NOTIONS OF DENSITY THAT IMPLY REPRESENTABILITY IN ALGEBRAIC LOGIC

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Abstract. Henkin and Tarski proved that an atomic cylindric algebra in which every atom is a rectangle must be representable (as a cylindric set algebra). This theorem and its analogues for quasi-polyadic algebras with and without equality are formulated in Henkin-Monk-Tarski [1985].

We introduce a natural and more general notion of ***rectangular density*** that can be applied to arbitrary cylindric and quasi-polyadic algebras, not just atomic ones. We then show that every rectangularly dense cylindric algebra is representable, and we extend this result to other classes of algebras of logic, for example quasi-polyadic algebras and substitution-cylindrification algebras with and without equality, relation algebras, and special Boolean monoids. The results of *op. cit.* mentioned above are special cases of our general theorems.

We point out an error in the proof of the Henkin-Monk-Tarski representation theorem for atomic equality-free quasi-polyadic algebras with rectangular atoms. The error consists in the implicit assumption of a property that does not, in general, hold. We then give a correct proof of their theorem.

Henkin and Tarski also introduced the notion of a *rich* cylindric algebra and proved in *op. cit.* that every rich cylindric algebra of finite dimension (or, more generally, of locally finite dimension) satisfying certain special identities is representable.

We introduce a modification of the notion of a rich algebra that, in our opinion, renders it more natural. In particular, under this modification richness becomes a density notion. Moreover, our notion of richness applies not only to algebras with equality, such as cylindric algebras, but also to algebras without equality. We show that a finite dimensional algebra is rich iff it is rectangularly dense and quasi-atomic; moreover, each of these conditions is also equivalent to a very natural condition of ***point density***. As a consequence, every finite dimensional (or locally finite dimensional) rich algebra of logic is representable. We do not have to assume the validity of any special identities to establish this representability. Not only does this give an improvement of the Henkin-Tarski representation theorem for rich cylindric algebras, it solves positively an open problem in *op. cit.* concerning the representability of finite dimensional rich quasi-polyadic algebras without equality.

Boolean algebra is an abstract algebraic theory that allows us to study the laws and theorems of propositional logic using modern algebraic methods. There

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are concrete set-theoretical Boolean algebras that consist of collections of sets and set-theoretical operations (say, union and complementation). There are also abstract Boolean algebras that are the models of a certain finite collection of axioms. Stone's representation theorem states that every abstract Boolean algebra can be represented as (i.e., is isomorphic to) a concrete Boolean algebra of sets. Thus, the axioms for Boolean algebra adequately capture the intuition of the concrete, set-theoretical models.

There are also abstract algebraic theories associated with more expressive parts of logic, for instance first-order logic. Examples include the relation algebras that date back to Peirce and Schröder, the cylindric algebras of Tarski and his collaborators, the polyadic and quasi-polyadic algebras of Halmos, and many others. In each of these domains, a situation analogous to that in Boolean algebra exists: there is a class of concrete, set-theoretical algebras and a more comprehensive class of abstract algebras that model a certain finite set (or, in infinite dimensions, a finite scheme) of equations. In each case the problem arises whether every abstract algebra can be represented as a concrete set-theoretical one. In contrast to the situation in Boolean algebra, the answer for algebras connected with first-order logic is in general negative. Not every abstract relation algebra is isomorphic to a set algebra of binary relations, not every cylindric algebra of dimension $\alpha \geq 2$ is isomorphic to a set algebra of α-ary relations, and so on.

These negative results lead to the important problem of finding various interesting criteria that *do* imply the representability of algebras of logic. For example, Jónsson-Tarski [1952] showed that an atomic relation algebra in which each atom is "functional" is representable. Using this result, Tarski [1955] was then able to prove the fundamental theorem that the class of all representable relation algebras can be axiomatized by an infinite set of equations.

In the domain of cylindric algebras, one of the main representation theorems involves the notion of a rectangle. In set-theoretical cylindric algebras of finite dimension α, a rectangle is the intersection of α many pairwise orthogonal cylinders. In dimension $\alpha = 2$ this corresponds to the usual notion of a rectangle. Henkin and Tarski gave an abstract definition of the notion of a rectangle and proved that *every atomic cylindric algebra* (of dimension at least 2) *in which all atoms are rectangles is representable* (see Henkin-Monk-Tarski [1985], Theorem 3.2.14).

In this paper we introduce a new notion: an algebra of logic is *rectangularly dense* if every non-zero element is above a non-zero rectangle. A particular example of such an algebra is an atomic cylindric algebra in which all atoms are rectangles. We show that many classes of rectangularly dense algebras of logic — in particular, rectangularly dense cylindric algebras — are representable. The proof for cylindric algebras breaks into three main steps. We show (in Lemma 3.10) that a simple, rectangularly dense cylindric algebra of finite dimension is in fact atomic, and hence representable by the Henkin-Tarski Theorem. We also show (in Corollary 2.10 and Theorem 2.11) that a rectangularly dense cylindric algebra of finite dimension is the directed union of subdirect products of countable, simple, rectangularly dense cylindric algebras, and hence representable. In this theorem we actually establish a much more general result, applying to arbitrary Boolean algebras with operators in a discriminator variety; one consequence of this theorem is that, e.g., the representability of the class of *simple* φ-dense algebras (for some notion φ of objects, such as rectangles) implies the representabilty of *all* φ-dense algebras. Finally, we show (in Theorem 3.11) how to make the passage from finite to infinite dimensions. We then use our representation theorems to prove that for many other classes of algebras of logic, rectangular density implies representability. In Section 4 we treat the case of quasi-polyadic algebras and substitution-cylindrification algebras with and without equality, and in Section 6 the case of relation algebras and special Boolean monoids.

In Section 5 we investigate two other notions of density: richness and point density. Both are related to notions that have already been studied in the literature and that are known to imply representability in certain special cases. The notion of richness that we shall introduce is a variant of a notion with the same name that was studied by Henkin and Tarski (see *op. cit.*, Section 3.2, or Henkin-Tarski [1961], Theorem 2.11). They proved that every rich algebra of finite dimension $\alpha \geq 2$ in which a certain collection of equations ε_{ij} $(i, j < \alpha$ and $i \neq j)$ is valid must be representable. Jónsson-Tarski [1952] essentially proved in Theorem 4.30 that every atomic relation algebra in which each atom is a "point" is representable. (However, they do not introduce the term "point".)

Our particular variant of the notion of richness, and our definition of point density, are formulated without any reference to diagonal (or identity) elements. Thus, in our formulation these notions can be applied not only to cylindric algebras and relation algebras, but also to other algebras of logic in which diagonal elements are not present, for example quasi-polyadic algebras and substitution cylindrification algebras without equality. We prove in Theorems 5.18 and 6.6 that in each of the finite dimensional algebras of logic $\mathfrak{A}$ under discussion, the three density notions essentially coincide. Specifically, $\mathfrak{A}$ is rich iff it is point dense iff it is rectangularly dense and quasi-atomic; moreover, in the presence of diagonal elements (e.g., in the case of cylindric algebras and relation algebras) we may drop "quasi-atomic": $\mathfrak{A}$ is rich iff it is point dense iff it is rectangularly dense.

This equivalence theorem helps to clarify the relationship between the two representation theorems of Henkin-Monk-Tarski [1985] and the representation theorem of Jónsson-Tarski [1952] mentioned above. It also shows that by modifying somewhat the definition of richness, one can obtain (from Theorem 3.11) a representation theorem for rich cylindric algebras without assuming the validity of the equations ε_{ij}. Finally, it shows that finite dimensional quasi-polyadic algebras (without equality) that are rich are representable. This solves positively Problem 5.6 in Henkin-Monk-Tarski [1985].

The original motivation behind the theorems in Sections 3, 4, and 6 lies not in algebraic logic, but in logic. Some of the authors were interested in the question of the completeness of several versions of first-order logic (corresponding to cylindric, quasi-polyadic, and substitution-cylindrification algebras) and of arrow logic (corresponding to relativized relation algebras and Boolean monoids). The analogue in algebraic logic of a completeness theorem is a representation theorem. To establish the completeness theorems, representation theorems for rectangularly dense algebras were needed. The known theorems were inadequate for the task because they required the additional hypothesis of atomicity. The interested reader is referred to Andréka-Németi-Sain [1994] and Mikulás [1995], [1996] for further details.

1. Preliminaries

We suppose the ordinal numbers to be defined so that each ordinal is equal to the set of its predecessors. The smallest infinite ordinal is denoted by "ω". The symbols "0" and "1" will be used in two different senses: as the names of the first two ordinals, and as the names of the least and greatest elements in a Boolean algebra. No confusion should arise from this dual use of notation. We shall sometimes use lower case Greek letters to denote ordinals, but usually we shall use i, j, k, ℓ, m, n, especially as subscripts of ordinally indexed operations.

The collection of all subsets of a set A will be denoted by $Sb(A)$. If f is a mapping from A to some set, and $a \in A$, then we shall sometimes write fa, instead of $f(a)$, to denote the value of f at a. The set of all mappings from A into a set B is denoted by ${}^{A}B$ and is called the *A^{th} power of B*.

Upper case German script letters $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}, \ldots$ denote algebras, and the corresponding upper case Roman letters A, B, $C, \ldots$ denote the corresponding universes. If $\mathfrak{A} = \langle A\,,\, O_i \rangle_{i \in I}$, then the operations O_i are called the *fundamental operations* of $\mathfrak{A}$. Two algebras $\mathfrak{A} = \langle A\,,\, O_i \rangle_{i \in I}$ and $\mathfrak{B} = \langle B\,,\, Q_j \rangle_{j \in J}$ are of the *same similarity type* provided that $I = J$ and that O_i and Q_i have the same rank (i.e., take the same number of arguments) for each $i \in I$. $\mathfrak{B}$ is a *reduct* of $\mathfrak{A}$ (to a subset J of I) if $Q_i = O_i$ for each $i \in J$. If $\mathfrak{B}$ is a reduct of $\mathfrak{A}$, then $\mathfrak{A}$ is called an *expansion* of $\mathfrak{B}$. A subalgebra of a reduct is called a *subreduct*. An algebra $\mathfrak{A}$ is *simple* provided that it has at least two elements and every homomorphism from $\mathfrak{A}$ into another algebra is either constant or one-one. $\mathfrak{A}$ is a *subdirect product* of a family of algebras $\langle \mathfrak{B}_i : i \in I \rangle$ if it is a subalgebra of the direct product and if, for each $i \in I$, the i^{th} projection function maps $\mathfrak{A}$ *onto* the factor algebra $\mathfrak{B}_i$. $\mathfrak{A}$ is *subdirectly irreducible* if it has at least two elements and if, for any isomorphism f between $\mathfrak{A}$ and a subdirect product of algebras, the composition of f with at least one of the projection functions is an isomorphism. By a well-known theorem of Birkhoff, every algebra is a subdirect product of subdirectly irreducible algebras.

A class of algebras is called a *variety* if it is axiomatizable by a set of equations. By another famous theorem of Birkhoff, a class of algebras is a variety iff it is closed under the formation of homomorphic images, subalgebras, and direct products.

We shall conceive of Boolean algebras as algebras of the form

$$\mathfrak{A} = \langle A\,,\, +^{\mathfrak{A}}\,,\, -^{\mathfrak{A}} \rangle\,.$$

When no confusion can arise, we shall drop the superscript references to $\mathfrak{A}$ on the operations. A similar remark applies in other, analogous situations. The binary operations $\cdot$ and $\oplus$ of Boolean multiplication and symmetric difference, the constants 0 and 1, and the partial ordering $\leq$ are defined in $\mathfrak{A}$ in the usual way. For example, we define $a \leq b$ iff $a + b = b$. This definition makes clear that, in Boolean algebra, every inequality can be expressed as an equation. The class of all Boolean algebras is denoted by BA. We shall also use this notation as an abbreviation for the phrase "Boolean algebra". If $X \subseteq A$, then $\sum^{\mathfrak{A}} X$ and $\prod^{\mathfrak{A}} X$, or simply $\sum X$ and $\prod X$, denote the supremum and infimum of X in $\mathfrak{A}$, whenever they exist. In particular, $\sum \varnothing = 0$ and $\prod \varnothing = 1$. A Boolean algebra is *complete* if the supremum and infimum of every set of elements exist, and *atomic* if every non-zero element is above an *atom*, i.e., a minimal non-zero element. A homomorphism f from a Boolean algebra $\mathfrak{A}$ into a Boolean algebra $\mathfrak{B}$ is said to be *complete* provided that, for every $X \subseteq A$, if $\sum X$ exists (in $\mathfrak{A}$), then $\sum\{f(x) : x \in X\}$ exists (in $\mathfrak{B}$) and $f(\sum X) = \sum\{f(x) : x \in X\}$.

In writing algebraic expressions we follow the usual conventions concerning the omission of parentheses: unary operations have the highest priority, then operations of multiplication, and then operations of addition. Thus, for example, the expression $a \cdot -b + c$ is to be read as $[a \cdot (-b)] + c$.

We assume that a first-order language (sometimes referred to as an *elementary language*) is correlated with each class of algebras of a given similarity type. The basic sentential connectives of the language are conjunction, disjunction, and negation, and there is a universal and an existential quantifier. A formula is *open* if it contains no quantifiers. An open formula that is the disjunction of one equation with a (possibly empty) finite collection of negations of equations is called a *conditional equation*. An open formula preceded by universal quantifiers is called a *universal* formula. A formula is *positive* if it contains no occurrences of the negation symbol.

Suppose that $\tau(x_0, \ldots, x_{n-1})$ is a term in the language of an algebra $\mathfrak{A}$ with variables among $x_0, \ldots, x_{n-1}$. Then $\tau^{\mathfrak{A}}$ denotes the n-ary operation defined by τ in $\mathfrak{A}$. Similarly, if $\varphi(x_0, \ldots, x_{n-1})$ is a formula in the language of $\mathfrak{A}$ with variables among $x_0, \ldots, x_{n-1}$, then $\varphi^{\mathfrak{A}}$ denotes the n-ary relation defined by φ in $\mathfrak{A}$. Of course this notation should also contain some reference to the rank n of the operation or relation being defined. However, it will always be clear from our discussion which n we have in mind. An operation f on A, say of rank n, is *term-definable* in $\mathfrak{A}$ if there is a term $\tau(x_0, \ldots, x_{n-1})$ such that f coincides with $\tau^{\mathfrak{A}}$. An algebra $\mathfrak{B}$ is a *term definitional extension* of $\mathfrak{A}$ if $\mathfrak{B}$ is an expansion of $\mathfrak{A}$ and if every fundamental operation of $\mathfrak{B}$ that does not occur among the fundamental operations of $\mathfrak{A}$ is term definable in $\mathfrak{A}$. An algebra $\mathfrak{B}$ is an *elementary subalgebra* (or an *elementary submodel*) of $\mathfrak{A}$ provided that $\mathfrak{B}$ is a subalgebra of $\mathfrak{A}$, and that, for each elementary formula $\varphi(x_0, \ldots, x_{n-1})$ in the language of $\mathfrak{A}$ and each sequence y of n elements from B, y is in $\varphi^{\mathfrak{A}}$ iff it is in $\varphi^{\mathfrak{B}}$.

A formula φ with one free variable is *preserved* (respectively, *strictly preserved*) *under homomorphisms on* $\mathfrak{A}$ if, for every homomorphism f from $\mathfrak{A}$ to an algebra $\mathfrak{B}$ and every element y of $\mathfrak{B}$ we have y in $\varphi^{\mathfrak{B}}$ if (respectively, if and only if) there is an element x in $\varphi^{\mathfrak{A}}$ such that $f(x) = y$.

2. Dense Boolean algebras with operators

We recall from Jónsson-Tarski [1951] the notion of a Boolean algebra with operators. A *Boolean algebra with operators*, or a BAO for short, is an algebra

$$\mathfrak{A} = \langle A\,,\, +\,,\, -\,,\, f_i \rangle_{i \in I}$$

such that $\langle A\,,\, +\,,\, - \rangle$ is a Boolean algebra and, for each $i \in I$, the *extra-Boolean operation* f_i is *additive* in every coordinate, i.e., if f_i has rank $n > 0$, then

$$f_i(x_0, \ldots, x_j + y_j, \ldots, x_{n-1}) = f_i(x_0, \ldots, x_j, \ldots, x_{n-1}) + f_i(x_0, \ldots, y_j, \ldots, x_{n-1})$$

for each $j < n$ and each sequence $x_0, \ldots, x_{n-1}, y_j$ of elements from $\mathfrak{A}$. The algebra $\mathfrak{A}$ is *normal* if every extra-Boolean operation f_i is *normal* in each argument, i.e., if f_i has rank $n > 0$, then

$$f_i(x_0, \ldots, x_j, \ldots, x_{n-1}) = 0$$

whenever $x_j = 0$ for some $j < n$. $\mathfrak{A}$ is *completely additive* if each of its extra-Boolean operations f_i (of positive rank n) is completely additive in every coordinate, i.e., for each $j < n$, each sequence of elements $x_0, \ldots, x_{j-1}, x_{j+1}, \ldots, x_{n-1}$ in A, and each set $X \subseteq A$ such that $\sum X$ exists, the sum

$$\sum\{f_i(x_0, \ldots, x_{j-1}, y, x_{j+1}, \ldots, x_{n-1}) : y \in X\}$$

exists and

$$f_i(x_0, \ldots, x_{j-1}, \sum X, x_{j+1}, \ldots, x_{n-1}) \\ = \sum\{f_i(x_0, \ldots, x_{j-1}, y, x_{j+1}, \ldots, x_{n-1}) : y \in X\}\,.$$

$\mathfrak{A}$ is *atomic* if its Boolean reduct is atomic, and *complete* if it is completely additive and its Boolean reduct is complete. Because they are additive, the operators f_i of a BAO are always *monotone* in the sense that

$$f_i(x_0, \ldots, x_{n-1}) \leq f_i(y_0, \ldots, y_{n-1})$$

whenever $x_j \leq y_j$ for all $j < n$.

A term in the language of a BAO is said to be *positive* if the complementation symbol does not occur in it; however, we do allow the defined symbols $\cdot$, 0, and 1 to occur. An equation in this language is *positive* if the right-hand and left-hand terms are both positive.

Boolean algebras with operators play a fundamental role in algebraic logic. Indeed, most of the algebraic structures encountered in algebraic versions of logics are BAOs. Another fundamental notion in algebraic logic is that of a discriminator operation.

A ternary operation t on a set A is a *ternary discriminator* if, for all x, y, z in A,

$$t(x, y, z) = \begin{cases} z & \text{if } x = y, \\ x & \text{if } x \neq y. \end{cases}$$

A unary operation c on A is a *unary discriminator* (with respect to elements 0 and 1 of A) if, for all x in A,

$$c(x) = \begin{cases} 0 & \text{if } x = 0, \\ 1 & \text{if } x \neq 0. \end{cases}$$

It is known that an algebra $\mathfrak{A}$ with a Boolean reduct $\langle A\,, +\,, -\rangle$ has a term-definable ternary discriminator iff it has a term-definable unary discriminator. Indeed, if t is a ternary discriminator, then

$$c(x) = -t(0, x, 1) \quad \text{for all } x \in A \tag{1}$$

defines a unary discriminator (here, 0 and 1 are the Boolean zero and unit), and we have

$$t(x, y, z) = x \cdot c(x \oplus y) + z \cdot -c(x \oplus y) \quad \text{for all } x, y, z \in A \tag{2}$$

(where $\cdot$ and $\oplus$ are the Boolean multiplication and symmetric difference operations). Conversely, if c is a unary discriminator, then (2) defines a ternary discriminator and (1) holds. (See Jipsen [1993], pp. 240–241.)

A variety V is a *discriminator variety* if there is a term that defines a ternary discriminator on the universe of each subdirectly irreducible algebra in V. It follows from the remarks of the previous paragraph that if V is, e.g., a variety of BAOs, then we can equivalently replace the word "ternary" by "unary" in the preceding definition.

It is a property of discriminator varieties that every subdirectly irreducible algebra is simple, and hence every algebra is a subdirect product of simple algebras. One consequence of this is that an equation or conditional equation holds in the variety iff it holds in all simple algebras of the variety.

We shall refer to a term that defines a ternary or unary discriminator in all subdirectly irreducible algebras of V as a *ternary* or *unary discriminator term for* V. McKenzie [1975] showed that the property of being a ternary discriminator term for V is expressible by a set of equations. Recently, Jipsen [1993], Theorem 3, showed that the property of being a unary discriminator term for a variety of normal Boolean algebras with operators is expressible by a simple set of equations. Corollary 2.2 below presents a variant of Jipsen's theorem.

Lemma 2.1. *Suppose that $\mathfrak{A}$ is a subdirectly irreducible, normal BAO and c a unary operation on A. Then the following conditions are equivalent.*

(i) *c defines a unary discriminator in $\mathfrak{A}$.*

(ii) *The following equations are identically satisfied in $\langle \mathfrak{A}\,, c\,\rangle$:*

(α) $x \le c(x)\,,$

(β) $c(c(x)) \le c(x)\,,$

(γ) $c(-c(x)) \le -c(x)\,,$

(δ) $f(x_0, \ldots, x_{n-1}) \le c(x_i)$ *for every extra-Boolean operation f of rank $n > 0$ and every $i < n$.*

Proof: Let $\mathfrak{A}$ be a subdirectly irreducible, normal BAO. Suppose, first, that condition (i) holds. Because $c(x) = 1$ for every non-zero x in $\mathfrak{A}$, we readily check that equations (α)–(δ) all hold in $\mathfrak{A}$. (The assumption of normality is needed to verify equation (δ) in the case when $x_i = 0$.)

Now suppose that condition (ii) holds. For a given extra-Boolean operation f of $\mathfrak{A}$ of rank $n > 0$ and a given $i < n$, define a unary operation g_i on A by stipulating that

$$g_i(x) = f(1, \ldots, 1, x, 1, \ldots, 1) \quad \text{for all } x \in A,$$

where x occurs as the i^{th} argument of f. Then

$$g_i(c(x)) \le c(x) \quad \text{for all } x \in A, \tag{1}$$

since

$$g_i(c(x)) \le c(c(x)) \le c(x)$$

by (β) and (δ). Also,

$$g_i(-c(x)) \le -c(x) \quad \text{for all } x \in A. \tag{2}$$

Indeed,

$$g_i(-c(x)) \leq c(-c(x)) \leq -c(x)$$

by (δ) (with "$-c(x)$" in place of "x_i"), the definition of g_i, and (γ).

$$c(0) = 0\,. \tag{3}$$

To prove (3) notice, first of all, that $c(1) = 1$ by condition (α); using this and (γ), we get

$$c(0) = c(-1) = c(-c(1)) \leq -c(1) = -1 = 0\,.$$

Jipsen, *ibid.*, shows that for a subdirectly irreducible algebra $\mathfrak{A}$, condition (i) is equivalent to the validity of equations (3), (ii)(α), and each instance of (1) and (2) (i.e., each instance arising from an extra Boolean operation f and an $i < n$). Since (α)–(δ) imply the validity of each instance of (1) and (2), this shows that (ii) implies (i). ∎

Using the fact that an equation holds in a variety iff it holds in all subdirectly irreducible algebras in the variety, we immediately draw the following conclusion from the preceding lemma.

Corollary 2.2. *Let V be a variety of normal BAOs and c a term with one free variable (in the language of V). Then c is a unary discriminator term for V iff equations (α)–(δ) of* 2.1 *hold in V.*

When a BAO has a unary discriminator, it is obviously unique, by the very form of its definition. Therefore, any term c such that equations (α)–(δ) hold in V must be unique up to term equivalence. Indeed, let c' be any other term for which these equations hold in V. Since both c and c' define unary discriminators in the subdirectly irreducibles, the equation $c(x) = c'(x)$ is identically satisfied in all subdirectly irreducibles of V, and hence in V itself. We may therefore refer to c as *the* unary discriminator term for V. In what follows we shall always denote this term (or the corresponding operation in an algebra of V) by c.

Corollary 2.3. *Let V be a variety of normal BAOs and c a term such that equations (α)–(δ) of Lemma* 2.1 *are valid in V. Then the following equations and conditional equations are also valid in V.*

(i) $c(x) = 0$ *iff* $x = 0$.
(ii) *If* $x \leq y$, *then* $c(x) \leq c(y)$.
(iii) $c(c(x)) = c(x)$.
(iv) $c(-c(x)) = -c(x)$.
(v) $c(x \cdot c(y)) = c(x) \cdot c(y)$.
(vi) $c(x + y) = c(x) + c(y)$.
(vii) $x \cdot c(y) = 0$ *iff* $y \cdot c(x) = 0$.

Proof: The implication from right to left in (i) is demonstrated in (3) of the proof of Lemma 2.1, and the reverse implication follows from equation (α). For (ii)–(vi), it suffices to check that they hold in all subdirectly irreducible algebras in V. As an example, we verify (v). Let $\mathfrak{A}$ be a subdirectly irreducible algebra in V and suppose that $x, y \in A$. If $y = 0$, then by (i) we have

$$c(x \cdot c(y)) = c(x \cdot 0) = 0 \tag{1}$$

and

$$c(x) \cdot c(y) = c(x) \cdot 0 = 0\,. \tag{2}$$

If $y \neq 0$, then using the fact that c must be a unary discriminator, we get

$$c(x \cdot c(y)) = c(x \cdot 1) = c(x) \tag{3}$$

and

$$c(x) \cdot c(y) = c(x) \cdot 1 = c(x)\,. \tag{4}$$

We leave the verification of (ii)–(iv) and (vi) to the reader. Part (vii) follows from (i) and (v):

$$\begin{aligned} x \cdot c(y) = 0 \quad &\text{iff} \quad c(x \cdot c(y)) = 0 && \text{by (i)},\\ &\text{iff} \quad c(x) \cdot c(y) = 0 && \text{by (v)},\\ &\text{iff} \quad c(c(x) \cdot y) = 0 && \text{by (v)},\\ &\text{iff} \quad c(x) \cdot y = 0 && \text{by (i)}. \end{aligned}$$

∎

An operation c on a set A satisfying conditions 2.1(ii)(α) and 2.3(ii),(iii) is called a *closure operation*. If, in addition, 2.3(iv) holds, then c is said to be *complemented*. Thus, we can reformulate Corollary 2.2 as follows:

A variety V of normal BAOs is a discriminator variety iff there is a term-definable complemented closure operation c satisfying 2.1(ii)(δ).

Just as in the case of rings and of Boolean algebras, there is the notion of an ideal of a Boolean algebra with operators $\mathfrak{A}$. A subset I of the universe of $\mathfrak{A}$ is an *ideal* if there is a congruence relation Θ on $\mathfrak{A}$ such that I is the *kernel* of Θ, i.e., I is the equivalence class of 0 under Θ. Ideals in the theory of BAOs function just as they do in the theory of rings. For example, we can form the quotient structure $\mathfrak{A}/I$ of $\mathfrak{A}$ modulo the ideal I and there is a canonical homomorphism of $\mathfrak{A}$ onto $\mathfrak{A}/I$ given by the mapping $x \longmapsto x/I$. If g is a homomorphism of $\mathfrak{A}$ onto $\mathfrak{B}$, then the kernel of g (i.e., the set of elements mapped to 0) forms an ideal I, and the corresponding quotient algebra $\mathfrak{A}/I$ is canonically isomorphic to $\mathfrak{B}$ via the mapping $x/I \longmapsto g(x)$.

Notice that an ideal must be closed under every term-definable operation (of non-zero rank) that is normal. In fact, suppose that I is an ideal of $\mathfrak{A}$, say it is the kernel of the congruence Θ. Let f be any term-definable normal operation of $\mathfrak{A}$, say of rank $n > 0$, and let $x_0, \ldots, x_{n-1}$ be any elements of A. If, for some fixed $i < n$, the element x_i is in I, then $\langle 0, x_i \rangle$ and $\langle x_j, x_j \rangle$ for $j < n$ with $j \neq i$ are all in Θ. Therefore,

$$\langle f(x_0, \ldots, x_{i-1}, 0, x_{i+1}, \ldots, x_{n-1}), f(x_0, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_{n-1}) \rangle$$

is in Θ, since f is term-definable. Because f is normal,

$$f(x_0, \ldots, x_{i-1}, 0, x_{i+1}, \ldots, x_{n-1}) = 0\,.$$

Therefore, $\langle 0, f(x_0, \ldots, x_{n-1}) \rangle$ is in Θ, so $f(x_0, \ldots, x_{n-1})$ is in I.

The following intrinsic characterization of ideals of Boolean algebras with normal operators is a special case of a result due to Ildikó Sain. (See Sain [1982], Proposition 7.4, for a formulation and proof of the key case when all the extra-Boolean operators are unary.)

Lemma 2.4. *A subset I of the universe of a Boolean algebra with normal operators $\mathfrak{A}$ is an ideal iff the following conditions are satisfied:*

(i) $0 \in I$;
(ii) *If $x, y \in I$, then $x + y \in I$;*
(iii) *If $x \in I$ and $y \leq x$, then $y \in I$;*
(iv) *For each extra-Boolean operation f of rank $n > 0$ and each sequence $x_0, \ldots, x_{n-1}$ of elements from A, if $x_i \in I$ for some i, then $f(x_0, \ldots, x_{n-1})$ is in I.*

To prove the principal lemma of the present section we shall need to derive a few easy facts about ideals of algebras in a discriminator variety of normal BAOs. These facts are well known for finite-dimensional cylindric algebras (see Henkin-Monk-Tarski [1971], Theorems 2.3.8–2.3.10 and 2.3.18), and the cylindric algebraic proofs essentially carry over to the present case.

Given a subset X of the universe of $\mathfrak{A}$, the *ideal generated by* X is defined to be the intersection of all the ideals that include X.

Lemma 2.5. *Let V be a discriminator variety of normal BAOs and $\mathfrak{A} \in V$.*

(i) *For any $X \subseteq A$, the ideal generated by X is the set*

$$\{x \in A : x \leq \textstyle\sum\{c(y) : y \in Y\} \text{ for some finite } Y \subseteq X\}.$$

(ii) *For any ideal J and any $z \in A$, the ideal generated by $J \cup \{z\}$ is the set*

$$\{x \in A : x \leq y + c(z) \text{ for some } y \in J\}.$$

Proof: We begin with the following observation:

(1) If J is an ideal and Y a finite subset of J, then $\sum\{c(w) : w \in Y\}$ is in J.

Indeed, since the operation c is term-definable, and normal by 2.3(i), J is closed under it (see the remarks preceding 2.4). Because J is also closed under finite sums, by 2.4(ii), we get (1).

Let I be the set displayed in (i).

(2) I is an ideal.

To prove (2) we use Sain's characterization. Since

$$0 = \textstyle\sum\{c(x) : x \in \varnothing\},$$

the element 0 is in I. It is equally easy to verify conditions 2.4(ii),(iii). Therefore, we turn our attention to verifying 2.4(iv). Let f be an extra-Boolean operation of $\mathfrak{A}$, say of rank $n > 0$, and let $x_0, \ldots, x_{n-1}$ be elements of A. Suppose, for a fixed $i < n$, that $x_i \in I$. By definition of I there is a finite set $Y \subseteq X$ such that

$$x_i \leq \textstyle\sum\{c(y) : y \in Y\}.$$

Then

$$\begin{aligned} f(x_0,\dots,x_{n-1}) &\le c(x_i) && \text{by 2.2,}\\ &\le c\big(\textstyle\sum\{c(y):y\in Y\}\big) && \text{by 2.3(ii),}\\ &\le \textstyle\sum\{cc(y):y\in Y\} && \text{by 2.3(vi),}\\ &\le \textstyle\sum\{c(y):y\in Y\} && \text{by 2.3(iii).}\end{aligned}$$

Thus, $f(x_0,\dots,x_{n-1})$ is in I by definition of I, which proves (2).

To prove (i), let J be any ideal including X. Then $I\subseteq J$ by (1) and 2.4(iii), so I is the smallest ideal including X.

Part (ii) follows from (i) (with $J\cup\{z\}$ in place of X), since by (1) each sum of the form $\sum\{c(w):w\in Y\}$, with $Y\subseteq J$, may be replaced by a single element of J. ∎

An ideal of $\mathfrak{A}$ is said to be *trivial* if 0 is its only element, *proper* if it is different from A, and *maximal* if it is proper and there are no other proper ideals that include it. As in the case of rings, the quotient algebra $\mathfrak{A}/I$ of a BAO $\mathfrak{A}$ is simple iff I is maximal.

Lemma 2.6. *Let V be a discriminator variety of normal BAOs and $\mathfrak{A}\in V$. Then an ideal I of $\mathfrak{A}$ is:*

(i) *improper iff, for some (or every) z, both z and $-c(z)$ are in I;*
(ii) *maximal iff, for every z, exactly one of z and $-c(z)$ is in I;*
(iii) *maximal or improper iff, for every z, at least one of z and $-c(z)$ is in I.*

Proof: We begin the implication from left to right in (i). If I is improper, then certainly z and $-c(z)$ are in I for some (and in fact every) z. For the reverse implication, suppose that z and $-c(z)$ are in I. Then $c(z)$ is in I (since c is normal and I is closed under term-definable, normal, unary operations). Therefore, the element $c(z)+-c(z)$, i.e., 1, is in I, so $I=A$.

To establish the implication from left to right in (ii), assume that I is maximal and that $z\notin I$. Set

$$J=\{x\in A: x\le y+-c(z)\text{ for some } y\in I\}.$$

Then J is an ideal by Lemma 2.5(ii) and Corollary 2.3(iv), and it certainly includes I and contains $-c(z)$. To see that J is proper, observe that $z\notin J$. Indeed, $z\cdot -c(z)=0$ since $z\le c(z)$. Hence, if we had $z\in J$, the inequality $z\le y+-c(z)$ would force $z\le y$ for some $y\in I$. But then we would have $z\in I$, a contradiction. Because J is a proper ideal including I, and I is maximal, we conclude that $J=I$ and therefore that $-c(z)\in I$, as desired. We cannot have both z and $-c(z)$ in I by part (i).

To prove the reverse implication, let I be an ideal that contains exactly one of $z,-c(z)$ for every $z\in A$. Suppose that J is an ideal that properly includes I, say z is in J but not in I. Then $-c(z)$ is in I—and hence also in J—by assumption. Therefore, $J=A$ by (i).

Part (iii) is an immediate consequence of (i) and (ii). ∎

We remark in passing that the preceding lemma remains true if we replace "z and $-c(z)$" by "$c(z)$ and $-c(z)$" in the statement.

Recall from Sikorski [1964], p. 37, that a subset X of a Boolean algebra $\mathfrak{B}$ is *dense* in $\mathfrak{B}$ if for every non-zero y in B there is a non-zero x in X such that $x \leq y$. This definition can be carried over without change to an arbitrary BAO $\mathfrak{A}$. It is well-known and easy to check that if X is dense in $\mathfrak{B}$, then every element of $\mathfrak{B}$ is the sum of the elements of X that are below it.

