THE EQUATIONAL THEORIES OF REPRESENTABLE POSITIVE CYLINDRIC AND RELATION ALGEBRAS ARE DECIDABLE.

First draft.

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THEOREM. *(i) The equational theory of* $Cs_n^+ = \{\langle \mathcal{P}({}^nU), \cup, \cap, \emptyset, {}^nU, C_i, D_{ij}\rangle_{i,j<n} :$ *U is a set*$\}$ *is decidable for any ordinal n.*

(ii) The equational theory of $Rs^+ = \{\langle \mathcal{P}(U \times U), \cup, \cap, \emptyset, U \times U, \circ, Id, \breve{}\ \rangle :$ *U is a set*$\}$ *is decidable.*

Proof: Let $n < \omega$. To any Cs_n^+–term τ we will associate an existential formula $\exists\overline{w}\psi$ with free variables $v_0, ..., v_{n-1}$, such that $\overline{v}$ is disjoint from $\overline{w}$ and ψ is quantifier–free, and further

(*) the meanings of τ and $\exists\overline{w}\psi$ are the same.

By this latter we mean the following. Assume that the variables occurring in τ are $x_1, ..., x_m \in X$. In our first–order language we will consider the elements $x \in X$ to be n–ary relation symbols. Let $\mathfrak{A} \in Cs_n^+$ with $A \subseteq \mathcal{P}({}^nU)$, let $k : X \longrightarrow A$ be an evaluation of the variables and let $\overline{u} = \langle u_0, ..., u_{n-1}\rangle \in {}^nU$. Then $\tau^{\mathfrak{A}}[k]$ denotes the n–ary relation on U (by evaluating the variables in τ along k), and $(U, k(x))_{x\in X}$ denotes the first–order model with universe U where the relation symbol $x \in X$ is realized by $k(x) \subseteq {}^nU$. Now (*) means that for any $\mathfrak{A}, k$ and $\overline{u}$ we have

$$\langle u_0, ..., u_{n-1}\rangle \in \tau^{\mathfrak{A}}[k] \quad \text{iff} \quad (U, k(x))_{x\in X} \models \exists\overline{w}\psi[u_0, ..., u_{n-1}].$$

Let $trans(\tau)$ denote the first–order formula associated to τ. We now define $trans$ by induction on τ (we will be a little sloppy in doing so): For $x \in X$ let $trans(x) = x(v_0, ..., v_{n-1})$. Assume that $trans(\tau) = \exists\overline{w}\psi$, $trans(\sigma) = \exists\overline{z}\varphi$. We may assume (by sloppyness) that $\overline{w}$ and $\overline{z}$ are disjoint. Now

$$trans(\tau + \sigma) = \exists\overline{w}\exists\overline{z}(\psi \vee \varphi)$$
$$trans(\tau \cdot \sigma) = \exists\overline{w}\exists\overline{z}(\psi \wedge \varphi)$$
$$trans(0) = (v_0 \neq v_0)$$
$$trans(1) = (v_0 = v_0)$$
$$trans(c_i\tau) = \exists v_j\exists\overline{w}\psi(v_i/v_j) \quad \text{where } v_j \text{ is a new variable}$$
$$trans(d_{ij}) = (v_i = v_j)$$

Now, $Cs_n^+ \not\models \tau = \sigma$ iff ($Cs_n^+ \not\models \tau \leq \sigma$ or $Cs_n^+ \not\models \sigma \leq \tau$). Thus, it is enough to decide whether $Cs_n^+ \models \tau \leq \sigma$. Let $trans(\tau) = \exists\overline{w}\psi$, $trans(\sigma) = \exists\overline{z}\varphi$ with $\overline{w}$ and $\overline{z}$ disjoint. Then $Cs_n^+ \not\models \tau \leq \sigma$ iff $\exists\overline{v}(\exists\overline{w}\psi \wedge \neg\exists\overline{z}\varphi)$ is satisfiable, but the latter formula is equivalent to the $\exists\forall$–formula $\exists\overline{v}\exists\overline{w}\forall\overline{z}(\psi \wedge \neg\varphi)$ and satisfiability of $\exists\forall$–formulas is decidable.

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The case of $n \geq \omega$ follows from the result in [HMTII] saying that $\langle \mathbf{SP}Cs_\alpha : \alpha \in Ord \rangle$ is a strong system of varieties definable by a schema (i.e. that for an equation e in the language of Cs_3 we have $Cs_3 \models e$ iff $Cs_\omega \models e$).

The relational algebraic case is completely analogous, therefore we omit its proof. **QED**.

REMARK. (i) The above proof fails if dual cylindrifications c_i^∂ $(=-c_i-)$ are added. (Dual cylindrifications are the algebraic counterparts of universal quantifiers.)

(ii) If we omit intersection but add transitive reflexive closure * , then we obtain the class of Kleene relation algebras. It is proved in Fischer–Ladner[77], that the equational theory of this class, i.e. that the equational theory of $\{\langle \mathcal{P}(U \times U), \cup, \emptyset, U \times U, \circ, Id, \breve{}, ^* \rangle : U \text{ is a set}\}$ is decidable. It would be interesting to know what the situation is if we allow both intersection and transitive closure.

PROBLEM 1. *Is the equational theory $Eq\mathcal{K}$ of $\mathcal{K} = \{\langle \mathcal{P}(U \times U), \cup, \cap, \emptyset, U \times U, \circ, Id, \breve{}, ^* \rangle :$ U is a set$\}$ decidable? Is $Eq\mathcal{K}$ the same as the equational theory of $\{\mathfrak{A} : \mathfrak{A}$ is finite, $\mathfrak{A} \subseteq \mathfrak{B}$ for some $\mathfrak{B} \in \mathcal{K}\}$?*

PROBLEM 2. *Let Cs_n^{++} be Cs_n^+ expanded with $\{c_i^\partial : i < n\}$. Is the equational theory $Eq(Cs_n^{++})$ decidable (for $n > 2$)?*

References

HMTII. Henkin,L. Monk.J.D. and Tarski,A., *Cylindric Algebras. Part II.* North-Holland, Amsterdam, 1985.

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