in Def.3.94 in §3.5 and the set of similarity linear transformations *ST* is defined in Notation 3.121 in §3.7.

Ax(group) $(\forall m, k, m', k' \in Obs)(\exists k'' \in Obs)$
$(\mathsf{f}_{mk} \circ \mathsf{f}_{m'k'} = \mathsf{f}_{mk''} \circ N \circ H \circ \tilde{\varphi}$, for some $N \in Newt$, $H \in ST$ and $\varphi \in Aut(\mathfrak{F}))$.

Intuitively this axiom says that the world-view transformations f_{mk} form a group under composition modulo a Newtonian transformation, a similarity transformation and an automorphism of $\mathfrak{F}$.

Definition 3.73

$$Bax(\mathrm{P1gr}) \stackrel{\mathrm{def}}{=} Bax(\mathrm{P1}) \cup \{\mathbf{Ax(group)}\}.$$

$\lhd$

This strengthening $Bax(\mathrm{P1gr})$ of $Bax(\mathrm{P1})$ is motivated by the following proposition.

PROPOSITION 3.74

(i) $Bax(\mathrm{P1}) \not\models \mathbf{Ax(group)}$, *but*

(ii) $Basax \models \mathbf{Ax(group)}$.

We omit the **proof**.

QUESTION 3.75

(i) *Is* $Bax \models \mathbf{Ax(group)}$ *true?*

(ii) *Is* $Bax^{-} \models \mathbf{Ax(group)}$ *true?*

$\lhd$

Consider the straightened form $\mathbf{Ax(group)}^{+}$ of $\mathbf{Ax(group)}$ which does not mention Newtonian and similarity transformations and automorphisms.

$\mathbf{Ax(group)}^{+}$ $(\forall m, k, m', k' \in Obs)(\exists k'' \in Obs)\mathsf{f}_{mk} \circ \mathsf{f}_{m'k'} = \mathsf{f}_{mk''}$.

It is a natural question to investigate $Basax + \mathbf{Ax(group)}^{+}$, etc. We will return to this question in section §3.7 "Symmetry axioms". In particular it would be nice to know if $Basax + \mathbf{Ax(group)}^{+} \models \mathbf{Ax\triangle 1}$, where $\mathbf{Ax\triangle 1}$ is introduced in §3.7.

3.4.4 Proof that *Bax* does not allow FTL observers

In this sub-section we will prove Thm.3.40 (§3.4.2) which says that *Bax* does not allow FTL observers.

In order to prove Thm.3.40 for $n = 3$, we need Lemmas 3.76, 3.77, 3.78 below.

LEMMA 3.76 *Assume* $n \geq 3$. *Then (i)-(iv) below hold.*

(i) $Bax \setminus \{\mathbf{AxE_{01}}\} \models \Big(Rng(w_m) \cap Rng(w_k) \neq \emptyset \ \Rightarrow \ (c_m = \infty \Leftrightarrow c_k = \infty)\Big)$.

(ii) $Bax \setminus \{\mathbf{AxE_{01}}\} \models \Big(m \stackrel{\odot}{\to} k \ \Rightarrow \ (c_m = \infty \Leftrightarrow c_k = \infty)\Big)$.

(iii) $Bax \setminus \{\mathbf{AxE_{01}}\} \models \Big(Rng(w_m) \cap Rng(w_k) \neq \emptyset \ \Rightarrow \ (c_m = 0 \Leftrightarrow c_k = 0)\Big)$.

(iv) $Bax \setminus \{\mathbf{AxE_{01}}\} \models \Big(m \stackrel{\odot}{\to} k \ \Rightarrow \ (c_m = 0 \Leftrightarrow c_k = 0)\Big)$.

Proof:
<u>*Proof of (i):*</u> Throughout the proof the reader is asked to consult Figure 39.

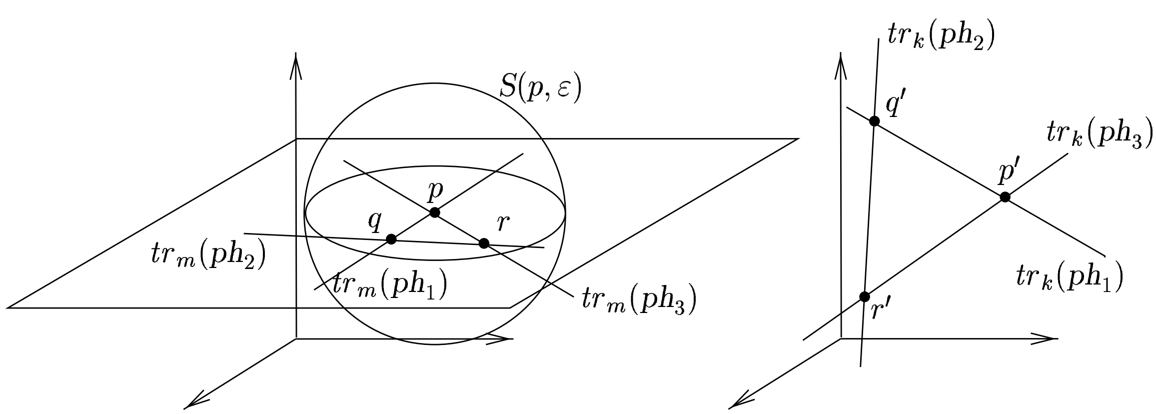


Figure 39: Illustration for the proof of Lemma 3.76(i).

Let $\mathfrak{M}$ be a frame model of $Bax \setminus \{\mathbf{AxE_{01}}\}$. Let $m, k \in Obs$ with

$$Rng(w_m) \cap Rng(w_k) \neq \emptyset \quad \text{and} \quad c_m = \infty.$$