We shall be concerned with the special case of the notion of density when X is a definable subset of $\mathfrak{A}$. In what follows φ will be a formula in some language (not necessarily first-order) appropriate for the algebra $\mathfrak{A}$. It is assumed to have a single free variable that ranges over individuals. Therefore, φ defines a subset $\varphi^{\mathfrak{A}}$ of the universe of $\mathfrak{A}$.

Definition 2.7. (Density) A BAO $\mathfrak{A}$ is said to be *φ-dense* if, for every non-zero y in A, there is a non-zero x in $\varphi^{\mathfrak{A}}$ that is below y. ∎

For a classic example, suppose that φ defines the notion of an atom. Then to say that an algebra is φ-dense just means that the algebra is atomic.

The reader may notice that the notion of φ-density actually makes sense for any algebraic structure with a partial ordering defined on its universe. Also, observe that if $\mathfrak{A}$ is φ-dense, then each element is the sum of the elements in $\varphi^{\mathfrak{A}}$ that are below it.

A problem with notions of density is that they are not necessarily preserved under homomorphisms. (See, for instance, Example 5.23.) Therefore, we cannot automatically conclude that every φ-dense algebra is a subdirect product of simple φ-dense algebras. The next lemma and its consequences, Corollary 2.10 and Theorem 2.11, are the principal tools that we shall use to overcome this difficulty (thereby avoiding the hypothesis of simplicity in some of our representation theorems). For example, in 2.11 we prove that, under suitable conditions on φ, if all the *simple* φ-dense algebras of a variety L are in a subvariety K, then *all* the φ-dense algebras of L are in K.

Lemma 2.8. *Let V be a discriminator variety of normal BAOs and $\mathfrak{A}$ a countable algebra in V. Suppose φ is a formula that is preserved under homomorphisms on $\mathfrak{A}$. If $\mathfrak{A}$ is φ-dense, then for each non-zero x in A there is a maximal ideal I excluding x such that $\mathfrak{A}/I$ is φ-dense.*

Proof: Fix a non-zero x in A, and let $\langle y_n : n \in \omega \rangle$ be an enumeration of the elements of A such that $x = y_0$ (here we use the assumption of countability). We easily construct by induction on n a sequence $\langle z_n : n \in \omega \rangle$ of non-zero elements below x that are in $\varphi^{\mathfrak{A}}$ and have the following properties:

$$z_0 \leq y_0 \tag{1}$$

and either

$$y_n \cdot c(z_n) = 0 \quad \text{and} \quad z_{n+1} = z_n \tag{2}$$

or else

$$y_n \cdot c(z_n) \neq 0 \quad \text{and} \quad z_{n+1} \leq y_n \cdot c(z_n)\,. \tag{3}$$

Indeed, since $x \neq 0$, we can apply the assumption of φ-density to obtain a non-zero z_0 in $\varphi^{\mathfrak{A}}$ that is below $x = y_0$. This verifies (1). Now suppose that z_n has been defined. If $y_n \cdot c(z_n) = 0$, we set $z_{n+1} = z_n$ as desired in (2). Suppose that $y_n \cdot c(z_n) \neq 0$. Then by φ-density there is a non-zero z_{n+1} in $\varphi^{\mathfrak{A}}$ that is below $y_n \cdot c(z_n)$, as called for in (3).

An easy argument by induction on integers, using the properties of a closure operation, shows that

$$0 < c(z_m) \leq c(z_n) \quad \text{whenever} \quad n \leq m\,. \tag{4}$$

Also,

$$y_0 \cdot c(z_n) \neq 0 \quad \text{for every} \quad n \in \omega. \tag{5}$$

To prove (5) observe that

$$0 < c(z_n) \leq c(z_0) \leq c(y_0)\,. \tag{6}$$

by (4) and (1). Therefore

$$\begin{aligned} c(y_0 \cdot c(z_n)) &= c(y_0) \cdot c(z_n) && \text{by 2.3(v),} \\ &= c(z_n) && \text{by (6),} \\ &\neq 0 && \text{by (6).} \end{aligned}$$

Applying Corollary 2.3(i), we conclude that $y_0 \cdot c(z_n) \neq 0$.

Let I be the ideal generated by the set $\{-c(z_n) : n \in \omega\}$. By Lemma 2.5(i) and Corollary 2.3(iv)

$$I = \left\{u \in A : u \leq \sum\{-c(z_n) : n \in Y\} \text{ for some finite } Y \subseteq \omega\right\}. \tag{7}$$

It is easy to check that, in fact,

$$I = \{u \in A : u \leq -c(z_n) \text{ for some } n \in \omega\}\,. \tag{8}$$

Indeed, the inclusion from right to left follows trivially from (7). For the reverse inclusion, suppose that Y is a finite subset of ω and that $u \leq \sum\{-c(z_n) : n \in Y\}$. Set $m = \max(Y)$. Then $c(z_m) \leq c(z_n)$ for each $n \in Y$, by (4), so

$$u \leq \sum\{-c(z_n) : n \in Y\} = -\prod\{c(z_n) : n \in Y\} = -c(z_m)\,.$$

This proves (8).

From (5), (8), and $x = y_0$ we immediately see that

$$x \notin I\,. \tag{9}$$

Before proceeding to the next step, we make some simple observations. Let v be an element of A that is not in I. By (8) we have

$$v \cdot c(z_n) \neq 0 \quad \text{for every} \quad n \in \omega\,. \tag{10}$$

Now $\langle y_n : n \in \omega\rangle$ is an enumeration of A, so for some $k \in \omega$ we have $v = y_k$. Thus, $y_k \cdot c(z_k) \neq 0$, by (10). We conclude from (3) that

$$0 < z_{k+1} \leq y_k \cdot c(z_k)\,. \tag{11}$$

We now show that

(12) $\quad I$ is maximal.

I is proper, by (9). Therefore, by Lemma 2.6(iii) it suffices to fix an arbitrary element v of A that is not in I and to show that $-c(v)$ must be in I. Let k be as above, i.e., $v = y_k$. Using (11) and Corollary 2.3(ii) we have

$$c(z_{k+1}) \leq c(y_k \cdot c(z_k)) \leq c(y_k) = c(v)\,.$$

It follows that $-c(v) \leq -c(z_{k+1})$, so $-c(v)$ is in I by (8). This completes the proof of (12).

It remains to show that the quotient algebra $\mathfrak{A}/I$ is φ-dense. We fix an arbitrary element v of A that is not in I (thus, v/I is an arbitrary non-zero element of $\mathfrak{A}/I$). Again, let k be such that $v = y_k$. Then (11) holds, and z_{k+1} is in $\varphi^{\mathfrak{A}}$ by choice of z_{k+1}. It follows from (11) that z_{k+1}/I is below v/I $(= y_k/I)$, and it is in $\varphi^{\mathfrak{A}/I}$ because φ is preserved under homomorphisms. (This is the unique place where we use the assumption that φ is preserved under homomorphisms.) Clearly, $-c(z_{k+1})$ is in I by (8), and I is proper by (9). Therefore, z_{k+1} is not in I, by Lemma 2.6(i). Thus, z_{k+1}/I is a non-zero element of $\mathfrak{A}/I$. The proof of the lemma is completed. ∎

Remark 2.9. The idea behind the proof of the preceding lemma is the following. We wish to construct a maximal ideal I so that $\mathfrak{A}/I$ is φ-dense. To achieve this we deal with the elements of $\mathfrak{A}$ one at a time. At stage n, if y_n has already been forced into the ideal (the condition $y_n \cdot c(z_n) = 0$ just means that $y_n \leq -c(z_n)$, so that y_n must be put into I), then we do not have to do anything: y_n/I will be zero in the quotient algebra. If y_n has not yet been forced into the ideal, then we force it to be non-zero, and at the same time maintain φ-density, in the quotient algebra by choosing a non-zero witness z_{n+1} in $\varphi^{\mathfrak{A}}$ below y_n and putting $-c(z_{n+1})$ into I to ensure that z_{n+1}/I is non-zero. The condition $z_{n+1} \leq c(z_n)$ ensures the consistency of I, so that it does not become improper. ∎

Corollary 2.10. *Let V be a discriminator variety of normal BAOs and $\mathfrak{A}$ a countable algebra in V. Suppose φ is a formula that is preserved under homomorphisms on $\mathfrak{A}$. If $\mathfrak{A}$ is φ-dense, then it is isomorphic to a subdirect product of simple φ-dense algebras.*

Proof: For each non-zero element x in A, let I_x be a maximal ideal of $\mathfrak{A}$ that excludes x and such that $\mathfrak{A}/I_x$ is φ-dense; such an ideal exists by the previous lemma. Since I_x is maximal, the quotient $\mathfrak{A}/I_x$ is simple. Let g be the canonical homomorphism of $\mathfrak{A}$ into the product $\mathfrak{B}$ of the algebras $\mathfrak{A}/I_x$ with $0 < x \in A$. Thus, g is defined by the stipulation

$$g(y) = \langle y/I_x : 0 < x \in A \rangle$$

for each y in A. For each non-zero y in A, the composition of the y^{th} projection function (from the product $\mathfrak{B}$ to the factor $\mathfrak{A}/I_y$) with g maps $\mathfrak{A}$ *onto* $\mathfrak{A}/I_y$. It remains to check that g is one-one, and for this it suffices to check that the kernel of g is $\{0\}$. Suppose $y \neq 0$. Then $y \notin I_y$ by Lemma 2.8, and therefore y/I_y is non-zero. Since this is one of the coordinates of $g(y)$, we conclude that the latter is non-zero. ∎

Theorem 2.11. *Let L be a discriminator variety of normal Boolean algebras with operators, K a subvariety of L, and φ an elementary, positive formula. If every simple φ-dense algebra of L is in K, then every φ-dense algebra of L is in K.*

Proof: Let $\mathfrak{A}$ be any φ-dense algebra of L. We first treat the case when $\mathfrak{A}$ is countable. In this case, since φ is positive and hence preserved under homomorphisms, we can apply the previous corollary to represent $\mathfrak{A}$ as a subdirect product of simple φ-dense algebras. Each of the latter is in L—since L is a variety—and therefore in K, by assumption. Since K is a variety, we conclude that $\mathfrak{A}$ is also in K.

Now suppose that $\mathfrak{A}$ is uncountable. Since φ is an elementary formula, we see that φ-density is an elementary property and hence is preserved under formation of elementary subalgebras. In particular, every countable elementary subalgebra of $\mathfrak{A}$ is φ-dense. We can now apply the first case to conclude that every countable elementary subalgebra of $\mathfrak{A}$ is in K. But it is a well-known, easy consequence of the Downward Löwenheim-Skolem-Tarski Theorem (see, e.g., Chang-Keisler [1973], Theorem 3.1.6) that every algebra is the directed union of its countable, elementary subalgebras. Thus, $\mathfrak{A}$ is the directed union of algebras in K. It follows that $\mathfrak{A}$ is in K, since K is a variety. ∎

We discuss briefly some extensions of Theorem 2.11. First of all, we do not fully use the hypotheses that L and K are varieties. Our proof still goes through if we assume only that L is closed under homomorphic images and elementary substructures (and generates a discriminator variety), and that K is closed under subalgebras, direct products, and directed unions.

The second extension concerns the assumption that the formula φ is elementary. We only use this assumption in applying the Downward Löwenheim-Skolem-Tarski Theorem. Now the latter theorem is known to hold in more general settings. For example, it holds in countable fragments of the infinitary language $\mathcal{L}_{\omega_1\omega}$ (in which countably infinite conjunctions and disjunctions are allowed). Thus, Theorem 2.11 actually holds when φ is, e.g., a positive formula of $\mathcal{L}_{\omega_1\omega}$.

We can also replace the single formula φ by a countable collection Φ of, e.g., first-order formulas. However, this extension is essentially included in the case when φ is a formula of $\mathcal{L}_{\omega_1\omega}$.

Another extension concerns a weakening of the notion of φ-density. Let φ and ψ be formulas in some language (not necessarily first-order) appropriate for a BAO $\mathfrak{A}$. As before, φ and ψ are assumed to have a single free variable ranging over individuals.

Definition 2.12. $\mathfrak{A}$ is *φ-dense in ψ* if, for every non-zero y in $\psi^{\mathfrak{A}}$, there is a non-zero x in $\varphi^{\mathfrak{A}}$ that is below y.

We get the notion of φ-density by taking ψ to be the formula $x = x$. However, we shall encounter other interesting examples where ψ does not define the entire universe. Clearly, φ-density implies φ-density in ψ for any formula ψ. The converse fails in general, even when ψ satisfies conditions (i)–(iii) of the next lemma.[1]

[1] For example, in the language of relation algebras (see Section 6), let ψ be the formula $x \leq 1'$ expressing the property of being below the identity element 1', and let φ be the formula $x;1;x \leq 1'$ expressing the property of being either 0 or a singleton below the identity. It is easy to check that ψ satisfies conditons (i)–(iii) of Theorem 2.15. Full set relation algebras over sets of cardinality at least 2 are φ-dense in ψ. However, they are certainly not φ-dense, since not every non-zero

Lemma 2.13. *Let V be a discriminator variety of normal BAOs and $\mathfrak{A}$ a countable algebra in V. Suppose φ is a formula that is preserved under homomorphisms on $\mathfrak{A}$ and suppose ψ is a formula with the following properties:*

(i) *ψ is strictly preserved under homomorphisms on $\mathfrak{A}$;*
(ii) *If y is in $\psi^{\mathfrak{A}}$, then for each non-zero x in A the element $y \cdot c(x)$ is in $\psi^{\mathfrak{A}}$;*
(iii) *For each non-zero x in A there is a y in $\psi^{\mathfrak{A}}$ such that $c(x) = c(y)$.*

If $\mathfrak{A}$ is φ-dense in ψ, then for each non-zero x in A there is a maximal ideal I excluding x such that $\mathfrak{A}/I$ is φ-dense in ψ.

Proof: We proceed as in the proof of Lemma 2.8, and we continue its numeration scheme. Fix a non-zero x in A, and let $\langle y_n : n \in \omega \rangle$ be an enumeration of the elements of $\psi^{\mathfrak{A}}$ such that $c(x) = c(y_0)$. Such an enumeration is possible because $\mathfrak{A}$ is countable and because ψ satisfies condition (iii). We construct a sequence $\langle z_n : n \in \omega \rangle$ of non-zero elements in $\varphi^{\mathfrak{A}}$ that are below x and that satisfy properties (1)–(3). Indeed, since $x \neq 0$ and $c(x) = c(y_0)$, we have $c(y_0) \neq 0$ and hence $y_0 \neq 0$. Using the assumption of φ-density in ψ we obtain a non-zero z_0 in $\varphi^{\mathfrak{A}}$ that is below y_0. Now suppose that z_n has been defined. If $y_n \cdot c(z_n) = 0$, we set $z_{n+1} = z_n$. If $y_n \cdot c(z_n) \neq 0$, then $y_n \cdot c(z_n)$ is a non-zero element of $\psi^{\mathfrak{A}}$ by condition (ii). Hence, by φ-density in ψ there is a non-zero z_{n+1} in $\varphi^{\mathfrak{A}}$ that is below $y_n \cdot c(z_n)$.

The proofs of (4)–(8) are exactly as before. To proceed, suppose that v is a non-zero element of A. By condition (iii) there is a $k \in \omega$ such that $c(v) = c(y_k)$. Using 2.3(vii) we see that

$$v \cdot c(z_n) \neq 0 \quad \text{iff} \quad z_n \cdot c(v) \neq 0 \quad \text{iff} \quad z_n \cdot c(y_k) \neq 0 \quad \text{iff} \quad y_k \cdot c(z_n) \neq 0 .$$

Thus,

$$v \cdot c(z_n) \neq 0 \quad \text{iff} \quad y_k \cdot c(z_n) \neq 0 . \tag{13}$$

Taking $v = x$ and $k = 0$ in (13) we obtain (9) from (5) and (8). The proofs of (11) and (12) are as before except that we choose k so that $c(v) = c(y_k)$ and we use (13) and (10) to obtain $y_k \cdot c(z_k) \neq 0$.

It remains to show that $\mathfrak{A}/I$ is φ-dense in ψ. Let w/I be a non-zero element of $\psi^{\mathfrak{A}/I}$. Since ψ satisfies condition (i) (see the last paragraph of the preliminaries for the relevant definition), there is a v in $\psi^{\mathfrak{A}}$ such that $v/I = w/I$. In particular, $v/I \neq 0$ and therefore $v \notin I$. Choose a k such that $v = y_k$; this is possible because $\langle y_n : n \in \omega \rangle$ enumerates $\psi^{\mathfrak{A}}$. The remainder of the proof is as in 2.8. ∎

Corollary 2.14. *Under the hypotheses of 2.13, if $\mathfrak{A}$ is φ-dense in ψ, then $\mathfrak{A}$ is isomorphic to a subdirect product of simple algebras that are φ-dense in ψ.*

The proof is identical to the proof of Corollary 2.10.

Theorem 2.15. *Let L be a discriminator variety of normal Boolean algebras with operators, K a subvariety of L, φ an elementary, positive formula, ψ an elementary formula, and $L_{\varphi\psi}$ the class of algebras of L that are φ-dense in ψ. Suppose that ψ satisfies the following conditions:*

(i) *ψ is strictly preserved under homomorphisms,*

---

element is above a non-zero element below the identity.

(ii) $\forall x \forall y[\psi(y) \rightarrow \psi(y \cdot c(x))]$ *holds in all algebras of* $L_{\varphi\psi}$,
(iii) $\forall x \exists y[\psi(y) \wedge (c(x) = c(y))]$ *holds in all algebras of* $L_{\varphi\psi}$.

If every simple algebra of $L_{\varphi\psi}$ *is in* K, *then every algebra of* $L_{\varphi\psi}$ *is in* K.

The proof is nearly identical to the proof of Theorem 2.11. We leave the details to the reader.

In general, in a product of Boolean algebras with operators, a sequence of atoms and zeros is no longer an atom. Still, such a sequence possesses many of the important properties of an atom. The definition of a quasi-atom is intended to capture these properties.

Definition 2.16. Suppose that $\mathfrak{A}$ is in a discriminator variety of normal BAOs.

(i) An element x is a *quasi-atom* if for every $y \leq x$ we have $y = x \cdot c(y)$.
(ii) $\mathfrak{A}$ is *quasi-atomic* if every non-zero element is above a non-zero quasi-atom.

∎

Notice that quasi-atomicity is a notion of density. The following lemma is an easy consequence of the definition, so we leave the proof to the reader.

Lemma 2.17. *Suppose that* $\mathfrak{A}$ *is in a discriminator variety of normal BAOs.*

(i) *An element* x *is a quasi-atom iff* $x \cdot y = x \cdot c(x \cdot y)$ *for all* y .
(ii) *If* $\mathfrak{A}$ *is simple, then* x *is a quasi-atom iff* $x = 0$ *or* x *is an atom. Therefore simple, quasi-atomic algebras are atomic.*

The next corollary shows that, in the case of countable algebras, the intuition behind the definition of a quasi-atom corresponds to the actual situation.

Corollary 2.18. *Let* V *be a discriminator variety of normal BAOs and* $\mathfrak{A}$ *a countable quasi-atomic algebra in* V. *Then* $\mathfrak{A}$ *is isomorphic to a subdirect product of simple, atomic algebras. Moreover, the isomorphism takes each quasi-atom of* $\mathfrak{A}$ *to a sequence in the product consisting of zeros and atoms.*

Proof: Let φ be the formula

$$\forall y[x \cdot y = x \cdot c(x \cdot y)]\,.$$

By part (i) of the preceding lemma, quasi-atomicity is the same thing as φ-density. Therefore, the first assertion of the corollary follows at once from Corollary 2.10 and part (ii) of the preceding lemma. Let f be an embedding of $\mathfrak{A}$ into a direct product $\mathfrak{B} = \prod_{\xi<\Xi}\mathfrak{B}_\xi$ of simple, atomic algebras $\mathfrak{B}_\xi$ such that $p_\xi \circ f$ maps $\mathfrak{A}$ onto $\mathfrak{B}_\xi$ (where p_ξ is the projection function from $\mathfrak{B}$ to $\mathfrak{B}_\xi$). Suppose that x is a quasi-atom in $\mathfrak{A}$. Since φ is preserved under "onto" homomorphisms, the coordinate $p_\xi \circ f(x)$ of the sequence $f(x)$ must be a quasi-atom and therefore either 0 or an atom by 2.17(ii). ∎

Problem 2.19. Can the hypothesis of countability be removed from Lemma 2.8 or at least from Corollaries 2.10 and 2.18?

Just as with φ-density, one can specialize the notion of a quasi-atom. Let ψ be a formula with one free variable. An element x of $\mathfrak{A}$ is a *quasi-atom in* $\psi^{\mathfrak{A}}$ provided that $x \in \psi^{\mathfrak{A}}$ and for every $y \in \psi^{\mathfrak{A}}$, if $y \leq x$, then $y = x \cdot c(y)$. We shall sometimes use this notion to simplify the statement of certain lemmas.

The next two examples are intended to illustrate the notion of quasi-atomicity. They show, in particular, that a quasi-atomic algebra need not have any atoms.

Example 2.20. Consider a variety V of normal BAOs in which each extra-Boolean operation f is (uniformly) definable by a term τ_f that involves only the Boolean operations; in other words, V is a term definitional extension of BA. The two element Boolean algebra $\mathfrak{A}$, expanded by the extra-Boolean operations $\tau_f^{\mathfrak{A}}$, is the unique subdirectly irreducible algebra in V. Since the extra Boolean operations are assumed to be normal, it is easy to check that the identity function is a unary discriminator in $\mathfrak{A}$. Thus, the term $c(x)$ given by $c(x) = x$ is a unary discriminator for V.

Let $\mathfrak{B}$ be any algebra of V. From the definition of c we see that an element x in $\mathfrak{B}$ is a quasi-atom iff

$$x \cdot y = x \cdot (x \cdot y)$$

for all $y \in B$. But every element x satisfies this formula. Thus, every element of $\mathfrak{B}$ is a quasi-atom. It follows that all algebras in V, including atomless algebras, are quasi-atomic.

Examples of such varieties include the variety of discrete cylindric algebras of a specified dimension and the variety of Boolean relation algebras. ∎

Example 2.21. For each $n, k \in \omega$ with $k < 2^n$ let $N_{n,k}$ be the set of natural numbers that are congruent to k modulo 2^n. For example,

$$N_{1,0} = \{2m : m \in \omega\} \qquad , \qquad N_{1,1} = \{2m+1 : m \in \omega\}\,,$$

and

$$\begin{aligned} N_{2,0} &= \{4m : m \in \omega\} \qquad , \qquad & N_{2,1} &= \{4m+1 : m \in \omega\}\,, \\ N_{2,2} &= \{4m+2 : m \in \omega\} \qquad , \qquad & N_{2,3} &= \{4m+3 : m \in \omega\}\,. \end{aligned}$$

For each n the sets $N_{n,k}$ with $k < 2^n$ form a partition of ω into 2^n many infinite sets. Moreover, $N_{n+1,k}$ and $N_{n+1,2^n+k}$ partition $N_{n,k}$ into two infinite sets.

Fix a normal BAO $\mathfrak{C}$ with a unary discriminator. Notice that the quasi-atoms of ${}^{\omega}\mathfrak{C}$ (the ω^{th} direct power of $\mathfrak{C}$) are just the functions f from ω to C such that each value of f is either an atom of $\mathfrak{C}$ or 0. We construct a subalgebra $\mathfrak{A}$ of ${}^{\omega}\mathfrak{C}$ as follows. For each $n \in \omega$, define B_n to be the set of functions f in ${}^{\omega}\mathfrak{C}$ such that f is constant on $N_{n,k}$ for each $k < 2^n$. For example, B_0 consists of the functions from ω to C that are constant, i.e., that assume just one value, and B_1 consists of the functions from ω to C that assume at most two values, one on the even numbers and one on the odd numbers. It is easy to check that B_n is a subuniverse of ${}^{\omega}\mathfrak{C}$ and that the mapping $f \longmapsto f \restriction 2^n$ (the restriction of f to 2^n) is an isomorphism from the corresponding subalgebra $\mathfrak{B}_n$ to ${}^{2^n}\mathfrak{C}$. Thus, the atoms of $\mathfrak{B}_n$ are the functions f such that, for some $k < 2^n$, the value of f on $N_{n,k}$ is an atom and its value off of $N_{n,k}$ is 0. In particular, each such atom is a non-zero quasi-atom of ${}^{\omega}\mathfrak{C}$. Also, $\mathfrak{B}_n$ is a subalgebra of $\mathfrak{B}_{n+1}$, and each non-zero element of $\mathfrak{B}_n$ gets split into non-zero pieces in $\mathfrak{B}_{n+1}$. In fact, for each non-zero f in $\mathfrak{B}_n$ define f_0 and f_1 in $\mathfrak{B}_{n+1}$ by specifying their values on each set $N_{n,k}$ as follows:

$$f_0(m) = \begin{cases} f(m) & \text{for } m \in N_{n+1,k}\,, \\ 0 & \text{for } m \in N_{n+1,2^n+k}\,, \end{cases}$$

and

$$f_1(m) = \begin{cases} 0 & \text{for } m \in N_{n+1,k}\,, \\ f(m) & \text{for } m \in N_{n+1,2^n+k}\,. \end{cases}$$

Then f_0 and f_1 are non-zero, pairwise disjoint, and sum to f. Moreover, if f is a quasi-atom of ${}^{\omega}\mathfrak{C}$, then the same is true of f_0 and f_1.

Let $\mathfrak{A}$ be the union of the chain $\{\mathfrak{B}_n : n \in \omega\}$. Then $\mathfrak{A}$ is a subalgebra of ${}^{\omega}\mathfrak{C}$ that is atomless. In fact, as we have just seen, every non-zero element of $\mathfrak{A}$ gets split into two disjoint, non-zero elements.

Suppose that $\mathfrak{C}$ is atomic. Then each $\mathfrak{B}_n$ is certainly atomic (because it is isomorphic to ${}^{2^n}\mathfrak{C}$). If g is any non-zero element of $\mathfrak{A}$, then g is in B_n for some n. Therefore, there is an atom of $\mathfrak{B}_n$ that is below g. We have seen that such an atom is a non-zero quasi-atom of ${}^{\omega}\mathfrak{C}$ and hence also of $\mathfrak{A}$. Thus, in addition to being atomless, $\mathfrak{A}$ is a quasi-atomic algebra. (See Givant [1994], Example 7.11, for a related construction. Notice that if $\mathfrak{C}$ is a finite Boolean algebra, then $\mathfrak{A}$ is just a countable, atomless Boolean algebra.) ∎

3. Rectangularly dense cylindric algebras

We now turn our attention to the representation theorem for rectangularly dense cylindric algebras. Throughout this and the following section we fix an ordinal α. We begin by recalling from Henkin-Monk-Tarski [1971] the definition of a cylindric algebra of dimension α.

Definition 3.1. A *cylindric algebra of dimension* α is an algebra

$$\mathfrak{A} = \langle A\,,\, +\,,\, -\,,\, c_i\,,\, d_{ij}\rangle_{i,j\in\alpha}$$

such that $\langle A\,,\, +\,,\, -\rangle$ is a Boolean algebra, d_{ij} is a constant and c_i is a unary operation for each $i, j \in \alpha$, and the following postulates are satisfied for all $x, y \in A$ and all $i, j, k \in \alpha$:

(C1) $c_i(0) = 0$,
(C2) $x \leq c_i(x)$,
(C3) $c_i(x \cdot c_i(y)) = c_i(x) \cdot c_i(y)$,
(C4) $c_i c_j(x) = c_j c_i(x)$,
(C5) $d_{ii} = 1$,
(C6) $d_{jk} = c_i(d_{ji} \cdot d_{ik})$ whenever $i \neq j, k$,
(C7) $c_i(d_{ij} \cdot x) \cdot c_i(d_{ij} \cdot -x) = 0$ whenever $i \neq j$.

The class of all cylindric algebras of dimension α is denoted by CA_α. We also use the notation "CA_α" as an abbreviation for the phrase "cylindric algebra of dimension α". The element d_{ij} is called the ij^{th} *diagonal* element, and the operation c_i is called the i^{th} *cylindrification*. ∎

To give an example of a CA_α, let U be any set and define $\mathfrak{C}_\alpha(U)$ to be the algebra

$$\langle Sb(^\alpha U)\,,\,\cup\,,\,\sim\,,\,C_i\,,\,D_{ij}\rangle_{i,j<\alpha}\,,$$

where $Sb(^\alpha U)$ is the collection of all subsets of $^\alpha U$, $\cup$ and $\sim$ are the set-theoretic operations of union and complementation with respect to $^\alpha U$, and for each $X \subseteq {}^\alpha U$ and each $i, j \in \alpha$ we have

$$\begin{aligned} C_i(X) &= \{u \in {}^\alpha U : \text{there is an } x \in X \text{ with } u_k = x_k \text{ for all } k \in \alpha \sim \{i\}\}\,, \\ D_{ij} &= \{u \in {}^\alpha U : u_i = u_j\}\,. \end{aligned}$$

It is easy to check that $\mathfrak{C}_\alpha(U)$ is a cylindric algebra. It is called the *full cylindric set algebra over U of dimension α.*

A CA_α is said to be *representable* if it can be embedded into a direct product of full cylindric set algebras of dimension α. The class of all such representable algebras is denoted by RCA_α. It is well known that RCA_α is a subvariety of CA_α (see Henkin-Monk-Tarski [1971], p. 171, and [1985], Corollary 3.1.108). For $\alpha = 0, 1$ the two varieties coincide, i.e., every cylindric algebra of dimension 0 or 1 is representable (see Henkin-Monk-Tarski [1971], p. 171). However, for $\alpha \geq 2$ the two varieties are distinct, i.e., there are cylindric algebras in every dimension ≥ 2 that are not representable. In fact, for $\alpha > 2$, RCA_α is not even finitely axiomatizable (and not even finitely schematizable when $\alpha \geq \omega$) over CA_α (see Henkin-Monk-Tarski [1985], Theorems 4.1.3 and 4.1.7).[2] Thus, the problem of finding simple criteria that imply the representability of a cylindric algebra becomes interesting and important.

Suppose that $\mathfrak{A}$ is a CA_α and that $\beta < \alpha$. By the *β-reduct* of $\mathfrak{A}$ we mean the algebra

$$\langle A\,,\,+\,,\,-\,,\,c_i\,,\,d_{ij}\rangle_{i,j\in\beta}\,.$$

It is easily seen to be a CA_β .

Let $N_\beta(U)$ be the set of elements X in $\mathfrak{C}_\alpha(U)$ such that for all $x \in X$ and all $y \in {}^\alpha U$, if y agrees with x on β, then $y \in X$. Thus, X is in $N_\beta(U)$ iff there is a set $Z \subseteq {}^\beta U$ such that X has the form

$$X = Z \times \underbrace{U \times U \times U \times \ldots}_{\alpha-\beta \text{ times}} \quad .$$

The set $N_\beta(U)$ is easily seen to be a subuniverse of the β-reduct of $\mathfrak{C}_\alpha(U)$. Let $\mathfrak{N}_\beta(U)$ be the corresponding subalgebra. It is not difficult to verify the well-known fact that $\mathfrak{C}_\beta(U)$ is canonically isomorphic to $\mathfrak{N}_\beta(U)$. From this it is a simple matter to check that every equation in the language of CA_β that holds in RCA_α must also hold in RCA_β . Indeed if ε is such an equation holding in RCA_α , then ε must hold in $\mathfrak{C}_\alpha(U)$ for every set U. Since $\mathfrak{C}_\beta(U)$ is isomorphic to a subalgebra of the β-reduct of $\mathfrak{C}_\alpha(U)$, we see that ε must hold in $\mathfrak{C}_\beta(U)$ for every set U. But every algebra in RCA_β is isomorphic to a subalgebra of a direct product of such full cylindric set algebras. Therefore ε must hold RCA_β . The converse is also true, but its proof is more involved; see Henkin-Monk-Tarski [1985], Theorem 3.1.126 or 3.1.127.

[2] These negative results were considerably strengthened in Andréka [a]. For example, no set of equations, whether finite or infinite, that involves only a finite number of variables can axiomatize RCA_α . For $\alpha < \omega$ this remains true even if we add an arbitrary number of unary operators or generalized quantifiers of arbitrary rank.

In the sequel, we will use the following abbreviations. For a finite subset

$$\Gamma = \{i_0, \dots, i_{n-1}\}$$

of α with $i_0 < \cdots < i_{n-1}$, we write $c_{(\Gamma)}(x)$ for $c_{i_0} \dots c_{i_{n-1}}(x)$. We make the convention that $c_{(\varnothing)}(x) = x$. For a finite dimension α we set

$$d = \prod\{d_{ij} : i < j < \alpha\}.$$

The element d is called the *main diagonal* and an element below d is called a *subdiagonal* element. An element x is said to be *closed under* $c_{(\Gamma)}$, or $c_{(\Gamma)}$-*closed*, if $c_{(\Gamma)}(x) = x$.

In the next lemma we list some simple, easy, and well-known laws regarding cylindric algebras that we shall need. The derivations of these laws can be found in Henkin-Monk-Tarski [1971], Chapter 1, Sections 2–3.

Lemma 3.2. *Let* $\mathfrak{A}$ *be a* CA_α *and* x, y *elements of* $\mathfrak{A}$.

(i) $c_i c_i(x) = c_i(x)$.
(ii) *If* $x \leq y$, *then* $c_i(x) \leq c_i(y)$.
(iii) $c_i(x + y) = c_i(x) + c_i(y)$.
(iv) $c_i(-c_i(x)) = -c_i(x)$.
(v) $c_i(x \cdot y) = x \cdot y$ *whenever* $c_i(x) = x$ *and* $c_i(y) = y$.
(vi) $c_i(d_{ij}) = 1$.
(vii) $c_i(d_{jk}) = d_{jk}$ *when* $i \neq j, k$.
(viii) $d_{ij} \cdot d_{jk} = d_{ij} \cdot d_{ik}$.
(ix) $d_{ij} \cdot c_i(x) = 0$ *iff* $x = 0$.
(x) $d_{ij} \cdot c_i(d_{ij} \cdot x) = d_{ij} \cdot x$ *when* $i \neq j$.
(xi) $x \cdot c_i(y) = 0$ *iff* $y \cdot c_i(x) = 0$.

From postulates (C1), (C2), (C4), Lemma 3.2(iii),(iv), and Corollary 2.2 it is seen that CA_α is a variety of normal Boolean algebras with complemented closure operations c_i, and for finite α, the term $c_{(\alpha)}$ is a unary discriminator for CA_α. Thus, for finite α, CA_α is a discriminator variety of normal BAOs, the subdirectly irreducible algebras of CA_α are simple, and a conditional equation holds in CA_α iff it holds in all simple algebras of CA_α.

Certain generalizations of the postulates and of the laws in Lemma 3.2 are completely obvious. For example, for any finite $\Delta \subseteq \alpha$ and any finite set J we have the following generalizations of (C3) and Lemma 3.2(v):

$$c_{(\Delta)}(x \cdot y) = c_{(\Delta)}(x) \cdot y \quad \text{whenever } c_i(y) = y \text{ for each } i \in \Delta,$$
$$c_{(\Delta)}(\textstyle\prod_{j \in J} y_j) = \prod_{i \in J} y_j \quad \text{whenever } c_i(y_j) = y_j \text{ for each } i \in \Delta \text{ and } j \in J.$$

We shall use such generalizations without further explanation, referring only to the corresponding postulate or law. The three generalizations that we formulate in Lemma 3.3 below concern products of diagonal elements and are slightly more involved. The first generalizes 3.2(viii) and the second 3.2(x). The third combines and generalizes (C6) and 3.2(vi),(vii). The three generalizations are stated and proved in Henkin-Monk-Tarski [1971], Theorems 1.8.5, 1.8.12(iv), and 1.8.6 respectively.

Lemma 3.3. *Let* $\mathfrak{A}$ *be a* CA_α *and* Δ, Γ *finite subsets of* α.

(i) $\prod_{i\in\Gamma} d_{ik} = \prod_{i,j\in\Gamma\cup\{k\}} d_{ij}$.
(ii) $c_{(\Delta)}(x)\cdot\prod_{i\in\Delta} d_{ik} = x$ *whenever* $k \in \alpha \sim \Delta$ *and* $x \leq \prod_{i\in\Delta} d_{ik}$.
(iii) $c_{(\Delta)}(\prod_{i,j\in\Gamma} d_{ij}) = \prod_{i,j\in\Gamma\sim\Delta} d_{ij}$.

Recall that d denotes the main diagonal of a finite-dimensional CA_α.

Lemma 3.4. *Let α be finite and suppose that x is a non-zero element of a CA_α. Then $d\cdot c_{(\alpha\sim\{i\})}(x) \neq 0$.*

Proof: By Lemma 3.3(iii) (with $\Gamma = \alpha$ and $\Delta = \alpha \sim \{i\}$) and (C5) we have

$$c_{(\alpha\sim\{i\})}(d) = d_{ii} = 1\,.$$

Therefore, $x\cdot c_{(\alpha\sim\{i\})}(d) \neq 0$. The desired conclusion follows from Lemma 3.2(xi). ∎

Lemma 3.5. *Suppose α is finite and $\mathfrak{A}$ a CA_α. Then for each sequence $\langle x_i : i \in \alpha\rangle$ of elements of $\mathfrak{A}$ and each $\Delta \subseteq \alpha$ we have*

(i) $c_{(\Delta)}\big(\prod_{i\in\alpha} c_{(\alpha\sim\{i\})}(x_i)\big) = \prod_{i\in\alpha\sim\Delta} c_{(\alpha\sim\{i\})}(x_i)\cdot\prod_{i\in\Delta} c_{(\alpha)}(x_i)$.

If $\mathfrak{A}$ is simple and the elements x_i non-zero, then

(ii) $c_{(\Delta)}\big(\prod_{i\in\alpha} c_{(\alpha\sim\{i\})}(x_i)\big) = \prod_{i\in\alpha\sim\Delta} c_{(\alpha\sim\{i\})}(x_i)$.

Proof: The proof is by induction on the size of Δ. When Δ is empty, the equation holds trivially. Suppose now that the equation holds whenever the set has size n, and assume that Δ has $n+1$ elements, say j is the smallest of these. For each i in α set

$$y_i = c_{(\alpha\sim\{i\})}(x_i)\,. \tag{1}$$

Observe that

$$c_j(y_i) = y_i \quad \text{for } j \neq i \tag{2}$$

by (1), (C4), and Lemma 3.2(i), and

$$c_i(y_i) = c_{(\alpha)}(x_i) \tag{3}$$

by (1). Therefore

$$\begin{aligned}
c_{(\Delta)}\big(\textstyle\prod_{i\in\alpha} y_i\big) &= c_j c_{(\Delta\sim\{j\})}\big(\textstyle\prod_{i\in\alpha} y_i\big) && \text{by convention,}\\
&= c_j\big[\big(\textstyle\prod_{i\in\alpha\sim(\Delta\sim\{j\})} y_i\big)\cdot\big(\textstyle\prod_{i\in\Delta\sim\{j\}} c_i(y_i)\big)\big] && \text{by the induction hypothesis,}\\
&= c_j(y_j)\cdot\textstyle\prod_{i\in\alpha\sim\Delta} y_i\cdot\prod_{i\in\Delta\sim\{j\}} c_i(y_i) && \text{by (2) and (C3),}\\
&= \textstyle\prod_{i\in\alpha\sim\Delta} y_i\cdot\prod_{i\in\Delta} c_i(y_i) && \text{by BA.}
\end{aligned}$$

In view of (1) and (3), this completes the proof of (i). Part (ii) follows at once from (i). ∎

Notice, in the statement of the previous lemma, that when $\Delta = \alpha$, the right-hand side of the displayed equation in (ii) is just 1.

Definition 3.6. Let $\mathfrak{A}$ be a CA_α.

(i) An element a of $\mathfrak{A}$ is said to be *rectangular* or a *rectangle* if

$$c_{(\Gamma)}(a) \cdot c_{(\Delta)}(a) = c_{(\Gamma\cap\Delta)}(a)$$

for all finite subsets Γ and Δ of α.

(ii) $\mathfrak{A}$ is *rectangularly dense* if below every non-zero element there is a non-zero rectangle.

▮

The notion of a rectangular element is from Henkin-Monk-Tarski [1971], Definition 1.10.6. In two-dimensional cylindric set algebras over a set U, rectangles are elements that have the form $X \times Y$ for some subsets X, Y of U (see the next lemma and Figure 1). In the Cartesian plane $U \times U$ such elements have the appearance of rectangles in the ordinary sense of the word (provided that the elements of U are geometrically represented in a suitable way). More generally, for finite α, rectangles in an α-dimensional cylindric set algebra are elements of the form $X_0 \times \cdots \times X_{\alpha-1}$ for some subsets $X_0, \ldots, X_{\alpha-1}$ of U. If X_i is a singleton for each $i < \alpha$, say $X_i = \{x_i\}$, then the rectangle $X_0 \times \cdots \times X_{\alpha-1}$ is the singleton $\{(x_0, \ldots, x_{\alpha-1})\}$. Thus, every singleton is a rectangle. In particular, $\mathfrak{C}_\alpha(U)$ is rectangularly dense, since every non-zero set is above a singleton. When $\alpha = 0, 1$ each element in a CA_α is trivially a rectangle; therefore, all cylindric algebras of dimension 0 or 1 are rectangularly dense.

FIGURE 1. A rectangle $X \times Y$.

Observe that, for finite α, the notion of rectangularity is definable by a finite conjunction of equations. Therefore, it is an elementary notion that is preserved under homomorphisms. Notice also that in a rectangularly dense algebra, every element is the sum of the rectangles below it.

Lemma 3.7. *Let α be finite and $\mathfrak{A}$ a CA_α. An element z is a rectangle iff there are $x_0, \ldots, x_{\alpha-1}$ such that $z = \prod_{i\in\alpha} c_{(\alpha\sim\{i\})}(x_i)$.*

Proof: For the implication from left to right, just set $x_i = z$ for each $i \in \alpha$ and use the definition of a rectangle:

$$\prod_{i\in\alpha} c_{(\alpha\sim\{i\})}(z) = c_{(\varnothing)}(z) = z\,.$$

For the reverse implication, observe that it can be expressed as a series of conditional equations. Since a conditional equation holds in all CA_α iff it holds in all simple CA_α, it suffices to verify the implication under the additional hypothesis that $\mathfrak{A}$ is simple. Set

$$y_i = c_{(\alpha\sim\{i\})}(x_i) \quad , \tag{1}$$

and suppose that

$$z = \prod_{i\in\alpha} y_i\,. \tag{2}$$

Let Γ and Δ be arbitrary subsets of α. Then

$$\begin{aligned}
c_{(\Gamma)}(z)\cdot c_{(\Delta)}(z) &= c_{(\Gamma)}\big(\textstyle\prod_{i\in\alpha} y_i\big)\cdot c_{(\Delta)}\big(\textstyle\prod_{i\in\alpha} y_i\big) && \text{by (2),}\\
&= \textstyle\prod_{i\in\alpha\sim\Gamma} y_i\cdot\prod_{i\in\alpha\sim\Delta} y_i && \text{by (1), 3.5(ii),}\\
&= \textstyle\prod_{i\in\alpha\sim(\Gamma\cap\Delta)} y_i && \text{by BA,}\\
&= c_{(\Gamma\cap\Delta)}\big(\textstyle\prod_{i\in\alpha} y_i\big) && \text{by (1), 3.5(ii),}\\
&= c_{(\Gamma\cap\Delta)}(z) && \text{by (2).}
\end{aligned}$$

Thus, z is a rectangle by Definition 3.6(i). ∎

The preceding lemma is closely related to Theorem 1.10.11 in Henkin-Monk-Tarski [1971]. Indeed, the equivalence of (i) and (iv) in that theorem is just the special case of our lemma when

$$x_0 = x_1 = \cdots = x_{n-1} = z\,.$$

On the other hand, let y_i be as defined in (1) of the preceding proof, and suppose that (2) holds. Then by (1) and Lemma 3.5(ii), using also the simplicity of $\mathfrak{A}$, we have

$$c_{(\alpha\sim\{i\})}(z) = c_{(\alpha\sim\{i\})}\big(\textstyle\prod_{j\in\alpha} y_j\big) = c_{(\alpha\sim\{i\})}\big(\textstyle\prod_{j\in\alpha} c_{(\alpha\sim\{j\})}(x_j)\big) = c_{(\alpha\sim\{i\})}(x_i) = y_i\,.$$

Therefore, our lemma also follows immediately from the theorem in Henkin-Monk-Tarski [1971].

Part (ii) of the next lemma occurs as Lemma 1.10.13(ii) in Henkin-Monk-Tarski [1971].

Lemma 3.8. *Suppose that $2 \le \alpha < \omega$.*

(i) *In a CA_α every rectangle below the main diagonal is a quasi-atom.*
(ii) *In a simple CA_α every non-zero rectangle below the main diagonal is an atom.*

Proof: To prove (i) suppose that x is a non-zero rectangle below d, and let y be below x. Since $x \cdot -y$ is below x, it must be below d. Therefore,

$$(1)\qquad d \cdot c_{(\alpha\sim\{0\})}(x \cdot -y) = x \cdot -y \quad\text{and}\quad d \cdot c_{(\alpha\sim\{i\})}(y) = y \quad \text{for } 0 < i < \alpha,$$

by Lemma 3.2(i),(ii). Set

$$(2)\qquad a = c_{(\alpha\sim\{0\})}(x \cdot -y) \cdot \textstyle\prod_{0<i<\alpha} c_{(\alpha\sim\{i\})}(y)\,.$$

Then

$$\begin{aligned} a &\leq c_{(\alpha\sim\{0\})}(x) \cdot \textstyle\prod_{0<i<\alpha} c_{(\alpha\sim\{i\})}(x) && \text{by (2) and monotony,}\\ &= x && \text{since } x \text{ is a rectangle,}\\ &\leq d && \text{by assumption.} \end{aligned}$$

Therefore,

$$\begin{aligned} a &= d \cdot a\\ &= d \cdot c_{(\alpha\sim\{0\})}(x \cdot -y) \cdot \textstyle\prod_{0<i<\alpha} c_{(\alpha\sim\{i\})}(y) && \text{by (2),}\\ &= (x \cdot -y) \cdot y && \text{by (1),}\\ &= 0 && \text{by BA.} \end{aligned}$$

We have shown that

$$c_{(\alpha\sim\{0\})}(x \cdot -y) \cdot \textstyle\prod_{0<i<\alpha} c_{(\alpha\sim\{i\})}(y) = 0\,.$$

Consequently,

$$\begin{aligned} 0 &= c_0 \ldots c_{\alpha-1}\big[c_{(\alpha\sim\{0\})}(x \cdot -y) \cdot \textstyle\prod_{0<i<\alpha} c_{(\alpha\sim\{i\})}(y)\big] && \text{by (C1),}\\ &= c_0 c_{(\alpha\sim\{0\})}(x \cdot -y) \cdot \textstyle\prod_{0<i<\alpha} c_i c_{(\alpha\sim\{i\})}(y) && \text{by (C3), 3.2(i),(v),}\\ &= c_{(\alpha)}(x \cdot -y) \cdot \textstyle\prod_{0<i<\alpha} c_{(\alpha)}(y) && \text{by definition of } c_{(\alpha)},\\ &= c_{(\alpha)}(x \cdot -y) \cdot c_{(\alpha)}(y) && \text{by Boolean algebra.} \end{aligned}$$

Applying 3.2(xi),(i) α times we conclude that

$$x \cdot -y \cdot c_{(\alpha)}(y) = 0\,.$$

Thus, $x \cdot c_{(\alpha)}(y) \leq y$. The reverse inclusion follows from the assumption that $y \leq x$ and from postulate (C2). This proves part (i).

Part (ii) follows from part (i) and Lemma 2.17(ii). ∎

Remark 3.9. The converse to 3.8 is false: there can be quasi-atoms (and even atoms) below the main diagonal that are not rectangles. In fact, let U be a set of cardinality at least 2 and let $\mathfrak{A}$ be the subalgebra of $\mathfrak{C}_\alpha(U)$ consisting of constants. Then d is itself an atom of $\mathfrak{A}$, but it is not a rectangle. ∎

Lemma 3.10. *Suppose that $2 \leq \alpha < \omega$. Then every rectangularly dense, simple CA_α is atomic.*

Proof: Suppose that $\mathfrak{A}$ is a rectangularly dense, simple CA_α. Let x be a non-zero element of $\mathfrak{A}$. We shall construct an atom below x. Since $\mathfrak{A}$ is rectangularly dense, there is a non-zero rectangle y below x. Then for each $i \in \alpha$ we have $d \cdot c_{(\alpha\sim\{i\})}(y) \neq 0$, by Lemma 3.4. Thus, we can choose a non-zero rectangle z_i below $d \cdot c_{(\alpha\sim\{i\})}(y)$. By the previous lemma and the simplicity of $\mathfrak{A}$ we have

(1) z_i is an atom.

Set

(2) $$w = \prod_{i\in\alpha} c_{(\alpha\sim\{i\})}(z_i)\,.$$

(See Figure 2.) Because $z_i \leq c_{(\alpha\sim\{i\})}(y)$, we have

$$c_{(\alpha\sim\{i\})}(z_i) \leq c_{(\alpha\sim\{i\})}c_{(\alpha\sim\{i\})}(y) = c_{(\alpha\sim\{i\})}(y)$$

by Lemma 3.2(i),(ii). Hence,

$$\begin{aligned} w &\leq \textstyle\prod_{i\in\alpha} c_{(\alpha\sim\{i\})}(y) && \text{by (2),}\\ &= y && \text{since } y \text{ is a rectangle,}\\ &\leq x && \text{by assumption.} \end{aligned}$$

Further, $c_{(\alpha)}(w) = 1$, by (2), Lemma 3.5(ii) (with $\Delta = \alpha$), and the simplicity of $\mathfrak{A}$. Consequently, $w \neq 0$ by (C1). Finally, w is a rectangle by Lemma 3.7. We have shown that

(3) w is a non-zero rectangle below x.

FIGURE 2. The construction of the atom w below x.

Our goal is to prove that

(4) w is an atom.

We first show:

(5) If u is a non-zero rectangle below w, then $w = u$.

Let u be a non-zero rectangle below w. Then

$$\begin{aligned} 0 &< d \cdot c_{(\alpha\sim\{i\})}(u) && \text{by 3.4,} \\ &\leq d \cdot c_{(\alpha\sim\{i\})}(w) && \text{by 3.2(ii) and } u \leq w, \\ &\leq d \cdot c_{(\alpha\sim\{i\})}\big(c_{(\alpha\sim\{i\})}(z_i)\big) && \text{by 3.2(ii) and (2),} \\ &= d \cdot c_{(\alpha\sim\{i\})}(z_i) && \text{by 3.2(i),} \\ &= z_i && \text{by 3.3(i),(ii) and } z_i \leq d. \end{aligned}$$

But z_i is an atom, by (1). Therefore, $z_i = d \cdot c_{(\alpha\sim\{i\})}(u)$. It follows that

$$\begin{aligned} c_{(\alpha\sim\{i\})}(z_i) &= c_{(\alpha\sim\{i\})}\big(d \cdot c_{(\alpha\sim\{i\})}(u)\big) \\ &= c_{(\alpha\sim\{i\})}(d) \cdot c_{(\alpha\sim\{i\})}(u) && \text{by (C3),} \\ &= 1 \cdot c_{(\alpha\sim\{i\})}(u) && \text{by (C5), 3.3(iii),} \\ &= c_{(\alpha\sim\{i\})}(u) && \text{by BA,} \end{aligned}$$

(In the above application of Lemma 3.3(iii) we take $\Delta = \alpha \sim \{i\}$ and $\Gamma = \alpha$.) Using the above equality together with (2) and the assumption that u is a rectangle, we see that

$$w = \textstyle\prod_{i\in\alpha} c_{(\alpha\sim\{i\})}(z_i) = \prod_{i\in\alpha} c_{(\alpha\sim\{i\})}(u) = u \quad ,$$

which proves (5).

Now assume that v is any non-zero element below w. Then there is a non-zero rectangle u below v, by rectangular density. By (5) we see that $w = u$. This forces $w = v$. We have proven (4). Together, (3) and (4) show that every non-zero element in $\mathfrak{A}$ is above an atom. The proof of the lemma is complete. ∎

We shall see in Theorem 5.18 below that the hypothesis of simplicity in the previous lemma can be dropped provided we replace "atomic" with "quasi-atomic", i.e., a rectangularly dense CA_α is always quasi-atomic. Moreover, we shall see there that the above proof really has two main parts: the proof that below every non-zero element x there is a non-zero "point" w, and the proof that every non-zero point w is an atom (or a quasi-atom in a non-simple algebra).

As mentioned in the introduction, Henkin and Tarski proved that an atomic CA_α with rectangular atoms is representable. The next theorem is a generalization of their result.

Theorem 3.11. *For $\alpha \geq 2$, every rectangularly dense CA_α is representable.*

Proof: First, assume that α is finite. Each simple, rectangularly dense CA_α is atomic, by the previous lemma. Therefore it is in RCA_α by the Henkin-Tarski representation theorem for atomic cylindric algebras with rectangular atoms (see Henkin-Monk-Tarski [1985], Theorem 3.2.14). It now follows from Theorem 2.11 (with $\mathsf{L} = \mathsf{CA}_\alpha$ and $\mathsf{K} = \mathsf{RCA}_\alpha$) that every rectangularly dense CA_α is in RCA_α.

Suppose that α is infinite, and let $\mathfrak{A}$ be a rectangularly dense CA_α. RCA_α is a variety. Therefore, to show that $\mathfrak{A}$ is in RCA_α it suffices to show that every equation true of RCA_α is true of $\mathfrak{A}$. Let ε be any such equation. Then ε contains only finitely many of the symbols that denote cylindrifications and diagonal elements in the language of CA_α. By reindexing these symbols, we may assume without loss of generality that ε is an equation in the language of CA_β for some finite $\beta \geq 2$. Let $\mathfrak{B}$ be the β-reduct of $\mathfrak{A}$, i.e., the reduct of $\mathfrak{B}$ to the language of CA_β. Then $\mathfrak{B}$ is also rectangularly dense, since the property of being a rectangle is preserved under formation of reducts. Therefore $\mathfrak{B}$ is in RCA_β by the finite dimensional case of the present theorem. Now every equation true of RCA_α is true of RCA_β (see the remarks concerning β-reducts that follow the definition of RCA_α). It follows that ε holds in $\mathfrak{B}$ and therefore also in its expansion $\mathfrak{A}$. This completes the proof of the theorem. ∎

Strictly speaking, the restriction $\alpha \geq 2$ in the statement of Theorem 3.11 is not necessary. In the case when $\alpha = 0, 1$ the theorem would assert that *every* CA_0 and CA_1 is representable (see the remark following Definition 3.6). As was pointed out after Definition 3.1, this assertion is known to be true. However, our proof does not cover these cases: it is based on Lemma 3.8, which requires the assumption $\alpha \geq 2$. A similar remark also applies to some of our later results, for example Theorems 4.6, 4.7, and 4.18. To emphasize that when $\alpha = 0, 1$ our theorems assert nothing new and our proofs do not apply, we shall always add the restriction $\alpha \geq 2$.

The reader will have noticed that the proof of our representation theorem uses the special case of the Henkin-Tarski representation theorem when the dimension is finite and the algebra is simple. As is pointed out in Discussion 3.2.15 of Henkin-Monk-Tarski [1985], this special case of their theorem has a substantially simpler and more intuitive proof than the general case. It will be helpful to present a brief sketch of this proof.

We assume that α is finite and $\mathfrak{A}$ is simple. Let U be the set of subdiagonal atoms of $\mathfrak{A}$. For each atom x of $\mathfrak{A}$ and each $i < \alpha$, the meet of the hyperplane $c_{(\alpha - \{i\})}(x)$ with the main diagonal, i.e., the element $d \cdot c_{(\alpha-\{i\})}(x)$, is a subdiagonal atom. Let's call it the i^{th} *coordinate atom of* x. We define a mapping h of the atoms of $\mathfrak{A}$ to elements of ${}^\alpha U$ by sending each atom x to the sequence of its coordinate atoms, i.e.,

$$h(x) = (x_0, \ldots, x_{\alpha-1}) .$$

Then h is one-one and onto, because any α-tuple $(y_0, \ldots, y_{\alpha-1})$ in ${}^\alpha U$ is the sequence of coordinate atoms of a unique atom of $\mathfrak{A}$, namely the element

$$x = \textstyle\prod_{i<\alpha} c_{(\alpha-\{i\})}(y_i) .$$

We extend h in the obvious way to an additive mapping of the elements of $\mathfrak{A}$ to elements of $\mathfrak{C}_\alpha(U)$. Certainly, h is a Boolean embedding, and it is not difficult to check that it preserves cylindrifications and diagonal elements as well. This completes our sketch.

Corollary 3.12. *For $\alpha \geq 2$, a CA_α is representable iff it is embeddable into a rectangularly dense CA_α.*

Proof: The difficult direction of the corollary, from right to left, follows from the previous theorem. For the converse direction, suppose that $\mathfrak{A}$ is a representable CA_α. Then $\mathfrak{A}$ is embeddable into the direct product $\mathfrak{B}$ of a family of full cylindric set algebras of dimension α. Now each such set algebra is rectangularly dense, since the singletons are always rectangles (see the remark after Definition 3.6). Moreover, the direct product of a family of rectangularly dense cylindric algebras is again rectangularly dense. This follows from two simple observations: (1) 0 is a rectangle in every cylindric algebra; (2) the property of being a rectangle is defined by a collection of equations, and hence an element of a direct product is a rectangle iff each coordinate element is a rectangle (in the coordinate algebra). Therefore, $\mathfrak{B}$ is rectangularly dense. ∎

Each of the representation theorems for rectangularly dense algebras that we shall formulate in the subsequent sections (Theorems 4.6, 4.7, 4.18, 6.7, and 6.13) has an "if and only if" corollary that is analogous in statement and proof to the preceding one. With few exceptions, we shall not bother to formulate them.

4. Rectangularly dense quasi-polyadic and substitution-cylindrification algebras

Polyadic and quasi-polyadic algebras (with and without equality) of dimension α were defined by Halmos in a series of papers in the 1950s (see Halmos [1962]). For finite dimensions the two notions coincide. An elegant definition of the notion of a quasi-polyadic algebra (with or without equality) that is equivalent to Halmos' original definition was given by Sain-Thompson [1991], Definition 1 and the remark on p. 548, and it is this latter definition that we shall use. Substitution-cylindrification algebras were introduced by Pinter [1973] under the name of *quantifier algebras*.

We shall extend Theorem 3.11 to quasi-polyadic and substitution-cylindrification algebras with and without equality. In each case the extension to algebras with equality is relatively routine. From this extension we are then able to derive the result for algebras without equality under the proviso that the substitution operations are completely additive. We shall show that this condition is met if the algebra is quasi-atomic. The representation theorems for algebras without equality should be viewed as the principal results of this section.

Representation theorems for atomic quasi-polyadic algebras of finite dimension with rectangular atoms are formulated in Henkin-Monk-Tarski [1985], Theorems 5.4.36 and 5.4.38. However, the proof of 5.4.38 is defective. It depends on Theorem 5.4.37, which asserts the existence of the completion of an arbitrary equality-free quasi-polyadic algebra $\mathfrak{A}$ of finite dimension. Unfortunately, such completions have only been shown to exist under the proviso that the substitution operations of the original algebra $\mathfrak{A}$ are completely additive. We correct this error by showing that in an atomic, rectangularly dense quasi-polyadic algebra the substitution operations are always completely additive.

Definition 4.1. A *quasi-polyadic algebra of dimension* α is an algebra

$$\mathfrak{A} = \langle A\,,\, +\,,\, -\,,\, c_i\,,\, s_{ij}\,,\, p_{ij}\,\rangle_{i,j\in\alpha}$$

such that $\langle A\,,\, +\,,\, -\rangle$ is a Boolean algebra, the operations c_i, s_{ij}, and p_{ij} are all unary, and the following postulates are satisfied for all $x, y \in A$ and all $i, j, k \in \alpha$:

(Q1) $s_{ii}(x) = x$,
(Q2) $p_{ii}(x) = x$,
(Q3) $p_{ij}(x) = p_{ji}(x)$,
(Q4) $x \leq c_i(x)$,
(Q5) $c_i(x + y) = c_i(x) + c_i(y)$,
(Q6) $s_{ij}c_i(x) = c_i(x)$,
(Q7) $c_i s_{ij}(x) = s_{ij}(x)$ whenever $i \neq j$,

(Q8) $s_{ij}c_k(x) = c_k s_{ij}(x)$ whenever $k \neq i, j$,
(Q9) $s_{ij}(x+y) = s_{ij}(x) + s_{ij}(y)$,
(Q10) $s_{ij}(-x) = -s_{ij}(x)$,
(Q11) $p_{ij}(x+y) = p_{ij}(x) + p_{ij}(y)$,
(Q12) $p_{ij}(-x) = -p_{ij}(x)$,
(Q13) $p_{ij}p_{ij}(x) = x$,
(Q14) $p_{ij}p_{ik}(x) = p_{jk}p_{ij}(x)$ whenever i, j, k are distinct ,
(Q15) $p_{ij}s_{ji}(x) = s_{ij}(x)$,

when $\alpha \geq 3$, and (Q1)–(Q15) and the following two additional postulates

(Q16) $c_i(x \cdot c_i(y)) = c_i(x) \cdot c_i(y)$,
(Q17) $c_i c_j(x) = c_j c_i(x)$,

when $\alpha \leq 2$. A *quasi-polyadic equality algebra of dimension* α is an algebra

$$\mathfrak{A} = \langle A\,,\,+\,,\,-\,,\,c_i\,,\,s_{ij}\,,\,p_{ij}\,,\,d_{ij}\,\rangle_{i,j\in\alpha}$$

such that the reduct

$$\mathfrak{A} = \langle A\,,\,+\,,\,-\,,\,c_i\,,\,s_{ij}\,,\,p_{ij}\,\rangle_{i,j\in\alpha}$$

is a quasi-polyadic algebra and the d_{ij} are constants satisfying the additional postulates

(Q18) $s_{ij}(d_{ij}) = 1$,
(Q19) $x \cdot d_{ij} \leq s_{ij}(x)$.

The classes of all quasi-polyadic algebras and all quasi-polyadic equality algebras of dimension α are denoted by QPA_α and QPEA_α respectively . We also use the notations "QPA_α" and "QPEA_α" as abbreviations for the phrases "quasi-polyadic algebra of dimension α" and "quasi-polyadic equality algebra of dimension α" respectively. The element d_{ij} is called the ij^{th} *diagonal* element, and the operations c_i , s_{ij} , and p_{ij} are called the i^{th} *cylindrification*, the ij^{th} *substitution*, and the ij^{th} *permutation* or *transposition*. ∎

From postulates (Q9) and (Q11) we easily see that whenever $x \leq y$ we have $s_{ij}(x) \leq s_{ij}(y)$ and $p_{ij}(x) \leq p_{ij}(y)$. We shall refer to these as *monotony laws* for quasi-polyadic algebras.

To give an example of a quasi-polyadic algebra and of a quasi-polyadic equality algebra, let U be any set and define $\mathfrak{Q}_\alpha(U)$ and $\mathfrak{P}_\alpha(U)$ to be the algebras

$$\langle Sb({}^\alpha U)\,,\,\cup\,,\,\sim\,,\,C_i\,,\,S_{ij}\,,\,P_{ij}\,\rangle_{i,j<\alpha}$$

and

$$\langle Sb({}^\alpha U)\,,\,\cup\,,\,\sim\,,\,C_i\,,\,S_{ij}\,,\,P_{ij}\,,\,D_{ij}\,\rangle_{i,j<\alpha}$$

respectively, where

$$\langle Sb({}^\alpha U)\,,\,\cup\,,\,\sim\,,\,C_i\,,\,D_{ij}\rangle_{i,j<\alpha}$$

is the full cylindric set algebra over U of dimension α, and the operations S_{ij} and P_{ij} are defined by

$S_{ij}(X) = X$ if $i = j$,
$S_{ij}(X) = \{u \in {}^\alpha U : \text{for some } x \in X \text{ with } x_i = x_j\,,\; u_k = x_k \text{ for } k \neq i\}$ if $i \neq j$,
$P_{ij}(X) = \{u \in {}^\alpha U : \text{for some } x \in X\,,\; u_k = x_k \text{ for } k \neq i, j \text{ and } u_i = x_j\,,\; u_j = x_i\}$

for each $X \subseteq {}^\alpha U$ and each i, j in α. It is easy to check that $\mathfrak{Q}_\alpha(U)$ (respectively $\mathfrak{P}_\alpha(U)$) is a quasi-polyadic algebra (respectively, a quasi-polyadic equality algebra) for every set U. It is called the *full quasi-polyadic* (respectively, the *full quasi-polyadic equality*) *set algebra over* U *of dimension* α. A quasi-polyadic algebra (respectively, a quasi-polyadic equality algebra) is said to be *representable* if it can be embedded into a direct product of full quasi-polyadic (respectively, quasi-polyadic equality) set algebras of dimension α.

The class RQPA_α (respectively, RQPEA_α) of all representable quasi-polyadic algebras (respectively, all representable quasi-polyadic equality algebras) of dimension α is a subvariety of QPA_α (respectively, of QPEA_α) for each $\alpha \geq 2$.[3] However, as in the cylindric algebraic case, the two varieties are distinct, and in fact, for $\alpha > 2$, RQPA_α (respectively, RQPEA_α) is not finitely axiomatizable and not even axiomatizable by a finite set of equational schemata (see Johnson [1969], Theorem 3.5, and Sain-Thompson [1991], Theorem 2).

Suppose that $\beta < \alpha$. Just as for cylindric algebras, every equation in the language of QPA_β (or QPEA_β) that holds in RQPA_α (or RQPEA_α) must hold in RQPA_β (or RQPEA_β). (See the remarks concerning β-reducts following the definition of RCA_α.)

It is well known and easy to check that both QPA_α and QPEA_α are varieties of normal Boolean algebras with operators. For finite α they are also discriminator varieties with the same discriminator term as CA_α.

We gather together some consequences of the axioms for quasi-polyadic algebras.

Lemma 4.2. *Let $\alpha \geq 2$ and $\mathfrak{A}$ a QPA_α or a $QPEA_\alpha$. Then the following are true for all $x, y \in A$ and all $i, j < \alpha$.*

(i) *The mapping $x \longmapsto s_{ij}(x)$ is an endomorphism, and $x \longmapsto p_{ij}(x)$ an automorphism, of the Boolean reduct $\langle A, +, - \rangle$ of $\mathfrak{A}$.*
(ii) *If $x \leq y$, then $c_i(x) \leq c_i(y)$.*
(iii) $c_i c_i(x) = c_i(x)$.
(iv) $c_i(-c_i(x)) = -c_i(x)$.
(v) $c_i(x \cdot c_i(y)) = c_i(x) \cdot c_i(y)$.
(vi) $x \cdot c_i(y) = 0$ *iff* $y \cdot c_i(x) = 0$.
(vii) $c_i c_j(x) = c_j c_i(x)$.
(viii) $s_{ij} s_{ij}(x) = s_{ij}(x)$.
(ix) $s_{ij} s_{ji}(x) = s_{ij}(x)$.
(x) *If $y \leq s_{ij}(x)$, then $c_i(y) \leq s_{ij}(x)$ when $i \neq j$.*
(xi) $c_j s_{ij}(x) = c_i s_{ji}(x)$.
(xii) $c_j p_{ij}(x) = p_{ij} c_i(x)$.
(xiii) $p_{ij} c_i c_j(x) = c_i c_j(x)$.
(xiv) $s_{ij} c_j(x) = p_{ij} c_j(x)$.
(xv) $s_{ij} p_{ij} c_i(x) = c_i(x)$.
(xvi) $p_{ij} s_{ki}(x) = s_{kj} p_{ij}(x)$ *when i, j, k are distinct.*

[3] It seems that an explicit statement and proof of these facts have not appeared in the literature. However, the proof given in Henkin-Monk-Tarski [1985] that RCA_α is a variety goes through with minor modifications to show that RQPA_α and RQPEA_α, as well as the classes RSCA_α and RSCEA_α defined below, are varieties. See, in particular, Lemmas 3.1.92, 3.1.93, and Theorems 3.1.103, 3.1.108 in *op. cit.* A simpler proof, in the context of discriminator varieties, is given Németi [1991].

(xvii) $s_{0i}(x) = p_{1i}s_{01}p_{1i}(x)$ *when* $i \neq 0$.
(xviii) $s_{i0}(x) = p_{i0}p_{1i}s_{01}p_{1i}(x)$ *when* $i \neq 0$.
(xix) $s_{ij}(x) = p_{0j}p_{i0}p_{1i}s_{01}p_{1i}p_{0j}(x)$ *for distinct* $i, j \neq 0$.
(xx) *If* $c_i(x) = x$, *then* $s_{ij}(x) = x$.

Proof: Part (i) follows at once from Axioms (Q9)–(Q13) (see Sain-Thompson [1991], p. 546). In particular, $s_{ij}(0) = 0$. From this and Axiom (Q7) we see that

$$c_i(0) = 0 \tag{1}$$

(see equation (5) in *op. cit.*, p. 546). The equivalence in (vi) is an immediate consequence of (v), Axiom (Q4), and (1). (For a proof see Henkin-Monk-Tarski [1971], Theorem 1.2.5.)

The validity of the remaining laws is a consequence of Theorem 1 in Sain-Thompson [1991], according to which two different formulations of the notion of a quasi-polyadic algebra are equivalent. It is possible to cite explicit passages in *op. cit.* where (ii)–(iv), (vii)–(ix), (xii) and (xvi) are established: laws (ii)–(iv) are established on pp. 545–546, equation (vii) in Claim 1.1, p. 548[4], equation (viii) in the proof of (J7) on p. 550, equations (ix) and (xvi) in Lemma 1.5(i),(v)[5] and equation (xii) in Lemma 1.6(i).[6]

Since (v), (x), (xi), (xiii)–(xv), and (xvii)–(xx) are not explicitly proved in *op. cit.*, we provide simple demonstrations. From (ii) and (iii) we easily get

$$c_i(x \cdot c_i(y)) \leq c_i c_i(y) = c_i(y). \tag{2}$$

Similarly, from (ii) and (iv) we get

$$c_i(x \cdot -c_i(y)) \leq c_i(-c_i(y)) = -c_i(y). \tag{3}$$

By BA and (Q5) we see that

$$c_i(x) = c_i(x \cdot c_i(y) + x \cdot -c_i(y)) = c_i(x \cdot c_i(y)) + c_i(x \cdot -c_i(y)).$$

Multiplying both sides by $c_i(y)$ and using Boolean distributivity gives us

$$c_i(x) \cdot c_i(y) = c_i(x \cdot c_i(y)) \cdot c_i(y) + c_i(x \cdot -c_i(y)) \cdot c_i(y).$$

Now the second summand is 0 by (3) and BA. The first summand is $c_i(x \cdot c_i(y))$ by (2) and BA. Thus, we arrive at (v).

Part (x) follows from (ii) and (Q7), and (xx) is a simple consequence of (Q6). For (xiii) we have

$$p_{ij}c_ic_j(x) = p_{ij}s_{ij}c_ic_j(x) \qquad \text{by (Q6)},$$

[4] The proof of this claim makes essential use of the hypothesis that $\alpha > 2$. However, in case $\alpha = 2$, (vii) is actually one of our axioms (see Definition 4.1). The first term in the penultimate line of p. 548 should read "c_ic_j" instead of "c_ic_i".

[5] Although Lemma 1.5 has the hypothesis that $\alpha > 2$, the hypothesis is not needed for the proof of part (i). In particular, the proof is valid when $\alpha = 2$.

[6] Again, the given hypothesis $\alpha > 2$ may be relaxed to $\alpha \geq 2$ and the proof remains valid. The formulation of Lemma 1.6(i) and its proof contain several typographic errors: the first term in the formulation should read "c_jp_{ij}", not "c_ip_{ij}"; the seventh and eighth terms in (25) should read "$p_{ji}s^i_jc_i$" and "$p_{ij}s^i_jc_i$" instead of "$p_{ij}s^j_ic_i$" and "$p_{ij}s^j_ic_i$".

$$\begin{aligned}
&= s_{ji}c_ic_j(x) && \text{by (Q15)},\\
&= s_{ji}c_jc_i(x) && \text{by (vii)},\\
&= c_jc_i(x) && \text{by (Q6)},\\
&= c_ic_j(x) && \text{by (vii)}.
\end{aligned}$$

For (xi) we have

$$\begin{aligned}
c_js_{ij}(x) &= c_jp_{ij}s_{ji}(x) && \text{by (Q15)},\\
&= p_{ij}c_is_{ji}(x) && \text{by (xii)},\\
&= p_{ij}c_ic_js_{ji}(x) && \text{by (Q7)},\\
&= c_ic_js_{ji}(x) && \text{by (xiii)},\\
&= c_is_{ji}(x) && \text{by (Q7)}.
\end{aligned}$$

For (xiv) we have

$$\begin{aligned}
s_{ij}c_j(x) &= p_{ij}s_{ji}c_j(x) && \text{by (Q15)},\\
&= p_{ij}c_j(x) && \text{by (Q6)}.
\end{aligned}$$

For (xv) we have

$$\begin{aligned}
s_{ij}p_{ij}c_i(x) &= s_{ij}c_jp_{ij}(x) && \text{by (xii)},\\
&= p_{ij}c_jp_{ij}(x) && \text{by (xiv)},\\
&= p_{ij}p_{ij}c_i(x) && \text{by (xii)},\\
&= c_i(x) && \text{by (Q13)}.
\end{aligned}$$

We now take up the proof of (xvii). The case $i = 1$ follows from (Q2). Suppose that $i \neq 0, 1$. Then by (xvi) (with i, j, k replaced by $1, i, 0$ respectively) we get

$$p_{1i}s_{01}(x) = s_{0i}p_{1i}(x)$$

for all x. Replacing x by $p_{1i}x$ in this equation and using (Q13) we see that

$$p_{1i}s_{01}p_{1i}(x) = s_{0i}p_{1i}p_{1i}(x) = s_{0i}(x)\,.$$

For (xviii) we use (Q15) and (xvii):

$$s_{i0}(x) = p_{i0}s_{0i}(x) = p_{i0}p_{1i}s_{01}p_{1i}(x)\,.$$

To prove (xix) we again use (xvi) (with i, j, k replaced by $0, j, i$ respectively):

$$p_{0j}s_{i0}(x) = s_{ij}p_{0j}(x)\,. \tag{4}$$

Hence,

$$\begin{aligned}
s_{ij}(x) &= s_{ij}p_{0j}p_{0j}(x) && \text{by (Q13)},\\
&= p_{0j}s_{i0}p_{0j}(x) && \text{by (4)},\\
&= p_{0j}p_{i0}p_{1i}s_{01}p_{1i}p_{0j}(x) && \text{by (xviii)}.
\end{aligned}$$

∎

Corollary 4.3. *Let $\alpha \geq 2$ and $\mathfrak{A}$ a QPA_α or a $QPEA_\alpha$. Then each permutation p_{ij} of $\mathfrak{A}$ is completely additive. If s_{01} is completely additive, then every other substitution s_{ij} of $\mathfrak{A}$ is completely additive.*

Proof: Since the permutation p_{ij} is a Boolean automorphism, it is easily seen to be self-conjugate in the sense that

$$(1) \qquad y \cdot p_{ij}(x) = 0 \quad \text{iff} \quad x \cdot p_{ij}(y) = 0\,.$$

Indeed, for any x, y, z we have $p_{ij}(z) = 0$ iff $z = 0$, and

$$p_{ij}(y \cdot p_{ij}(x)) = p_{ij}(y) \cdot p_{ij} p_{ij}(x) = p_{ij}(y) \cdot x\,,$$

by Lemma 4.2(i) and (Q13). Together, these two observations yield (1).

Since a self-conjugate operation is known to be completely additive, by Jónsson-Tarski [1951], Theorem 1.14, this proves the first assertion of the corollary. For the second, suppose that s_{01} is completely additive. In view of the first part of the corollary, it follows from Lemma 4.2(xvii)–(xix) and (Q1) that every substitution can be written as a composition of unary, completely additive operations, namely the permutations and s_{01}. By Theorem 1.9 of *op. cit.*, the composition of two (unary) completely additive functions is again completely additive. Hence, every substitution is completely additive. ∎

The reduct of a QPEA_α $\mathfrak{A}$ to the cylindric algebraic operations is a cylindric algebra (see Henkin-Monk-Tarski [1985], Theorem 5.4.3) and is called the *cylindric reduct* of $\mathfrak{A}$. Similiarly, the reduct of $\mathfrak{A}$ to the quasi-polyadic operations is called the *quasi-polyadic reduct* of $\mathfrak{A}$.

Lemma 4.4. *Let α be finite and $\mathfrak{A}$ a $QPEA_\alpha$. A subset of A is an ideal of $\mathfrak{A}$ iff it is an ideal of the cylindric reduct of $\mathfrak{A}$. Hence, $\mathfrak{A}$ is simple iff its cylindric reduct is simple.*

Proof: Let $\mathfrak{A}$ be a QPEA_α and $\mathfrak{B}$ its cylindric reduct. A subset I of A that is an ideal of $\mathfrak{A}$ is obviously an ideal of $\mathfrak{B}$. Suppose now that I is an ideal of $\mathfrak{B}$.

(1) $\quad s_{ij}(x)$ and $p_{ij}(x)$ are in I whenever $x \in I$.

Indeed, let $x \in I$. Then

$$\begin{aligned} s_{ij}(x) &\leq s_{ij} c_i c_{(\alpha \sim \{i\})}(x) && \text{by (Q4) and monotony,} \\ &= c_{(\alpha)}(x) && \text{by (Q6) and 4.2(vii).} \end{aligned}$$

Therefore, $s_{ij}(x)$ is in I by Lemma 2.4(iii),(iv) applied to $\mathfrak{B}$. Also,

$$\begin{aligned} p_{ij}(x) &\leq p_{ij} c_j c_{(\alpha \sim \{j\})}(x) && \text{by (Q4) and monotony,} \\ &= s_{ij} c_j c_{(\alpha \sim \{j\})}(x) && \text{by 4.2(xiv),} \\ &= s_{ij} c_i c_{(\alpha \sim \{i\})}(x) && \text{by 4.2(vii),} \\ &= c_{(\alpha)}(x) && \text{by (Q6).} \end{aligned}$$

Hence, $p_{ij}(x)$ is in I by Lemma 2.4(iii),(iv) applied to $\mathfrak{B}$. This proves (1). Using (1), we see that I satisfies Sain's conditions for being an ideal of $\mathfrak{A}$ (see Lemma 2.4). ∎

The notions of a rectangle and of a rectangularly dense algebra (see Definition 3.6) carry over without change from CA_α to QPA_α and QPEA_α.

Recall that for finite α a polyadic algebra (respectively, a polyadic equality algebra) is the same thing as a quasi-polyadic algebra (respectively, a quasi-polyadic equality algebra). Thus, any result concerning finite dimensional polyadic algebras and polyadic equality algebras applies automatically to quasi-polyadic algebras and quasi-polyadic equality algebras. We shall use this observation several times below without further mention.

Corollary 4.5. *Let $2 \leq \alpha < \omega$. Then every rectangularly dense, simple $QPEA_\alpha$ is atomic.*

Proof: Let $\mathfrak{A}$ be a simple, rectangularly dense QPEA_α and $\mathfrak{B}$ its cylindric reduct. Then $\mathfrak{B}$ is a simple cylindric algebra, by the preceding lemma. Furthermore, $\mathfrak{B}$ is rectangularly dense, since the definition of this notion involves only the operations of the cylindric reduct. Therefore $\mathfrak{B}$ is atomic, by Lemma 3.10. Because the definition of the notion of being atomic involves only the Boolean operations, $\mathfrak{A}$ must also be atomic. ∎

Theorem 4.6. *For $\alpha \geq 2$, every rectangularly dense $QPEA_\alpha$ is representable.*

The proof of the theorem is nearly identical to the proof of the cylindric algebraic version, Theorem 3.11. Instead of Lemma 3.10 and Theorem 3.2.14 in Henkin-Monk-Tarski [1985], we use Corollary 4.5 and Theorem 5.4.36 in *op. cit.*The latter states that, for finite $\alpha \geq 3$, every atomic QPEA_α with rectangular atoms is representable. One can check that the theorem and its proof are valid also in the case when $\alpha = 2$. We leave the details of the proof to the reader.

We now turn to the task of extending Theorem 4.6 to quasi-polyadic algebras. First, recall the notion of a completion of a BAO. Let $\mathfrak{A}$ and $\mathfrak{B}$ be Boolean algebras with operators of the same similarity type. Then $\mathfrak{B}$ is a *completion*[7] of $\mathfrak{A}$ provided that: (i) the Boolean reduct of $\mathfrak{B}$ (but not necessarily $\mathfrak{B}$ itself) is complete and $\mathfrak{A}$ is a subalgebra of $\mathfrak{B}$; (ii) for every y in B, we have (in $\mathfrak{B}$)

$$y = \sum\{x \in A : x \leq y\}\,;$$

(iii) for every extra-Boolean operator f of $\mathfrak{B}$, say of rank n, and for all $y_0, \ldots, y_{n-1}$ in B we have (in $\mathfrak{B}$)

$$f(y_0, \ldots, y_{n-1}) = \sum\{f(x_0, \ldots, x_{n-1}) : x_i \in A \text{ and } x_i \leq y_i \text{ for all } i < n\}\,.$$

(The point of the caution in (i) is that the extra-Boolean operations of $\mathfrak{B}$ are not required to be completely additive.) A general study of completions of Boolean

[7] An equivalent formulation of condition (ii) is that A is dense in $\mathfrak{B}$. Condition (iii) actually follows from conditions (i) and (ii) (and is therefore superfluous) when the extra-Boolean operators of $\mathfrak{B}$ are completely additive. This is the case for cylindric algebras, quasi-polyadic algebras with equality, and relation algebras. In Henkin-Monk-Tarski [1971], Remark 2.7.22, another condition is added to the definition of a completion, namely it is required that $\sum^{\mathfrak{B}} X = \sum^{\mathfrak{A}} X$ for every $X \subseteq A$ such that $\sum^{\mathfrak{A}} X$ exists. However, the proof of 23.1 in Sikorski [1964], p.93, shows that this condition is implied by conditions (i) and (ii).

algebras with operators can be found in Monk [1970]. He shows that every completely additive BAO $\mathfrak{A}$ has a completion $\mathfrak{B}$, and this completion is unique up to isomorphisms over $\mathfrak{A}$. Moreover, any positive equation holding in $\mathfrak{A}$ holds in $\mathfrak{B}$.

Suppose that $\mathfrak{A}$ is a QPA_α in which s_{01} is completely additive. Then by Corollary 4.3, $\mathfrak{A}$ is completely additive, so it has a completion $\mathfrak{B}$. Now $\mathfrak{B}$ must be a QPA_α as well. Indeed, all the equations defining quasi-polyadic algebras, with the exception of (Q10) and (Q12), are positive and hence hold in $\mathfrak{B}$. The fact that substitutions and permutations preserve $+$, $\cdot$, 0, and 1 is expressible by positive equations, so this is true in $\mathfrak{B}$ as well. To verify that (Q10) holds in $\mathfrak{B}$, suppose that $y = -x$. Then $x + y = 1$ and $x \cdot y = 0$. Therefore,

$$s_{ij}(x) + s_{ij}(y) = s_{ij}(x + y) = s_{ij}(1) = 1\,,$$

and similarly

$$s_{ij}(x) \cdot s_{ij}(y) = 0\,,$$

i.e., $s_{ij}(y) = -s_{ij}(x)$. An analogous argument applies in the case of (Q12). (A similar argument is used by Sain-Thompson [1991], Proposition 9, to show that the *perfect extension* of a quasi-polyadic algebra is again a quasi-polyadic algebra. Halmos [1962], pp. 230–232, shows directly that the perfect extension of a polyadic algebra is itself a polyadic algebra.)

Sain-Thompson [1991], pp. 561–562, shows that the perfect extension of a QPA_α has a first-order definitional extension that is a QPEA_α. (Hence, every QPA_α is a subreduct of a QPEA_α.) The same argument shows that the completion $\mathfrak{B}$ has a first-order definitional extension that is a QPEA_α. We give the argument here for the sake of completeness.

$$s_{ij}\big(\prod X\big) = \prod\{s_{ij}(x) : x \in X\} \quad \text{for every } X \subseteq B\,. \tag{1}$$

Indeed,

$$\begin{aligned} s_{ij}\big(\prod X\big) &= s_{ij}\big(-\sum\{-x : x \in X\}\big) && \text{by Boolean algebra,}\\ &= -s_{ij}\big(\sum\{-x : x \in X\}\big) && \text{by 4.2(i),}\\ &= -\sum\{s_{ij}(-x) : x \in X\} && \text{by complete additivity,}\\ &= -\sum\{-s_{ij}(x) : x \in X\} && \text{by 4.2(i),}\\ &= \prod\{s_{ij}(x) : x \in X\} && \text{by Boolean algebra.} \end{aligned}$$

Now define elements d_{ij} in $\mathfrak{B}$ by stipulating

$$d_{ij} = \prod\{y : s_{ij}(y) = 1\}\,, \tag{2}$$

and notice that this definition (due to Halmos [1962], p. 228) is first-order: it is the conjunction of the two conditions

$$\forall y(s_{ij}(y) = 1 \to x \leq y) \qquad , \qquad \forall z[\forall y(s_{ij}(y) = 1 \to z \leq y) \to z \leq x]\,.$$

Then

$$\begin{aligned} s_{ij}(d_{ij}) &= s_{ij}\left(\prod\{y : s_{ij}(y) = 1\}\right) && \text{by (2),} \\ &= \prod\{s_{ij}(y) : s_{ij}(y) = 1\} && \text{by (1),} \\ &= 1 && \text{by BA.} \end{aligned}$$

Thus (Q18) holds. Also, for any x we have

$$\begin{aligned} s_{ij}(-x \cdot d_{ij} + s_{ij}(x)) &= -s_{ij}(x) \cdot s_{ij}(d_{ij}) + s_{ij}(s_{ij}(x)) && \text{by 4.2(i),} \\ &= -s_{ij}(x) + s_{ij}(x) && \text{by (Q18) and 4.2(viii),} \\ &= 1 && \text{by BA.} \end{aligned}$$

Thus, $-x \cdot d_{ij} + s_{ij}(x)$ is one of the factors y in the definition (2) of d_{ij}. It follows that

$$d_{ij} \leq -x \cdot d_{ij} + s_{ij}(x),$$

and therefore

$$x \cdot d_{ij} \leq -x \cdot d_{ij} + s_{ij}(x).$$

But $x \cdot d_{ij}$ and $-x \cdot d_{ij}$ are disjoint, so $x \cdot d_{ij} \leq s_{ij}(x)$. This shows that (Q19) holds. It follows that the expansion $\mathfrak{C}$ of $\mathfrak{B}$ obtained by adding the constants d_{ij} is a QPEA_α.

Now assume that $\mathfrak{A}$ is also rectangularly dense. By the remarks of the previous paragraphs, its completion $\mathfrak{B}$ is the reduct of a QPEA_α $\mathfrak{C}$. Using condition (ii) in the definition of a completion, we easily check that $\mathfrak{B}$, and hence also $\mathfrak{C}$, inherit the rectangular density of $\mathfrak{A}$. Therefore $\mathfrak{C}$ is representable, by Theorem 4.6. This gives us a representation of $\mathfrak{B}$ and hence also of $\mathfrak{A}$. We have shown that

Theorem 4.7. *For $\alpha \geq 2$, every rectangularly dense QPA_α in which s_{01} is completely additive is representable.*

It is worth emphasizing why we have used completions instead of perfect extensions in the previous argument. In general, the completion of a rectangularly dense algebra is rectangularly dense. However, the perfect extension of a rectangularly dense algebra may not be rectangularly dense.

In the case of a quasi-atomic QPA_α the substitutions are always completely additive, as the next lemma shows.

Lemma 4.8. *Let $\mathfrak{A}$ be a quasi-atomic, rectangularly dense QPA_α. Then each substitution is completely additive.*

Proof: Let $\mathfrak{A}$ be a quasi-atomic, rectangularly dense QPA_α, and suppose X is a subset of A such that the sum $w = \sum X$ exists. We must show that $s_{ij}(w)$ is the least upper bound of $\{s_{ij}(x) : x \in X\}$. When $i = j$ this is trivial by (Q1). Thus, we may assume that $i \neq j$. Since s_{ij} is a Boolean endomorphism, $s_{ij}(w)$ is certainly an upper bound for $\{s_{ij}(x) : x \in X\}$. Suppose that z is any other upper bound, with the goal of showing that $s_{ij}(w) \leq z$. Since $\mathfrak{A}$ is quasi-atomic, it suffices to show that every quasi-atom below $s_{ij}(w)$ is below z.

Assume that

(1) y is a non-zero quasi-atom

and

(2) $y \leq s_{ij}(w)$,

with the goal of proving

(3) $y \cdot z \neq 0$.

We begin by showing that

(4) $c_i(y)$ is a quasi-atom in the set $\{x : c_i(x) = x\}$.

Indeed, suppose that

$$x = c_i(x) \leq c_i(y).$$

Then

$$\begin{aligned}
x \cdot y &= y \cdot c_{(\alpha)}(x \cdot y) && \text{by 2.17(i), since } y \text{ is a quasi-atom,} \\
&= y \cdot c_{(\alpha\sim\{i\})} c_i(x \cdot y) && \text{by 4.2(vii),} \\
&= y \cdot c_{(\alpha\sim\{i\})}(x \cdot c_i(y)) && \text{by 4.2(v) and } c_i(x) = x, \\
&= y \cdot c_{(\alpha\sim\{i\})}(x) && \text{since } x \leq c_i(y), \\
&= y \cdot c_{(\alpha\sim\{i\})} c_i(x) && \text{since } c_i(x) = x, \\
&= y \cdot c_{(\alpha)}(x) && \text{by 4.2(vii).}
\end{aligned}$$

Applying c_i to both sides, and using $c_i(x) = x$ and 4.2(v), we get

$$x \cdot c_i(y) = c_i(y) \cdot c_{(\alpha)}(x).$$

But $x \leq c_i(y)$. Thus we arrive at (4).

Next, we establish that

(5) $s_{ji} c_i(y)$ is a quasi-atom in the set $\{x : c_j(x) = x\}$.

To prove (5), set $v = c_i(y)$. Certainly $s_{ji}(v)$ is c_j-closed, by (Q7). To show that it is a quasi-atom in the set of c_j-closed elements, let x be such that $c_j(x) = x$ and $x \leq s_{ji}(v)$. Applying s_{ij} to both sides, we get

$$\begin{aligned}
s_{ij}(x) &\leq s_{ij} s_{ji}(v) && \text{by 4.2(i),} \\
&= s_{ij}(v) && \text{by 4.2(ix),} \\
&= v && \text{by 4.2(xx).}
\end{aligned}$$

Because $s_{ij}(x)$ is a c_i-closed element, by (Q7), and v is a quasi-atom in the set of c_i-closed elements, by (4), we obtain

$$s_{ij}(x) = v \cdot c_{(\alpha)} s_{ij}(x).$$

Applying s_{ji} to both sides and proceeding stepwise, we get

$$\begin{aligned}
s_{ji}s_{ij}(x) &= s_{ji}(v)\cdot s_{ji}c_{(\alpha)}s_{ij}(x) && \text{by 4.2(i)},\\
s_{ji}(x) &= s_{ji}(v)\cdot c_{(\alpha)}s_{ij}(x) && \text{by 4.2(ix),(xx)},\\
s_{ji}(x) &= s_{ji}(v)\cdot c_{(\alpha)}c_j s_{ij}(x) && \text{by 4.2(vii),(iii)},\\
s_{ji}(x) &= s_{ji}(v)\cdot c_{(\alpha)}c_i s_{ji}(x) && \text{by 4.2(xi)},\\
s_{ji}(x) &= s_{ji}(v)\cdot c_{(\alpha)}s_{ji}(x) && \text{by 4.2(vii),(iii)},\\
x &= s_{ji}(v)\cdot c_{(\alpha)}(x) && \text{by 4.2(xx)}.
\end{aligned}$$

This proves (5).

Set

(6) $$u = c_i(y)\cdot s_{ji}c_i(y)\,.$$

(7) $v = u\cdot c_{(\alpha)}(v)$ for every rectangle $v \le u$.

To establish (7), suppose that v is a rectangle below u. Then

$$c_i(v) \le c_i(u) \le c_i(y)$$

by (6) and 4.2(ii). Therefore

(8) $$c_i(v) = c_i(y)\cdot c_{(\alpha)}(v)$$

by (4). Similarly,

$$c_j(v) \le c_j(u) \le s_{ji}c_i(y)$$

by 4.2(ii) and (Q7), so

(9) $$c_j(v) = s_{ji}c_i(y)\cdot c_{(\alpha)}(v)$$

by (5). Hence,

$$\begin{aligned}
v &= c_i(v)\cdot c_j(v) && \text{since } v \text{ is a rectangle and } i \neq j,\\
&= c_i(y)\cdot c_{(\alpha)}(v)\cdot s_{ji}c_i(y)\cdot c_{(\alpha)}(v) && \text{by (8), (9)},\\
&= u\cdot c_{(\alpha)}(v) && \text{by (6)}.
\end{aligned}$$

This proves (7).

With the help of (7) we establish

(10) u is a quasi-atom

as follows. Let $x \le u$ and take V to be the set of rectangles below x. Since $\mathfrak{A}$ is rectangularly dense we have

(11) $$x = \sum V\,.$$

Because $c_{(\alpha)}$ is completely additive (since it is self-conjugate — see 2.3(vii) and the proof of 4.3), (11) gives

(12) $$c_{(\alpha)}(x) = \sum\{c_{(\alpha)}(v) : v \in V\}\,.$$

Therefore

$$\begin{aligned} u \cdot c_{(\alpha)}(x) &= u \cdot \textstyle\sum\{c_{(\alpha)}(v) : v \in V\} && \text{by (12),} \\ &= \textstyle\sum\{u \cdot c_{(\alpha)}(v) : v \in V\} && \text{by BA,} \\ &= \textstyle\sum\{v : v \in V\} && \text{by (7),} \\ &= x && \text{by (11).} \end{aligned}$$

This proves (10).

Next, we show that

$$s_{ij}(u) = c_i(y)\,. \tag{13}$$

Indeed,

$$\begin{aligned} s_{ij}(u) &= s_{ij}(c_i(y) \cdot s_{ji}c_i(y)) && \text{by (6),} \\ &= s_{ij}c_i(y) \cdot s_{ij}s_{ji}c_i(y) && \text{by 4.2(i),} \\ &= s_{ij}c_i(y) && \text{by 4.2(ix) and BA,} \\ &= c_i(y) && \text{by (Q6).} \end{aligned}$$

From (13) we conclude

$$s_{ij}(u \cdot w) = c_i(y) \tag{14}$$

as follows:

$$\begin{aligned} s_{ij}(u \cdot w) &= s_{ij}(u) \cdot s_{ij}(w) && \text{by 4.2(i),} \\ &= c_i(y) \cdot s_{ij}(w) && \text{by (13),} \\ &= c_i(y) && \text{by (2) and 4.2(x).} \end{aligned}$$

Finally, from (6) and 4.2(ii) we see that $c_i(u \cdot x) \le c_i(y)$ for any x. Therefore, by (4) we have

$$c_i(u \cdot x) = c_i(y) \cdot c_{(\alpha)}(u \cdot x) \quad \text{for every } x\,. \tag{15}$$

We are ready to prove (3). Since $c_i(y) \neq 0$ by (1) and (Q4), we have $s_{ij}(u \cdot w) \neq 0$ by (14). It follows that $u \cdot w \neq 0$. Because w is the supremum of X, there must be an $x \in X$ such that $u \cdot x \neq 0$. Now u is a quasi-atom by (10), so

$$u \cdot x = u \cdot c_{(\alpha)}(u \cdot x)$$

by 2.17(i). Applying s_{ij} to both sides we get

$$\begin{aligned} s_{ij}(u) \cdot s_{ij}(x) &= s_{ij}(u) \cdot c_{(\alpha)}(u \cdot x) && \text{by 4.2(i), (Q6),} \\ c_i(y) \cdot s_{ij}(x) &= c_i(y) \cdot c_{(\alpha)}(u \cdot x) && \text{by (13),} \\ c_i(y) \cdot s_{ij}(x) &= c_i(u \cdot x) && \text{by (15),} \\ c_i(y) \cdot s_{ij}(x) &\neq 0 && \text{since } u \cdot x \neq 0, \\ y \cdot c_i s_{ij}(x) &\neq 0 && \text{by 4.2(vi),} \\ y \cdot s_{ij}(x) &\neq 0 && \text{by (Q7),} \\ y \cdot z &\neq 0 && \text{since } s_{ij}(x) \le z\,. \end{aligned}$$

This proves (3).

It follows from (3) that $s_{ij}(w) \leq z$, i.e., $s_{ij}(w) \cdot -z = 0$. Otherwise there would be a non-zero quasi-atom y below $s_{ij}(w) \cdot -z$, and this would contradict (3). ∎

Remark 4.9. We shall see below that the assumptions of quasi-atomicity (atomicity in the simple case) and of rectangular density are both necessary in the preceding lemma. However, it is possible to give the lemma a somewhat stronger formulation. By Corollary 4.3, it suffices to prove that s_{01} is completely additive. Therefore, everywhere in the proof we may replace "i" with "0" and "j" with "1". Doing this, we notice that only one instance of rectangularity is needed, namely $v = c_0(v) \cdot c_1(v)$, just after (9). Let us say that an element v is a *01-rectangle* if it satisfies this condition, and let us say that $\mathfrak{A}$ is *01-rectangularly dense* if every non-zero element is above a non-zero 01-rectangle. Then the modified proof of Lemma 4.8 shows the following: *If $\mathfrak{A}$ is quasi-atomic and 01-rectangularly dense, then s_{01} — and hence every substitution of $\mathfrak{A}$ — is completely additive.*

Corollary 4.10. *Let $\mathfrak{A}$ be a QPA_α, where $\alpha \geq 2$. Then the following are equivalent:*

(i) *$\mathfrak{A}$ is representable;*
(ii) *$\mathfrak{A}$ is embeddable in an atomic, rectangularly dense QPA_α;*
(iii) *$\mathfrak{A}$ is embeddable in a quasi-atomic, rectangularly dense QPA_α;*
(iv) *$\mathfrak{A}$ is embeddable in a rectangularly dense QPA_α in which s_{01} is completely additive;*
(v) *$\mathfrak{A}$ is embeddable in a rectangularly dense, completely additive QPA_α.*

Proof: Every representable QPA_α is embeddable into the direct product of full quasi-polyadic set algebras of dimension α. Each of these full set algebras is atomic and rectangularly dense. Hence, as is easy to check, the product is atomic and rectangularly dense. This establishes the implication from (i) to (ii). The implication from (ii) to (iii) is trivial and the implication from (iii) to (v) is just Lemma 4.8 (see also Corollary 4.3). The implication from (v) to (iv) is obvious and the implication from (iv) to (i) is proved in Theorem 4.7. ∎

Remark 4.11. Theorem 5.4.38 in Henkin-Monk-Tarski [1985] asserts that for finite $\alpha \geq 3$ a QPA_α is representable iff it is embeddable into an atomic QPA_α with rectangular atoms. It follows from the definition of rectangular density that an algebra is atomic and rectangularly dense iff it is atomic and has rectangular atoms. Thus, the equivalence of (i) and (ii) in Corollary 4.10 is essentially an extension of 5.4.38 to the infinite dimensional case. However, the proof of 5.4.38 is based on Theorem 5.4.37 in *op. cit.*, which asserts that every finite dimensional quasi-polyadic algebra has a completion. The proof of the latter theorem is defective since it assumes implicitly that the substitution operations are completely additive. (Thus, it is still unknown whether every QPA_α has a completion, even in case when α is finite.) However, it is a consequence of Lemma 4.8 above that

(1) An atomic, rectangularly dense QPA_α has a completion.

In the proof of Theorem 5.4.38, instead of referring to 5.4.37, it suffices to refer to (1). The rest of the proof remains unchanged and is very similiar to our proof (except that it refers to Halmos [1962] instead of to Sain-Thompson [1991]). ∎

A stronger theorem than Corollary 4.10 is known in the case $\alpha = 2$. Theorem 5.4.33 in Henkin-Monk-Tarski [1985] says that every QPA_2 is representable.

We now give two examples which show that in a quasi-polyadic algebra that is rectangularly dense but not atomic, or that is atomic but not rectangularly dense, the substitutions need not be completely additive.

Example 4.12. To show that the assumption of atomicity in necessary in Lemma 4.8 we construct an example of an (incomplete) atomless, rectangularly dense quasi-polyadic set algebra of dimension 2 in which substitutions are not completely additive.

Let $\mathfrak{B}$ be an atomless Boolean set algebra, say with unit U, that has the following separation property:

(1) For any distinct $u, v \in U$ there is an $X \in B$ such that $u \in X$ and $v \in \sim X$.

(For instance, $\mathfrak{B}$ can be taken to be the Stone representation of some atomless Boolean algebra.) Put

$$R = \{X \times Y : X, Y \in B\} \qquad \text{and} \qquad A = \{\bigcup S : S \subseteq R \text{ and } |S| < \omega\}.$$

Thus, R is a set of rectangles whose sides are subsets of U (and elements of B), and A is the collection of finite unions of these rectangles.

(2) A is a subuniverse of $\mathfrak{Q}_2(U)$, the full 2-dimensional quasi-polyadic set algebra on U.

Obviously A is closed under finite unions. It is also closed under finite intersections. Indeed, R is closed under finite intersections since

$$(X_1 \times Y_1) \cap (X_2 \times Y_2) = (X_1 \cap X_2) \times (Y_1 \cap Y_2).$$

Hence, if S_1 and S_2 are finite subsets of R, then

$$S_3 = \{W \cap Z : W \in S_1 \text{ and } Z \in S_2\}$$

is also a finite subset of R, and we have

$$(\bigcup S_1) \cap (\bigcup S_2) = \bigcup S_3$$

by the distributive law. To check that A is closed under complementation, observe first of all that the complement of each rectangle from R is in A:

$$\sim (X \times Y) = [\sim X \times U] \cup [U \times \sim Y].$$

If $S \subseteq R$ is finite, then

$$\sim \bigcup S = \bigcap \{\sim Z : Z \in S\}$$

by De Morgan's laws. Since each $\sim Z$ is in A, by the previous remark, and since A is closed under intersections, we conclude that $\sim \bigcup S$ is in A. Thus, A is closed under the Boolean operations.

To show that A is closed under cylindrifications, substitutions, and permutations, consider first the case of a rectangle $X \times Y$ from R. Then, e.g.,

$$C_0(X \times Y) = U \times Y \quad , \quad C_1(X \times Y) = X \times U,$$
$$S_{01}(X \times Y) = U \times (X \cap Y) \quad , \quad S_{10}(X \times Y) = (X \cap Y) \times U,$$

$$P_{01}(X \times Y) = P_{10}(X \times Y) = Y \times X\,,$$

i.e., R is closed under cylindrifications, substitutions, and permutations. Since these operations are additive, it follows from the definition of A that A is closed under them as well. This completes the proof of (2).

In view of (2) we can form the subalgebra $\mathfrak{A}$ of $\mathfrak{Q}_2(U)$ with universe A. It is obviously rectangularly dense since its universe consists of unions of rectangles. It remains to show that, e.g., S_{01} is not completely additive.

(3) The only subset of D_{01} in $\mathfrak{A}$ is $\varnothing$.

To prove (3) it suffices, by rectangular density, to show that no non-zero rectangle in R is included in D_{01}. Indeed, suppose that $X \times Y$ is a non-empty subset of D_{01}. Then for any $u \in X$ and $v \in Y$ we have $u = v$. Thus, $X = Y = \{u\}$ for some $u \in U$. But then X and Y cannot be in $\mathfrak{B}$ since the latter is atomless and X and Y are atoms.

Set

$$S = \{X \times \sim X : X \in B\}\,. \tag{4}$$

Observe that

$$\bigcup S = \sim D_{01}\,. \tag{5}$$

Indeed, if (u, v) is in $\bigcup S$, say (u, v) is in $X \times \sim X$, then $u \neq v$. Therefore, $\bigcup S$ is a subset of $\sim D_{01}$. For the reverse inclusion, suppose that $u, v \in U$ are distinct. By the separation property (1) there is an $X \in B$ such that (u, v) is in $X \times \sim X$. Hence, (u, v) is in $\bigcup S$, by (4).

$$\textstyle\sum^{\mathfrak{A}} S = U \times U\,. \tag{6}$$

Certainly $U \times U$ is an upper bound for S in $\mathfrak{A}$. Suppose that Z is any other upper bound for S in $\mathfrak{A}$. Then $\bigcup S \subseteq Z$. By (5) this means that $\sim D_{01} \subseteq Z$, i.e., $\sim Z \subseteq D_{01}$. In view of (3) we see that $\sim Z = \varnothing$, i.e., $Z = U \times U$.

Now using the definition of S_{01} we see that

$$S_{01}(U \times U) = U \times (U \cap U) = U \times U\,.$$

On the other hand,

$$S_{01}(X \times \sim X) = U \times (X \cap \sim X) = \varnothing$$

for each $X \in B$. Thus,

$$S_{01}\big(\textstyle\sum^{\mathfrak{A}} S\big) = U \times U \qquad \text{and} \qquad \textstyle\sum^{\mathfrak{A}}\{S_{01}(Z) : Z \in S\} = \varnothing\,.$$

This competes the proof that S_{01} is not completely additive in $\mathfrak{A}$. ∎

It is not difficult to extend the above example to arbitrary dimensions $\alpha \geq 2$. We define R to be the set of α-dimensional rectangles

$$X_0 \times X_1 \times \cdots \times X_\xi \times \ldots \quad , \quad \xi < \alpha\,,$$

such that $X_\xi \in B$ for every $\xi < \alpha$ and $X_\xi = U$ for all but finitely many ξ. The definition of A remains unchanged. Take the set S in (4) to be the collection of rectangles

$$X \times \sim X \times U \times U \times \dots$$

such that $X \in B$.

Example 4.13. To show that the assumption of rectangular density is necessary in Lemma 4.8 we construct an example of a complete, atomic quasi-polyadic set algebra of dimension 2 in which substitutions are not completely additive. *A fortiori*, the algebra is not rectangularly dense, by Lemma 4.8.

Let U be an infinite set and $\langle Q_n : n \in \omega \rangle$ a sequence of binary relations on U with the following properties:

(1) $\langle Q_n : n \in \omega \rangle$ is a partition of $U \times U$,

i.e., the relations are non-empty, pairwise disjoint, and have $U \times U$ as their union;

(2) Q_0 is the identity relation on U,

i.e., $Q_0 = \{(x, x) : x \in U\}$;

(3) Each Q_n has U as its domain and range,

i.e., for every $x \in U$ there are $y, z \in U$ such that (x, y) and (z, x) are in Q_n;

(4) Each Q_n is symmetric,

i.e., if (x, y) is in Q_n, then so is (y, x). (At the end of the example we define a specific sequence of relations with the above properties.)

Fix a non-principal ultrafilter F on the set ω^+ of positive integers. For each subset X of ω^+ we define a binary relation R_X on U as follows:

$$R_X = \begin{cases} \bigcup\{Q_n : n \in X\} & \text{if } X \notin F, \\ \bigcup\{Q_n : n \in X \cup \{0\}\} & \text{if } X \in F. \end{cases} \tag{5}$$

Set

$$A = \{R_X : X \subseteq \omega^+\}. \tag{6}$$

We begin by showing that

(7) A is a subuniverse of $\mathfrak{Q}_2(U)$.

To prove (7), let X, Y be subsets of ω^+. If either X or Y is in F, then so is $X \cup Y$ (since F is a filter). Hence,

$$R_X \cup R_Y = \bigcup\{Q_n : n \in X\} \cup \bigcup\{Q_n : n \in Y\} \cup Q_0 = R_{X \cup Y}.$$

If neither X nor Y is in F, then $X \cup Y$ is also not in F (since F is an ultrafilter). Hence,

$$R_X \cup R_Y = \bigcup\{Q_n : n \in X\} \cup \bigcup\{Q_n : n \in Y\} = R_{X \cup Y}.$$

Thus, A is closed under finite unions.

Now suppose that X is the complement of Y in ω^+. Since F is an ultrafilter, exactly one of them, say X, is in F. In view of (5) and (1) we have

$$\sim R_X = \sim \bigcup\{Q_n : n \in X \cup \{0\}\} = \bigcup\{Q_n : n \in Y\} = R_Y\,,$$

and hence also $\sim R_Y = R_X$. Therefore, A is closed under complementation.

To show closure under cylindrifications and permutations, notice first of all that for $i, j < 2$ we have $C_i(Q_n) = U \times U$ by (3), and $P_{ij}(Q_n) = Q_n$ by (4). Because cylindrifications and permutations in $\mathfrak{Q}_2(U)$ are completely additive, it follows that

$$C_i(R_X) = \begin{cases} U \times U & \text{if } X \neq \varnothing\,, \\ \varnothing & \text{if } X = \varnothing\,, \end{cases}$$

and that each permutation is the identity operation on A.

Turning to the non-trivial substitutions, we have

$$S_{01}(R_X) = C_0(Q_0 \cap \bigcup\{Q_n : n \in X\}) = C_0(\varnothing) = \varnothing \quad \text{if } X \notin F \tag{8}$$

and

$$S_{01}(R_X) = C_0(Q_0 \cap \bigcup\{Q_n : n \in X \cup \{0\}\}) = C_0(Q_0) = U \times U \quad \text{if } X \in F\,,$$

by definition of S_{01}, (1), and (2). Similarly,

$$S_{10}(R_X) = \begin{cases} \varnothing & \text{if } X \notin F\,, \\ U \times U & \text{if } X \in F\,. \end{cases}$$

This completes the proof of (7).

Let $\mathfrak{A}$ be the subalgebra of $\mathfrak{Q}_2(U)$ with universe A. Then $\mathfrak{A}$ is a quasi-polyadic set algebra of dimension 2. Moreover, it is atomic and its atoms are just the elements

$$R_{\{n\}} = Q_n \quad \text{for } n \in \omega^+\,;$$

to show this, one makes use of the assumption that F is non-principal, i.e.,

$$\{n\} \notin F \quad \text{for } n \in \omega^+\,. \tag{9}$$

Finally, $\mathfrak{A}$ is complete. To prove this, notice that by (6) and (7) an arbitrary sum in $\mathfrak{A}$ is the supremum of a set

$$\{R_{X_\xi} : \xi \in \Xi\}\,, \tag{10}$$

where $\langle X_\xi : \xi \in \Xi \rangle$ is a sequence of subsets of ω^+. Put $X = \bigcup\{X_\xi : \xi \in \Xi\}$. It suffices to show that

$$R_X = \sum\{R_{X_\xi} : \xi \in \Xi\}\,. \tag{11}$$

It is clear from (5) that R_X is an upper bound for the set (10). From (5)–(7) we see that any upper bound for (10) must have the form R_Y, where $R_{X_\xi} \subseteq R_Y$ and hence $X_\xi \subseteq Y$ for each $\xi \in \Xi$. It follows that $X \subseteq Y$ by definition of X. In particular, if $X \in F$, then $Y \in F$ (since F is a filter). Therefore we have $R_X \subseteq R_Y$, by (5). This proves (11).

Applying the preceding reasoning to the sequence $\langle R_{\{n\}} : n \in \omega^+ \rangle$, we see that

$$\sum\{R_{\{n\}} : n \in \omega^+\} = R_{\omega^+} = \bigcup\{Q_n : n \in \omega\} = U \times U\,.$$

Now

$$\sum\{S_{01}(R_{\{n\}}) : n \in \omega^+\} = \sum\{\varnothing : n \in \omega^+\} = \varnothing\,,$$

by (8) and (9). On the other hand,

$$S_{01}(U \times U) = U \times U\,.$$

Therefore

$$S_{01}(\sum\{R_{\{n\}} : n \in \omega^+\}) \neq \sum\{S_{01}(R_{\{n\}}) : n \in \omega^+\}\,.$$

This shows that S_{01} is not completely additive.

To complete the example, we must define a set U and a sequence $\langle Q_n : n \in \omega\rangle$ of binary relations on U with properties (1)–(4). For simplicity take U to be the set of integers and let P be the predecessor function. For $n \in \omega^+$ take Q_n to be the union of P^n—the composition of P with itself n times—and the inverse of P^n, and take Q_0 to be the identity function on U. ∎

With the help of the methods used to prove Theorem 4.7 it is possible to establish other, related representation theorems. We give an example using the substitution-cylindrification algebras that were first introduced in Pinter [1973] under the name of *quantifier algebras*.

Definition 4.14. A *substitution-cylindrification algebra of dimension* α is an algebra

$$\mathfrak{A} = \langle A\,,\, +\,,\, -\,,\, c_i\,,\, s_{ij}\,\rangle_{i,j\in\alpha}$$

such that $\langle A\,,\, +\,,\, -\rangle$ is a Boolean algebra, the operations c_i and s_{ij} are unary, and postulates (Q1), (Q4)–(Q10), and

$$s_{ij}\, s_{ki}(x) = s_{ij}\, s_{kj}(x) \quad \text{for all } i, j, k < \alpha$$

are valid in $\mathfrak{A}$ when $\alpha \geq 3$; when $\alpha \leq 2$ we require in addition that (Q16) and (Q17) are valid in $\mathfrak{A}$. A *substitution-cylindrification equality algebra of dimension* α is an algebra

$$\mathfrak{A} = \langle A\,,\, +\,,\, -\,,\, c_i\,,\, s_{ij}\,,\, d_{ij}\,\rangle_{i,j\in\alpha}$$

such that the reduct

$$\mathfrak{A} = \langle A\,,\, +\,,\, -\,,\, c_i\,,\, s_{ij}\,\rangle_{i,j\in\alpha}$$

is a substitution-cylindrification algebra and the d_{ij} are constants satisfying the additional postulates (Q18) and (Q19). The classes of all substitution-cylindrification algebras and all substitution-cylindrification equality algebras of dimension α are denoted by SCA_α and SCEA_α respectively. We also use these notations as abbreviations for the phrases "substitution-cylindrification algebra of dimension α" and "substitution-cylindrification equality algebra of dimension α" respectively. ∎

The notions of a *full* substitution cylindrification *set* algebra (with or without equality), a *representable* substitution cylindrification algebra, a *rectangularly dense* substitution cylindrification algebra, and the *cylindric reduct* of a substitution cylindrification algebra are the obvious analogues of the quasi-polyadic notions. The classes RSCA_α and RSCEA_α of all representable SCA_α and SCEA_α are varieties. Also, for $\beta < \alpha$, every equation in the language of SCA_β (or SCEA_β) that holds in RSCA_α (or RSCEA_α) must hold in RSCA_β (or RSCEA_β). (See the analogous remarks concerning QPA_α and QPEA_α.) The next two lemmas are intended to clarify the relationship between cylindric algebras and substitution cylindrification algebras. The first of them is due to Pinter [1973], p. 366.

Lemma 4.15. *The class $SCEA_\alpha$ is a term definitional extension of CA_α.*

Proof: Let $\mathfrak{A}$ be in SCEA_α and $\mathfrak{B}$ its cylindric reduct. Then $\mathfrak{B}$ is in CA_α and each substitution s_{ij} of $\mathfrak{A}$ can be defined by the formula

$$s_{ij}(x) = \begin{cases} x & \text{if } i = j, \\ c_i(d_{ij} \cdot x) & \text{if } i \neq j. \end{cases}$$

Conversely, if $\mathfrak{A}$ is in CA_α and we define s_{ij} as above, then $\langle \mathfrak{A}, s_{ij} \rangle_{i,j<\alpha}$ is in SCEA_α. Indeed, the postulates for SCEA_α are all valid in CA_α under the given definition of substitution, by Henkin-Monk-Tarski [1971], Section 1.5. ∎

Lemma 4.16. *Every SCA_α is a subreduct of a CA_α (more precisely, a subreduct of the definitional extension of a CA_α to a $SCEA_\alpha$).*

Proof: The proof is very similar to the proof of Theorem 4.7. Let $\mathfrak{A}$ be a SCA_α and $\mathfrak{B}$ its perfect extension in the sense of Jónsson-Tarski [1951]. From the definition of a perfect extension we know, in particular, that $\mathfrak{B}$ is a complete, atomic BAO and that $\mathfrak{A}$ is a subalgebra of $\mathfrak{B}$. The proof we gave to show that the completion of a QPA_α is again a QPA_α carries over to the case of the perfect extension of a SCA_α and shows that

(1) $\mathfrak{B}$ is a SCA_α.

Also, the proof that the completion of a QPA_α has a definitional extension which is a QPEA_α carries over to show that

(2) $\mathfrak{B}$ has a definitional extension which is a SCEA_α.

The key point to observe in carrying over this proof to (2) is that substitutions are completely additive in a perfect extension (as opposed to the situation in a completion, where they need not be completely additive). This is a consequence of the additivity of substitutions (see (Q9)) and the fact that the canonical extension of an additive operation (in a perfect extension) is completely additive (see Jónsson-Tarski [1951], Theorem 2.4). To carry over the proof to (2) one must also check that (viii) and the first part of (i) in 4.2 hold in SCA_α, and this is easy.

Let $\mathfrak{C}$ be the definitional extension of $\mathfrak{B}$ to a SCEA_α that is guaranteed by (2). By the previous lemma, $\mathfrak{C}$ is a term definitional extension of a CA_α $\mathfrak{D}$. Thus, $\mathfrak{A}$ is a subalgebra of $\mathfrak{B}$, which is a reduct of $\mathfrak{C}$, which is a definitional extension of $\mathfrak{D}$. This proves the assertion of the lemma. ∎

Corollary 4.17. *An equation in the language of SCA_α is true of SCA_α iff it is true of CA_α (with substitutions defined as usual).*

Proof: Suppose that ε is an equation in the language of SCA_α. Assume first that ε is valid in SCA_α. Let $\mathfrak{A}$ be any CA_α, $\mathfrak{B}$ the definitional extension of $\mathfrak{A}$ to a SCEA_α guaranteed by Lemma 4.15, and $\mathfrak{C}$ the reduct of $\mathfrak{B}$ to a SCA_α. Then ε holds in $\mathfrak{C}$ by assumption and therefore also in $\mathfrak{A}$ (more precisely, in $\mathfrak{B}$).

Now assume that ε is valid in CA_α. Let $\mathfrak{C}$ be an arbitrary SCA_α, $\mathfrak{B}$ the definitional extension to a SCEA_α of the perfect extension of $\mathfrak{C}$, and $\mathfrak{A}$ the reduct of $\mathfrak{B}$ to CA_α. Then ε holds in $\mathfrak{A}$ (more precisely, in $\mathfrak{B}$) by assumption. Therefore it must hold in the subreduct $\mathfrak{C}$. ∎

Thus, when we wish to use a law that is valid in SCA_α we may refer to the corresponding law for CA_α.

Theorem 4.18. *Let $\alpha \geq 2$.*

(i) *Every α-dimensional rectangularly dense substitution cylindrification algebra with equality is representable.*
(ii) *Every α-dimensional rectangularly dense substitution cylindrification algebra without equality that has completely additive substitutions is representable.*

Proof: We sketch the proof. First, suppose that $\mathfrak{A}$ is a SCEA_α, and let $\mathfrak{B}$ be its cylindric reduct. Then $\mathfrak{B}$ is a cylindric algebra and $\mathfrak{A}$ is a term definitional extension of $\mathfrak{B}$, by Lemma 4.15. Assume that $\mathfrak{A}$ is rectangularly dense. Then certainly $\mathfrak{B}$ is rectangularly dense and hence representable by Theorem 3.11. The representation must preserve term-definable operations, so we also have a representation of $\mathfrak{A}$.

Now suppose that $\mathfrak{A}$ is a SCA_α in which the substitutions are completely additive. Then by the results of Monk [1970] $\mathfrak{A}$ has a completion $\mathfrak{B}$. One readily checks that $\mathfrak{B}$ is also a SCA_α. Also, just as in the case of quasi-polyadic algebras (see the proof of Theorem 4.7), one can show that $\mathfrak{B}$ is the reduct of a SCEA_α, $\mathfrak{C}$. Assuming the rectangular density of $\mathfrak{A}$, we easily check that $\mathfrak{B}$ and $\mathfrak{C}$ inherit this rectangular density. By the results of the preceding paragraph, $\mathfrak{C}$ is representable. This gives us a representation of $\mathfrak{B}$ and hence also of $\mathfrak{A}$. ∎

The laws in Lemma 4.2(i)–(xi),(xx) are valid in CA_α, by the results of Sections 1.2 and 1.5 in Henkin-Monk-Tarski [1971] (see, in particular, Theorems 1.5.8 and 1.5.9). Hence, they are valid in SCA_α, by Corollary 4.17. Therefore, the proof that substitutions are completely additive in a quasi-atomic, rectangularly dense QPA_α carries over without change to the case of SCA_α.

Lemma 4.19. *Let $\mathfrak{A}$ be a quasi-atomic, rectangularly dense SCA_α. Then each substitution is completely additive.*

Consequently, we arrive at the following characterization of representability in SCA_α.

Corollary 4.20. *Let $\mathfrak{A}$ be a SCA_α, where $\alpha \geq 2$. Then the following are equivalent:*

(i) $\mathfrak{A}$ *is representable;*
(ii) $\mathfrak{A}$ *is embeddable in an atomic, rectangularly dense SCA_α;*
(iii) $\mathfrak{A}$ *is embeddable in a quasi-atomic, rectangularly dense SCA_α;*
(iv) $\mathfrak{A}$ *is embeddable in a rectangularly dense, completely additive SCA_α.*

Problem 4.21. Can the hypothesis of the complete additivity of s_{01} in Theorem 4.7 and of the complete additivity of all the (non-trivial) substitutions in Theorem 4.18 be dropped? That is to say, is every rectangularly dense QPA_α or SCA_α representable?[8]

Problem 4.22. Does every QPA_α and every SCA_α have a completion?

Notice that a positive solution to the second problem would give a positive solution to the first problem as well.

5. Rich cylindric and quasi-polyadic algebras

In Henkin-Monk-Tarski [1985] the concept of an i-thin element is introduced, and with its help the notions of a rich cylindric algebra and a rich quasi-polyadic equality algebra are defined (see pp. 60, 242, and in particular, Definition 3.2.1). It is shown in Theorems 3.2.13 and 5.4.34 that a cylindric algebra or a quasi-polyadic equality algebra of finite dimension $\alpha \geq 2$ is representable iff it is embeddable into a rich cylindric algebra or quasi-polyadic equality algebra (of the same dimension) in which the so-called Henkin equations are universally valid. (For quasi-polyadic equality algebras the theorem is only formulated for $\alpha \geq 3$, but the proof goes through for the case $\alpha = 2$ as well.) The proof is an algebraic version of the proof of the Completeness Theorem for first-order logic (see *op. cit.*, pp. 59–64). The authors then ask in Problem 5.6 whether this result can be extended to quasi-polyadic algebras (without equality). (Actually, they refer to polyadic algebras instead of quasi-polyadic algebras, but recall that for finite dimensions the two notions coincide.) Since the concept of an i-thin element is defined with the help of diagonal elements, part of the problem consists in finding an appropriate definition of i-thinness when diagonal elements are not available.

We shall introduce a form of the definition of an i-thin element and of a rich algebra that does not depend on the presence of diagonal elements, but only on cylindrifications. Therefore, it can be applied to any of the classes of algebras studied in Sections 3 and 4. From this definition it will be seen that richness can be viewed as a density notion, similar in spirit to rectangular density. We shall then give two distinct characterizations of richness, one in terms of rectangular density and the other in terms of point density. As a consequence of these characterizations, we shall show that (for the classes studied in Sections 3 and 4) a rich algebra of finite (or locally finite) dimension is representable. Thus, we will obtain a form of the Henkin-Tarski representation theorem that does not require the presence of the Henkin equations. We will also obtain a positive solution to Problem 5.6. Finally, we shall discuss the relationship of our notions of thinness and richness to those in *op. cit.*, and we shall give some historic remarks.

Recall that the *dimension set* of an element x is the set

$$\Delta x = \{i < \alpha : c_i(x) \neq x\}.$$

[8] Quite recently, Andréka, Givant, and Németi have obtained a positive solution to this problem. The case $\alpha < \omega$ requires a new argument; the case $\alpha \geq \omega$ follows from the case $\alpha < \omega$ by the argument in the second paragraph of the proof of Theorem 3.11.

Recall also that an algebra is said to be *locally finite dimensional* if Δx is finite for each element x of the algebra. Observe that a finite dimensional algebra is always locally finite dimensional.

In what follows, assume that $\mathfrak{A}$ is a CA_α, a QPEA_α, a QPA_α, or an SCA_α. (We ignore SCEA_α because this is just a definitional extension of CA_α.) When establishing abstract properties of $\mathfrak{A}$ we shall need to refer to various laws concerning cylindrifications and substitutions that are valid not only in CA_α, but also in QPA_α, QPEA_α, and SCA_α. Among these laws are postulates (Q1) and (Q4)–(Q10), and Lemma 4.2(ii)–(xi) and the first part of (i).

Definition 5.1. Let x be an element of $\mathfrak{A}$ and suppose that $i < \alpha$. Then x is *i-thin* if

(i) $c_j(x) = x$ for all $j < \alpha$ with $j \neq i$, i.e., $\Delta x \subseteq \{i\}$,
(ii) $x \cdot c_i(x \cdot y) \leq y$ for every y in $\mathfrak{A}$.

∎

Thin elements play the role in algebraic logic that individual constants play in model theory and, in particular, in Henkin's proof of the Completeness Theorem for first-order logic; see *op. cit.*, pp. 59–63. If we think of x as representing a formula, then condition (i) asserts that x has at most one free variable and condition (ii) says that for every formula y, if there is an element satisfying both x and y (with a given set of parameters for the other variables of y), then every element satisfying x must satisfy y (with the given parameters). In other words, no formula can distinguish the elements that satisfy x from one another. The Leibniz law asserts that two things are equal iff they are indistinguishable from one another. From this point of view a thin element corresponds to a formula that is satisfied by at most one element.

The next lemma sheds some light on the set-theoretic intuition behind the definition of thin elements.

Lemma 5.2. *Let $i < \alpha < \omega$ and suppose that $\mathfrak{A}$ is the full set algebra on a non-empty set U. Then an element X of $\mathfrak{A}$ is i-thin iff $X = \varnothing$ or there is a $u \in U$ such that*

$$X = \{x \in {}^{\alpha}U : x_i = u\}. \tag{i}$$

Proof: The empty set obviously satisfies the conditions of Definition 5.1. Suppose, now, that for a fixed $u \in U$ the set X has the form (i). It is easy to verify condition 5.1(i), so we concentrate on (ii). First some notation: for a sequence $x \in {}^{\alpha}U$ and an element $u \in U$, let $x[i\backslash u]$ denote the sequence obtained from x by replacing the i^{th} coordinate x_i with u. Now let Y be an arbitrary subset of ${}^{\alpha}U$. Using the definitions of X and of set-theoretic cylindrification, we have

$$X \cap Y = \{x \in Y : x_i = u\} \quad \text{and} \quad C_i(X \cap Y) = \{x \in {}^{\alpha}U : x[i\backslash u] \in Y\},$$

so

$$\begin{aligned} X \cap C_i(X \cap Y) &= \{x \in C_i(X \cap Y) : x_i = u\} \\ &= \{x \in {}^{\alpha}U : x_i = u \text{ and } x \in Y\} \\ &= X \cap Y. \end{aligned}$$

Thus, condition 5.1(ii) is satisfied.

Now suppose that X is an i-thin element in $\mathfrak{A}$. From condition 5.1(i) we see that X must have the form

$$X = \{x \in {}^{\alpha}U : x_i \in Z\}$$

for some set $Z \subseteq U$. (Here we are using the assumption that α is finite.) If Z is empty, then clearly so is X. Suppose that Z is not empty. Fix an element $u \in Z$ and put

$$Y = \{x \in {}^{\alpha}U : x_i = u\}.$$

Then

$$C_i(Y) = {}^{\alpha}U \quad \text{and} \quad X \cap Y = Y. \tag{1}$$

Using (1) and the assumption that X satisfies condition (ii), we obtain $X \subseteq Y$. The reverse inclusion follows from the second equation in (1). Thus, X has the form (i). ∎

From the preceding lemma we see that in an α-dimensional space (with α finite), i-thin elements are either empty or have dimension $\alpha - 1$. Thus, from the point of view of dimension they are thin (or flat) in comparison with elements of dimension α, much as lines are thin in a two dimensional world and planes are thin in a three dimensional world. It is from this geometric perspective that the name "i-thin" derives. (See Figures 3 and 4.)

FIGURE 3. 0-thin and 1-thin elements in two dimensions.

Thin elements are not quite quasi-atoms, but in the set of elements with dimension set included in $\{i\}$ they are quasi-atoms. The next lemma expresses this fact.

FIGURE 4. A 2-thin element in three dimensions.

Lemma 5.3. *Suppose that $\mathfrak{A}$ has finite dimension $\alpha \geq 1$ and that x is an i-thin element. If $w \leq x$ and $\Delta w \subseteq \{i\}$, then w is i-thin and in fact $w = x \cdot c_{(\alpha)}(w)$. Thus, x is a quasi-atom in the set $\{y : \Delta y \subseteq \{i\}\}$.*

Proof: Suppose that w and x satisfy the hypotheses of the lemma. Then

$$\begin{aligned} x \cdot c_{(\alpha)}(w) &= x \cdot c_i(w) && \text{since } \Delta w \subseteq \{i\}, \\ &= x \cdot c_i(x \cdot w) && \text{since } w \leq x, \\ &\leq w && \text{since } x \text{ is } i\text{-thin}, \\ &\leq x \cdot c_{(\alpha)}(w) && \text{by (Q4) and } w \leq x. \end{aligned}$$

This shows that $w = x \cdot c_{(\alpha)}(w)$. To check condition 5.1(ii), suppose that y is an arbitrary element. Then

$$w \cdot c_i(w \cdot y) \leq x \cdot c_i(x \cdot y) \leq y$$

by monotony and the i-thinness of x. ∎

We remark in passing that not all quasi-atoms in the set $\{y : \Delta \subseteq \{i\}\}$ need be i-thin. For example, in the algebra $\mathfrak{B}$ from Example 5.23 below, the element ${}^{\alpha}U$ is a quasi-atom in the set

$$\{y : \Delta y \subseteq \{i\}\} = \{\varnothing, {}^{\alpha}U\},$$

but it is not i-thin.

The following definition is a modification of Definition 3.2.1(ii) in Henkin-Monk-Tarski [1985], and is based on our notion (as opposed to their notion) of 0-thinness.

Definition 5.4. Suppose that $\mathfrak{A}$ has dimension $\alpha \geq 1$. Then $\mathfrak{A}$ is *rich* if, for every non-zero y with $\Delta y \subseteq \{0\}$, there is a non-zero 0-thin x below y. ∎

Full set algebras are natural examples of rich algebras.

Lemma 5.5. *If $\mathfrak{A}$ is a full set algebra of finite dimension $\alpha \geq 1$, then $\mathfrak{A}$ is rich.*

Proof: Let Y be a non-zero element of $\mathfrak{A}$ such that $C_j(Y) = Y$ for every non-zero $j < \alpha$. Then there is a non-empty set $Z \subseteq U$ such that Y has the form

$$Y = \{x \in {}^{\alpha}U : x_0 \in Z\}. \tag{1}$$

Choose any $u \in Z$ and set

$$X = \{x \in {}^{\alpha}U : x_0 = u\}.$$

Then X is a 0-thin element of $\mathfrak{A}$ by Lemma 5.2, and $X \subseteq Y$ by (1). Thus, $\mathfrak{A}$ is rich. ∎

Remark 5.6. Let $\psi(x)$ be the conjunction of the equations that together express the assertion $\Delta x \subseteq \{0\}$, let $\vartheta(x)$ be the formula

$$\forall y[x \cdot c_0(x \cdot y) \leq y],$$

and let φ be the conjunction of ψ and ϑ. Thus, φ is a positive formula expressing the notion of 0-thinness. It is clear that richness coincides with the notion of φ-density in ψ. Let's verify that ψ satisfies the three conditions of Theorem 2.15. Since ψ is positive, it is preserved under homomorphisms. To check strict preservation, let f map $\mathfrak{A}$ homomorphically onto $\mathfrak{B}$ and suppose that y is in $\psi^{\mathfrak{B}}$, i.e., $\Delta y \subseteq \{0\}$. Let z be an element in $\mathfrak{A}$ such that $f(z) = y$, and set $x = c_{(\alpha\sim\{0\})}(z)$. Then x is obviously in $\psi^{\mathfrak{A}}$. Further,

$$f(x) = f(c_{(\alpha\sim\{0\})}(z)) = c_{(\alpha\sim\{0\})}(f(z)) = c_{(\alpha\sim\{0\})}(y) = y,$$

since $\Delta y \subseteq \{0\}$. This verifies condition (i) in 2.15. To verify (ii) suppose that $y \in \psi^{\mathfrak{A}}$ and $x \in A$. For each i with $0 < i < \alpha$ we have

$$c_i(y \cdot c_{(\alpha)}(x)) = c_i(y) \cdot c_{(\alpha)}(x) = y \cdot c_{(\alpha)}(x)$$

by 4.2(v) and the assumpution $\Delta y \subseteq \{0\}$. This shows that the element $y \cdot c_{(\alpha)}(x)$ is in $\psi^{\mathfrak{A}}$. To verify condition (iii) let $x \in A$ and set $y = c_{(\alpha\sim\{0\})}(x)$. Then y is in $\psi^{\mathfrak{A}}$ and

$$c_{(\alpha)}(y) = c_{(\alpha)}c_{(\alpha\sim\{0\})}(x) = c_{(\alpha)}(x).$$

∎

The term "rich" was probably employed by Henkin and Tarski to express that an algebra possesses a sufficient number of elements acting like constants (much as a first-order theory is sometimes called "rich" if it possesses a sufficient number of constants acting as witnesses to existential assertions).[9] But from our perspective a better terminology might be 0-*thin density* (in the set of elements with dimension set included in $\{0\}$). It is natural to ask whether 0-thin density implies i-thin density for each $i < \alpha$. Our first task (which is accomplished in Corollary 5.9) is to prove that this is indeed the case.

Lemma 5.7. *Suppose that $\mathfrak{A}$ is rich and has finite dimension $\alpha \geq 1$. Then every element whose dimension set is included in $\{0\}$ is the sum of the 0-thin elements below it.*

Proof: Let y be any element such that $\Delta y \subseteq \{0\}$, and set

$$X = \{x : x \text{ is 0-thin and } x \leq y\}.$$

Clearly y is an upper bound for X.

Suppose that z is an element such that $y \cdot -z \neq 0$. Then

$$0 \neq c_{(\alpha\sim\{0\})}(y \cdot -z) = y \cdot c_{(\alpha\sim\{0\})}(-z)\,,$$

by 4.2(v) and $\Delta y \subseteq \{0\}$. Because $\mathfrak{A}$ is rich there is a non-zero 0-thin element $w \leq y \cdot c_{(\alpha\sim\{0\})}(-z)$. Since $w \leq y$, we have $w \in X$. Since $0 < w \leq c_{(\alpha\sim\{0\})}(-z)$, we have

$$\begin{aligned} &0 \neq w \cdot c_{(\alpha\sim\{0\})}(-z) && \text{by BA,}\\ &0 \neq -z \cdot c_{(\alpha\sim\{0\})}(w) && \text{by 4.2(vi),}\\ &0 \neq -z \cdot w && \text{since } \Delta w \subseteq \{0\}. \end{aligned}$$

Because $w \in X$ and $w \not\leq z$, the element z cannot be an upper bound for X. Thus, if z is any upper bound for X, then $y \cdot -z = 0$ and therefore $y \leq z$. ∎

Lemma 5.8. *Suppose that $\mathfrak{A}$ is rich and of finite dimension $\alpha \geq 1$. If x is a 0-thin element in $\mathfrak{A}$, then $s_{0i}(x)$ is i-thin.*

Proof: When $i = 0$ we have $s_{0i}(x) = x$, so the lemma is trivial. Suppose that $0 \neq i$. We verify the two conditions of i-thinness. For $k \in \alpha - \{0, i\}$ we have

$$c_k s_{0i}(x) = s_{0i} c_k(x) = s_{0i}(x)$$

by (Q8) and $\Delta x \subseteq \{0\}$. Also,

$$c_0 s_{0i}(x) = s_{0i}(x)$$

by (Q7). This verifies condition 5.1(i).

Before verifying condition 5.1(ii) we make some preliminary observations.

(1) If $y \leq s_{0i}(x)$, then $s_{i0}c_0(y) \leq x$.

[9] The approach of Halmos [1962], p. 143, to constants is not via elements of the algebra — in our case, thin elements — but via systems of Boolean endomorphisms. His definition of a "rich" (polyadic) algebra is based on his notion of a constant (see *op. cit.*, p. 157).

Indeed, suppose that $y \le s_{0i}(x)$. Then

$$c_0(y) \le c_0 s_{0i}(x) = s_{0i}(x) \tag{2}$$

by 4.2(ii) and (Q7). Hence,

$$\begin{aligned} s_{i0}c_0(y) &\le s_{i0}s_{0i}(x) && \text{by 4.2(i),} \\ &= s_{i0}(x) && \text{by 4.2(ix),} \\ &= x && \text{by 4.2(xx), since } \\ & && \Delta x \subseteq \{0\}. \end{aligned}$$

This proves (1).

(3) If $y \le s_{0i}(x)$, then $s_{0i}(x) \cdot c_0 c_i(y) = c_0(y)$.

For the proof, suppose that $y \le s_{0i}(x)$. Then

$$c_0(y) \le s_{0i}(x) \cdot c_0 c_i(y)$$

by (2), (Q4), and 4.2(ii). For the reverse inequality we have

$$\begin{aligned} x \cdot c_0 c_i(y) &= x \cdot c_i c_0(y) && \text{by 4.2(vii),} \\ &= x \cdot c_i s_{0i} c_0(y) && \text{by (Q6),} \\ &= x \cdot c_0 s_{i0} c_0(y) && \text{by 4.2(xi),} \\ &= x \cdot c_0(x \cdot s_{i0} c_0(y)) && \text{by (1),} \\ &\le s_{i0} c_0(y) && \text{since } x \text{ is 0-thin.} \end{aligned}$$

Applying s_{0i} to both sides yields

$$\begin{aligned} s_{0i}(x \cdot c_0 c_i(y)) &\le s_{0i} s_{i0} c_0(y) && \text{by 4.2(i),} \\ s_{0i}(x) \cdot s_{0i} c_0 c_i(y) &\le s_{0i} c_0(y) && \text{by 4.2(i),(ix),} \\ s_{0i}(x) \cdot c_0 c_i(y) &\le c_0(y) && \text{by (Q6).} \end{aligned}$$

This proves (3).

To verify condition 5.1(ii) let y be an arbitrary element of $\mathfrak{A}$ and set

$$z = s_{0i}(x) \cdot c_i(s_{0i}(x) \cdot y). \tag{4}$$

We must show that $z \le y$.

(5) If w is 0-thin, then $w \cdot z \le y$.

Indeed,

$$\begin{aligned} w \cdot z &= w \cdot s_{0i}(x) \cdot w \cdot c_i(s_{0i}(x) \cdot y) && \text{by (4) and BA,} \\ &= w \cdot s_{0i}(x) \cdot c_i(w \cdot s_{0i}(x) \cdot y) && \text{since } \Delta w \subseteq \{0\} \text{ and } i \ne 0, \\ &\le w \cdot s_{0i}(x) \cdot c_0 c_i(w \cdot s_{0i}(x) \cdot y) && \text{by (Q4),} \\ &= w \cdot c_0(w \cdot s_{0i}(x) \cdot y) && \text{by (3) with ``} w \cdot s_{0i}(x) \cdot y \text{'' for ``} y \text{'',} \\ &\le s_{0i}(x) \cdot y && \text{since } w \text{ is 0-thin,} \\ &\le y && \text{by BA.} \end{aligned}$$

FIGURE 5. A 0-thin element x and its s_{01} substitution.

Now let X be the set of 0-thin elements below $c_{(\alpha\sim\{0\})}(z)$. Then

(6) $$c_{(\alpha\sim\{0\})}(z) = \sum X$$

by the richness of $\mathfrak{A}$ and the preceding lemma. Hence,

$$\begin{aligned} z &= z \cdot c_{(\alpha\sim\{0\})}(z) && \text{by (Q4),} \\ &= z \cdot \sum X && \text{by (6),} \\ &= \sum\{z \cdot w : w \in X\} && \text{by BA,} \\ &\leq y && \text{by (5).} \end{aligned}$$

∎

Corollary 5.9. *Suppose that $\mathfrak{A}$ is rich and has finite dimension $\alpha \geq 1$. Then for every non-zero y with $\Delta y \subseteq \{i\}$ there is a non-zero i-thin x below y.*

Proof: If $i = 0$, then this follows at once from the definition of richness. Suppose that $i \neq 0$. Then

(1) $$s_{0i}(y) = y$$

by 4.2(xx) and the assumptions $\Delta y \subseteq \{i\}$ and $i \neq 0$. Now

$$c_k s_{i0}(y) = s_{i0} c_k(y) = s_{i0}(y) \quad \text{for } k \neq 0, i,$$

by (Q8) and $\Delta y \subseteq \{i\}$, and

$$c_i s_{i0}(y) = s_{i0}(y)$$

by (Q7). In other words, $\Delta s_{i0}(y) \subseteq \{0\}$. Hence, using the assumption that $\mathfrak{A}$ is rich, we can find a non-zero 0-thin element $x \leq s_{i0}(y)$. Applying s_{0i} to both sides, we obtain

$$\begin{array}{ll} s_{0i}(x) \leq s_{0i}s_{i0}(y) & 4.2(\mathrm{i}), \\ s_{0i}(x) \leq s_{0i}(y) & \text{by } 4.2(\mathrm{ix}), \\ s_{0i}(x) \leq y & \text{by } (1). \end{array}$$

The element $s_{0i}(x)$ is i-thin by Lemma 5.8. Further, $s_{0i}(x) \neq 0$ since

$$0 \neq x = s_{i0}(x) = s_{i0}s_{0i}(x)$$

by $\Delta x \subseteq \{0\}$ and 4.2(ix),(xx). Therefore, the conclusion of the lemma is satisfied by $s_{0i}(x)$ (in place of x). ∎

The notion of a *point* in relation algebras occurs implicitly in Theorem 4.30 of Jónsson-Tarski [1952] and was studied by several subsequent researchers (see, for example, Maddux [1991][10]). We now introduce a notion of a point that is appropriate for finite dimensional algebras with cylindrification operations.

Definition 5.10. Suppose that $\mathfrak{A}$ is of finite dimension $\alpha \geq 1$. An element x is a *point* if $c_{(\alpha\sim\{i\})}(x)$ is i-thin for each $i < \alpha$. ∎

Lemma 5.11. *Suppose $\mathfrak{A}$ has finite dimension $\alpha \geq 1$. Then every point is a rectangle.*

Proof: Assume that x is a point. Then

(1) $\quad c_{(\alpha\sim\{i\})}(x)$ is i-thin for each $i < \alpha$,

by definition. From (Q4) we see that

(2) $\quad c_{(\Delta)}(x) \leq c_{(\alpha\sim\{i\})}(x)$ for $\Delta \subseteq \alpha - \{i\}$.

Using 4.2(vii), (2), and (1), we obtain

(3) $\quad c_{(\alpha\sim\{i\})}(x) \cdot c_{(\Delta\cup\{i\})}(x) \leq c_{(\Delta)}(x)$ for $\Delta \subseteq \alpha - \{i\}$

as follows:

$$\begin{aligned} c_{(\alpha\sim\{i\})}(x) \cdot c_{(\Delta\cup\{i\})}(x) &= c_{(\alpha\sim\{i\})}(x) \cdot c_i c_{(\Delta)}(x) \\ &= c_{(\alpha\sim\{i\})}(x) \cdot c_i[c_{(\alpha\sim\{i\})}(x) \cdot c_{(\Delta)}(x)] \\ &\leq c_{(\Delta)}(x)\,. \end{aligned}$$

[10] In Maddux's work a narrower notion of a point is adopted, one that corresponds to points lying below the main diagonal, i.e., below the identity element. (One might call these *diagonal points* or *subidentity points*.) However, the more general notion of a point is implicit in his Theorems 40 and 41.

We now apply (3) α times, starting with $\Delta = \alpha \sim \{0,1\}$ and $i = 1$, then taking $\Delta = \alpha \sim \{0,1,2\}$ and $i = 2$, and so on, to get

$$\begin{aligned}\textstyle\prod_{i<\alpha} c_{(\alpha\sim\{i\})}(x) &\leq \textstyle\prod_{1\leq i<\alpha} c_{(\alpha\sim\{i\})}(x) \cdot c_{(\alpha\sim\{0,1\})}(x)\\ &\leq \textstyle\prod_{2\leq i<\alpha} c_{(\alpha\sim\{i\})}(x) \cdot c_{(\alpha\sim\{0,1,2\})}(x)\\ &\leq \cdots\\ &\leq c_{(\alpha\sim\alpha)}(x)\\ &= x\,.\end{aligned}$$

Thus,

$$\textstyle\prod_{i<\alpha} c_{(\alpha\sim\{i\})}(x) \leq x\,.$$

The reverse inclusion follows from (Q4). Hence, x is a rectangle by Lemma 3.7. (See Figure 6.) ∎

FIGURE 6. A point x is a rectangle: it is the intersection of the 0-thin element $c_{12}(x)$, the 1-thin element $c_{02}(x)$, and the 2-thin element $c_{01}(x)$.

The terminology "point" derives from the fact that in a full set algebra an element is a point iff it is empty or a singleton:

Lemma 5.12. *Suppose $\mathfrak{A}$ is the full set algebra of finite dimension $\alpha \geq 1$ on a set U. Then an element X in $\mathfrak{A}$ is a point iff $X = \varnothing$ or there are $u_0, \ldots, u_{\alpha-1}$ in U such that $X = \{(u_0, \ldots, u_{\alpha-1})\}$.*

Proof: Suppose X is a non-empty point in $\mathfrak{A}$. Then $C_{(\alpha\sim\{i\})}(X)$ is non-empty and i-thin by definition. By Lemma 5.2 there is an element u_i in U such that

(1) $$C_{(\alpha\sim\{i\})}(X) = \{w \in {}^{\alpha}U : w_i = u_i\}.$$

Therefore,

$$\begin{aligned} X &= \textstyle\bigcap_{i<\alpha} C_{(\alpha\sim\{i\})}(X) && \text{by 5.11,} \\ &= \textstyle\bigcap_{i<\alpha} \{w \in {}^{\alpha}U : w_i = u_i\} && \text{by (1),} \\ &= \{(u_0, \ldots, u_{\alpha-1})\}. \end{aligned}$$

For the converse, it is easy to check that the empty set is a point. Suppose that $X = \{(u_0, \ldots, u_{\alpha-1})\}$. Then

$$C_{(\alpha\sim\{i\})}(X) = \{w \in {}^{\alpha}U : w_i = u_i\}$$

by definition of set cylindrification, and this element is i-thin by Lemma 5.2. Therefore X is a point. ∎

In analogy with Lemma 3.7 we have the following characterization of points.

Lemma 5.13. *An element x is a point iff there exist elements $y_0, \ldots, y_{\alpha-1}$ such that y_i is i-thin for each $i < \alpha$ and $x = \prod_{i<\alpha} y_i$.*

Proof: If x is a point, then set $y_i = c_{(\alpha\sim\{i\})}(x)$. By Definition 5.10 the element y_i is i-thin and by Lemma 5.11 we have $x = \prod_{i<\alpha} y_i$.

For the reverse implication, suppose that y_i is i-thin and that $x = \prod_{i<\alpha} y_i$. Then

$$c_{(\alpha\sim\{i\})}(x) \leq c_{(\alpha\sim\{i\})}(y_i) = y_i,$$

since $\Delta y_i \subseteq \{i\}$. Hence, $c_{(\alpha\sim\{i\})}(x)$ is i-thin for each i, by Lemma 5.3. By Definition 5.10, x is a point. ∎

The next two lemmas elucidate the relationship between points and quasi-atoms.

Lemma 5.14. *Suppose that $\mathfrak{A}$ has finite dimension $\alpha \geq 1$. Then every point is a quasi-atom and every element below a point is again a point. In particular, if $\mathfrak{A}$ is simple, then every non-zero point is an atom.*

Proof: Suppose

(1) x is a point.

Let $y \leq x$. Then $c_{(\alpha\sim\{i\})}(y) \leq c_{(\alpha\sim\{i\})}(x)$ for each $i < \alpha$. Therefore, by Lemma 5.3 $c_{(\alpha\sim\{i\})}(y)$ is i-thin for each $i < \alpha$. Hence,

(2) y is a point.

By the same lemma, $c_{(\alpha\sim\{i\})}(x)$ is a quasi-atom in the set of elements with dimension set included in $\{i\}$. Therefore

(3) $$c_{(\alpha\sim\{i\})}(y) = c_{(\alpha\sim\{i\})}(x) \cdot c_{(\alpha)}(y).$$

Hence,

$$\begin{aligned}
y &= \textstyle\prod_{i<\alpha} c_{(\alpha\sim\{i\})}(y) && \text{by (2) and 5.11,}\\
&= \textstyle\prod_{i<\alpha} c_{(\alpha\sim\{i\})}(x)\cdot c_{(\alpha)}(y) && \text{by (3),}\\
&= x\cdot c_{(\alpha)}(y) && \text{by (1) and 5.11.}
\end{aligned}$$

It follows that x is a quasi-atom. The final assertion of the lemma follows from Lemma 2.17(ii). ∎

A kind of converse to the preceding lemma is also true.

Lemma 5.15. *Suppose $\mathfrak{A}$ has finite dimension $\alpha \geq 1$. If $\mathfrak{A}$ is rectangularly dense, then every quasi-atom x is a point.*

Proof: Assume that $\mathfrak{A}$ is rectangularly dense. Let x be a quasi-atom and fix an $i < \alpha$. We must show that $c_{(\alpha\sim\{i\})}(x)$ is i-thin. It clearly suffices to show that for any given y we have

$$c_{(\alpha\sim\{i\})}(x)\cdot c_i(c_{(\alpha\sim\{i\})}(x)\cdot y) \leq y\,. \tag{1}$$

Suppose that z is a non-zero rectangle below $c_{(\alpha\sim\{i\})}(x)\cdot c_i(c_{(\alpha\sim\{i\})}(x)\cdot y)$.

$$c_{(\alpha\sim\{i\})}(z) = c_{(\alpha\sim\{i\})}(x)\cdot c_{(\alpha)}(z)\,. \tag{2}$$

To see this, notice that

$$c_{(\alpha\sim\{i\})}(z) \leq c_{(\alpha\sim\{i\})}(x) \tag{3}$$

by 4.2(ii),(iii),(vii), and the assumption that $z \leq c_{(\alpha\sim\{i\})}(x)$. Because x is a quasi-atom, we get

$$\begin{aligned}
x\cdot c_{(\alpha\sim\{i\})}(z) &= x\cdot c_{(\alpha)}(x\cdot c_{(\alpha\sim\{i\})}(z)) && \text{by 2.17(i),}\\
&= x\cdot c_i c_{(\alpha\sim\{i\})}(x\cdot c_{(\alpha\sim\{i\})}(z)) && \text{by 4.2(vii),}\\
&= x\cdot c_i(c_{(\alpha\sim\{i\})}(x)\cdot c_{(\alpha\sim\{i\})}(z)) && \text{by 4.2(v),}\\
&= x\cdot c_i(c_{(\alpha\sim\{i\})}(z)) && \text{by (3),}\\
&= x\cdot c_{(\alpha)}(z) && \text{by 4.2(vii).}
\end{aligned}$$

Applying $c_{(\alpha\sim\{i\})}$ to both sides, we see from 4.2(v) that

$$c_{(\alpha\sim\{i\})}(x)\cdot c_{(\alpha\sim\{i\})}(z) = c_{(\alpha\sim\{i\})}(x)\cdot c_{(\alpha)}(z)\,.$$

In view of (3), this gives us (2).

$$z\cdot y \neq 0\,. \tag{4}$$

Indeed,

$$0 < z \leq c_i(c_{(\alpha\sim\{i\})}(x)\cdot y) \qquad \text{and} \qquad z \leq c_{(\alpha)}(z)$$

by assumption and (Q4), so

$$\begin{aligned}
0 &< z\\
&\leq c_i(c_{(\alpha\sim\{i\})}(x)\cdot y)\cdot c_{(\alpha)}(z) && \text{by BA,}\\
&= c_i(c_{(\alpha\sim\{i\})}(x)\cdot c_{(\alpha)}(z)\cdot y) && \text{by 4.2(v),}\\
&= c_i(c_{(\alpha\sim\{i\})}(z)\cdot y) && \text{by (2).}
\end{aligned}$$

Therefore,

$$\begin{aligned} z \cdot c_i(c_{(\alpha\sim\{i\})}(z) \cdot y) &\neq 0 && \text{by BA,} \\ c_i(z) \cdot c_{(\alpha\sim\{i\})}(z) \cdot y &\neq 0 && \text{by 4.2(vi),} \\ z \cdot y &\neq 0 && \text{since } z \text{ is a rectangle.} \end{aligned}$$

This proves (4).

We have shown that for any non-zero rectangle z below

$$c_{(\alpha\sim\{i\})}(x) \cdot c_i(c_{(\alpha\sim\{i\})}(x) \cdot y)$$

we have $z \cdot y \neq 0$. Hence, for such a z we cannot have $z \leq -y$. In other words there can be no non-zero rectangles below

$$(5) \qquad c_{(\alpha\sim\{i\})}(x) \cdot c_i(c_{(\alpha\sim\{i\})}(x) \cdot y) \cdot -y\,.$$

Since $\mathfrak{A}$ is assumed to be rectangularly dense, that means that (5) is 0. Assertion (1) is proved. ∎

Definition 5.16. Suppose $\mathfrak{A}$ has finite dimension $\alpha \geq 1$.

(i) $\mathfrak{A}$ is *point dense* iff every non-zero element is above a non-zero point.

(ii) $\mathfrak{A}$ is *diagonally point dense* if every non-zero subdiagonal element is above a non-zero point.

∎

Remark 5.17. If we take φ to be the (positive) formula that defines a point, then it becomes clear that point density is just φ-density.

Diagonal point density is another example of the notion of φ-density in ψ. Here ψ is the formula $x \leq d$ defining the subdiagonal elements and φ is the conjunction of ψ and the formula defining points. Notice that ψ satisfies conditions (i)–(iii) of Theorem 2.15. Indeed, since ψ is positive, it is preserved under homomorphisms. Now let f map $\mathfrak{A}$ homomorphically onto $\mathfrak{B}$ and suppose that $y \in \psi^{\mathfrak{B}}$. Choose $z \in A$ so that $f(z) = y$. Set $x = d \cdot z$. Then $x \in \psi^{\mathfrak{A}}$ and

$$f(x) = f(d \cdot z) = f(d) \cdot f(z) = d \cdot y = y\,.$$

This verifies condition (i). Condition (ii) is trivially satisfied. For condition (iii) let $x \in A$ and set $y = d \cdot c_{(\alpha)}(x)$. Then

$$c_{(\alpha)}(y) = c_{(\alpha)}(d \cdot c_{(\alpha)}(x)) = c_{(\alpha)}(d) \cdot c_{(\alpha)}(x) = 1 \cdot c_{(\alpha)}(x) = c_{(\alpha)}(x)\,,$$

by 4.2(v) and 3.3(iii). ∎

Theorem 5.18. *Suppose that $\mathfrak{A}$ is a CA_α, $QPEA_\alpha$, QPA_α, or SCA_α with finite dimension $\alpha \geq 1$. Then the following conditions are equivalent:*

(i) *$\mathfrak{A}$ is rich;*

(ii) *$\mathfrak{A}$ is point dense;*

(iii) *$\mathfrak{A}$ is rectangularly dense and quasi-atomic.*

If $\mathfrak{A}$ is a CA_α or a $QPEA_\alpha$, then each of the above conditions is equivalent to each of the following conditions:

(iv) *$\mathfrak{A}$ is rectangularly dense;*

(v) *$\mathfrak{A}$ is diagonally point dense.*

In case $\mathfrak{A}$ is simple we may replace "quasi-atomic" by "atomic" in (iii).

Proof: For the implication from (i) to (ii) suppose that $\mathfrak{A}$ is rich. To show that $\mathfrak{A}$ is point dense, fix a non-zero y. We define by induction on $k \leq \alpha$ a sequence $x_0, \ldots, x_{k-1}$ of elements such that

(1) $\quad x_i$ is i-thin

for each $i < k$ and

$$y \cdot \textstyle\prod_{i<k} x_i \neq 0\,. \tag{2}$$

When $k = 0$ both (1) and (2) hold vacuously because $y \neq 0$. Suppose now that $0 \leq k < \alpha$ and that for $i < k$ the elements x_i have been defined so that (1) and (2) hold. From (Q4) and (2) we see that

$$c_{(\alpha\sim\{k\})}(y \cdot \textstyle\prod_{j<k} x_j) \neq 0\,. \tag{3}$$

Because $\mathfrak{A}$ is rich, we can use Corollary 5.9 to find a non-zero k-thin element x_k below the left-hand side of (3). Hence, (1) holds for $i = k$ and

$$x_k \cdot c_{(\alpha\sim\{k\})}(y \cdot \textstyle\prod_{j<k} x_j) \neq 0\,.$$

Applying 4.2(vi) repeatedly we get

$$y \cdot \textstyle\prod_{j<k} x_j \cdot c_{(\alpha\sim\{k\})}(x_k) \neq 0\,.$$

But $c_{(\alpha\sim\{k\})}(x_k) = x_k$ since $\Delta x_k \subseteq \{k\}$. Thus, we arrive at (2), with "$k+1$" in place of "k".

Set

$$w = \textstyle\prod_{i<\alpha} x_i\,.$$

Then w is a point by (1) and Lemma 5.13, so $w \cdot y$ is a point by Lemma 5.14. Since $w \cdot y \neq 0$, by (2) with $k = \alpha$, we see that $w \cdot y$ is a non-zero point beneath y. (See Figure 7.)

To establish the implication from (ii) to (iii), suppose that $\mathfrak{A}$ is point dense. Every point is a rectangle by Lemma 5.11. Moreover, every point is a quasi-atom by Lemma 5.14. Therefore, $\mathfrak{A}$ is rectangularly dense and quasi-atomic. If $\mathfrak{A}$ is simple, then every non-zero point is an atom by the same lemma, so $\mathfrak{A}$ is atomic.

To establish the implication from (iii) to (i), suppose that $\mathfrak{A}$ is rectangularly dense and quasi-atomic. To show that $\mathfrak{A}$ is rich, let y be any non–zero element with $\Delta y \subseteq \{0\}$. Because $\mathfrak{A}$ is quasi-atomic, there is a non-zero quasi-atom x below y. Then $c_{(\alpha\sim\{0\})}(x)$ is below y by 4.2(ii) and the assumption that $\Delta y \subseteq \{0\}$. Moreover, $c_{(\alpha\sim\{0\})}(x)$ is non-zero by (Q4) and 0-thin by Lemma 5.15 and the definition of a point. Thus, every non-zero element with dimension set included in $\{0\}$ has a non-zero 0-thin element below it.

Assume now that $\mathfrak{A}$ is a CA_α or a QPEA_α. Clearly, (iii) implies (iv). To establish the implication from (iv) to (v), suppose that $\mathfrak{A}$ is rectangularly dense. Let y be any non-zero subdiagonal element. By rectangular density, there is a non-zero rectangle x below y. From Lemma 3.8(i) we see that x must be a quasi-atom, and therefore a point, by Lemma 5.15. Thus, $\mathfrak{A}$ is diagonally point dense.

Now suppose that (v) holds, with the goal of establishing (i). Let y be a non-zero element of $\mathfrak{A}$ whose dimension set is included in $\{0\}$. Then $c_{(\alpha\sim\{0\})}(y) = y$,

FIGURE 7. Constructing a point w below a non-zero element y in a rich algebra.

so $y \cdot d \neq 0$ by Lemma 3.4. Invoking diagonal density, we obtain a non-zero point x below $y \cdot d$. Using (Q4) and 4.2(ii), we see that

$$0 < c_{(\alpha\sim\{0\})}(x) \leq c_{(\alpha\sim\{0\})}(y) = y\,.$$

Since x is a point, we conclude with the help of Definition 5.10 that $c_{(\alpha\sim\{0\})}(x)$ is a non-zero 0-thin element below y. Hence, $\mathfrak{A}$ is rich. This completes the proof of the theorem. ∎

Remark 5.19. From the previous theorem we see, e.g., that a simple, rich algebra is always atomic. Lemmas 5.15 and 5.14 give us a characterization of these atoms: they are precisely the non-zero points.

In contrast to the situation for CA_α and QPEA_α, in the case of QPA_α and SCA_α rectangular density does not imply quasi-atomicity. This is clear from Example 4.12, where we construct an atomless, rectangularly dense QPA_α $\mathfrak{A}$. Because $\mathfrak{A}$ is simple, a non-zero quasi-atom is the same thing as an atom. Hence, $\mathfrak{A}$ has no non-zero quasi-atoms. ∎

Theorem 5.20. *Suppose that $\mathfrak{A}$ is a CA_α, $QPEA_\alpha$, QPA_α, or SCA_α with (possibly infinite) dimension $\alpha \geq 2$. If $\mathfrak{A}$ is rich, and if either $\alpha < \omega$ or else $\alpha \geq \omega$ and $\mathfrak{A}$ is locally finite dimensional, then $\mathfrak{A}$ is representable.*

Proof: Assume that $\mathfrak{A}$ is rich. If $2 \leq \alpha < \omega$, then $\mathfrak{A}$ is rectangularly dense and quasi-atomic by Theorem 5.18. Therefore, it is representable by Theorem 3.11 in the CA_α case, 4.6 in the QPEA_α case, 4.10 in the QPA_α case, and 4.20 in the SCA_α case.

Now suppose that $\alpha = \omega$ and that $\mathfrak{A}$ is locally finite dimensional. For each n with $2 \leq n < \alpha$ let B_n be the set of elements whose dimension set is included in

n. Then B_n is easily seen to be a subuniverse of the n-reduct of $\mathfrak{A}$, i.e., the reduct of $\mathfrak{A}$ to the Boolean operations and to the extra-Boolean operations and constants whose indices are included in n (see Henkin-Monk-Tarski [1971], Theorem 2.6.27.) Let $\mathfrak{B}_n$ be the corresponding algebra. (This is usually called the *neat* n-reduct of $\mathfrak{A}$).

(1) Every 0-thin element of $\mathfrak{A}$ is in $\mathfrak{B}_n$ and is 0-thin in $\mathfrak{B}_n$.

To prove (1), let x be a 0-thin element of $\mathfrak{A}$. Then $\Delta x \subseteq \{0\} \subseteq n$, so x is in $\mathfrak{B}_n$. To show that x remains 0-thin in $\mathfrak{B}_n$, let y be any element of $\mathfrak{B}_n$. Then in $\mathfrak{A}$ we have

$$x \cdot c_0(x \cdot y) \leq y\,,$$

and this equation must of course continue to hold in the subreduct $\mathfrak{B}_n$.

(2) $\mathfrak{B}_n$ is rich.

For the proof, let y be a non-zero element of $\mathfrak{B}_n$ such that $\Delta y \subseteq \{0\}$ in $\mathfrak{B}_n$. Thus, $c_i(y) = y$ for every i such that $0 < i < n$. On the other hand, $c_i(y) = y$ whenever $n \leq i < \alpha$, by definition of $\mathfrak{B}_n$. Therefore, $\Delta y \subseteq \{0\}$ in $\mathfrak{A}$. Because $\mathfrak{A}$ is rich there must be a non-zero 0-thin element x below y in $\mathfrak{A}$. Then x is in $\mathfrak{B}_n$ by (1), and in $\mathfrak{B}_n$ it remains a non-zero 0-thin element below y.

From (2) and the finite dimensional case of the theorem that has already been proved, we conclude that

(3) $\mathfrak{B}_n$ is representable.

Let K_α and K_n be the classes of representable algebras of dimension α and of dimension n respectively. Thus, e.g., $\mathsf{K}_\alpha = \mathsf{RCA}_\alpha$ when $\mathfrak{A}$ is a CA_α, $\mathsf{K}_\alpha = \mathsf{RQPA}_\alpha$ when $\mathfrak{A}$ is a QPA_α, etc. Then K_α is a variety. (See the remarks in Sections 3 and 4 regarding the classes of representable algebras.) To show that $\mathfrak{A}$ is in K_α it therefore suffices to show that every equation true of K_α is also true of $\mathfrak{A}$. Let ε be any such equation, say with variables among $v_0, \ldots, v_{k-1}$. Let $(y_0, \ldots, y_{k-1})$ be a sequence of arbitrary elements of $\mathfrak{A}$. Because $\mathfrak{A}$ is locally finite dimensional, we can find a finite $n \geq 2$ such that $\Delta y_i \subseteq n$ for each $i < k$ and such that every extra-Boolean operation symbol of ε has its indices in n. Then $y_0, \ldots, y_{k-1}$ are in $\mathfrak{B}_n$ and ε is an equation in the language of K_n. Because ε holds in K_α it must hold in K_n. (A proof of this in the cylindric algebraic case is given in the paragraph following the definition of RCA_α. As was noted in Section 4, this proof can easily be extended to cover the cases QPEA_α, QPA_α, and SCA_α.) Now $\mathfrak{B}_n$ is in K_n by (3). Therefore ε holds in $\mathfrak{B}_n$. In particular, $(y_0, \ldots, y_{k-1})$ satisfies ε in $\mathfrak{B}_n$. Since ε is an equation and $\mathfrak{B}_n$ is a subreduct of $\mathfrak{A}$, we see that $(y_0, \ldots, y_{k-1})$ must satisfy ε in $\mathfrak{A}$ as well. It follows that ε is true of $\mathfrak{A}$, as was to be shown.

The proof in the case when $\alpha > \omega$ is essentially the same as in the case $\alpha = \omega$, but is notationally more complicated. We leave the details to the reader. ∎

In case $\alpha \geq \omega$, stronger results than the previous theorem are already known. In the remarks following Theorem 5.27 we shall explain what these results are and why we have bothered to include the case $\alpha \geq \omega$ in our theorem.

Corollary 5.21. *Suppose that $\mathfrak{A}$ is a CA_α, $QPEA_\alpha$, QPA_α, or SCA_α with finite dimension $\alpha \geq 2$. Then $\mathfrak{A}$ is representable iff it is embeddable into a rich algebra (from the same class).*

Proof: Suppose that $\mathfrak{A}$ is embeddable into a rich algebra $\mathfrak{B}$. Then $\mathfrak{B}$ is representable by the previous theorem, so $\mathfrak{A}$ is representable.

Now suppose that $\mathfrak{A}$ is representable. Then it is embeddable into the direct product of full set algebras. Each such full set algebra is rich by Lemma 5.5, and it is not difficult to check that the product of rich algebras is again rich. (Indeed, 0-thinness is defined by a collection of equations and a single equation preceded by a universal quantifier.) Such formulas are satisfied by a sequence $\langle x_\xi : \xi \in \Xi \rangle$ in a product $\mathfrak{B} = \prod_{\xi\in\Xi} \mathfrak{B}_\xi$ iff each coordinate x_ξ satisfies them in the appropriate factor algebra $\mathfrak{B}_\xi$. Thus, a sequence is 0-thin iff each coordinate is 0-thin. Similarly, a sequence $\langle y_\xi : \xi \in \Xi \rangle$ has dimension set included in $\{0\}$ iff this is true of each coordinate. Suppose $y = \langle y_\xi : \xi \in \Xi \rangle$ is non-zero and $\Delta y \subseteq \{0\}$. For each non-zero y_ξ (of which there must be at least one) choose a non-zero 0-thin x_ξ below y_ξ and otherwise set $x_\xi = 0$. Then $x = \langle x_\xi : \xi \in \Xi \rangle$ is a non-zero 0-thin element below y in $\mathfrak{B}$.) Thus $\mathfrak{A}$ is embeddable into a rich algebra. ∎

The remaining remarks in this section are historical, and concern principally the relationship of our notions and theorems for rich algebras to those of Henkin and Tarski. The following definition is due to Henkin and Tarski, and occurs as Definition 3.2.1 in Henkin-Monk-Tarski [1985].

Definition 5.22. Let $\mathfrak{A}$ be a CA_α with $2 \leq \alpha$. An element x is *i-thin* provided that

(i) $c_j(x) = x$ for $j \neq i$,
(ii) $x \cdot s_{ij}(x) \leq d_{ij}$ for some $j \neq i$,
(iii) $c_i(x) = 1$.

$\mathfrak{A}$ is *rich* provided that, for every non-zero y in $\mathfrak{A}$ with $\Delta y \subseteq \{0\}$, there is a 0-thin x such that $x \cdot c_0(y) \leq y$. ∎

Let us denote by ε_{ij} the equation

$$c_j[x \cdot y \cdot c_i(x \cdot -y)] \cdot -c_i[c_j(x) \cdot -d_{ij}] = 0\,.$$

Henkin and Tarski proved that for $2 \leq \alpha < \omega$ every rich CA_α $\mathfrak{A}$ in which the equations ε_{ij} are valid for distinct i, j is representable (see Theorem 2.11(iii) in Henkin-Tarski [1961] for an announcement of this theorem).[11] Condition (iii) in their definition of an i-thin element does not play a role in the proof when $\mathfrak{A}$ is simple. To obtain the general theorem from the simple case, one represents $\mathfrak{A}$ as a subdirect product of simple algebras. These simple factors are also rich and satisfy the same equations ε_{ij}. Hence, they are representable, by the simple case of the theorem. It follows that $\mathfrak{A}$ is representable. To prove that the simple factors

[11] It is possible to replace ε_{ij} with the following equation, which has only one variable:

$$d_{ij} \cdot c_j[-x \cdot c_i(x)] \cdot -c_i[c_j(x) \cdot -d_{ij}] = 0\,.$$

In fact, de Rijke-Venema [1995] have shown that for $i \neq j$ and $2 \leq \alpha < \omega$ this equation is valid in a CA_α $\mathfrak{A}$ iff ε_{ij} is valid in $\mathfrak{A}$.

are rich, one must ensure that the image of a non-zero 0-thin element under a homomorphism is non-zero. This is just what condition (iii) guarantees.

In a direct proof of our version of the Henkin-Tarski theorem, condition (iii) is not needed because we do not use an arbitrary subdirect decomposition of $\mathfrak{A}$. Instead, we use the decomposition guaranteed by Corollary 2.10. Thus, we do not have to consider arbitrary simple homomorphic images of $\mathfrak{A}$, but rather only those that retain the 0-thin density character of $\mathfrak{A}$.

We are now in a position to understand the difference between the notions of richness defined in 5.22 and in 5.4. We believe that Henkin and Tarski would have liked to define richness (as we have) to mean that below every non-zero y with $\Delta y \subseteq \{0\}$ there is a non-zero 0-thin x. However, the condition $c_0(x) = 1$ in their definition forces a 0-thin element in a subdirect product to have a non-zero coordinate in every factor algebra. On the other hand, such a subdirect product may have many non-zero elements y with $\Delta y \subseteq \{0\}$ that have zero coordinates in some of the factor algebras. To get around this difficulty one must require that in each factor where the coordinate y_ξ of y is non-zero, the corresponding coordinate x_ξ of x is below y_ξ. This is precisely what is expressed by the condition $x \cdot c_{(\alpha)}(y) \leq y$. (Essentially, $c_{(\alpha)}(y)$ is the union of the universes of all those factors where y_ξ is non-zero.) Since $\Delta y \subseteq \{0\}$, we may replace the term "$c_{(\alpha)}(y)$" by "$c_0(y)$" in this condition. In this way we arrive at the Henkin-Tarski definition of a rich algebra.

Adding the condition $c_i(x) = 1$ to the definition of an i-thin element was a way for Henkin and Tarski to circumvent a difficulty that they encountered. However, it was not without its price. The next example shows that this condition restricts the number of algebras that can satisfy the definition of richness. To illustrate this quite clearly we shall construct a cylindric set algebra $\mathfrak{A}$ that is rich in the sense of 5.4 but does not possess a single thin element in the sense of 5.22, and hence cannot be rich in that sense. As an interesting additional property of $\mathfrak{A}$, we shall exhibit a homomorphic image of $\mathfrak{A}$ that is not rich in the sense of 5.4 (or, equivalently, that is not rectangularly dense). Thus, the notion of richness in this sense is not preserved under homomorphic images.

Example 5.23. Fix a finite $\alpha \geq 2$ and a set U of cardinality at least 2. Let $\mathfrak{B}$ be the minimal subalgebra of $\mathfrak{C}_\alpha(U)$, i.e., the subalgebra of constants. Any subalgebra of $\mathfrak{C}_2(U)$, and in particular $\mathfrak{B}$, is simple. Therefore, by Theorem 2.1.17 in Henkin-Monk-Tarski [1971] (the cylindric algebraic analogue of quantifier elimination for the theory of equality), the elements of $\mathfrak{B}$ are just the Boolean combinations of diagonal elements. Using this, a rather straightforward set-theoretic argument shows:

(1) The only elements X in $\mathfrak{B}$ with $|\Delta X| \leq 1$ are $\varnothing$ and ${}^\alpha U$.

We mention in passing that $\varnothing$ is the only element of $\mathfrak{B}$ that is i-thin (for any i). In fact, by (1), the only possible i-thin elements of $\mathfrak{B}$ are $\varnothing$ and ${}^\alpha U$; it is easy to check, using Definition 5.1, that ${}^\alpha U$ is not i-thin. Moreover, there is no non-zero i-thin element below ${}^\alpha U$. Thus, $\mathfrak{B}$ is not rich in the sense of Definition 5.4.

We single out a special subalgebra $\mathfrak{A}$ of the ω^{th} direct power of $\mathfrak{C}_\alpha(U)$, namely the ω^{th} *direct sum* (also called the ω^{th} *weak direct power*) of $\mathfrak{C}_\alpha(U)$ (see Sain [1982a] or Givant [1994]). The universe of $\mathfrak{A}$ consists of all those sequences that are eventually a constant. More precisely, it consists of the sequences $\langle X_n : n \in \omega \rangle$ of subsets X_n of ${}^\alpha U$ such that for some cofinite set $\Gamma \subseteq \omega$ we have

(2) $X_n \in B$ and $X_m = X_n$ for all $m, n \in \Gamma$.

(3) $\mathfrak{A}$ is rich in the sense of 5.4

Indeed, let $Y = \langle Y_n : n \in \omega \rangle$ be a non-zero sequence in $\mathfrak{A}$ with $\Delta Y \subseteq \{0\}$. Then $\Delta Y_n \subseteq \{0\}$ for every $n \in \omega$ and $Y_m \neq 0$ for some fixed $m \in \omega$. Suppose that $\langle u_i : i < \alpha \rangle$ is an element of Y_m. Since $\Delta Y_m \subseteq \{0\}$, the set

$$Z = \{x \in {}^\alpha U : x_0 = u_0\}$$

is included in Y_m. Define $X = \langle X_n : n \in \omega \rangle$ by stipulating that

$$X_n = \begin{cases} Z & \text{if } n = m\,, \\ \varnothing & \text{if } n \neq m\,. \end{cases}$$

Then X is certainly in $\mathfrak{A}$ by definition of $\mathfrak{A}$; it is non-zero since $X_m \neq \varnothing$; it is below Y since $X_n \subseteq Y_n$ for all $n \in \omega$; and it is 0-thin since each coordinate is 0-thin in $\mathfrak{C}_\alpha(U)$, by 5.2. This proves (3).

(4) No element of $\mathfrak{A}$ is i-thin in the sense of 5.22.

To prove (4) fix $i < \alpha$ and suppose that $X = \langle X_n : n \in \omega \rangle$ is an element of $\mathfrak{A}$ that satisfies conditions (i) and (iii) of 5.22. We shall show that it cannot satisfy condition (ii). Since X is in $\mathfrak{A}$, there is a cofinite subset Γ of ω such that (2) holds. Since X satisfies condition (i) we have $\Delta X_n \subseteq \{i\}$ for all $n \in \omega$; therefore,

(5) For each $n \in \Gamma$ the set X_n is either $\varnothing$ or ${}^\alpha U$

by (1) and (2). Since X satisfies condition (iii) we have

(6) $C_i(X_n) = {}^\alpha U$ for all $n \in \omega$.

Now $C_i(\varnothing) = \varnothing \neq {}^\alpha U$. Therefore (5) and (6) together show us that $X_n = {}^\alpha U$ for every $n \in \Gamma$. But ${}^\alpha U$ does not satisfy condition (ii), since $|U| \geq 2$ (see Lemma 5.2). Because of the form of condition (ii), a sequence in a direct power will satisfy it iff each coordinate satisfies it in the factor algebra. Therefore X cannot satisfy condition (ii).

From (4) we conclude that $\mathfrak{A}$ is not rich in the sense of 5.22.

Notice that $\mathfrak{B}$ is a homomorphic image of $\mathfrak{A}$. Indeed, the function that maps each sequence $X = \langle X_n : n \in \omega \rangle$ in $\mathfrak{A}$ to the set that occurs cofinitely often in X is easily seen to be a homomorphism from $\mathfrak{A}$ onto $\mathfrak{B}$. In view of the remark after (1), this shows that $\mathfrak{A}$ has homomorphic images that are not rich in the sense of 5.4. ∎

In our opinion, one of the conclusions of the above discussion is that, in view of Corollary 2.10 and Theorem 2.11, condition (iii) should be dropped from the Henkin-Tarski definition of an i-thin element and the definition of a rich algebra should be modified accordingly. Let us call an element *weakly i-thin* if it satisfies conditions (i) and (ii) of Definition 5.22. An algebra will be called *weakly rich* if below every non-zero element y with $\Delta y \subseteq \{0\}$ there is a non-zero weakly 0-thin element. From now on, unless stated otherwise, whenever we refer to the notions of an i-thin element and a rich algebra (without the modifier "weakly") we mean them in the sense of 5.1 and 5.4 respectively.

The definition of a weakly i-thin element differs from that of an i-thin element because the condition

$$x \cdot s_{ij}(x) \leq d_{ij} \tag{1}$$

is weaker than the condition

$$x \cdot c_i(x \cdot y) \leq y \quad \text{for every } y. \tag{2}$$

Indeed, suppose that (2) holds. Taking $y = d_{ij}$ and using the definition of substitution in CA_α, we get (1). The reverse implication is not in general true, as we shall see in Example 5.24. However, it becomes true for fixed, distinct i, j if we assume ε_{ij}. Indeed, it is proved in Lemma 3.2.3(v) of Henkin-Monk-Tarski [1985] that if ε_{ij} is valid in a CA_α $\mathfrak{A}$, then an element of $\mathfrak{A}$ satisfying (1) must also satisfy (2). Thus, on the basis of the equations ε_{ij} the notions of i-thinness and weak i-thinness agree.

Whereas the notion of an i-thin element is based on a logical intuition, that of a weakly i-thin element is based on a set-theoretic intuition. In a full cylindric set algebra an element X is weakly i-thin if, for some $j \neq i$, when we replace all the j^{th} coordinates of sequences in X by the elements from the i^{th} coordinates, then the i^{th} and j^{th} coordinates always agree. Under condition (i) this implies that all the sequences in X have the same i^{th} coordinate. Thus, the notion of a weakly i-thin element projects thinness back to the diagonal elements.

Example 5.24. We construct a simple CA_2 $\mathfrak{A}$ which is weakly rich but not rich. In $\mathfrak{A}$ we will find weakly i-thin elements that are not i-thin. The universe of $\mathfrak{A}$ is the collection of all subsets of the set $\{0, 1, 2, 3, 4\}$. The Boolean operations are just the set-theoretic ones. The diagonal elements d_{00} and d_{11} are the unit and $d_{01} = d_{10} = \{0, 2\}$. The cylindrifications are completely determined by their action on the atoms (see Figure 8.):

$$\begin{aligned} &c_0(\{0\}) = c_0(\{3\}) = c_0(\{4\}) = \{0, 3, 4\}, \\ &c_0(\{1\}) = c_0(\{2\}) = \{1, 2\}, \\ &c_1(\{2\}) = c_1(\{3\}) = c_1(\{4\}) = \{2, 3, 4\}, \\ &c_1(\{0\}) = c_1(\{1\}) = \{0, 1\}. \end{aligned}$$

It is not difficult to check that $\mathfrak{A}$ is a simple CA_2. Its cylinders $\{0, 1\}$ and $\{2, 3, 4\}$ are weakly 0-thin, and $\{1, 2\}$ and $\{0, 3, 4\}$ are weakly 1-thin. For example, let $x = \{2, 3, 4\}$ with the goal of showing that x is weakly 0-thin. Certainly $c_1(x) = x$. Also,

$$\begin{aligned} x \cdot s_{01}(x) = x \cdot c_0(x \cdot d_{01}) &= \{2, 3, 4\} \cap c_0(\{2, 3, 4\} \cap \{0, 2\}) \\ &= \{2, 3, 4\} \cap c_0(\{2\}) = \{2, 3, 4\} \cap \{1, 2\} = \{2\} \subseteq d_{01}. \end{aligned}$$

Because the weakly 0-thin elements coincide with the elements different from the unit that are c_0-closed, we see that $\mathfrak{A}$ is weakly rich.

FIGURE 8. A weakly rich CA_2 that is not rich.

The element x defined above is not 0-thin. To see this take $y = \{3\}$. Then

$$\begin{aligned} x \cdot c_0(x \cdot y) &= \{2,3,4\} \cap c_0(\{2,3,4\} \cap \{3\}) \\ &= \{2,3,4\} \cap c_0(\{3\}) = \{2,3,4\} \cap \{0,3,4\} = \{3,4\}\,, \end{aligned}$$

and this is not a subset of y. In a similar fashion, taking $z = \{0,3,4\}$ and the same y, one can show that z is weakly 1-thin but not 1-thin. Thus, weak i-thinness does not imply i-thinness.

The cylinder x is not 0-thin, nor is there is a 0-thin element below it. In fact, x is an atom in the set of c_1-closed cylinders. Therefore, the algebra $\mathfrak{A}$ is not rich. Nor can it be represented. In a representable algebra the equations ε_{ij} always hold for distinct i, j and therefore the two notions of thinness coincide. However, in $\mathfrak{A}$ the two notions do not coincide.

The intuition behind the above example is the following. We have taken the full cylincric set algebra $\mathfrak{C}_2(2)$ on a base set of two elements and *split* one of the atoms that is off the diagonal into two (we split 3 into 3 and 4). In all other respects, $\mathfrak{A}$ behaves like $\mathfrak{C}_2(2)$. Because the definition of a weakly i-thin element projects thinness back to the diagonal (where $\mathfrak{A}$ is "normal"), the weakly thin elements of $\mathfrak{B}$ remain weakly thin in their modified form in $\mathfrak{A}$. However, thinness (as opposed to weak thinness) looks at the interaction of x not only with substitution instances of x, but with all elements y. Thus, it can notice that there is some abnormality "in the corner". ∎

It is natural to ask if one can project some notion of point back to the diagonal and in this way obtain a notion of point density that is equivalent to weak richness (without using the equations ε_{ij}). The next definition and the subsequent theorem achieve this goal.

Definition 5.25. Let α be finite and $\mathfrak{A}$ a CA_α. An element x is a *weak diagonal point* if it is a subdiagonal rectangle. $\mathfrak{A}$ is *weakly diagonally point dense* if every non-zero subdiagonal element is above a non-zero weak diagonal point.

Lemma 5.26. *Let $\alpha \geq 2$ be finite and $\mathfrak{A}$ a CA_α.*

(i) *If $x \leq d_{ij}$, then $s_{ij} c_{(\alpha \sim \{i\})}(x) = c_{(\alpha \sim \{j\})}(x)$.*

(ii) *If x is a non-zero weak diagonal point, then $c_{(\alpha\sim\{i\})}(x)$ is a non-zero weakly i-thin element.*

(iii) *If x is a non-zero weakly i-thin element, then $\prod_{j<\alpha} s_{ij}(x)$ is a non-zero weak diagonal point.*

Proof: Since $c_{(\alpha\sim\{ij\})}(d_{ij}) = d_{ij}$, 3.2(iii) gives:

(1) If $x \leq d_{ij}$, then $c_{(\alpha\sim\{ij\})}(x) \leq d_{ij}$.

Using (1) we prove (i). When $i = j$ the equality to be proved is a tautology. Suppose that $i \neq j$.

$$
\begin{aligned}
s_{ij}c_{(\alpha\sim\{i\})}(x) &= c_i(d_{ij} \cdot c_{(\alpha\sim\{i\})}(x)) && \text{by definition of } s_{ij}\,, \\
&= c_i(d_{ij} \cdot c_j c_{(\alpha\sim\{ij\})}(x)) && \text{by (C4),} \\
&= c_i(c_{(\alpha\sim\{ij\})}(x)) && \text{by (1) and 3.3(ii) with } \Delta = \{j\},\ k = i, \text{ and} \\
& && \text{“}c_{(\alpha\sim\{ij\})}(x)\text{” for “}x\text{”,} \\
&= c_{(\alpha\sim\{j\})}(x) && \text{by (C4).}
\end{aligned}
$$

To prove (ii), suppose that x is a non-zero weak diagonal point, i.e., a non-zero subdiagonal rectangle. Then for $j \neq i$ we have

$$
\begin{aligned}
c_{(\alpha\sim\{i\})}(x) \cdot s_{ij}c_{(\alpha\sim\{i\})}(x) &= c_{(\alpha\sim\{i\})}(x) \cdot c_{(\alpha\sim\{j\})}(x) && \text{by (i),} \\
&= c_{(\alpha\sim\{ij\})}(x) && \text{since } x \text{ is a rectangle,} \\
&\leq d_{ij} && \text{by (1).}
\end{aligned}
$$

Since $\Delta c_{(\alpha\sim\{i\})}(x) \subseteq \{i\}$, this shows that $c_{(\alpha\sim\{i\})}(x)$ is weakly i-thin. If $x \neq 0$, then of course $c_{(\alpha\sim\{i\})}(x) \neq 0$ by (C2).

Taking up the proof of (iii), suppose that x is a non-zero weakly i-thin element. From Lemma 3.2.3(i) in Henkin-Monk-Tarski [1971] we obtain

$$x \cdot s_{ij}(x) \leq d_{ij}\,. \tag{2}$$

Since $s_{ii}(x) = x$, we see from (2) that

$$\prod_{j<\alpha} s_{ij}(x) \leq \prod_{j<\alpha} d_{ij} = d\,.$$

Thus, $\prod_{j<\alpha} s_{ij}(x)$ is a subdiagonal element. Because $d_{ij} \cdot x \leq x$, we obviously have

$$c_i(d_{ij} \cdot x) \leq c_i(x)\,. \tag{3}$$

Therefore,

$$
\begin{aligned}
s_{ij}(x) &= c_i(x) \cdot c_i(d_{ij} \cdot x) && \text{by (3) and the definition of } s_{ij}\,, \\
&= c_i(x \cdot c_i(d_{ij} \cdot x)) && \text{by (C3),} \\
&= c_i(x \cdot c_i(d_{ij} \cdot c_{(\alpha\sim\{ij\})}(x))) && \text{since } \Delta x \subseteq \{i\}, \\
&= c_i(x \cdot c_i c_{(\alpha\sim\{ij\})}(d_{ij} \cdot x)) && \text{by (C3) since } \Delta d_{ij} \subseteq \{i, j\}, \\
&= c_i c_{(\alpha\sim\{ij\})}(x \cdot c_i(d_{ij} \cdot x)) && \text{by (C4) and (C3),}
\end{aligned}
$$

$$\begin{aligned} &= c_{(\alpha\sim\{j\})}(x \cdot c_i(d_{ij} \cdot x)) && \text{by (C4),} \\ &= c_{(\alpha\sim\{j\})}(x \cdot s_{ij}(x)) && \text{by definition of } s_{ij}\,. \end{aligned}$$

Taking the product over all $j < \alpha$ we arrive at

$$\prod_{j<\alpha} s_{ij}(x) = \prod_{j<\alpha} c_{(\alpha\sim\{j\})}(x \cdot s_{ij}(x))\,.$$

In view of 3.7, this shows that $\prod_{j<\alpha} s_{ij}(x)$ is a rectangle.

The above string of equalities also shows that

$$c_{(\alpha\sim\{j\})} s_{ij}(x) = s_{ij}(x)\,. \tag{4}$$

In addition,

$$c_j s_{ij}(x) = c_i(x)\,. \tag{5}$$

For $i = j$ this is obvious. Suppose $i \neq j$. Then

$$\begin{aligned} c_j s_{ij}(x) &= c_j c_i(d_{ij} \cdot x) && \text{by definition of } s_{ij}\,, \\ &= c_i c_j(d_{ij} \cdot x) && \text{by (C4)}\,, \\ &= c_i(c_j(d_{ij}) \cdot x) && \text{by (C3) and } c_j(x) = x\,, \\ &= c_i(x) && \text{by 3.2(vi) and BA}\,. \end{aligned}$$

Therefore,

$$\begin{aligned} c_{(\alpha)}(\textstyle\prod_{j<\alpha} s_{ij}(x)) &= c_{(\alpha)}(\textstyle\prod_{j<\alpha} c_{(\alpha\sim\{j\})} s_{ij}(x)) && \text{by (4),} \\ &= \textstyle\prod_{j<\alpha} c_{(\alpha)} s_{ij}(x) && \text{by 3.5(i) with } \Delta = \alpha, \\ &= \textstyle\prod_{j<\alpha} c_j c_{(\alpha\sim\{j\})} s_{ij}(x) && \text{by (C4),} \\ &= \textstyle\prod_{j<\alpha} c_j s_{ij}(x) && \text{by (4),} \\ &= \textstyle\prod_{j<\alpha} c_i(x) && \text{by (5),} \\ &= c_i(x) && \text{by BA,} \\ &\geq x && \text{by (C2),} \\ &> 0 && \text{by assumption.} \end{aligned}$$

This forces $\prod_{j<\alpha} s_{ij}(x)$ to be non-zero.

Summarizing, we have shown that $\prod_{j<\alpha} s_{ij}(x)$ is a non-zero subdiagonal rectangle, which is just what was to be proved. ∎

Theorem 5.27. *Let $\alpha \geq 2$ be finite and $\mathfrak{A}$ a CA_α. Then $\mathfrak{A}$ is weakly rich iff it is weakly diagonally point dense.*

Proof: Suppose first that $\mathfrak{A}$ is weakly diagonally point dense. To show that $\mathfrak{A}$ is weakly rich let y be a non-zero element such that $\Delta y \subseteq \{0\}$. We must construct a weakly 0-thin element below y. By Lemma 3.4 and $y = c_{(\alpha\sim\{0\})}(y)$ we have

$$d \cdot y = d \cdot c_{(\alpha\sim\{0\})}(y) \neq 0\,.$$

Therefore, by assumption, there is a non-zero rectangle x below $d \cdot y$. Then

$$c_{(\alpha\sim\{0\})}(x) \leq c_{(\alpha\sim\{0\})}(y) = y\,.$$

Moreover, $c_{(\alpha\sim\{0\})}(x)$ is a non-zero, weakly 0-thin element, by part (ii) of the previous lemma.

Now assume that $\mathfrak{A}$ is weakly rich. To show that it is weakly diagonally point dense, let y be a non-zero subdiagonal element. By our assumption, there is a non-zero, weakly 0-thin element x below $c_{(\alpha\sim\{0\})}(y)$. Then

$$\begin{aligned} s_{0i}(x) &\leq s_{0i}c_{(\alpha\sim\{0\})}(y) && \text{by 4.2(i)}, \\ &= c_{(\alpha\sim\{i\})}(y) && \text{by 5.26(i) and } y \leq d. \end{aligned}$$

Therefore

$$\textstyle\prod_{i<\alpha} s_{0i}(x) \leq \prod_{i<\alpha} c_{(\alpha\sim\{i\})}(y)\,.$$

Now

$$\textstyle\prod_{i<\alpha} s_{0i}(x) \leq d$$

by 5.26(iii). Also,

$$d \cdot c_{(\alpha\sim\{i\})}(y) = y$$

by 3.3(ii) with ($\Delta = \alpha \sim \{i\}$ and $k = i$). Combining these three equations, we obtain

$$\textstyle\prod_{i<\alpha} s_{0i}(x) \leq d \cdot \prod_{i<\alpha} c_{(\alpha\sim\{i\})}(y) = y\,.$$

Since $\prod_{i<\alpha} s_{0i}(x)$ is a non-zero, weak diagonal point, by 5.26(iii), we are done. ∎

In this paragraph we assume that α is infinite. Tarski proved that every locally finite dimensional CA_α is representable (see Henkin-Monk-Tarski [1971], Theorem 3.2.8, and see Tarski [1952] for the announcement of this result); the hypothesis of richness is not needed. Halmos proved that every locally finite dimensional polyadic algebra is representable (see, in particular, Theorems 16.9 and 17.1 in Chapter V of Halmos [1962], originally published in 1956). A consequence of this theorem is an analogous representation theorem for QPA_α and SCA_α. For the case of QPA_α this follows from Theorem 7.6 in Chapter 5 of *op. cit.* For the case of SCA_α we use also a theorem of Galler [1957] according to which a locally finite dimensional SCA_α is a reduct of a (locally finite dimensional) polyadic algebra of dimension α (see Pinter [1973], p. 365). (A direct proof that locally finite dimensional QPA_α and SCA_α are representable is given in Andréka-Gergely-Németi [1977], Theorem (A.3).) An argument similar to Galler's shows that every locally finite dimensional QPEA_α is the reduct of a locally finite dimensional polyadic equality algebra of dimension α. (This also follows readily from results of Sain-Thompson [1991].) Since Halmos [1962] proved that the latter algebras are representable (see Theorem 6.9 of Chapter 7, and the remarks preceding the theorem), it follows that every locally finite dimensional QPEA_α is representable.

Theorem 5.20 above establishes the representability of rich algebras of finite dimension at least 2 and of rich algebras of infinite dimension that are locally finite dimensional. One may ask why we have included the latter case in our theorem when a much stronger result was established decades ago.

The proof of Tarski's theorem in Henkin-Monk-Tarski [1985] proceeds in four main steps. Assume that $\alpha \geq \omega$. The notion of richness referred to here is that of Definition 5.22.

(1) Every simple, rich, locally finite dimensional algebra in which the equations ε_{ij} are valid for all distinct $i, j < \alpha$ is embeddable into a full cylindric set algebra of dimension α (see Theorem 3.2.5 in *op. cit.*).

(2) A locally finite dimensional CA_α can be embedded into a rich, locally finite dimensional CA_α (see Lemma 3.2.7 in *op. cit.*).

(3) Every homomorphic image of a rich, locally finite dimensional CA_α is rich and locally finite dimensional (see Lemma 3.2.6 in *op. cit.*).

(4) The equations ε_{ij} for distinct $i, j < \alpha$ are valid in every locally finite dimensional CA_α (see Theorem 1.11.7 in Henkin-Monk-Tarski [1971]).

Given an arbitrary locally finite dimensional CA_α $\mathfrak{A}$, one first passes to a rich, locally finite dimensional extension $\mathfrak{B}$ by using (2), then to a subdirect product of simple, rich, locally finite dimensional CA_α by using Birkhoff's subdirect decomposition theorem and (3), and then to a representation of these simple factors by using (4) and (1).

In view of the above approach to the proof of Tarski's theorem, and in particular in view of the important role played in the proof by (1), it seems to us of some interest to give the locally finite dimensional case (for $\alpha \geq \omega$) in Theorem 5.20 as well. If one could establish the analogue of (2) for the notion of richness in 5.4 (and for each of the classes CA_α, QPA_α, QPEA_α, and SCA_α), then one would obtain as a consequence of 5.20 a general proof of the representation theorem for locally finite dimensional algebras of logic with infinite dimension.

Theorem 5.20 asserts, in particular, that every rich QPA_α of finite dimension $\alpha \geq 2$ is representable. This gives a positive solution to Problem 5.6 in Henkin-Monk-Tarski [1985]. Monk [1991] announced that Richard Thompson had found a solution to this problem. Unfortunately, this solution has never been published or even described in print. Thompson has communicated to us that he does not recall the details of his solution nor can he locate his notes on the subject. However, he does recollect that his solution was simple and followed closely the lines of the proof of Theorem 5.4.34 in *op. cit.* In attempting to reconstruct his solution on our own, we ran into the same difficulty that we encountered in Section 4: the substitution operations of a QPA_α need not be completely additive.

6. Rectangularly dense relation algebras and their generalizations

The first algebras of logic to be investigated after Boolean algebras were the algebras of binary relations of Peirce and Schröder. The study of an abstract version of these algebras, called *relation algebras*, was initiated by Tarski [1941].

Definition 6.1. A *relation algebra* is an algebra

$$\mathfrak{A} = \langle A\,,\ +\,,\ -\,,\ ;\,,\ ^\smile\,,\ 1'\rangle$$

such that $\langle A\,, +\,, -\rangle$ is a Boolean algebra, the operations ; and $^\smile$ of *relative product* (or *composition*) and *conversion* are, respectively, binary and unary, 1' is a distinguished constant called the *identity element*, and the following postulates are satisfied for all $x, y, z \in A$:

(R1) $x\,;(y\,;z)=(x\,;y)\,;z$,
(R2) $x\,;(y+z)=(x\,;y)+(x\,;z) \quad , \quad (y+z)\,;x=(y\,;x)+(z\,;x)$,
(R3) $x\,;1'=x \quad , \quad 1'\,;x=x$,
(R4) $x^{\smile\smile}=x$,
(R5) $(x+y)^{\smile}=x^{\smile}+y^{\smile}$,
(R6) $(x\,;y)^{\smile}=y^{\smile}\,;x^{\smile}$,
(R7) $x^{\smile}\,;[-(x\,;y)]\leq -y$.

The class of all relation algebras is denoted by RA. We also use this notation as an abbreviation for the phrase "relation algebra". ∎

Remark 6.2. It is well known that the second equations in (R2) and (R3) are derivable from the remaining postulate, and are therefore redundant. We have included them among the basic postulates of the theory in order to facilitate later references, in particular the definition of a Boolean monoid. ∎

For an example of a relation algebra, let U be any set and define $\mathfrak{R}(U)$ to be the algebra

$$\langle Sb(U\times U)\,,\,\cup\,,\,\sim\,,\,|\,,\,^{-1}\,,\,Id_U\rangle \quad ,$$

where $|$ and $^{-1}$ are, respectively, the set-theoretic operations of composing two relations and taking the inverse of a relation, and Id_U is the identity relation on U. It is easy to check that $\mathfrak{R}(U)$ is a relation algebra; it is called the *full set relation algebra over U*. A relation algebra is said to be *representable* if it can be embedded into a direct product of full set relation algebras. The class RRA of all representable relation algebras is a subvariety of RA (see Tarski [1955]), but it is not finitely axiomatizable over the latter (see Monk [1964]). RA is a discriminator variety of BAOs with the unary discriminator term $1\,;x\,;1$.

We summarize some well-known laws that are valid in all relation algebras. Derivations of these laws can be found in Chin-Tarski [1951] and Givant [1994].

Lemma 6.3. *Let $\mathfrak{A}$ be an RA and x,y,z,w elements of $\mathfrak{A}$.*

(i) $0^{\smile}=0 \quad , \quad 1^{\smile}=1 \quad , \quad 1'^{\smile}=1'$.
(ii) $1\,;1=1 \quad , \quad 0\,;x=0$.
(iii) $(x\cdot y)^{\smile}=x^{\smile}\cdot y^{\smile}$.
(iv) *If $z\leq x$ and $w\leq y$, then $z\,;w\leq x\,;y$.*
(v) *If $x\leq 1'$, then $x^{\smile}=x$ and $x\,;x=x$.*
(vi) $1\,;[1'\cdot(1\,;x)]=1\,;x$.
(vii) $(x\,;1)\cdot(1\,;y)=x\,;1\,;y$.
(viii) $(x\,;y)\cdot z\leq x\,;[y\cdot(x^{\smile}\,;z)]$ *and* $(x\,;y)\cdot z\leq[(z\,;y^{\smile})\cdot x]\,;y$.

Laws (R1), (R2), (R3), and 6.3(iv) are respectively referred to as the *associative law*, the *distributive laws*, the *identity laws*, and the *monotony law*. In omitting parentheses, we follow the convention that ; has priority over $\cdot$.

In each RA $\mathfrak{A}$, we can define two unary operations c_0 and c_1 and constants d_{ij} for $i,j\in\{0,1\}$ as follows:

$$c_0(x)=1\,;x \quad , \quad c_1(x)=x\,;1 \quad , \quad d_{01}=d_{10}=1' \quad , \quad d_{00}=d_{11}=1\,.$$

It is well known that the algebra

$$\langle A\,,\,+\,,\,-\,,\,c_0\,,\,c_1\,,\,d_{ij}\rangle_{i,j\in\{0,1\}}$$

is a CA_2. It is called the *cylindric reduct* of $\mathfrak{A}$. Since $\mathfrak{A}$ and its cylindric reduct $\mathfrak{B}$ have the same unary discriminator, we see that $\mathfrak{A}$ is simple iff $\mathfrak{B}$ is simple.

In view of the preceding observation, we can translate each of the notions defined in Sections 3 and 5 into the langugage of relation algebras.

Definition 6.4. Let $\mathfrak{A}$ be an RA and x an element of $\mathfrak{A}$.

(i) x is a *rectangle* if $x = (x\,;1)\cdot(1\,;x)$.
(ii) x is 0-*thin* if $x = x\,;1$ and $x\cdot[1\,;(x\cdot y)] \leq y$ for all y. The definition of a 1-*thin element* is just the dual definition.
(iii) x is *weakly* 0-*thin* if $x = x\,;1$ and $x\cdot[1\,;(x\cdot 1')] \leq 1'$. The definition of a *weakly* 1-*thin element* is just the dual definition.
(iv) x is a *point* if $x\,;1$ is 0-thin and $1\,;x$ is 1-thin.
(v) x is a *diagonal point* if x is a point below 1'.
(vi) x is a *weak diagonal point* if x is a rectangle below 1'.
(vii) x is a *quasi-atom* if $y \leq x$ always implies $y = x\cdot(1\,;y\,;1)$.

Each of the above notions (i)–(vi) has equivalent forms in the theory of relation algebras that are more concise or more appealing.

Lemma 6.5. *Let $\mathfrak{A}$ be an RA and x an element of $\mathfrak{A}$.*

(i) *The following are equivalent:*
 (α) *x is a rectangle,*
 (β) $x = x\,;1\,;x$,
 (γ) *There are y, z such that* $x = y\,;1\,;z$,
 (δ) *There are $y, z \leq 1'$ such that* $x = y\,;1\,;z$.
(ii) *The following are equivalent:*
 (α) *x is 0-thin,*
 (β) *x is weakly 0-thin,*
 (γ) $x = x\,;1$ *and* $x\,;1\,;x^{\smile} \leq 1'$,
 (δ) $x = x\,;1$ *and* $x\cdot x^{\smile} \leq 1'$,
 (ϵ) $x = x\,;1$ *and* $x\,;x^{\smile} \leq 1'$.
(iii) *The following are equivalent:*
 (α) *x is a point,*
 (β) $x\,;1\,;x^{\smile} \leq 1'$ *and* $x^{\smile}\,;1\,;x \leq 1'$.
(iv) *The following are equivalent:*
 (α) *x is a diagonal point,*
 (β) *x is a weak diagonal point,*
 (γ) $x\,;1\,;x \leq 1'$.

Proof: Part (i) is formulated and proved in Givant [1994], Lemma 1.12. To prove (ii), assume that

$$x = x\,;1\,. \tag{1}$$

By (R6) and 6.3(i) we get

$$x^{\smile} = 1\,;x^{\smile}\,. \tag{2}$$

Then

$$x\,;x^{\smile} = x\,;1\,;x^{\smile} \qquad \text{by (1)},$$

$$
\begin{aligned}
&= (x\,;1)\cdot(1\,;x^{\smile}) && \text{by 6.3(vii),}\\
&= x\cdot x^{\smile} && \text{by (1), (2),}\\
&= x\cdot(1\,;x^{\smile}) && \text{by (2),}\\
&= x\cdot\bigl(1\,;[1'\cdot(1\,;x^{\smile})]\bigr) && \text{by 6.3(vi),}\\
&= x\cdot[1\,;(1'\cdot x^{\smile})] && \text{by (2),}\\
&= x\cdot[1\,;(1'\cdot x)] && \text{by 6.3(v),(iii),(i).}
\end{aligned}
$$

From this string of equalities we immediately obtain the equivalence of (β), (γ), (δ), and (ϵ). The implication from (α) to (β) follows by taking $y = 1'$. To show that (δ) implies (α), we use the second equation in 6.3(viii) (with "1" for "x", "$x \cdot y$" for "y", and "x" for "z"):

$$
\begin{aligned}
x\cdot[1\,;(x\cdot y)] &\leq \bigl([x\,;(x\cdot y)^{\smile}]\cdot 1\bigr)\,;(x\cdot y)\\
&= x\,;(x^{\smile}\cdot y^{\smile})\,;(x\cdot y) && \text{by BA and 6.3(iii),}\\
&= x\,;x^{\smile}\,;y && \text{by monotony,}\\
&\leq 1'\,;y && \text{by } (\delta),\\
&= y && \text{by (R3).}
\end{aligned}
$$

Turning to (iii), suppose first that x is a point. By (ii)(β),

$$
\begin{aligned}
x\,;1\,;x^{\smile} &= (x\,;1)\,;1\,;(1\,;x^{\smile}) && \text{by 6.3(ii),}\\
&= (x\,;1)\,;1\,;(x\,;1)^{\smile} && \text{by (R6) and 6.3(i),}\\
&\leq 1' && \text{by 6.4(iv) and (ii)}(\gamma),\\
& && \text{with ``}x\,;1\text{'' for ``}x\text{''.}
\end{aligned}
$$

A similar argument using the 1-thinness of $1\,; x$ and the dual of (ii)(γ) for 1-thin elements shows that $x^{\smile}; 1; x \leq 1'$. Now suppose that (iii)(β) holds, and set $y = x; 1$. Using (iii)(β) and 6.3(i) we easily check that $y\,; 1 = 1$ and $y\,; 1\,; y^{\smile} \leq 1'$. Thus, y is 0-thin. A similar argument shows that $1\,; x$ is 1-thin. Hence, x is a point.

Finally, we prove (iv). Under any of the three assumptions we have $x \leq 1'$. Therefore

(1) $$x = x^{\smile} = x\,;x\,.$$

In particular,

$$x\,;1\,;x = x\,;1\,;x^{\smile} = x^{\smile}\,;1\,;x\,.$$

This shows that (iv)(γ) is equivalent to (iii)(β), and hence to (iv)(α). The implication from (iv)(β) to (iv)(γ) follows at once from (i)(β). To prove that (iv)(γ) implies (iv)(β) we must show that $x\,; 1\,; x \leq x$ (the reverse inclusion follows from monotony):

$$\begin{aligned} x\,;1\,;x &= (x\,;1\,;x)\cdot 1' && \text{by (iv)}(\gamma), \\ &= x\,;[(1\,;x)\cdot(x^{\smile}\,;1')] && \text{by the first equation in 6.3(viii) with “}1\,;x\text{” for “}y\text{”}, \\ &\le x\,;x^{\smile} && \text{by (R3) and monotony,} \\ &= x && \text{by (1).} \end{aligned}$$

∎

For each of the notions in Definition 6.4 we obtain a corresponding notion of density, just as in the cylindric algebraic case. We shall use the cylindric algebraic name to describe it.

Theorem 6.6. *Let $\mathfrak{A}$ be an RA. Then the following are equivalent:*

(i) $\mathfrak{A}$ *is rectangularly dense*,
(ii) $\mathfrak{A}$ *is rich*,
(iii) $\mathfrak{A}$ *is weakly rich*,
(iv) $\mathfrak{A}$ *is point dense*,
(v) $\mathfrak{A}$ *is diagonally point dense*,
(vi) $\mathfrak{A}$ *is weakly diagonally point dense.*

If any one of these conditions holds, then $\mathfrak{A}$ is quasi-atomic, and atomic in case it is simple.

Proof: Let $\mathfrak{B}$ be the cylindric reduct of $\mathfrak{A}$. Then (i), (ii), (iv), and (v) are equivalent in $\mathfrak{B}$ by Theorem 5.18, (iii) and (vi) are equivalent by Theorem 5.27, and (ii) implies (iii) by the remarks preceding Example 5.24. Moreover, if (ii) holds in $\mathfrak{B}$ then it is quasi-atomic, and atomic in case it is simple, by Theorem 5.18. Because $\mathfrak{B}$ and $\mathfrak{A}$ have the same elements, and $\mathfrak{A}$ is simple iff $\mathfrak{B}$ is simple, we see that these equivalences and implications remain valid in $\mathfrak{A}$. Finally, since the notions of 0-thin and weakly 0-thin are equivalent by 6.5(ii), we see that (iii) implies (ii). ∎

Maddux [1991] defines a *point* in a relation algebra to be a non-zero element x such that $x\,;1\,;x \le 1'$. He calls a relation algebra *point-dense* if every non-zero element below 1' is above a point. In view of Lemma 6.5(iv), we see that Maddux's notion of point density corresponds exactly to our notion of diagonal point density. Maddux proves in *op. cit.* (as part of a more general theorem) that a relation algebra which is (diagonally) point dense is representable. From this we immediately conclude:

Theorem 6.7. *A relation algebra that satisfies one of the conditions* (i)–(vi) *in the preceding theorem is representable.*

Lemma 6.5 sheds light on the relationship between the diagonally point dense part of Maddux's theorem and Theorem 4.30 in Jónsson-Tarski [1952]. Jónsson and Tarski show that an atomic relation algebra in which each atom satisfies $x^{\smile};1;x \le 1'$ must be representable. Since $x^{\smile}$ is an atom whenever x is an atom, we see that, under the hypotheses of their theorem, every atom also satisfies $x\,;1\,;x^{\smile} \le 1'$. In view of 6.5(iii), we can reformulate the Jónsson-Tarski theorem as follows: *every* *atomic relation algebra in which each atom is a point is representable.* Now a special case of one of the steps in Maddux's proof (his Theorem 48) shows that, for simple relation algebras, diagonal point density implies atomicity; the proof actually shows that each atom is a point. Thus, a simple, diagonally point dense relation algebra satisfies the hypothesis of the Jónsson-Tarski theorem; from this its representability follows at once.

We did not use Theorem 3.11 to prove the preceding theorem as we did in the case of quasi-polyadic algebras. Instead, we obtained it as an almost immediate consequence of the diagonally point dense case of Maddux's theorem. Of course, one can also prove it quite easily using 3.11. In the process, one obtains both the Jónsson-Tarski theorem and the special case of Maddux's theorem as immediate corollaries. In fact, using 3.11 leads to somewhat stronger results (whose proofs are correspondingly more involved). To formulate one example of such a result we introduce the following definition.

Definition 6.8. A *Boolean* (or *Boolean-ordered*) *monoid* is an algebra

$$\mathfrak{A} = \langle A\,,\ +\,,\ -\,,\ ;\,,\ 1'\rangle$$

(of the same type as a relation algebra without conversion) such that $\langle A\,, +\,, -\rangle$ is a Boolean algebra and the laws (R1)–(R3) hold. If, in addition, the laws

(R8) $-(x\,;1)\,;1 = -(x\,;1) \quad , \quad 1\,;-(1\,;x) = -(1\,;x)$

hold, then the algebra is called *special*. The classes of all special Boolean monoids is denoted by BM^*. We also use this notation as an abbreviation for the phrase "special Boolean monoid". ∎

The name "Boolean-ordered monoid" derives from the fact that $\langle A\,, ;\,, 1'\rangle$ is a monoid (i.e., a semigroup with an identity element) which is a Boolean algebra under the operations $+$ and $-$, and the operation ; not only respects the Boolean order (i.e., the monotony law holds) but in fact is distributive over Boolean addition. Related structures such as lattice-ordered groups, lattice-ordered monoids, distributive lattice-ordered semigroups, etc. have been studied in the literature. See, for example, Birkhoff [1967], Chapters XIII and XIV (in particular, the remarks on pages 292 and 323) and Andréka [1991]. An apparently weaker notion of a Boolean monoid in which the monotony law replaces the distributive laws is considered in Pratt [1990]. The study of Boolean monoids has been stimulated in part by their connection with logics of the dynamic trend, for example, dynamic logic and arrow logic (see, e.g., Pratt [1990], Mikulás [1995], and Marx-Masuch-Polos [1996]).

The notions of a *full set* BM^* (the algebra of all binary relations on some set under the obvious set-theoretic operations), a *representable* BM^*, the *cylindric reduct* of a BM^*, a *rectangle* in a BM^*, and a *rectangularly dense* BM^* are the obvious analogues of the relation algebraic notions.

All of the laws in the next lemma are very well known in the context of relation algebras. We show that they also hold in the context of special Boolean monoids. Several of the laws we shall list have a dual form. For example, the dual of (iv) is

$$1\,;x = 0 \quad \text{iff} \quad x = 0\,.$$

The proof of the dual is just the dualized version of the proof of the given law. In what follows we shall not bother to formulate such duals. Moreover, when using these duals in other proofs, we shall simply refer to the given law in 6.9.

Lemma 6.9. *The following laws are valid in all special Boolean monoids.*

(i) $x\,;y \leq u\,;v$ *whenever* $x \leq u$ *and* $y \leq v$.
(ii) $x \leq x\,;1$.
(iii) $1\,;1 = 1$.
(iv) $x\,;1 = 0$ *iff* $x = 0$.
(v) $[x \cdot (y\,;1)]\,;1 = (x\,;1) \cdot (y\,;1)$.
(vi) $x\,;1 = [(x\,;1) \cdot 1']\,;1$.
(vii) *If* $x \leq 1'$, *then* $(x\,;1) \cdot 1' = x$.
(viii) *If* $x \leq 1'$ *and* $x' = -x \cdot 1'$, *then* $(x\,;1) \cdot (x'\,;1) = 0$.
(ix) *If* $x, y \leq 1'$, *then* $x\,;y = x \cdot y$.
(x) $x \cdot (y\,;1) \neq 0$ *iff* $(x\,;1) \cdot y \neq 0$.
(xi) *If* x, y *are atoms, then* $x \leq y\,;1$ *iff* $y \leq x\,;1$.
(xii) *If* x *is an atom, then so are* $(x\,;1) \cdot 1'$ *and* $(1\,;x) \cdot 1'$.
(xiii) $x\,;y = 0$ *iff* $[(1\,;x) \cdot 1'] \cdot [(y\,;1) \cdot 1'] = 0$.

Proof: Part (i) is an almost immediate consequence of distributivity. Indeed, suppose that $x \leq u$ and $y \leq v$. Then

$$u\,;v = u\,;(y + v) = u\,;y + u\,;v\,.$$

Therefore, $u\,;y \leq u\,;v$. Similarly, $x\,;y \leq u\,;y$. Combining these two inequalities, we get the desired result.

Part (ii) follows from the identity laws and monotony (part (i)):

$$x = x\,;1' \leq x\,;1\,.$$

Part (iii) follows at once from (ii) with $x = 1$.

To prove (iv), assume first that $x = 0$. Then by (iii) and (R8) we have:

$$x\,;1 = 0\,;1 = [-(1\,;1)]\,;1 = -(1\,;1) = 0\,.$$

Now assume that $x\,;1 = 0$. Using the identity laws and monotony, we have

$$x = x\,;1' \leq x\,;1 = 0\,.$$

The proof of part (v) is more involved. Notice that $(x;1) \cdot -(y;1)$ and $(x;1) \cdot (y;1)$ form a partition of $x\,;1$ in the sense that they are disjoint and sum to $x\,;1$. We now show that same is true of the elements $[x \cdot -(y\,;1)]\,;1$ and $[x \cdot (y\,;1)]\,;1$, and that these elements are below $(x\,;1) \cdot -(y\,;1)$ and $(x\,;1) \cdot (y\,;1)$ respectively. From this it follows at once that these two partitions must in fact be the same partition. In particular,

$$[x \cdot (y\,;1)]\,;1 = (x\,;1) \cdot (y\,;1)\,.$$

By monotony and (R8) we have

$$[x \cdot -(y\,;1)]\,;1 \leq x\,;1$$

and

$$[x \cdot -(y\,;1)]\,;1 \leq -(y\,;1)\,;1 = -(y\,;1)\,,$$

so

$$[x \cdot -(y\,;1)]\,;1 \leq (x\,;1) \cdot -(y\,;1)\,. \tag{1}$$

Similarly,

$$[x \cdot (y\,;1)]\,;1 \leq (x\,;1) \cdot (y\,;1)\,. \tag{2}$$

In particular, since the elements on the right-hand sides of (1) and (2) are disjoint, so are the elements on the left. Further, using distributivity and BA we see that

$$\begin{aligned} [x \cdot -(y\,;1)]\,;1 + [x \cdot (y\,;1)]\,;1 &= [x \cdot -(y\,;1) + x \cdot (y\,;1)]\,;1 \\ &= (x \cdot [-(y\,;1) + (y\,;1)])\,;1 \\ &= (x \cdot 1)\,;1 \\ &= x\,;1\,. \end{aligned}$$

This completes the proof of (v).

Part (vi) is an immediate consequence of (v) and the identity laws:

$$[(x\,;1) \cdot 1']\,;1 = (x\,;1) \cdot (1'\,;1) = (x\,;1) \cdot 1 = x\,;1\,.$$

For (vii)–(ix) assume that $x \leq 1'$ and set $x' = -x \cdot 1'$. Then

$$\begin{aligned} (x\,;1) \cdot 1' &= [x\,;(1' + -1')] \cdot 1' && \text{by BA,} \\ &= (x\,;1' + x\,;-1') \cdot 1' && \text{by distributivity,} \\ &= (x + x\,;-1') \cdot 1' && \text{by the identity laws,} \\ &\leq (x + 1'\,;-1') \cdot 1' && \text{by monotony and } x \leq 1', \\ &= (x + -1') \cdot 1' && \text{by the identity law,} \\ &= x \cdot 1' && \text{by BA,} \\ &= x && \text{since } x \leq 1'. \end{aligned}$$

The reverse inequality follows from (ii) and the assumption that $x \leq 1'$. This proves (vii).

Before proving (viii) and (ix) we establish

$$x \cdot (x'\,;1) = 0\,, \tag{3}$$

$$x\,;x' = 0\,. \tag{4}$$

Indeed, using $x, x' \leq 1'$ and part (vii) (applied to x') we have

$$x \cdot (x'\,;1) \leq 1' \cdot (x'\,;1) = x'\,,$$

and trivially

$$x \cdot (x'\,;1) \leq x\,,$$

so

$$x \cdot (x'\,;1) \leq x \cdot x' = 0\,.$$

Similarly, by monotony and the identity laws we have

$$x\,;x' \leq x\,;1' = x \qquad \text{and} \qquad x\,;x' \leq 1'\,;x' = x'\,,$$

so

$$x\,;x' \le x \cdot x' = 0\,.$$

Turning to (viii), we have

$$\begin{aligned}(x\,;1)\cdot(x'\,;1) &= [x\cdot(x'\,;1)]\,;1 && \text{by (v)},\\ &= 0\,;1 && \text{by (3)},\\ &= 0 && \text{by (iv)}.\end{aligned}$$

For (ix) we first establish the special case when $x = y$, i.e.,

$$x\,;x = x\,. \tag{5}$$

Indeed, by the identity laws, distributivity, and (4) we have

$$x = x\,;1\text{'} = x\,;(x+x') = x\,;x + x\,;x' = x\,;x\,.$$

Turning to the general case of (ix), suppose that $y \le 1\text{'}$. Then

$$x\cdot y = (x\cdot y)\,;(x\cdot y) \le x\,;y\,,$$

by (5) and monotony. On the other hand,

$$x\,;y \le x\,;1\text{'} = x\,,$$

by monotony and the identity laws. Similarly, $x\,;y \le y$. Therefore, $x\,;y \le x\cdot y$. This completes the proof of (ix).

Part (x) follows from (iv) and (v):

$$\begin{aligned} x\cdot(y\,;1)\ne 0 \quad &\text{iff} \quad [x\cdot(y\,;1)]\,;1 \ne 0 && \text{by (iv)},\\ &\text{iff} \quad (x\,;1)\cdot(y\,;1)\ne 0 && \text{by (v)},\\ &\text{iff} \quad [(x\,;1)\cdot y]\,;1\ne 0 && \text{by (v)},\\ &\text{iff} \quad (x\,;1)\cdot y \ne 0 && \text{by (iv)}.\end{aligned}$$

Part (xi) is an immediate consequence of (x).

To prove (xii), assume that x is an atom. Then $x;1 \ne 0$ by (ii), so $[(x;1)\cdot 1\text{'}];1 \ne 0$ by (vi). Hence, $(x\,;1)\cdot 1\text{'} \ne 0$ by (iv). Let y be a non-zero element below $(x\,;1)\cdot 1\text{'}$. Our goal is to show that $(x\,;1)\cdot 1\text{'} \le y$. We have

$$\begin{aligned} y\cdot(x\,;1) &= (y\cdot 1\text{'})\cdot(x\,;1) && \text{since } y\le 1\text{'},\\ &= y && \text{since } y \le (x\,;1)\cdot 1\text{'},\\ &\ne 0 && \text{by assumption}.\end{aligned}$$

Therefore $(y\,;1)\cdot x \ne 0$ by (x). Hence,

$$\begin{aligned} x &\le y\,;1 && \text{since } x \text{ is an atom},\\ x\,;1 &\le y\,;1 && \text{by monotony and (iii)},\\ (x\,;1)\cdot 1\text{'} &\le (y\,;1)\cdot 1\text{'} && \text{by BA},\\ (x\,;1)\cdot 1\text{'} &\le y && \text{by (vii)}.\end{aligned}$$

Finally, part (xiii) follows at once from (several applications of) (iv) and the following observation:

$$
\begin{aligned}
1 \,;\, x \,;\, y \,;\, 1 &= 1 \,;\, [(1 \,;\, x) \cdot 1']\,;\, [(y \,;\, 1) \cdot 1'] \,;\, 1 && \text{by (vi)},\\
&= 1 \,;\, \big([(1 \,;\, x) \cdot 1')] \cdot [(y \,;\, 1) \cdot 1']\big) \,;\, 1 && \text{by (ix)}.
\end{aligned}
$$

∎

Lemma 6.10. (i) *BM^* is a discriminator variety of normal Boolean algebras with operators. In fact, the term $1\,;x\,;1$ is a unary discriminator for BM^*.*
(ii) *The cylindric reduct of a special Boolean monoid is a CA_2.*

Proof: Relative product is the only extra-Boolean operation of rank > 0 in a BM^*. It is additive by the distributive laws, and normal by 6.9(iv). In view of Corollary 2.2, to show that $1\,;x\,;1$ is a unary discriminator for BM^* we must show that it satisfies the five conditions (α)–(δ) of Lemma 2.1(ii). The validity of these conditions follows from 6.9(ii),(iii), (R8), and monotony respectively. For example for (δ) we must check that, for all x, y in a special Boolean monoid we have

(1) $$x\,;y < 1\,;x\,;1 \qquad \text{and} \qquad x\,;y \leq 1\,;y\,;1\,.$$

Now $x \leq 1\,;x$ by 6.9(ii), and $y \leq 1$, so the first inequality in (1) follows from the monotony law. The second is proved similarly. This completes our proof of (i).

To prove (ii), let $\mathfrak{A}$ be a BM^*, and suppose that $\mathfrak{B}$ is its cylindric reduct. We must verify that axioms (C1)–(C7) are valid in $\mathfrak{B}$. This is quite easy using the postulates for special Boolean monoids and Lemma 6.9. For example, to verify the validity of (C7) in $\mathfrak{B}$, we first translate it into an equation in the language of special Boolean monoids and then check the validity of this equation in $\mathfrak{A}$. In the case when $i = 1$ and $j = 0$, (C7) translates into the equation

$$[(x \cdot 1')\,;1] \cdot [(-x \cdot 1')\,;1] = 0\,,$$

and this is valid in $\mathfrak{A}$ by 6.9(viii). The validity of the translation of (C1) follows from 6.9(iv), that of (C2) from 6.9(ii), that of (C3) from 6.9(v), that of (C4) from the associative law, and that of (C5) from the definition of d_{ii}. Finally, we check the validity of (C6) in the case when $i = 1$. We must have $j = k = 0$, since $i \neq j, k$. Thus, (C6) translates into the equation $1 = 1'\,;1$, and this is just a special case of the identity laws. ∎

Lemma 6.11. *Let $\mathfrak{A}$ be a special Boolean monoid. A set I is an ideal in $\mathfrak{A}$ iff it is an ideal in the cylindric reduct of $\mathfrak{A}$. In particular, $\mathfrak{A}$ is simple iff the cylindric reduct of $\mathfrak{A}$ is simple.*

Proof: Let $\mathfrak{A}$ be a special Boolean monoid and $\mathfrak{B}$ its cylindric reduct. Suppose, first, that I is an ideal of $\mathfrak{A}$ and that $x \in I$. Then $x\,;1$ and $1\,;x$ are in I by 2.4(iv), i.e., $c_1(x)$ and $c_0(x)$ are in I. Thus, I is an ideal of $\mathfrak{B}$ by 2.4. Now suppose that I is an ideal of $\mathfrak{B}$ and that $x \in I$. Then $c_1(x)$ and $c_0(x)$ are in I by 2.4(iv), i.e., $x\,;1$ and $1\,;x$ are in I. But $x\,;y \leq x\,;1$ and $y\,;x \leq 1\,;x$ by monotony, so $x\,;y$ and $y\,;x$ are in I by 2.4(iii). Therefore I is an ideal of $\mathfrak{A}$ by 2.4. This proves the first assertion. The second assertion is an immediate consequence of the first. ∎

Lemma 6.12. *A simple, rectangularly dense, special Boolean monoid is representable.*

Proof: Let $\mathfrak{A}$ be a simple, rectangularly dense BM^* and $\mathfrak{B}$ its cylindric reduct. Then $\mathfrak{B}$ is a simple, rectangularly dense CA_2, by Lemmas 6.11 and 6.10(ii). Hence, $\mathfrak{B}$ is atomic and representable, by Lemma 3.10 and Theorem 3.11. In fact, taking U to be the set of atoms below $1' = d_{01} = d$, the function f mapping A into $Sb(U \times U)$ that is determined by

$$f(x) = \{(w\,;1\cdot 1',\ 1\,;w\cdot 1') : w \text{ is an atom below } x\}$$

is an embedding of $\mathfrak{B}$ into $\mathfrak{C}_2(U)$ (see the remarks after Theorem 3.11). Notice that by its very definition, f preserves (as unions) all Boolean sums in $\mathfrak{B}$ that exist, i.e., it is a complete embedding.

Let $\mathfrak{D}$ be the full set BM^* on U. Notice that $\mathfrak{D}$ is also simple and rectangularly dense (since its cylindric reduct, $\mathfrak{C}_2(U)$, is). We wish to show that f embeds $\mathfrak{A}$ into $\mathfrak{D}$. Since f preserves the cylindric algebraic operations, it preserves in particular the Boolean operations and constants, and maps the constant 1' to Id_U (these are the main diagonals of $\mathfrak{B}$ and $\mathfrak{C}_2(U)$ respectively). It remains to show that f preserves relative products. In view of our previous remarks, it suffices to prove that in a simple (and hence atomic) rectangularly dense BM^* the relative product operation is definable in terms of the cylindric algebraic operations.

(1) If u, v are atoms, then so is $(u\,;1)\cdot(1\,;v)$.

To prove (1), suppose that u, v are atoms. Then

$$\begin{aligned}
[(u\,;1)\cdot(1\,;v)]\,;1 &= (u\,;1)\cdot(1\,;v\,;1) && \text{by 6.9(v)},\\
&= (u\,;1)\cdot 1 && \text{by simplicity, since } v \neq 0,\\
&= u\,;1 && \text{by BA},\\
&\geq u && \text{by 6.9(ii)},\\
&> 0 && \text{since } u \text{ is an atom.}
\end{aligned}$$

Thus $(u\,;1)\cdot(1\,;v) \neq 0$, by 6.9(iv). Now let w be any non-zero element below $(u\,;1)\cdot(1\,;v)$. By rectangular density there is a non-zero rectangle e below w. In particular, $e \leq u\,;1$, so

$$\begin{aligned}
e\cdot(u\,;1) &\neq 0 && \text{by BA},\\
(e\,;1)\cdot u &\neq 0 && \text{by 6.9(x)},\\
u &\leq e\,;1 && \text{since } u \text{ is an atom},\\
u\,;1 &\leq e\,;1 && \text{by 6.9(i),(iii)}.
\end{aligned}$$

Similarly, $1\,;v \leq 1\,;e$. Therefore, using also the assumption that e is a rectangle, we get

$$(u\,;1)\cdot(1\,;v) \leq (e\,;1)\cdot(1\,;e) = e\,.$$

This of course forces

$$e = w = (u\,;1)\cdot(1\,;v)\,,$$

which is what we wanted to prove.

(2) For atoms u, v we have

$$u\,;v = \begin{cases} (u\,;1)\cdot(1\,;v) & \text{if} \quad (1\,;u)\cdot 1' = (v\,;1)\cdot 1' \,, \\ 0 & \text{if} \quad (1\,;u)\cdot 1' \neq (v\,;1)\cdot 1' \,. \end{cases}$$

Indeed, suppose that u, v are atoms. If $(1\,;u)\cdot 1' \neq (v\,;1)\cdot 1'$, then

$$[(1\,;u)\cdot 1']\cdot[(v\,;1)\cdot 1'] = 0\,,$$

since $(1\,;u)\cdot 1'$ and $(v\,;1)\cdot 1'$ are atoms by 6.9(xii). Therefore $u\,;v = 0$ by 6.9(xiii). If $(1\,;u)\cdot 1' = (v\,;1)\cdot 1'$, then $u\,;v \neq 0$, by 6.9(xiii). But $u\,;v \leq u\,;1$ and $u\,;v \leq 1\,;v$ by monotony, so $u\,;v \leq (u\,;1)\cdot(1\,;v)$. Since the latter element is an atom, by (1), we conclude that equality actually holds in the previous equation.

In cylindric algebraic notation, the equation in (2) becomes:

$$u\,;v = \begin{cases} c_1(u)\cdot c_0(v) & \text{if} \quad c_0(u)\cdot d = c_1(v)\cdot d \,, \\ 0 & \text{if} \quad c_0(u)\cdot d \neq c_1(v)\cdot d \,. \end{cases}$$

Thus, (2) shows that, for atoms, relative product is definable in terms of the cylindric algebraic operations.

(3) For arbitrary elements x, y we have

$$x\,;y = \sum\{u\,;v : u, v \text{ atoms and } u \leq x\,,\ v \leq y\}\,.$$

Let w be an atom below $x\,;y$. We shall construct atoms u and v below x and y respectively such that

(4) $$w = u\,;v\,.$$

Since $x\,;y \neq 0$ (because $w \leq x\,;y$), we see that

$$[(1\,;x)\cdot 1']\cdot[(y\,;1)\cdot 1'] \neq 0$$

by 6.9(xiii). Because the BM^* with which we are dealing is assumed to be simple, and hence atomic, there is an atom z below $[(1\,;x)\cdot 1']\cdot[(y\,;1)\cdot 1']$. By 6.9(vii) we have

(5) $$z = (z\,;1)\cdot 1' = (1\,;z)\cdot 1'\,.$$

Set

(6) $$u = (w\,;1)\cdot(1\,;z) \qquad \text{and} \qquad v = (z\,;1)\cdot(1\,;w)\,.$$

Then

(7) $$u \quad , \quad v \quad , \quad (u\,;1)\cdot(1\,;v) \quad \text{are atoms}$$

by (1).

Now

$$\begin{aligned}(1\,;u)\cdot 1' &= (1\,;[(w\,;1)\cdot(1\,;z)])\cdot 1' && \text{by (6),}\\ &= [(1\,;w\,;1)\cdot(1\,;z)]\cdot 1' && \text{by 6.9(v),}\\ &= (1\,;z)\cdot 1' && \text{by simplicity and BA.}\\ &= z && \text{by (5).}\end{aligned}$$

Similarly, $(v\,;1)\cdot 1' = z$, so

(8) $$(1\,;u)\cdot 1' = (v\,;1)\cdot 1' = z\,.$$

Since z is non-zero, we see from (8), 6.9(xiii), and monotony that

$$0 < u\,;v \leq (u\,;1)\cdot(1\,;v)\,.$$

But $(u\,;1)\cdot(1\,;v)$ is an atom, by (7). Hence,

(9) $$u\,;v = (u\,;1)\cdot(1\,;v)\,.$$

Applying simplicity and 6.9(v), we get that

$$u\,;1 = [(w\,;1)\cdot(1\,;z)]\,;1 = (w\,;1)\cdot(1\,;z\,;1) = w\,;1\,.$$

Similarly,

$$1\,;v = 1\,;w\,.$$

Therefore, using also (9) and monotony, we obtain

$$0 < w \leq (w\,;1)\cdot(1\,;w) = (u\,;1)\cdot(1\,;v) = u\,;v\,.$$

But $u\,;v$ is an atom, by (7) and (9). Therefore, we arrive at (4).

Since the algebra under consideration is atomic, $x\,;y$ is the sum of the atoms below it. By (4), each such atom can be written in the form $u\,;v$ for some atoms u and v beneath x and y respectively. Therefore, $x\,;y$ is a sum of *some* elements of the form $u\,;v$ with u, v atoms, $u \leq x$, and $v \leq y$. But each such element $u\,;v$ is clearly below $x\,;y$ by monotony. Thus, $x\,;y$ is the sum of *all* such elements. This completes the proof of (3).

Each of $\mathfrak{A}$ and $\mathfrak{D}$ is a simple, rectangularly dense BM^*. Therefore, equations (2) and (3) apply to both algebras. In particular, in each algebra a relative product is the sum of relative products of certain atoms, and each such relative product of atoms is definable by the cylindric algebraic operations. Now f preserves arbitrary Boolean sums (by its very definition) and the cylindric algebraic operations. Hence, f must preserve relative products. ∎

Our proof of Lemma 6.12 has relied on the proof of Henkin and Tarski that an atomic CA_2 with rectangular atoms is representable. However, a direct proof of the lemma is easily given. In fact, all of the necessary ingredients are already present in our argument. Using (1) and some similar arguments, one first shows that

(10) f maps the set of atoms of $\mathfrak{A}$ one-one onto the set of atoms of $\mathfrak{D}$.

It follows that f preserves all of the Boolean operations. Also, f preserves 1', since $f(u) = (u, u)$ for every subidentity atom u. The next step is to show that

(11) f preserves the relative product of two atoms.

The proof uses (2) applied to both $\mathfrak{A}$ and $\mathfrak{D}$, (10), and the following equations:

$$[(U \times U)|f(u)] \cap Id_U = ((1\,;u)\cdot 1\text{'}, (1\,;u)\cdot 1\text{'}),$$
$$[f(u)|(U \times U)] \cap Id_U = ((u\,;1)\cdot 1\text{'}, (u\,;1)\cdot 1\text{'})$$

for every atom u of $\mathfrak{A}$. From these equations it follows that

$$[(U \times U)|f(u)] \cap Id_U = [f(v)|(U \times U)] \cap Id_U \quad \text{iff} \quad (1\,;u)\cdot 1\text{'} = (v\,;1)\cdot 1\text{'}$$

for atoms u, v of $\mathfrak{A}$. Finally, one uses (3) and the complete additivity of f to prove that f preserves arbitrary relative products.

Theorem 6.13. *A rectangularly dense, special Boolean monoid is representable.*

The proof is identical to the proof in the finite dimensional cylindric algebraic case of Theorem 3.11 except that we use the preceding lemma instead of the Henkin-Tarski representation theorem. We leave the details to the reader.

Using the previous result, one readily establishes the next theorem (see Mikulás [1995], p. 112).

Theorem 6.14. *Let*

$$\mathfrak{A} = \langle A\,,\ +\,,\ -\,,\ ;\,,\ {}^{\smile}\,,\ 1\text{'}\rangle$$

be an algebra of the same similarity type as relation algebras. Suppose that $\langle A\,,\ +\,,\ -\rangle$ *is a Boolean algebra and that laws* (R1)–(R6), (R8) *are valid in* $\mathfrak{A}$. *If* $\mathfrak{A}$ *is rectangularly dense, then it is isomorphic to a set relation algebra.*

In particular, it follows that in the presence of (R1)–(R6), the law (R8) (which can be viewed as a very special case of (R7)) and rectangular density together imply (R7).

Example 6.15. Axiom (R8) is necessary in the proof of Theorem 6.13. To show this, let V be the $\leq$ relation on the integers restricted to the set $U = \{0, 1, 2\}$. Let $\mathfrak{B}$ be the full set BM^* on U and $\mathfrak{A}$ the relativization of $\mathfrak{B}$ to V. Thus, the universe of $\mathfrak{A}$ consists of all subsets of V and the operations of $\mathfrak{A}$ are the relativizations to V of the (set-theoretic) operations of $\mathfrak{B}$. For example, for R, S in $\mathfrak{A}$ we define

$$R\,;^{\mathfrak{A}} S = (R|S) \cap V\,.$$

It is not hard to verify that (R1)–(R3) hold in $\mathfrak{A}$. Further, $\mathfrak{A}$ is rectangularly dense (in fact, it is point dense). But (R8) fails in $\mathfrak{A}$. For example, if $R = \{(0, 2)\}$, then

$$-(1\,;R) = V\sim [(V|R) \cap V] = V\sim \{(0, 2)\}$$

and

$$1\,;[-(1\,;R)] = \big(V\,|\,[V\sim \{(0, 2)\}]\big) \cap V = V\,.$$

Since (R8) holds in all set BM^*, it follows that $\mathfrak{A}$ is not representable. ∎

7. Some further open problems

The representability of algebras of logic that are rectangularly dense seems to be a general phenomenon, that is, it appears to hold for many different kinds of algebras in which we can formulate a notion of *rectangle*. Therefore, in our opinion there should be a characteristic cause for the representability that finds its expression in the methodology of the proof: there should be a "method" for building a representation directly from a densely ordered collection of rectangles of an algebra, a method that is independent of the choice of fundamental operations of the algebra.

Unfortunately, no such method of direct representation is currently known. Each of the representation theorems for rectangularly dense algebras in this paper has been established by a reduction to the Henkin-Tarski representation theorem for atomic cylindric algebras with rectangular atoms or by using the idea of the proof of that theorem. This reduction, as well as the proof of the Henkin-Tarski Theorem itself, requires the presence of diagonal elements (perhaps in some appropriate extension). Can this reduction to the atomic case, and in particular can the dependence on diagonal elements, be avoided?

Summarizing, the problem is to (1) find a general and direct method for constructing a representation of rectangularly dense algebras from the rectangles themselves, without using diagonal elements and without reducing the construction to the atomic case, and (2) explore the areas of applicability of this method.

As a test case for such a method, we propose the (still unsolved) problem of showing that rectangularly dense diagonal-free cylindric algebras are representable. A *diagonal-free cylindric algebra of dimension* α is an algebra

$$\mathfrak{A} = \langle A\,,\,+\,,\,-\,,\,c_i\rangle_{i\in\alpha}$$

such that $\langle A\,,\,+\,,\,-\rangle$ is a Boolean algebra and the c_i are unary operations satisfying postulates (C1)–(C4) (see Henkin-Monk-Tarski [1971], Definition 1.1.2). The definition of a rectangle and of rectangular density is the same as in the cylindric algebraic case. Since diagonal elements are not present and cannot be defined in appropriate extensions, it does not seem that the methods developed in the present paper can be used to solve the problem.[12]

Turning to another problem, suppose that $\mathfrak{A}$ is, e.g., a CA_α or an RA (or one of the other algebras of logic considered in this paper). We shall say that a subalgebra $\mathfrak{B}$ of $\mathfrak{A}$ is *dense in* $\mathfrak{A}$ if the set B is dense in $\mathfrak{A}$, i.e., if every non-zero element of $\mathfrak{A}$ is above a non-zero element of $\mathfrak{B}$. As was mentioned earlier, this is equivalent to the assumption that every element of $\mathfrak{A}$ is the sum of the elements of $\mathfrak{B}$ that are below it.

Problem 7.1. If $\mathfrak{B}$ is dense in $\mathfrak{A}$ and representable, is $\mathfrak{A}$ necessarily representable?

[12] A proof that rectangularly dense, diagonal-free cylindric algebras are representable was recently obtained by Venema [1996]. A different proof was found a few months later by Andréka-Givant, who also obtained a positive solution to the general methodological problem. Venema's approach is to build a representation using a step-by-step construction. In contrast to this, the approach of Andréka-Givant is to use the Boolean algebras of cylinders to construct directly a representation. (See Example 4.12 for another application of this idea.)

A positive solution to Problem 7.1 would provide us with a generalization of Theorem 3.11, the representation theorem for rectangularly dense CA_α, at least in the finite dimensional case. To see this, suppose that $\mathfrak{A}$ is a rectangularly dense CA_α with $2 \leq \alpha < \omega$. Let $\mathfrak{B}$ be the subalgebra of $\mathfrak{A}$ generated by the set of *monadic elements*, i.e., by the set

$$\{x \in A : |\Delta x| \leq 1\}.$$

Notice that, by Lemma 3.7, every rectangle of $\mathfrak{A}$ is in $\mathfrak{B}$. Since the set of rectangles is assumed to be dense in $\mathfrak{A}$, the algebra $\mathfrak{B}$ must be dense in $\mathfrak{A}$. Now Monk [1964a] proved that every monadically generated CA_α is representable. Thus, $\mathfrak{B}$ is representable and dense in $\mathfrak{A}$. Applying the assumed positive solution to Problem 7.1, we conclude that $\mathfrak{A}$ is representable.

Interestingly, for classes K of algebras with completely additive extra-Boolean operations and such that K is closed under completions (i.e., the completion of each algebra in K is again in K), Problem 7.1 is equivalent to a problem formulated in Section 2 of Monk [1970]:

Problem 7.2. Is the completion of a representable algebra necessarily representable?[13]

Indeed, suppose that Problem 7.2 has a positive solution. Let $\mathfrak{B}$ be a dense subalgebra of $\mathfrak{A}$ that is representable. Take $\mathfrak{C}$ to be the completion of $\mathfrak{A}$. Since $\mathfrak{B}$ is dense in $\mathfrak{A}$, it is dense in $\mathfrak{C}$. From this it is easy to check that $\mathfrak{C}$ must be the completion of $\mathfrak{B}$ (use conditions (ii) and (iii) of the definition of a completion, given after Theorem 4.6, and the complete additivity of the extra-Boolean operations). Because $\mathfrak{B}$ is representable, we conclude from our assumption that $\mathfrak{C}$ is representable. Therefore $\mathfrak{A}$ is also representable.

Now suppose that Problem 7.1 has a positive solution. Let $\mathfrak{B}$ be a representable algebra and $\mathfrak{A}$ the completion of $\mathfrak{B}$. It is easy to check (using condition (ii) of the definition of a completion) that $\mathfrak{B}$ is dense in $\mathfrak{A}$. Hence, by our assumption, the representability of $\mathfrak{B}$ implies that of $\mathfrak{A}$.

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[13] Quite recently, Hodkinson [1997] has given, in Corollary 8.10, an example of two relation algebras, one representable, the other not, that have the same atom structure. Using this example, it is not difficult to conclude that there must a representable relation algebra whose completion is not representable. Thus, Problems 7.1 and 7.2 both have negative solutions.

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